Discrete Mathematics

SETS, FUNCTIONS, SEQUENCES & SUMS

Basic Structures

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Department of Mathematics

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Competency Goals

1. Identify and apply basic terminologies in set theory, including finding the power set and Cartesian product, and representing subsets using binary notation.
2. Perform set operations and verify the equality of two sets.
3. Determine whether a given rule defines a function, execute operations on functions, and manipulate functions involving floor and ceiling operations.
4. Analyze whether a function is injective, surjective, or bijective, and explain how these properties are used to compare sets.
5. Derive a formula for a sequence and evaluate finite sums, incorporating special summation techniques where applicable.

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What's next?

Introduction to Sets

- A **set** is an unordered collection of objects.
- An object of a set is called an **element** or a **member**, of that set.
- The **cardinality** of the set A is the number of distinct elements of A, denoted by $|A|$.
- The **empty set**, denoted by $\emptyset$, is the set whose cardinality is 0.

Example.

1. The set $\{a, cat, catches, a, mouse\}$ has 4 elements.
2. The set $\{a, b, c, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \emptyset\}$ has 8 elements.

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Subsets

- If x is an element of A, we write $x \in A$. Otherwise, we write $x \notin A$.
- If all elements of A are also elements of B, we write $A \subseteq B$, and A is called a **subset** of B.
- If A is a **proper subset** of B, meaning $A \subseteq B$ and $A \neq B$, we write $A \subset B$.
- The empty set $\emptyset$ is a subset of any set; and the set A is a subset of itself, $A \subseteq A$.

Example. Which of the following statements are true?

1. $x \in \{x\}$	True		5. $\emptyset \in \{\emptyset\}$	True
2. $x \subseteq \{x\}$	False		6. $\emptyset \subset \{\emptyset\}$	True
3. $\{a, b\} \subseteq \{a, b, \{a, b\}, c\}$	True		7. $\{a, b, c\} \subseteq \{a, b, c\}$	True
4. $\{a, b\} \in \{a, b, \{a, b\}, c\}$	True		8. $\{a, b, c\} \in \{a, b, c\}$	False

Cartesian Product

The **Cartesian product** of two sets A and B, denoted $A \times B$, is the set of all ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

In general, the Cartesian product of n sets $A_1, A_2, \ldots, A_n$ is defined as

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in A_i, i = \overline{1, n}\}.$$

Example. Let $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Then,

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

Question. Given $A_1 = \{a, b\}, A_2 = \{1, 2, 3\}$ and $A_3 = \{x, y\}$. Determine all elements of the set $A_1 \times A_2 \times A_3$.

Power Sets

The **power set** of the set A, denoted by $P(A)$, is the set of all subsets of A.

Example. Let $S = \{a, b, c\}$. Then

$$P(S) = \{a, b, c, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}, \emptyset\}.$$

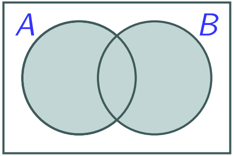
Theorem

1. If $|A| = m$ and $|B| = n$, then $|A \times B| = m \times n$.
2. If $|A_1| = k_1, |A_2| = k_2, ..., |A_n| = k_n$, then $|A_1 \times A_2 \times \cdots \times A_n| = k_1 \times k_2 \times \cdots \times k_n$.
3. If $|A| = n$, then $|P(A)| = 2^n$.

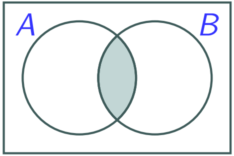
Set Operations

1. **Union** of A and B is
$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$
2. **Intersection** of A and B is
$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$
3. **Difference** of A and B is
$$A - B = \{x \mid (x \in A) \wedge (x \notin B)\}.$$
4. **Symmetric difference** of A and B is
$$A \oplus B = \{x \mid (x \in A) \oplus (x \in B)\}$$
$$A \oplus B = \{x \mid (x \in A \cup B) \wedge (x \notin A \cap B)\}.$$
5. **Complement** of A with respect to the universal set U is
$$\bar{A} = U - A = \{x \mid (x \in U) \wedge (x \notin A)\}.$$

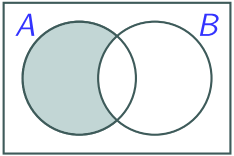
Set Operations: Venn Diagram



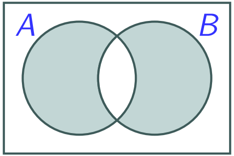
$A \cup B$



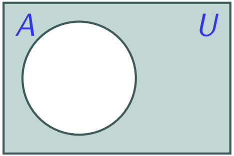
$A \cap B$



$A - B$



$A \oplus B$



$\bar{A}$

Set Identities

- Complementation law

$$\overline{\overline{A}} = A$$

- Identity laws

$$A \cup \emptyset = A, A \cap U = A$$

- Domination laws

$$A \cup U = U, \quad A \cap \emptyset = \emptyset$$

- Complement laws

$$A \cup \overline{A} = U, \quad A \cap \overline{A} = \emptyset$$

Set Identities (cont')

- Idempotent laws

$$A \cup A = A, \quad A \cap A = A$$

- Commutative laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- Associative laws

$$(A \cup B) \cup C = A \cup (B \cup C)$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

Set Identities (cont')

- Distributive laws

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

- Absorption laws

$$A \cup (A \cap B) = A$$

$$A \cap (A \cup B) = A$$

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B},$$

$$\overline{A \cap B} = \overline{A} \cup \overline{B}$$

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Computer Representation of Sets

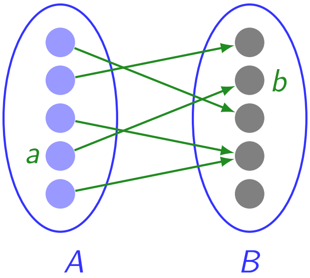
Let U be a universal set. Fix an ordering of elements of U as $a_1, a_2, \ldots, a_n$. If A is a subset of U, represent A with a bit string of length n, where the ith bit is 1 if $a_i \in A$ and 0 if $a_i \notin A$.

Example. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

- The subset $A = \{1, 3, 5, 7, 9\}$ is represented as the bit string 1010101010.
- The subset $B = \{1, 8, 9\}$ is represented as the bit string 1000000110.
- We have
$$A \cup B = 1010101010 \vee 1000000110 = 1010101110$$
which implies $A \cup B = \{1, 3, 5, 7, 8, 9\}$.
- We have
$$A \cap B = 1010101010 \wedge 1000000110 = 1000000010$$
which implies $A \cap B = \{1, 9\}$.

What's next?

Functions



- A **function** f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.
- The set A is called **domain** and B is called **codomain** of f.
- If $f(a) = b$, we say b is the **image** of a and a is a **preimage** of b.

Let S be a subset of A. The set $f(S) = \{b \in B \mid \exists a \in A,\ f(a) = b\}$ is called the **image of** S, and the set $f^{-1}(S) = \{a \in A \mid f(a) \in S\}$ is called the **preimage of** S. The set $f(A)$ is called the **range** of f.

Some Important Functions

In mathematics, computer science, and engineering, particularly when discretizing real numbers, implementing algorithms, or working with integer-valued indices, there are two functions which are widely used:

- Floor Function
- Ceil Function

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Floor Function

Given a real number x. **Floor function** is the function that gives as output the greatest integer less than or equal to x, denoted $\lfloor x \rfloor$ or **floor**(x).

$$\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}.$$

In other words, it rounds x down to the nearest integer.

Examples.

- $\lfloor 3.7 \rfloor = 3$
- $\lfloor -2.3 \rfloor = -3$
- $\lfloor 5 \rfloor = 5$ (since 5 is already an integer)

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Ceil Function

Given a real number x. **Ceil function** is the function that maps to the least integer greater than or equal to x, denoted $\lceil x \rceil$ or $\mathbf{ceil}(x)$.

$$\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geq x\}.$$

In other words, it rounds x up to the nearest integer.

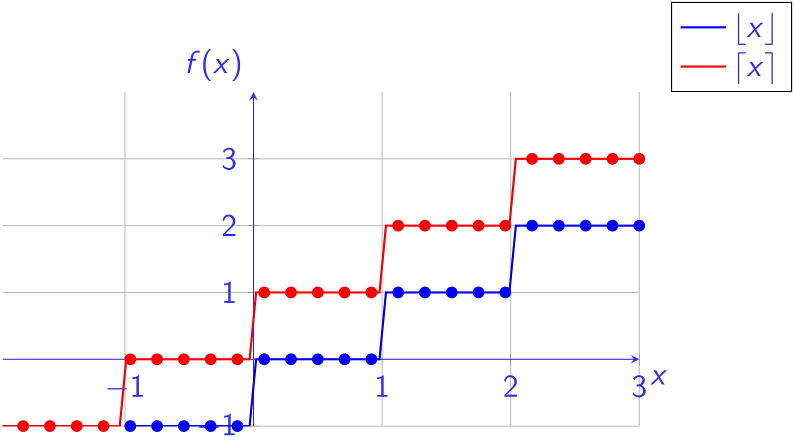
Examples.

- $\lceil 3.2 \rceil = 4$: Since 4 is the smallest integer $\geq$ 3.2.
- $\lceil -1.7 \rceil = -1$: Since -1 is the smallest integer $\geq$ -1.7.
- $\lceil 5 \rceil = 5$: If x is already an integer, the ceiling function returns x itself.

Properties of Floor and Ceil Functions

1. For any real number x,
$$\lceil x \rceil = -\lfloor -x \rfloor$$
2. For any real number x,
$$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1.$$
3. The floor and ceiling function are non-decreasing function; that is, if $x \le y$, then
   - $\lfloor x \rfloor \le \lfloor y \rfloor$.
   - $\lceil x \rceil \le \lceil y \rceil$.

Graphs of Floor and Ceil Functions



Question

Which statements are true for all real numbers x, y and all integers n?

- $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$
- $\lfloor x + n \rfloor = \lfloor x \rfloor + \lfloor n \rfloor$
- $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$
- $\lceil x + n \rceil = \lceil x \rceil + \lceil n \rceil$

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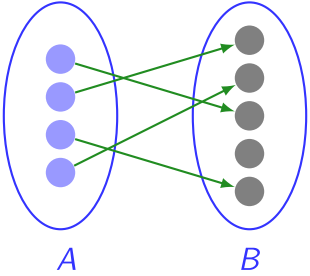
One-to-one

The function $f : A \to B$ is **one-to-one** if $f(a_1) \neq f(a_2)$ for all $a_1 \neq a_2$ in A.



- A function $f : A \to B$ is said to be one-to-one if $f(x_1) = f(x_2) \implies x_1 = x_2$ for all $x_1, x_2 \in A$.
- A one-to-one function is also called an **injection**.
- We call a function **injective** if it is one-to-one.

Note. A function that is NOT one-to-one is referred to as **many-to-one**.

Problem Solving: One-to-one or Not?

Prove a Function is One-to-one

To conclude that the function $f : A \to B$ is one-to-one, we proceed as follows:

1. Assume $f(x_1) = f(x_2)$.
2. Show that it must be true that $x_1 = x_2$.

Prove a Function is not One-to-one

To conclude that the function $f : A \to B$ is not one-to-one, we take a counterexample where $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

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Question

Example. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x - 1$ is one-to-one. Indeed, assume that $f(x_1) = f(x_2)$ which means

$$2x_1 - 1 = 2x_2 - 1.$$

Therefore, $x_1 = x_2$. Hence, the function f is one-to-one.

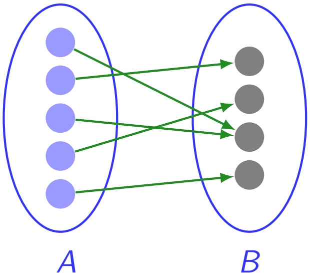
Question. Which functions are one-to-one?

1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$
2. $f : \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = x^2.$
3. $f : \mathbb{Z} \to \mathbb{Z}, \quad f(n) = \left\lfloor \dfrac{n+1}{2} \right\rfloor.$
4. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad f(m, n) = m + n.$
5. $f : \mathbb{Z} \to \mathbb{Z},$

$$f(n) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

Onto

The function $f : A \to B$ is **onto** if for each b in B, there is a in A such that $f(a) = b$. In other words, the function $f : A \to B$ is onto if $f(A) = B$.



Note. An onto function is also called **surjection**, and we say it is **surjective**.

Problem Solving: Onto or Not?

Prove a Function is Onto

To conclude that the function $f : A \to B$ is onto, we proceed as follows:

1. Let y be any element in the codomain B.
2. Figure out an element in the domain A that is a preimage of y.
3. Choose x equal to the value you found.
4. Demonstrate that x is indeed an element of the domain A.
5. Show that $f(x) = y$.

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Example

Example. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 5x + 1$. Show that f is onto.
Solution. Let y be any element of $\mathbb{R}$. Choose $x = \frac{y-1}{5}$. It is easy to see that the real numbers are closed under subtraction and non-zero division, i.e., $x \in \mathbb{R}$. Also, $f(x) = f(\frac{y-1}{5}) = 5.\frac{y-1}{5} + 1 = y$.

Therefore, we found an $x \in \mathbb{R}$ such that $f(x) = y$. In other words, given an arbitrary element of the codomain, we have shown a preimage in the domain. We conclude that f is onto.

Questions

1. Check whether the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -n & \text{if } n < 0 \end{cases}$$

   is one-to-one or onto?

2. Which of the following functions are onto?

   1. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2.$
   2. $f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3.$
   3. $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = 2\lfloor x \rfloor.$
   4. $f : \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lfloor 2x \rfloor.$
   5. $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad f(m, n) = m + n.$

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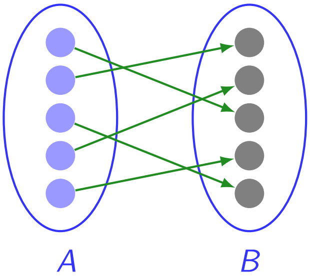
Bijection

The function $f : A \to B$ is a **bijection** (or one-to-one correspondence) if it is both one-to-one and onto.



Note. If a function is a bijection, we say that it is **bijective**.

Problem Solving: Bijection or Not?

Prove a Function is Bijective

To conclude that the function $f : A \to B$ is bijective, we proceed as follows:

1. f is injective.
2. f is surjective.

Example. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$ is bijective.

Solution. We proceed as follows:

1. Show that f is injective.
   Assume that $f(x_1) = f(x_2)$. Then $3x_1 - 5 = 3x_2 - 5$ which implies $x_1 = x_2$.
2. Show that f is surjective.
   Let $y \in \mathbb{R}$ be an arbitrary. Choose $x = \frac{y+5}{3} \in \mathbb{R}$. Then

$$f(x) = f(\frac{y+5}{3}) = 3 \cdot \frac{y+5}{3} - 5 = y.$$

Questions

1. Which of the following functions are bijection?

   1. $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^2$.
   2. $f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$.
   3. $f: \mathbb{R} \to \mathbb{Z}, \quad f(x) = \lfloor 2x \rfloor$.
   4. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad f(m, n) = m + n$.
   5. $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}, \quad f(m, n) = (m, m + n)$.

2. Does there exist

   1. a bijection/ one-to-one/ onto function from a set of 7 elements to a set of 5 elements? from a set of 5 elements to a set of 7 elements?
   2. a bijection from the set of even integers to the set of odd integers?
   3. a bijection from the set of odd inetegers to the set of all integers?
   4. a bijection from the set of all real numbers to the set of positive real numbers?

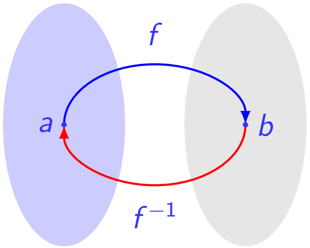
What's next?

Let $f : A \to B$ be a bijective function. Then, its **inverse function** is the function

$$\begin{aligned} f^{-1} : B &\to A \\ b &\mapsto f^{-1}(b) = a. \end{aligned}$$

Note. In the above definition,

$$f^{-1}(b) = a \quad \text{equivalent} \quad b = f(a).$$



Find f^{-1}

We can find the inverse function f^{-1} by following these steps:

1. Check if the function f is a bijective function.
   - If f is not a bijective function, stop, f^{-1} does not exist.
   - If f is a bijective function, we continue.
2. Since $f^{-1}(y) = x \Leftrightarrow y = f(x)$, then we solve for x and express x in terms of y.
3. The resulting expression is $f^{-1}(y)$.

Example. Find the inverse function of the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x - 5$.

Solution. Since f is a bijective function, there exists f^{-1}. Put $y = 3x - 5$ which implies $x = \frac{y+5}{3}$. Therefore, $f^{-1} : \mathbb{R} \to \mathbb{R}, \quad y \mapsto f^{-1}(y) = \frac{y+5}{3}$.

Question

Find the inverse functions of the following functions:

1. $f: [-3, +\infty) \to [0, +\infty), \quad f(x) = \sqrt{x+3}.$
2. $g: \mathbb{R} \to (0, +\infty), \quad g(x) = e^x.$
3. $h: \mathbb{R} \to \mathbb{R}, \quad h(x) = \begin{cases} 3x & \text{if } x \leq 1 \\ 2x+1 & \text{if } x > 1. \end{cases}$
4. $k: [-1, 1] \to [-2, 2], \quad k(x) = x^3 - 3x$

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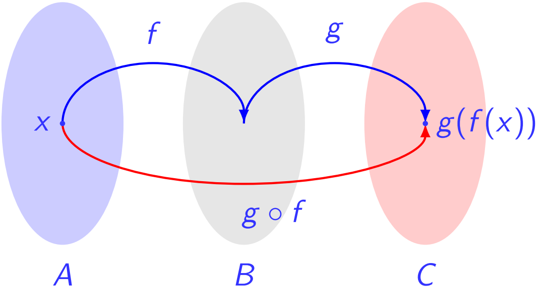
Composite function

Let $f : A \to B_1$ and $g : B \to C$ where $B_1 \subseteq B$. Then, the **composite function**, denoted as $g \circ f$, is defined by

$$\begin{aligned} g \circ f : A &\to C \\ x &\mapsto (g \circ f)(x) = g(f(x)). \end{aligned}$$



Example

Example. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = x^{2023}$ and $g(x) = 2023x + 1$. Find $g \circ f$ and $f \circ g$.

Solution. We have

$$g \circ f : \mathbb{R} \to \mathbb{R}, (g \circ f)(x) = g(f(x)) = 2023f(x) + 1 = 2023x^{2023} + 1,$$

$$f \circ g : \mathbb{R} \to \mathbb{R}, (f \circ g)(x) = f(g(x)) = [g(x)]^{2023} = (2023x + 1)^{2023}.$$

Question. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 2x + 1, & \text{if } x \geq 0 \\ 1 - 2x, & \text{otherwise;} \end{cases} \quad \text{and } g(x) = x - 3.$$

Find $f \circ g$ and $g \circ f$.

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What's next?

Sequences

A **sequence** is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \ldots\}$ or the set $\{1, 2, 3, \ldots\}$) to a set S. We use the notation a_n to denote the image of the integer n. We call a_n a **term** of the sequence.

Example. Consider the sequence $\{a_n\}$, where $a_n = \dfrac{1}{n}$. The list of the terms of this sequence, beginning with a_1, namely,

$$a_1, a_2, a_3, a_4, \ldots,$$

starts with

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

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Geometric & Arithmetic Progressions

A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the initial term a and the common ratio r are real numbers.

An **arithmetic progression** is a sequence of the form

$$a, a+d, a+2d, \ldots, a+nd, \ldots$$

where the initial term a and the common difference d are real numbers.

Example.

- *Geometric progression.* $1, 2, 4, 8, \ldots$
- *Arithmetic progression.* $1, 3, 5, 7, \ldots$

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Questions

1. In the given AP series find the number of terms?

$$5, 8, 11, 14, 17, 20, \ldots, 50.$$

2. In the given AP series the term at position 11 would be?

$$5, 8, 11, 14, 17, 20, \ldots, 50.$$

3. For the given Arithmetic progression find the position of first negative term?

$$50, 47, 44, 41, \ldots$$

Questions (cont')

1. In the given Geometric progression find the number of terms.

$$32, 256, 2048, 16384, \ldots, 2^{50}.$$

2. In the given Geometric progression the term at position 11 would be ... .

$$32, 256, 2048, 16384, \ldots, 2^{50}.$$

3. For the given Geometric progression find the position of first fractional term?

$$2^{50}, 2^{47}, 2^{44}, \ldots$$

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Recurrence relation

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.
A sequence is called a **solution of a recurrence relation** if its terms satisfy the recurrence relation.

Example. The **Fibonacci sequence**, $f_0, f_1, f_2, \ldots$, is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2} \text{ for } n = 2, 3, 4, \ldots.$$

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Practical Questions

1. Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.
   - $a_n = -2a_{n-1},\ a_0 = -1$
   - $a_n = a_{n-1} - a_{n-2},\ a_0 = 2,\ a_1 = -1$
   - $a_n = 3a_{n-1}^2 - 1,\ a_0 = 1$
   - $a_n = na_{n-1} + a_{n-2}^2,\ a_0 = -1,\ a_1 = 0$
   - $a_n = a_{n-1} - a_{n-2} + a_{n-3},\ a_0 = 1,\ a_1 = 1,\ a_2 = 2$
2. Let a_n be the nth term of the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, \ldots,$$

   constructed by including the integer k exactly k times. Show that $a_n = \left\lfloor \sqrt{2n} + \frac{1}{2} \right\rfloor$.
3. Find a general formula for a_n of each sequence:
   - $1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, \ldots$
   - $1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, \ldots$
   - $1, 0, 2, 0, 4, 0, 8, 0, 16, 0, \ldots$
   - $1, 10, 11, 100, 101, \ldots$

What's next?

Summations

To **express the sum** of the terms $a_m, a_{m+1}, \ldots, a_n$ from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^{n} a_j, \text{ or } \sum_{m \leq j \leq n} a_j.$$

Note. Here, the variable j is called the **index of summation**, and the choice of the letter j as the variable is arbitrary.

$$\sum_{j=m}^{n} a_j = \sum_{i=m}^{n} a_i = \sum_{k=m}^{n} a_k$$

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Summations in real-life

Suppose you have a savings account that earns a fixed annual interest rate $r\%$, and you deposit a fixed amount D every year. Calculate the total amount of interest earned after several years n.

Details:

- You deposit \$1,000 at the end of each year.
- The annual interest rate is 5%.
- You want to find the total interest earned after 10 years.

Solution Using Summation:

$$I = D\sum_{i=1}^{n}\left[(1+r)^{n-i} - 1\right]$$

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Illustration of infinite sum

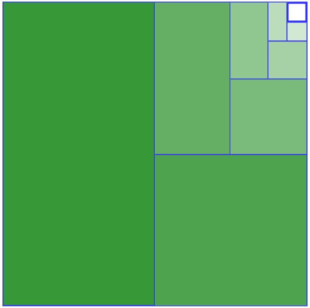

Figure: $\dfrac{1}{2} + \dfrac{1}{2^2} + \dfrac{1}{2^3} + \cdots$

Some Useful Summation Formulae

- $\sum_{i=1}^{10} i a_i = a_1 + 2a_2 + \cdots + 10a_{10}$
- $\sum_{i=1}^{10} a_{2n}^2 = a_2^2 + a_4^2 + \cdots + a_{10}^2$
- $\sum_{i=1}^{10} a_{2i-1} = a_1 + a_3 + \cdots + a_{19}$
- $\sum_{i=1}^{10} 1 = \underbrace{1 + 1 + \cdots + 1}_{\text{10 terms}}$

- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r} \quad r \neq 1$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}, \quad |x| < 1$
- $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$
- $\sum_{i=1}^{\infty} i x^{i-1} = \frac{1}{(x-1)^2}, \quad |x| < 1$

Properties of Summations

1. *Linearity:*

$$\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i, \quad \sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i$$

2. *Index Shift:*

$$\sum_{i=m}^{n} a_i = \sum_{i=m+k}^{n+k} a_{i-k}$$

3. *Splitting a Summation:*

$$\sum_{i=m}^{n} a_i = \sum_{i=m}^{k} a_i + \sum_{i=k+1}^{n} a_i$$

4. *Telescoping Series:*

$$\sum_{i=1}^{n}(b_i - b_{i+1}) = b_1 - b_{n+1}$$

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Questions

1. What are the values of these sums?
   - $\displaystyle\sum_{i=1}^{5}(2i + i^2)$
   - $\displaystyle\sum_{i=1}^{5}\frac{1}{i(i+1)}$
   - $\displaystyle\sum_{i=1}^{5}\frac{3^i}{4^{i+1}}$
   - $\displaystyle\sum_{i=1}^{5}(2i - 1)$
2. Compute each of these double sums.
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5}(2i + j)$
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5}(i - j)$
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5} ij$
   - $\displaystyle\sum_{i=1}^{5}\sum_{j=1}^{5} i^2 j^3$
3. Find a formula for $\displaystyle\sum_{k=1}^{n}\left\lfloor\sqrt{k}\right\rfloor$.

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