Discrete Mathematics

GRAPH THEORY

A brief introduction

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Department of Mathematics

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Competency Goals

1. Analyze and classify the structure of a graph, apply the handshaking theorem, and identify special types of graphs such as bipartite graphs.
2. Represent graphs using incidence and adjacency matrices, and determine if two graphs are isomorphic.
3. Assess the connectivity of a graph, identify its connected components, and count the number of paths within the graph.
4. Determine the existence of Eulerian and Hamiltonian paths or circuits in a graph, and find these paths or circuits when they exist.
5. Apply Dijkstra's algorithm to find the shortest paths and distances in a weighted graph.

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4. Connectivity
5. Euler and Hamilton Paths
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What's next?

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Graphs and Graph Models

A **graph** $G = (V, E)$ consists of V a non-empty set of **vertices** (or nodes) and E, a set of **edges**. Each edge has one or two vertices associated with it, called **endpoints**. An edge is said to **connect** its endpoints.

Note.

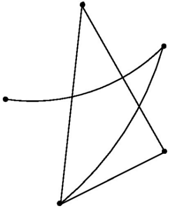
1. The set of vertices V of a graph G may be **infinite**.
2. A graph with an infinite vertex set or an infinite number of edges is called an **infinite graph**.
3. A graph with a finite vertex set and a finite edge set is called a **finite graph**.
4. In this course, we will consider only finite graphs.

Undirected graphs

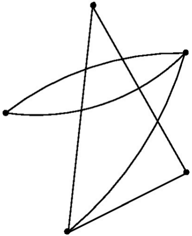
Graph classification

1. **Undirected graphs**: Simple graphs, Multigraphs and Pseudographs.
2. **Directed graphs**: Simple directed graphs, Directed multigraphs.

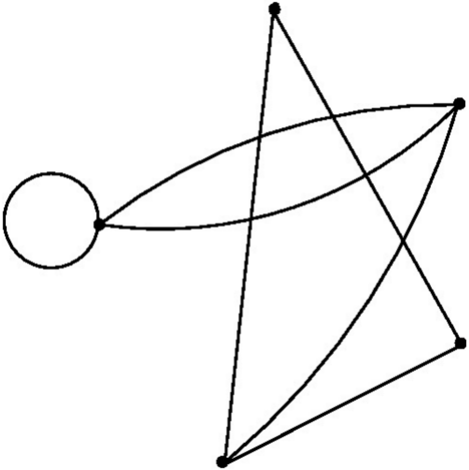
Simple graphs: For any two vertices there is at most one edge connecting them, and there are no loops.



Multigraphs: There are possibly **multiple edges**, but no loops.



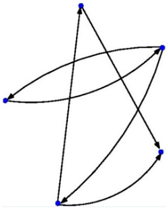
Pseudographs: There are possibly multiple edges and loops.



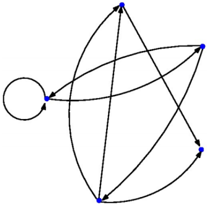
Directed Graphs

A **directed graph** (or digraph) (V, E) consists of a nonempty set of vertices V and a set of **directed edges** (or arcs) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to start at u and end at v.

Simple directed graphs: There are no loops and no multiple directed edges.



Directed multigraphs: There are possibly multiple directed edges and loops.



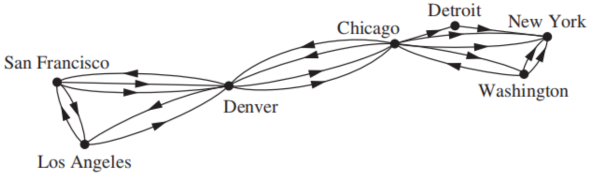


Figure. A Computer Network with Multiple One-Way Links.

TABLE 1 Graph Terminology.

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes

Note. A graph with both directed and undirected edges is called a **mixed graph**.

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Graph Terminologies

- In an undirected graph, two vertices u and v are called **adjacent** (or neighbors) if they are endpoints of an edge e, and e is called **incident** with u and v.
- The **degree of a vertex** in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.

The Handshaking Theorem for Undirected Graphs

Let $G = (V, E)$ be an undirected graph with e edges. Then,

$$\sum_{v \in V} \deg(v) = 2e.$$

Therefore, the number of vertices of odd degrees is an even number.

Note. A vertex of degree zero is called **isolated**. A vertex is **pendant** if and only if it has degree one.

Example and Question

Example. How many edges are there in a graph with 10 vertices each of degree six?

Solution. Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2e = 60$ where e is the number of edges. Therefore, $e = 30$.

Question. Do there exist simple graphs with 5 vertices of degrees:

1. $1, 2, 3, 3, 4$.
2. $1, 2, 3, 3, 3$.
3. $1, 2, 3, 4, 4$.

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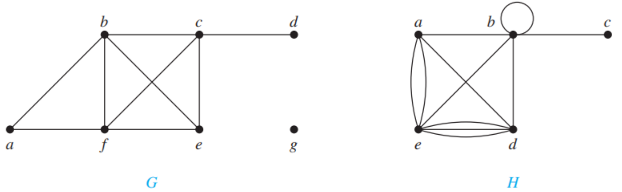
Neighbors of vertices

What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in the following figures?



Solution.

- In G, the neighborhoods of these vertices are $N(a) = \{b, f\}$, $N(b) = \{a, c, e, f\}$, $N(c) = \{b, d, e, f\}$, $N(d) = \{c\}$, $N(e) = \{b, c, f\}$, $N(f) = \{a, b, c, e\}$, and $N(g) = \varnothing$.
- In H, the neighborhoods of these vertices are $N(a) = \{b, d, e\}$, $N(b) = \{a, b, c, d, e\}$, $N(c) = \{b\}$, $N(d) = \{a, b, e\}$, and $N(e) = \{a, b, d\}$.

In-degree & Out-degree

- If $e = (u, v)$ is an edge of a directed graph, u is said to be **adjacent to** v and v is **adjacent from** u. The vertex u is called the **initial** and v is called the **terminal** or **end** vertex of e.
- The **in-degree** of a vertex v in a directed graph, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The **out-degree** of u, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex.

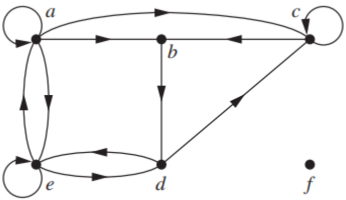
Note. A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

The Handshaking Theorem for Directed Graphs

Let $G = (V, E)$ be a graph with directed edges. Then,

$$\sum_{v \in V} \deg^+(v) = \sum_{v \in V} \deg^-(v) = |E|.$$

Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in the figure as follows



G

Solution.

- The in-degrees in G are

$$\deg^-(a) = 2,\ \deg^-(b) = 2,\ \deg^-(c) = 3,$$
$$\deg^-(d) = 2,\ \deg^-(e) = 3,\ \deg^-(f) = 0.$$

- The out-degrees in G are

$$\deg^+(a) = 4,\ \deg^+(b) = 1,\ \deg^+(c) = 2,\ \deg^+(d) = 2,\ \deg^+(e) = 3,\ \deg^+(f) = 0.$$

Complete Graphs K_n $n \geq 1$: n vertices, any two distinct vertices are connected by only one edge.

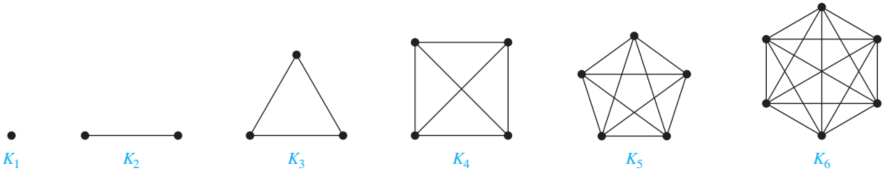


Figure. The Graphs K_n for $1 \leq n \leq 6$.

Cycles

Cycles C_n $n \geq 3$: n vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1$.

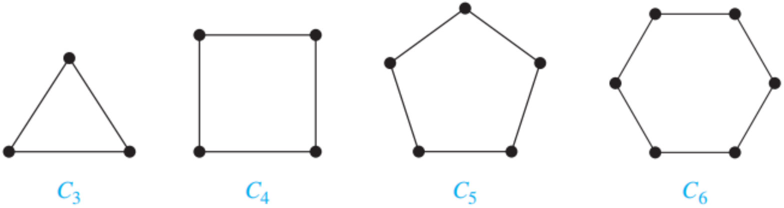


Figure. The Cycles C_3, C_4, C_5, and C_6.

Wheels W_n, $n \geq 3$: Add one vertex to C_n and connect it with the remaining vertices.

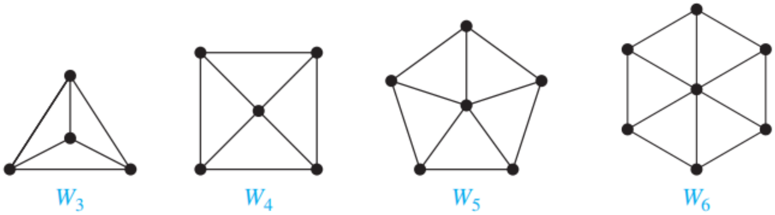


Figure. The Wheels W_3, W_4, W_5, and W_6.

***n*-Cubes** Q_n, $n \geq 1$: 2^n vertices, and the edges are drawn by the rule: represent each vertex by a bit string of length n, and two vertices are connected if their bit strings differ in exactly one position.

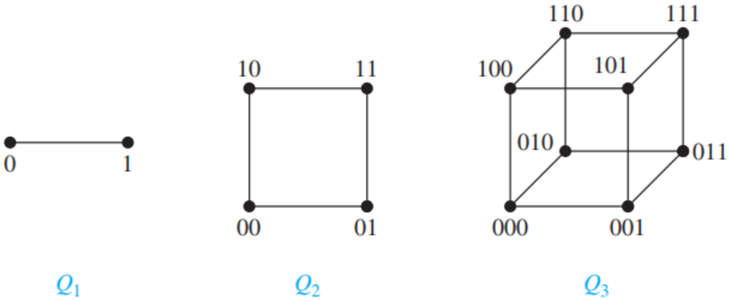


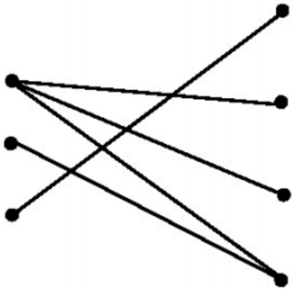
Figure. The n-cube Q_n, $n = 1, 2, 3$.

Question

How many edges each of the graphs K_n, C_n, W_n, Q_n has?
Hint. content...

Bipartite graphs

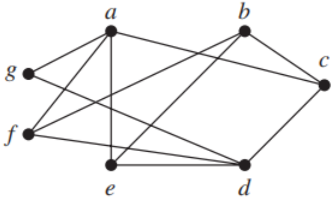
A simple graph G is called **bipartite** if the vertex set can be divided in two disjoint subsets such that each edge connects one vertex from one of these two subsets to another vertex of the other subset



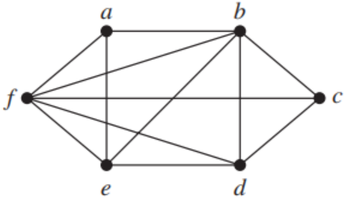
Question. Which graphs K_n, C_n, W_n, Q_n are bipartite?

Question

Are the graphs G and H bipartite?



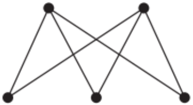
G



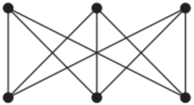
H

Complete Bipartite

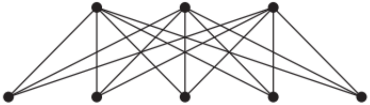
The **complete bipartite** graph K_{mn} is the graph whose vertex set is divided to two disjoint subsets of m and n vertices, such that two vertices are connected if and only if they do not belong to the same subset.



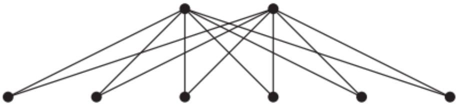
$K_{2,3}$



$K_{3,3}$



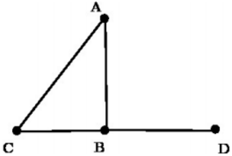
$K_{3,5}$

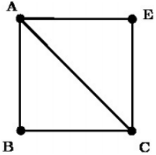


$K_{2,6}$

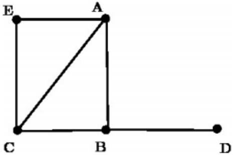
The **union** of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph, denoted as $G_1 \cup G_2$, with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$.

Example. Union of the following two graphs





is the graph



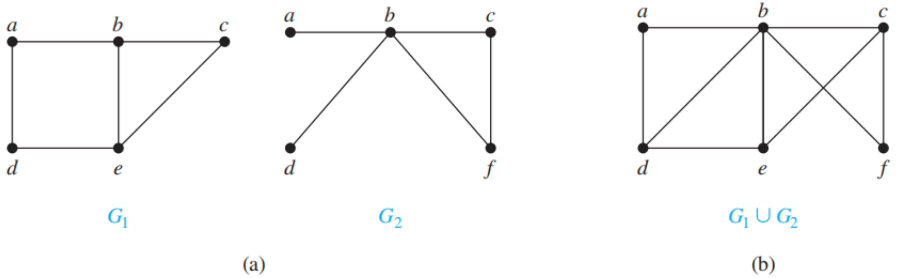
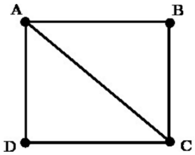


Figure. (a) The Simple Graph G_1 and G_2, (b) Their Union $G_1 \cup G_2$.

1. A **subgraph** of a graph $G = (V, E)$ is a graph $H = (W, F)$ where $W \subseteq V$ and $F \subseteq E$.
2. A subgraph H of G is a **proper subgraph** of G if $H \neq G$.

Question. Given the following graph (G)

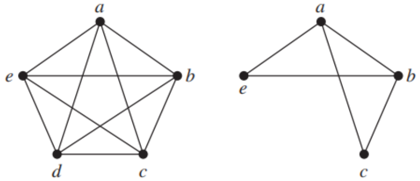


Find all subgraphs of (G).

Remarks

Given a graph $G = (V, E)$, a vertex $v \in V$ and an edge $e \in E$.

1. We can produce a **subgraph** of G by removing the edge e. The resulting subgraph is
$$G - e = (V, E - \{e\}).$$
2. We remove v and all edges incident to it from $G = (V, E)$, we obtain a subgraph,
$$G - v = (V - \{v\}, E')$$
where E' is the set of edges of G not incident to v.
3. We can produce a **new larger graph** by adding a new edge e, connecting two previously incident vertices, to the graph G. The resulting is $G + e = (V, E \cup \{e\})$.



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Representing Graphs. To represent a graph with no multiple edges is to use **adjacency lists**, which specify the vertices that are adjacent to each vertex of the graph.

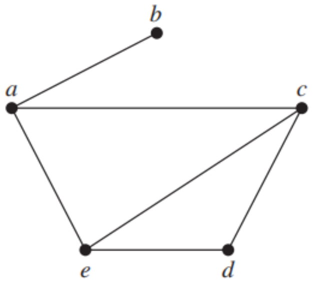

Figure. A Simple Graph.

TABLE 1 An Adjacency List for a Simple Graph.

Vertex	*Adjacent Vertices*
a	b, c, e
b	a
c	a, d, e
d	c, e
e	a, c, d

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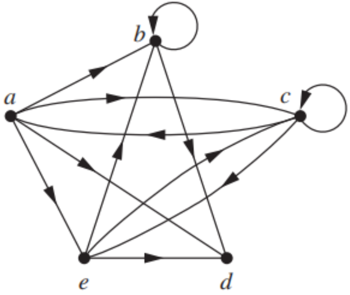


Figure. A Directed Graph.

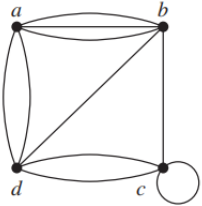
TABLE 2 An Adjacency List for a Directed Graph.

Initial Vertex	*Terminal Vertices*
a	b, c, d, e
b	b, d
c	a, c, e
d	
e	b, c, d

Let G be a pseudograph with vertices $v_1, v_2, \ldots, v_n$. We can represent G by a square matrix $[a_{ij}]$ of order n, whose entries are determined as follows

$$a_{ij} = \text{the number of edges in } G \text{ connecting } v_i \text{ and } v_j.$$

Example. Use an adjacency matrix to represent the pseudograph in the figure belows.
Solution. The adjacency matrix using the ordering of vertices a, b, c, d is

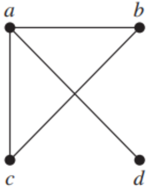


$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}.$$

More Examples

Example 1. Use an adjacency matrix to represent the pseudograph in the figure belows.

Solution. The adjacency matrix using the ordering of vertices a, b, c, d is

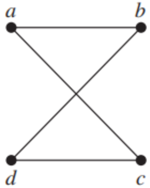


$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Example 2. Draw a graph with the given adjacency matrix with respect to the ordering of vertices a, b, c, d.

Solution.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$



Adjacency & Incident matrices for Directed graphs

Adjacency matrices for Directed graphs

Let G be a directed graph with vertices $v_1, v_2, \ldots, v_n$. We can represent G by a square matrix $[a_{ij}]$ of order n, whose entries are determined as follows

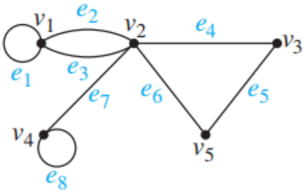
$$a_{ij} = \text{the number of edges in } G \text{ whose initial vertex is } v_i \text{ and whose end vertex is } v_j.$$

Incident matrices

Let G be a pseudograph with vertices $v_1, v_2, \ldots, v_n$ and edges $e_1, e_2, \ldots, e_m$. We can represent G by an incident matrix $[a_{ij}]$ of size $n \times m$, whose entries are determined as follows

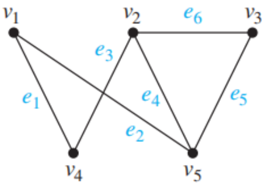
$$a_{ij} = \begin{cases} 1 & \text{if } e_j \text{ is incident to } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}.$$

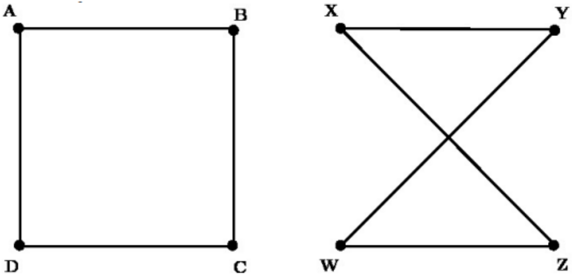
Example 2.



$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}.$$

Two graphs G and H are **isomorphic**, denoted $G \cong H$, if there is a bijection f between the vertex sets of two graphs with the property that u and v are adjacent in G if and only if $f(u)$ and $f(v)$ are adjacent in V.

Example.



Isomorphism:

$f(A) = X$

$f(B) = Y$

$f(C) = W$

$f(D) = Z$

How to Determine If Two Graphs are Isomorphic?

We need to answer four questions as follows:

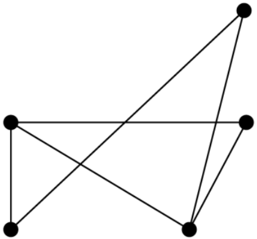
1. Are the number of vertices in both graphs the same?
2. Are the number of edges in both graphs the same?
3. Is the degree sequence in both graphs the same?
4. If the vertices in one graph can form a cycle of length k, can we find the same cycle length in the other graph?

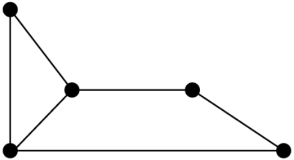
$\rightarrow$ If we can answer yes to all four of the above questions, the graphs are isomorphic.

Note. The graphs are isomorphic or in other words, they are the equivalent graphs just in different forms.

Example

Show that the following two graphs are isomorphic.

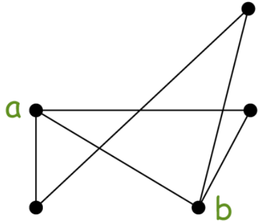


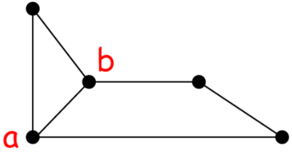


Solution.

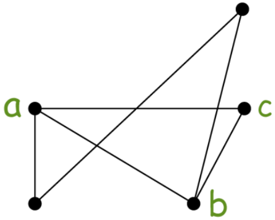
- We check vertices and degrees, both graphs have 5 vertices and the degree sequence in ascending order is $(2, 2, 2, 3, 3)$.

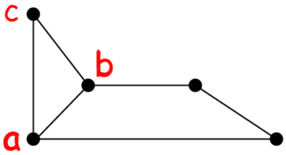
- We start labeling vertices by beginning with the vertices of degree 3, marking a and b.



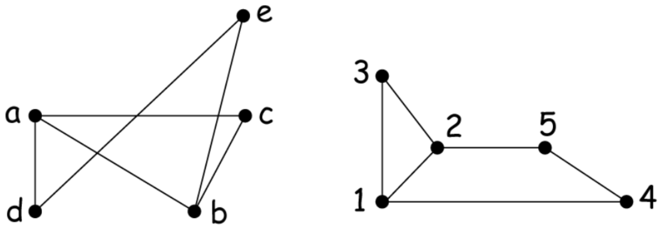


- Notice that in both graphs, there is a vertex that is adjacent to both a and b, so we label this vertex c in both graphs.





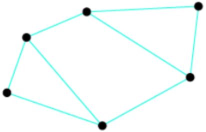
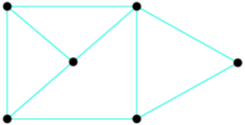
- Finally, we define the isomorphism by relabeling each graph and verifying one-to-correspondence.



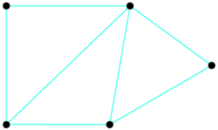
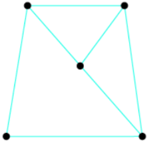
where $a = f(1),\ b = f(2),\ c = f(3),\ d = f(4),\ e = f(5).$

More Examples

Example 1. Determine whether two graphs are isomorphic.

Graph G	Graph H	Degree Sequence
		G: (2,2,3,3,3,3) H: (2,3,3,3,3,3) $G \not\cong H$

Example 2. Determine whether two graphs are isomorphic.

Graph J	Graph K	Degree Sequence
		J: (2,2,3,3,4) K: (2,3,3,3,3) $J \not\cong K$

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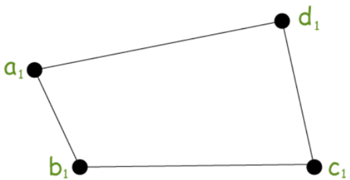
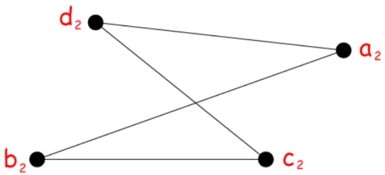
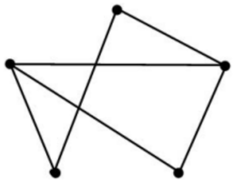
Determine whether the following pairs of graphs are isomorphic.

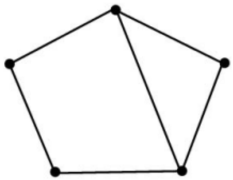
1.





2.



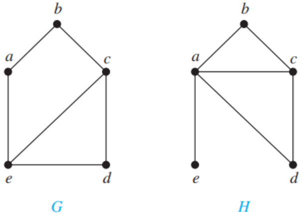


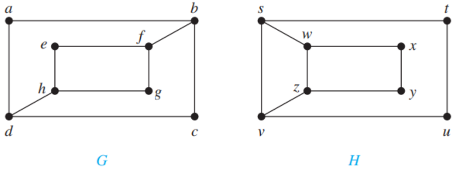
Graph Invariants

1. If there are no correspondences between two vertex sets preserving the adjacency, then the two graphs are not isomorphic.
2. In some cases, it is easy to conclude that the given graphs are not isomorphic if we can find some properties of the graphs that are not the same.
3. These properties are called **graph invariants**. They can be
   - The number of vertices.
   - The number of edges.
   - Degrees.
   - Circuits, paths (to be studied later).
   - ...

Question 1

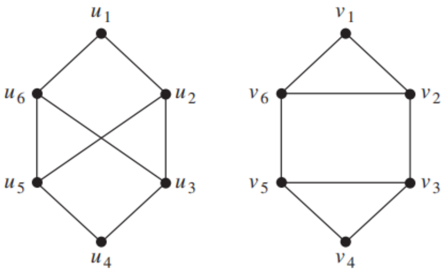
Are the following pairs of graph isomorphic?



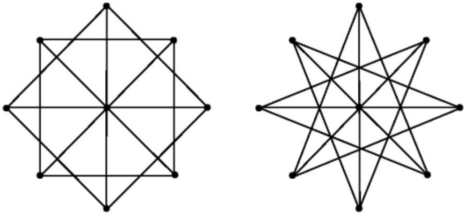


Question 2

Use graph invariants of paths and circuits to check if the two graphs are isomorphic.



G *H*



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What's next?

1. Graphs and Graph Models
2. Graph Terminologies & Special Types of Graphs
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4. Connectivity
5. Euler and Hamilton Paths
6. Shortest-Path Problems

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Paths - Circuits - Simple

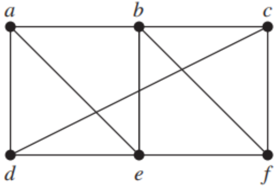
1. Let G be an undirected graph. A **path** of length n from u to v is a sequence of edges $x_0x_1, x_1x_2, \ldots, x_{n-1}x_n$, in which x_i and x_j are not necessarily distinct, and $x_0 = u$, $x_n = v$.

2. A path is called a **circuit** if it starts and ends at the same vertex, and has length greater than 0.

3. A path, or a circuit, is **simple** if it does not contain the same edge more than once.

Example

In the following simple graph,



- a, d, c, f, e is a **simple path** of length 4 since $\{a, d\}, \{d, c\}, \{c, f\}, \{f, e\}$ are all edges.
  $\rightarrow$ d, e, c, a is not a path, because e, c is not an edge.
- b, c, f, e, b is a **circuit** of length 4 since $\{b, c\}, \{c, f\}, \{f, e\}, \{e, b\}$ are edges, and this path begins and ends at b.
- The path a, b, e, d, a, b, which is of length 5, is **not simple** since it contains the edge $\{a, b\}$ twice.

Connectedness in Undirected Graphs

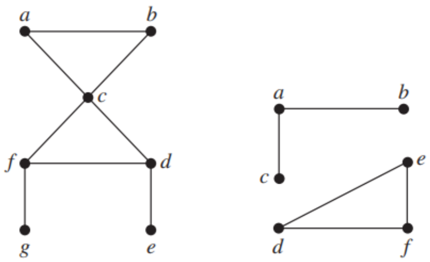
1. An undirected graph is **connected** if there is a path between any pair of distinct vertices of the graph.
2. A **connected component** of an undirected graph is a maximal subgraph that is connected.
3. A **cut vertex**, or an **articulation point**, is a vertex that if we remove it and all the edges incident with it we will obtain a subgraph having more connected components than the original graph.
4. A **cut edge**, or a **bridge**, is an edge that if we remove it we will obtain a subgraph having more connected components than the original graph.

Remarks

1. An undirected graph that is not connected is called **disconnected**.
2. We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.
3. The removal of a cut vertex from a connected graph produces a subgraph that is not connected.

Theorem. There is a simple path between every pair of distinct vertices of a connected undirected graph.

Example 1



G_1 G_2

- The graph G_1 is connected, since for every pair of distinct vertices there is a path between them.
- The graph G_2 is not connected because there is no path between vertices a and d.

Example 2

What are the connected components of the graph H?

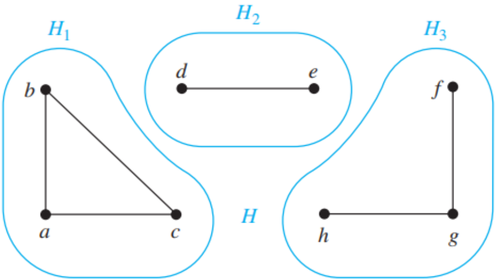
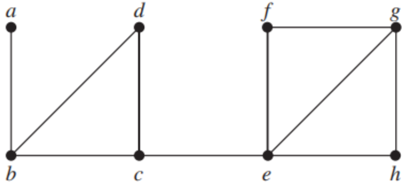


Figure. The Graph H and Its Connected Components H_1, H_2, H_3.

Solution. The graph H is the union of three disjoint connected subgraphs H_1, H_2 and H_3. These three subgraphs are the connected components of H.

Example 3

Find all cut vertices and cut edges in the graph G_1.



G_1

Solution.

- The cut vertices of G_1 are b, c and e.
  $\rightarrow$ The removal of one of these vertices (and its adjacent edges) disconnects the graph.
- The cut edges are $\{a, b\}$ and $\{c, e\}$.
  $\rightarrow$ Removing either one of these edges disconnects G_1.

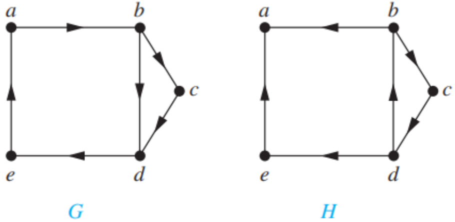
Vertex Connectivity

Not all graphs have cut vertices. For instance, the complete graph K_n where $n \geq 3$, has no cut vertices. Indeed, when removing a vertex from K_n and all edges incident to it, the resulting subgraph is the complete graph K_{n-1}, a connected graph.
$\rightarrow$ Connected graphs without cut vertices are called **nonseparable graphs**.

Connectedness in Directed Graphs

1. A directed graph is **strongly connected** if for all pairs of vertices u and v there is a path from u to v and vice versa.
2. A directed graph is **weakly connected** if the underlying undirected graph is connected.
3. A **strongly connected component** of a directed graph G is a maximal subgraph of G that is strongly connected.

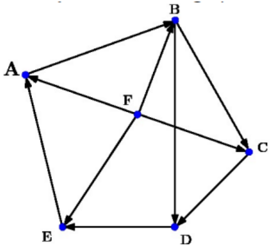
Example



- G is strongly connected since there is a path between any two vertices in this directed graph. Hence, G is also weakly connected.
- H is not strongly connected since there is no directed path from a to b. However, H is weakly connected, because there is a path between any two vertices in the underlying undirected graph of H.
- H has three strongly connected components, consisting of the vertex a, the vertex e and the subgraph including the vertices b, c, d and edges $(b, c), (c, d), (d, b)$.

Question

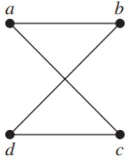
Determine whether the given graph is strongly connected, weakly connected, and find the number of strongly connected components.



Counting Paths Between Vertices

Let G be a graph (undirected, directed) whose adjacency matrix with respect to the ordering of vertices $v_1, v_2, \ldots, v_n$ is A. The number of paths of length r from v_i to v_j is the (i, j)-entry of the matrix A^r.

Example. How many paths of length four are there from a and d in the following graph?

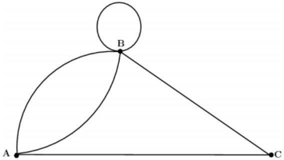


Solution (previous example). The adjacency matrix of G ordering the vertices as a, b, c, d is

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{implies} \quad A^4 = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix}.$$

Hence, the number of paths of length four from a to d is the $(1,4)$-entry of A^4, is 8.

Question. Count the number of paths of length 3 between A and C in the following graph.



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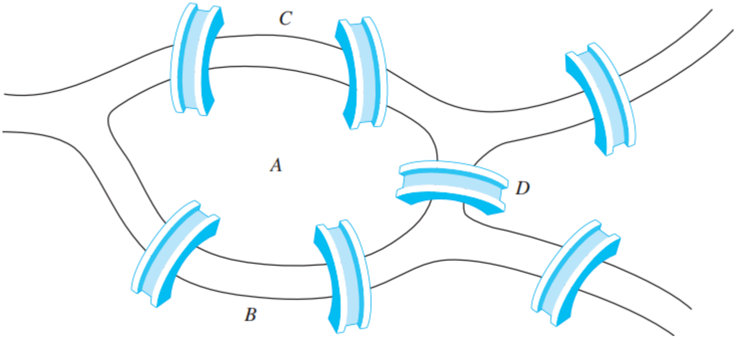
The Seven Bridges of Königsberg



Is this possible to start at some location, travel across all bridges without crossing any bridge twice, then return to the starting point?

Euler Paths and Circuits

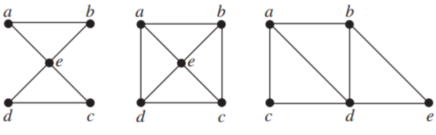
Euler Paths and Euler Circuits

1. A simple path containing all edges of a graph is called **Euler path**.
2. A simple circuit containing all edges of a graph is called **Euler circuit**.

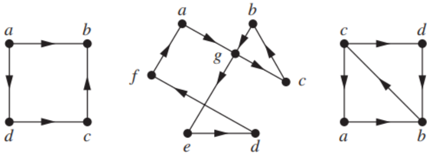
Conditions for existence of Euler paths and circuits

1. A connected multigraph G has Euler circuits if and only if every vertex has even degree.
2. If G does not have Euler circuits, then it has Euler paths if and only if it has exactly two vertices of odd degrees.

Example



G_1 G_2 G_3



H_1 H_2 H_3

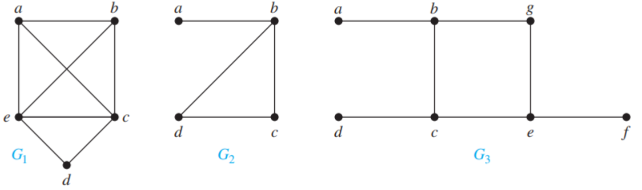
- G_1 has an Euler circuit, eg. a, e, c, d, e, b, a.
- Neither G_2 nor G_3 has an Euler circuit.
- G_3 has an Euler path, namely, a, c, d, e, b, d, a, b, but G_2 does not.
- H_2 has an Euler circuit, eg. $a, g, c, b, g, e, d, f, a$.
- Neither H_1 nor H_3 has an Euler circuit.
- H_3 has an Euler path, namely, c, a, b, c, d, b, but H_1 does not.

Question. For what values of m, n do the special graphs $K_n, C_n, W_n, Q_n, K_{mn}$ have Euler paths or circuits?

Hamilton Paths and Hamilton Circuits

1. A simple path that passes through all vertices exactly once is called **Hamilton path**.
2. A simple circuit that passes through all vertices exactly once is called **Hamilton circuit**.

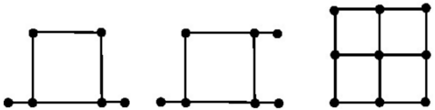
Example. Which of the simple graphs in the following figure have a Hamilton circuit or, if not, a Hamilton path?



Conditions for the Existence Hamilton Circuits

A graph with a vertex of degree one cannot have a Hamilton circuit, because in a Hamilton circuit, each vertex is incident with two edges in the circuit.

Question 1. Find Hamilton paths/circuits of the graphs.



Question 2. For what values of m, n do the special graphs K_n, C_n, W_n, Q_n, K_{mn} has Hamilton paths/circuits?

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Theorems

Dirac's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that the degree of every vertex in G is at least $n/2$, then G has a Hamilton circuit.

Ore's Theorem

If G is a simple graph with n vertices with $n \geq 3$ such that $\deg(u) + \deg(v) \geq n$ for every pair of nonadjacent vertices u and v in G, then G has a Hamilton circuit.

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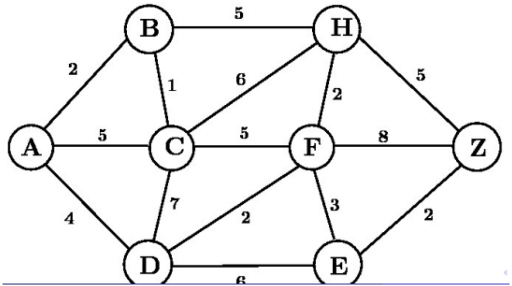
Shortest-Path Problems

The shortest path problem is foundational in many areas, making it essential in both theoretical research and practical applications.

- Navigation systems: Finding the quickest route between two locations.
- Network routing: Determining the most efficient way to send data across a network.
- Logistics: Optimizing delivery routes to minimize travel time or distance.

1. A graph that has a number assigned to each edge is called a **weighted graph**.
2. The **length** of a path in a weighted graph is the sum of all weights of the edges of this path.

Question. Find the path of shortest length between two vertices A-Z in a weighted graph.

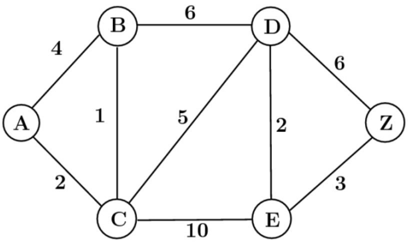


Dijkstra's Algorithm

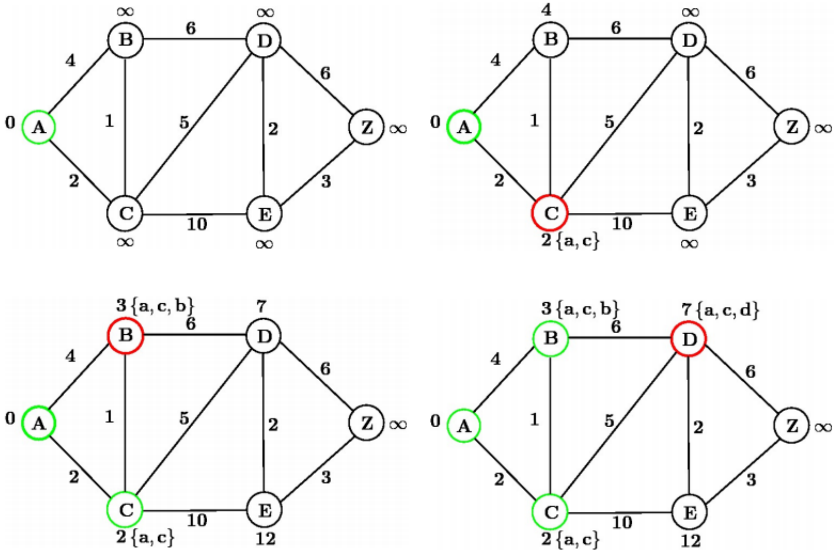
Let G be a weighted graph. To find the shortest path between A and Z in G, Dijkstra's algorithm:

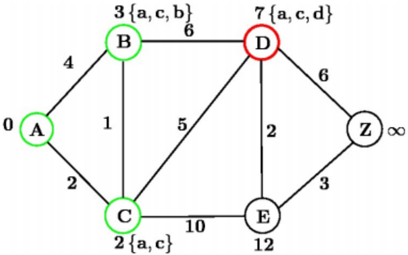
1. Finds the length of the shortest path from A to the first vertex.
2. Finds the length of the shortest path from A to the second vertex.
3. Finds the length of the shortest path from A to the third vertex.
4. ...
5. Continue the process until Z is reached.

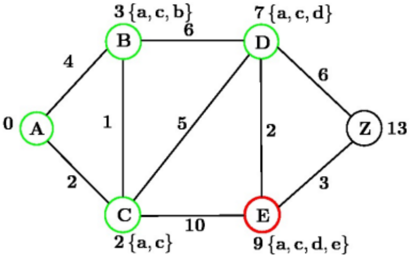
Find the shortest path from A to Z in the weighted graph.

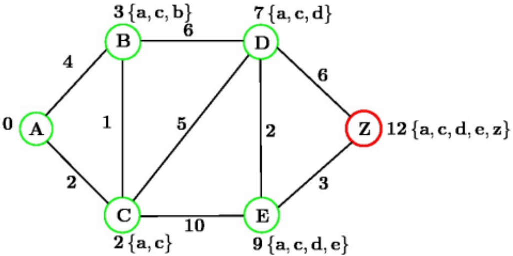


Solution.




3 {a, c, b}
7 {a, c, d}
B
D
6
4
6
0
A
1
5
2
Z
∞
2
3
C
E
10
2 {a, c}
12


3 {a, c, b}
7 {a, c, d}
B
D
6
4
6
0
A
1
5
2
Z
13
2
3
C
E
10
2 {a, c}
9 {a, c, d, e}


3 {a, c, b}
7 {a, c, d}
B
D
6
4
6
0
A
1
5
2
Z
12 {a, c, d, e, z}
2
3
C
E
10
2 {a, c}
9 {a, c, d, e}