Discrete Mathematics

COUNTING TECHNIQUES

Theory and Practice of Enumeration

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Department of Mathematics

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Competency Goals

1. Use basic rules for counting
2. Use recurrence relations for counting, including divide-and-conquer recurrence relations.

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The Basics of Counting

The Product Rule

Suppose that a procedure can be broken down into a sequence of two tasks. If there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 \cdot n_2$ ways to do the procedure.

n_1 choices
Task 1: Choose a main course (e.g., pasta, salad)

→

n_2 choices
Task 2: Choose a drink (e.g., water, juice)

→

n_3 choices
Task 3: Choose a dessert (e.g., cake, fruit)

$$\text{Total combinations} = n_1 \times n_2 \times n_3$$

Example

A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

Solution. The procedure of assigning offices to these two employees consists of

- assigning an office to Sanchez, which can be done in 12 ways,
- then assigning an office to Patel different from the office assigned to Sanchez, which can be done in 11 ways.

By the product rule, there are $12 \cdot 11 = 132$ ways to assign offices to these two employees.

Questions

1. The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

2. How many different bit strings of length seven are there?

3. How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

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Counting Functions

Example. How many functions are there from a set with m elements to a set with n elements?

Solution. A function corresponds to a choice of one of the n elements in the codomain for each of the m elements in the domain. Hence, by the product rule there are $nn \cdots n = n^m$ functions from a set with m elements to one with n elements.
$\rightarrow$ For instance, there are $5^3 = 125$ different functions from a set with 3 elements to a set with 5 elements.

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Counting One-to-One Functions

How many one-to-one functions are there from a set with m elements to one with n elements?

Solution. Note that when $m > n$ there are no one-to-one functions from a set with m elements to a set with n elements.

Now let $m \leq n$. Suppose the elements in the domain are $a_1, a_2, \ldots, a_m$.

- There are n ways to choose the value of the function at a_1.
- Because the function is one-to-one, the value of the function at a_2 can be picked in $n - 1$ ways.
- In general, the value of the function at a_k can be chosen in $n - k + 1$ ways.

By the product rule, there are

$$n(n-1)(n-2)\cdots(n-m+1)$$

one-to-one functions from a set with m elements to one with n elements.

$\rightarrow$ For instance, there are $5 \cdot 4 \cdot 3 = 60$ one-to-one functions from a set with 3 elements

Inclusion-exclusion Principle

Theorem

- Let A and B be two disjoint sets. Then, $|A \cup B| = |A| + |B|$.
- Let A and B be two arbitrary sets. Then,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Questions.

1. How many bit strings of length eight either start with a 1 bit or end with the two bits 00?
2. A computer company receives 350 applications from computer graduates for a job planning a line of new Web servers. Suppose that 220 of these applicants majored in computer science, 147 majored in business, and 51 majored both in computer science and in business. How many of these applicants majored neither in computer science nor in business?

Counting Factors

Counting Numbers

The number of positive integers not exceeding n and divisible by k is $\left\lfloor \frac{n}{k} \right\rfloor$.

Questions.

1. The number of positive integers not exceeding 1000 and divisible by 12.
2. The number of positive integers less than 1000, greater than 100 and divisible by 12.
3. The number of positive integers not exceeding 1000 and divisible by 12 or 8.
4. The number of positive integers less than 1000 divisible by 12 but not divisible by 8.

More Complex Counting Problems

Generalization The Sum Rule

Let $A_1, A_2, \ldots, A_m$ be m sets. Assume that $A_i \cap A_j = \emptyset$ where $i, j = 1, 2, \ldots, m$ and $i \neq j$. Then,

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \cdots + |A_m|.$$

Example. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

Solution. Let P be the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. We have

$$P_6 = 36^6 - 26^6, \quad P_7 = 36^7 - 26^7, \quad P_8 = 36^8 - 26^8.$$

Hence, $P = P_6 + P_7 + P_8$.

The Division Rule

Assume the finite set A is the union of n pairwise disjoint subsets each with d elements. Then,

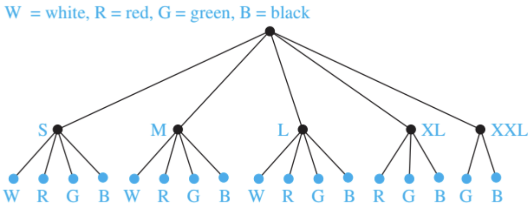
$$n = \frac{|A|}{d}.$$

Example. How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?

Solution. There are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.

Tree Diagrams

Example. Suppose that "I Love New Jersey" T-shirts come in five different sizes: S, M, L, XL, and XXL. Further suppose that each size comes in four colors, white, red, green, and black, except for XL, which comes only in red, green, and black, and XXL, which comes only in green and black. How many different shirts does a souvenir shop have to stock to have at least one of each available size and color of the T-shirt?



Solution. The tree diagram as follows displays all possible size and color pairs. It follows that the souvenir shop owner needs to stock 17 different T-shirts.

Tree Diagrams (cont')

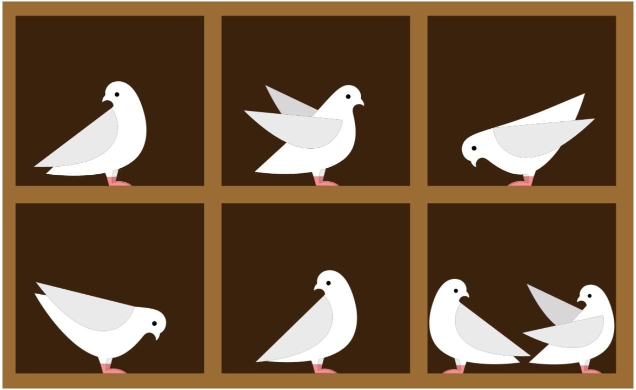
Question 1. How many bit strings of length four do not have two consecutive 1s?

Question 2. A playoff between two teams consists of at most five games. The first team that wins three games wins the playoff. In how many different ways can the playoff occur?

What's next?

Pigeonhole Principle

If N objects are placed into k boxes, then there is at least one box that contains at leasts $\lceil N/k \rceil$ objects.



Example and Question

Example. Among 100 people there are at least $\lceil 100/12 \rceil = 9$ people born in the same month.

Question.

1. How many people must be selected to guarantee that there are at least 10 people born in the same month?

2. How many cards must be selected from a standard deck of 52 cards to guarantee that there are at least 3 cards of the same suit? at least 3 hearts?

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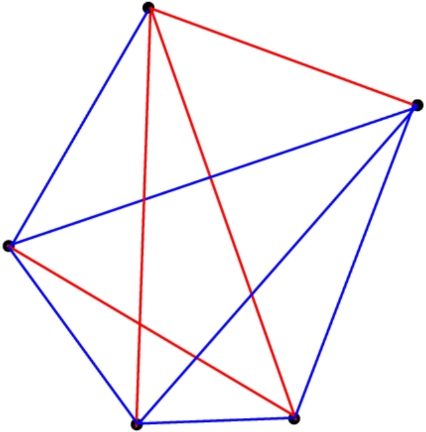
Two-color Graph

A **graph of 2 colors** (blue, red) is a graph so that between two distinct vertices there is one edge, and this edge is either blue or red.



Ramsey number $R(m, n)$

Let m, n be positive integers greater than or equal to 2. The **Ramsey number** $R(m, n)$ is the least number of vertices of a graph of 2 colors such that we can always find a subgraph of m vertices whose edges are all blue, or a subgraph of n vertices whose edges are all red.

Problem. Find $R(2, 3), R(2, n), R(3, 3)$.

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What's next?

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects. We also are interested in ordered arrangements of some of the elements of a set. An ordered arrangement of r elements of a set is called an r-permutation.

Theorem. If n is a positive integer and r is an integer with $1 \leq r \leq n$, then there are

$$P(n, r) = n(n-1)(n-2)\cdots(n-r+1)$$

r-permutations of a set with n distinct elements.

Questions

Question 1. How many different ways can the letters in the word "MATH" be arranged?

Question 2. In how many ways can we select three students from a group of five to stand in line for a picture? In how many ways can we arrange all five of these students in a line for a picture?

Question 3. How many permutations of the letters ABCDEFGH contain the string ABC ?

Combinations

An **r-combination** of elements of a set is an unordered selection of r elements from the set. Thus, an r-combination is simply a subset of the set with r elements.

Theorem. The number of r-combinations of a set with n elements, where n is a nonnegative integer and r is an integer with $0 \leq r \leq n$, equals

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Questions

Question 1. How many poker hands of five cards can be dealt from a standard deck of 52 cards? Also, how many ways are there to select 47 cards from a standard deck of 52 cards?

Question 2. A group of 30 people have been trained as astronauts to go on the first mission to Mars. How many ways are there to select a crew of six people to go on this mission (assuming that all crew members have the same job)?

Permutations with Repetition

Example. How many strings of length r can be formed from the uppercase letters of the English alphabet?

Solution. By the product rule, because there are 26 uppercase English letters, and because each letter can be used repeatedly, we see that there are 26^r strings of uppercase English letters of length r.

Theorem. The number of r-permutations of a set of n objects with **repetition** allowed is n^r.

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Example. How many ways are there to select four pieces of fruit from a bowl containing apples, oranges, and pears if the order in which the pieces are selected does not matter, only the type of fruit and not the individual piece matters, and there are at least four pieces of each type of fruit in the bowl?

Solution. To solve this problem we list all the ways possible to select the fruit. There are 15 ways:

4 apples	4 oranges	4 pears
3 apples, 1 orange	3 apples, 1 pear	3 oranges, 1 apple
3 oranges, 1 pear	3 pears, 1 apple	3 pears, 1 orange
2 apples, 2 oranges	2 apples, 2 pears	2 oranges, 2 pears
2 apples, 1 orange, 1 pear	2 oranges, 1 apple, 1 pear	2 pears, 1 apple, 1 orange

Theorem. There are $C(n+r-1, r) = C(n+r-1, n-1)$ r-combinations from a set with n elements when **repetition** of elements is allowed.

Example

How many solutions does the equation

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2, and x_3 are nonnegative integers?

Solution. We note that a solution corresponds to a way of selecting 11 items from a set with three elements so that x_1 items of type one, x_2 items of type two, and x_3 items of type three are chosen. Hence, the number of solutions is equal to the number of 11-combinations with repetition allowed from a set with three elements. From the previous theorem, it follows that there are

$$C(3+11-1, 11) = C(13, 11) = C(13, 2) = 78$$

solutions.

What's next?

Applications of Recurrence Relations

Compound Interes. A person deposited $\$10,000$ in a saving account at the rate of 11% a year with interest compounded annually. How much will be in the account after 30 years?

Fibonacci Numbers. A young pair of rabits (one of each sex) is placed on an island. A pair of rabbits does not breed untill they are 2 month old. After they are 2 month old, each month they produce a pair. Find the recurrence relation for the number of pairs of rabbits after n months (that is, at the end of the nth month).

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The Tower of Hanoi

64 disks are placed on the first of three pegs in order of size (as shown in the picture). A disk is allowed to move from one peg to another as long as a disk is never placed on a disk of smaller size. Find the least number of moves required to move all disks to another peg.

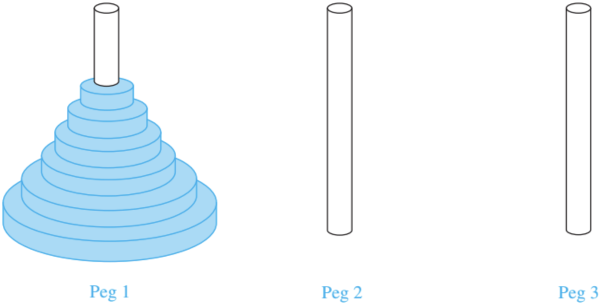

Figure. The Initial Position in the Tower of Hanoi.

What's next?

Recall the Merge sort algorithm.

Merge Sort Algorithm

Procedure mergesort $(L = a_1, a_2, \ldots, a_n)$
if $n > 1$ **then**
  $m := \lfloor n/2 \rfloor$
  $L_1 = a_1, a_2, \ldots, a_m$
  $L_2 := a_{m+1}, a_{m+2}, \ldots, a_n$
  $L :=$ merge(mergesort(L_1), mergesort(L_2))
Print(L)

Note. Let $f(n)$ be the number of comparisons used in the algorithm. Then,

$$f(1) = 1 \text{ and } f(n) = 2f(n/2) + n.$$

Divide-and-Conquer Recurrence Relation

Let $f(n)$ be a function on the set of integers. A divide-and-conquer recurrence relation for f has the form

$$f(n) = af(n/b) + g(n)$$

where $g(n)$ is some function and a, b are real numbers.

Question. Let $f(n)$ be such that $f(1) = 2$ and $f(n) = f(n/3) + 1$. Find $f(81)$, $f(3^k)$.

Master Theorem

Let f be an increasing function that satisfies

$$f(n) = af(n/b) + cn^d$$

for all $n = b^k$ where k is a positive integer, $a \geq 1$ and $b > 1$ be positive integers, and c, d are positive real numbers. Then,

$$f(n) = \begin{cases} O(n^d) & \text{if } a < b^d \\ O(n^d \log n) & \text{if } a = b^d \\ O(n^{\log_b a}) & \text{if } a > b^d. \end{cases}$$

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Fast Multiplication Algorithm

The conventional algorithm of multiplying two integers whose binary expansions has length n has complexity $O(n^2)$. The following algorithm has a better complexity.
Let $a = (a_{2n-1}a_{2n-2}\dots a_0)_2$ and $b = (b_{2n-1}b_{2n-2}\dots b_0)_2$.
Let $a = 2^n A_1 + A_0$ and $b = 2^n B_1 + B_0$. Then,

$$ab = (2^{2n} + 2^n)A_1B_1 + 2^n(A_1 - A_0)(B_0 - B_1) + (2^n + 1)A_0B_0.$$

Let $f(n)$ be the total number of bit operations used in this algorithm for integers of length n then

$$f(2n) = 3f(n) + Cn$$

where C is a constant.
Using the Master Theorem, we obtain $f(n) \approx O(n^{1.6})$.