Lecture 3

DISCRETE RANDOM VARIABLES PROBABILITY DISTRIBUTION

Education

FPT UNIVERSITY

Department of Mathematics

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Contents

1. Discrete random variables

Discrete random variables

A **discrete random variable** is a random variable with a finite or countable infinite range.

Example.

1. Roll a dice twice: Let X be the number of times 4 comes up. Then $X = 0, 1,$ or $2.$
2. Toss a coin 5 times: Let X be the number of heads. Then $X = 0, 1, 2, 3, 4,$ or $5.$
3. $X =$ The number of stocks in the Dow Jones Industrial. Average that have share price increases on a given day, then X is a discrete random variable because its share price increases can be counted.

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Determining a Discrete Random Variable

Let X be a discrete random variable with possible outcomes $x_1, x_2, \ldots, x_n$.

1. Find the probability of each possible outcome.
2. Check that each probability is between 0 and 1 and that the sum is 1.
3. Summarizing results in following table, we obtain the **probability distribution** of X.

X	x_1	x_2	$\cdots$	x_n
$P(x)$	p_1	p_2	$\cdots$	p_n

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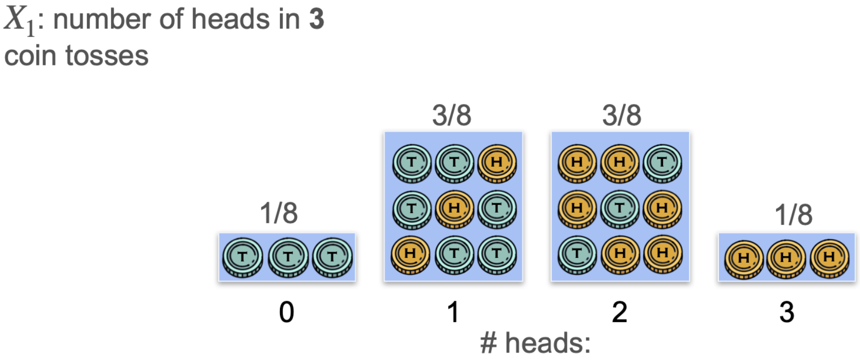
Probability distribution: Example

Let the random variable X_1 denote the number of heads in three tosses of a fair coin. Determine the probability distribution of X_1.

Hint.



Probability mass function (pmf)

For a discrete random variable X with possible values $x_1, x_2, ..., x_n$, the **Probability Mass Function (PMF)** is typically denoted as $P(X = x)$ or $f(x)$, where x is a specific value that X can take.

Here are some key properties of the PMF:

- *Probability*
$$f(x_i) = P(X = x_i).$$
- *Non-negativity* $f(x) \geq 0$ for all values of x.
- *Normalization*
$$\sum_{i=1}^{n} f(x_i) = 1.$$

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Example

Suppose that a days production of 100 manufactured parts contains 10 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. *What is the probability mass function of X?*

Solution.

$$f(x) = \begin{cases} \frac{89}{110}, & \text{if } x = 0 \\ \frac{2}{11}, & \text{if } x = 1 \\ \frac{1}{110}, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

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Cumulative distribution function (cdf)

The **cumulative distribution function** of a discrete random variable X, denoted as $F(x)$, is given by

$$F(x) = P(X \leq x) = \sum_{x_i \leq x} f(x_i).$$

For a discrete random variable X, $F(x)$ satisfies the following properties:

(i) $0 \leq F(x) \leq 1$

(ii) If $x \leq y$, then $F(x) \leq F(y)$

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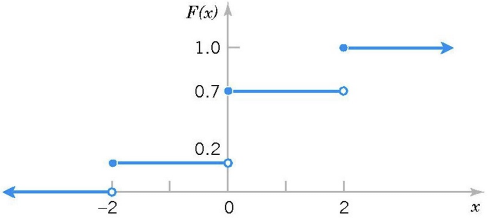
Example

Determine the probability mass function of X from the following cumulative distribution function:

$$F(x) = \begin{cases} 0, & \text{if } x < -2 \\ 0.2, & \text{if } -2 \le x < 0 \\ 0.7, & \text{if } 0 \le x < 2 \\ 1, & \text{if } x \ge 2 \end{cases}$$

Hint.



2. Mean and Variance

Mean and Variance

1. The **mean** or **expected value** of the discrete random variable X with probability mass function $P(X = x_i) = p_i$ for all possible values x_i, denoted as μ or $E(X)$, is given by:

$$\mu = E(X) = \sum_i x_i p_i$$

2. The **variance** of X, denoted as σ^2 or $V(X)$ is:

$$\sigma^2 = V(X) = E(X - \mu)^2 = \sum_i x_i^2 p_i - \mu^2$$

3. The **standard deviation** of X is $\sigma = \sqrt{V(X)}$

Example. The number of messages sent per hour over a computer network has the following distribution:

X	10	11	12	13	14	15
$f(x)$	0.08	0.15	0.30	0.20	0.20	0.07

Determine the mean and standard deviation of the number of messages sent per hour.

Hint.

$$\mu = 10 \times 0.08 + 11 \times 0.15 + \cdots + 15 \times 0.07 = 12.5$$

$$\sigma^2 = 10^2 \times 0.08 + 11^2 \times 0.15 + \cdots + 15^2 \times 0.07 - 12.5^2 = 1.85$$

Remark.

$$E(aX + b) = aE(X) + b$$

$$V(aX + b) = a^2 V(X)$$

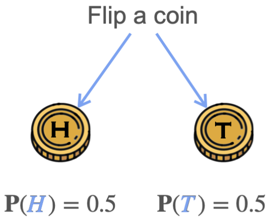
3. Discrete Distributions

Discrete uniform distribution

A random variable X has a **discrete uniform distribution** if each of the n values in its range, say, $x_1, x_2, \cdots, x_n$ has equal probability. Then,

$$f(x_i) = P(X = x_i) = \frac{1}{n}$$

Example.



Discrete uniform distribution: Properties

Theorem (Mean and Variance)

Suppose X is a discrete uniform random variable on the consecutive integers $a, a+1, \ldots, b$ for $a \le b$. The mean and variance of X are given by

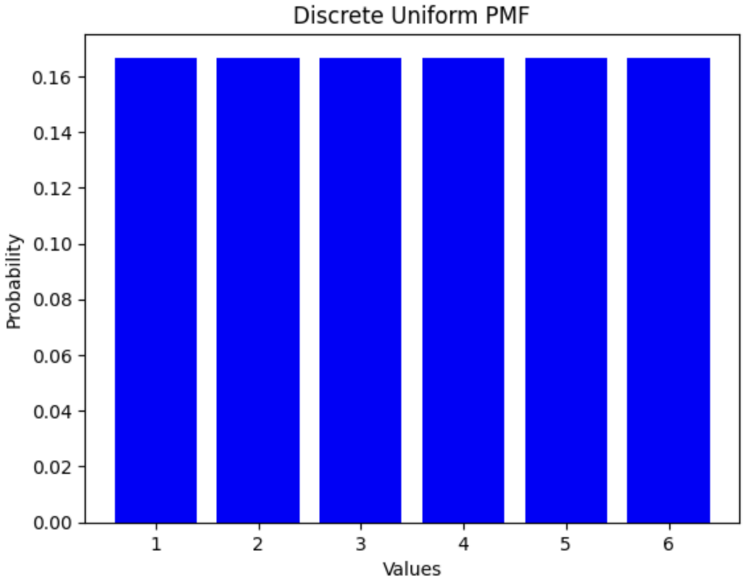
$$\mu = E(X) = \frac{b+a}{2}, \qquad \sigma^2 = V(X) = \frac{(b-a+1)^2 - 1}{12}$$

Example.

Let X be the number on the dice.

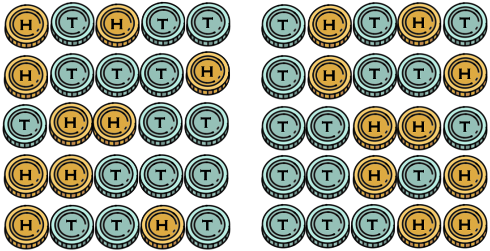
$$E(X) = \frac{1+6}{2} = 3.5$$

$$V(X) = \frac{(6-1+1)^2-1}{12} = 2.91(6)$$


Discrete Uniform PMF
Probability
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
0.16
Values
1
2
3
4
5
6

Binomial distribution: Example

What is the probability that if I flip 5 coins, 2 of them land in heads?



$$10 = \frac{5!}{2!(5-2)!} = \binom{5}{2}$$

Binomial coefficient

Number of ways you can get 2 heads in 5 coin tosses

Binomial distribution

A random experiment consists of n trials such that:

(i) The trials are independent

(ii) Each trial results in only two possible outcomes, labeled as success andfailure

(iii) The probability of a success in each trial, denoted as p, remains constant

The random variable $X =$ the number of successes in n trials has a **binomial distribution** with parameters p and n.

Theorem

Let X be a binomial distribution with parameters p and n. The probability mass function of X is

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \; x = 0, 1, 2, ..., n$$

Binomial distribution

Theorem (Mean and Variance)

$$\mu = E(X) = np, \qquad \sigma^2 = V(X) = np(1-p)$$

Quiz. Each sample of water has a 20% chance of containing a particular organic pollutant. Assume that the samples are independent with regard to the presence of the pollutant. Let $X =$ the number of samples that contain the pollutant in the next 18 samples analyzed.

(a) Find $P(X = 2)$.

(b) Determine the probability that at least four samples contain the pollutant.

(c) Determine the probability that $3 \leq X < 7$.

(d) Find the mean and standard deviation of X.

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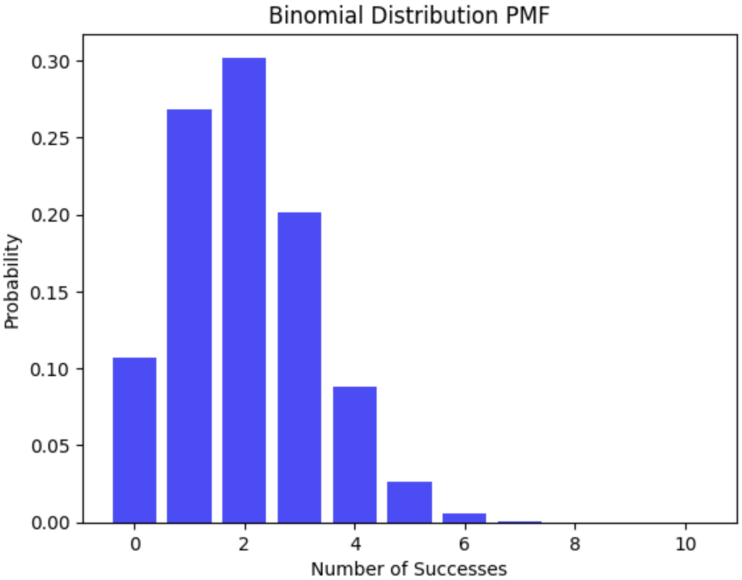

Binomial Distribution PMF
Probability
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
Number of Successes

Geometric distribution

Example. The probability of a successful optical alignment in a assembly of an optical data storage product is 0.8. Assume the trials are independent. *What is the probability that the first successful alignment requires exactly four trials?*
Hint. Let $X =$ the number of trials to the first success.

$$P(X = 4) = P(FFFS)$$

Definition

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable $X =$ the number of trials until the first success. Then X has a **geometric distribution** with parameter p, and the probability mass function of X is

$$f(x) = (1-p)^{x-1}p \text{ for } x = 1, 2, \ldots$$

Geometric distribution

Theorem (Mean and Variance)

If X is a geometric random variable with parameter p then

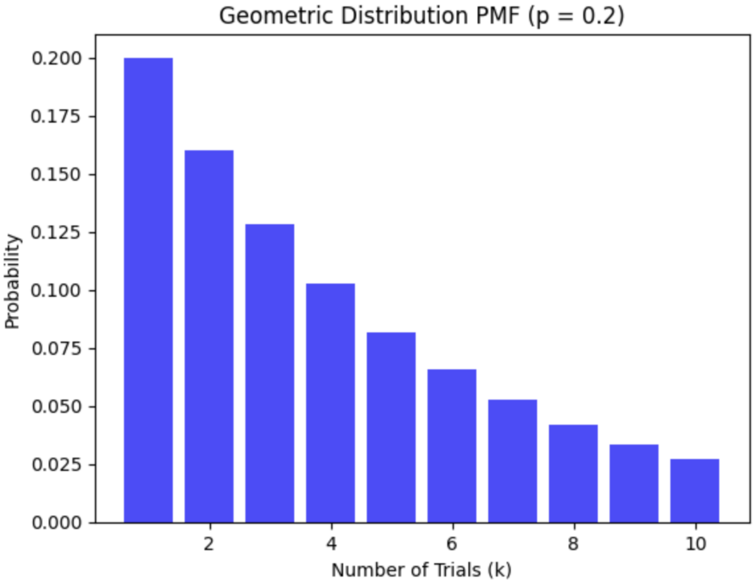
$$\mu = E(X) = \frac{1}{p}, \qquad \sigma^2 = V(X) = \frac{1-p}{p^2}$$

Example. Assume that each of your calls to a popular radio station has a probability of 0.2 of connecting, that is, of not obtaining a busy signal. Assume that your calls are independent.

a) What is the probability that your first call that connects is your tenth call?

b) What is the probability that it requires more than five calls for you to connect?

c) What is the mean number of calls needed to connect?


Geometric Distribution PMF (p = 0.2)
Probability
0.200
0.175
0.150
0.125
0.100
0.075
0.050
0.025
0.000
2
4
6
8
10
Number of Trials (k)

Negative Binomial distribution

Definition

In a series of Bernoulli trials (independent trials with constant probability p of a success), let the random variable $X =$ the number of trials until the first r successes occur. Then X has a **Negative Binomial distribution** with parameter p, and the probability mass function of X is

$$f(x) = \binom{x-1}{r-1}(1-p)^{x-r}p^r \text{ for } x = r, r+1, r+2, \ldots$$

Example. Find the probability that a man flipping a coin gets the fourth head on the ninth flip.

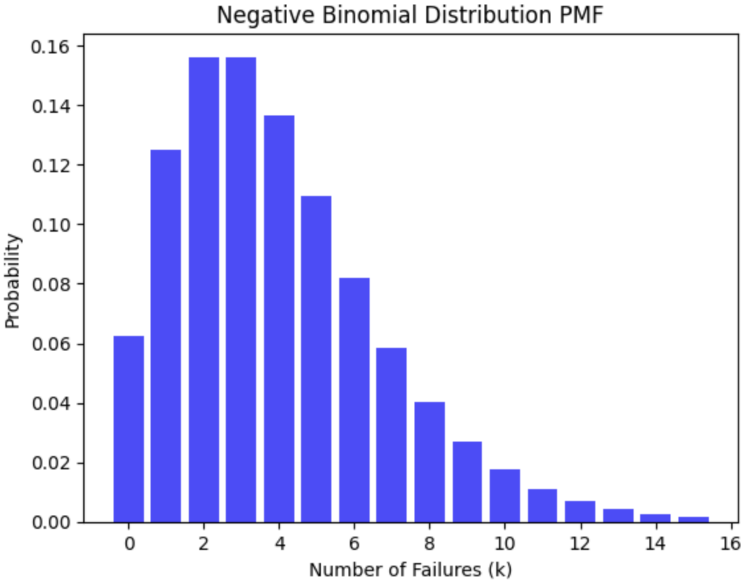
Negative Binomial distribution

Theorem (Mean and Variance)

If X is a Negative Binomial Distribution with parameters p and r then

$$\mu = E(X) = \frac{r}{p}, \qquad \sigma^2 = V(X) = \frac{r(1-p)}{p^2}$$

Example. Find the mean and standard deviation of the number of flips until that man gets four heads.


Negative Binomial Distribution PMF
Probability
0.16
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
0
2
4
6
8
10
12
14
16
Number of Failures (k)

Hyper-geometric distribution

Definition

A set of N objects contains: K objects classified as successes; $N-K$ objects classified as failures. A sample of size n objects is selected randomly (without replacement) from the N objects, where $K \leq N$, $n \leq N$. Let the random variable $X =$ the number of successes in the sample. Then X has a **hyper-geometric distribution** and the probability mass function of X is:

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

for $x = \max\{0, n+K-N\}$ to $\min\{K, n\}$.

Hyper-geometric distribution

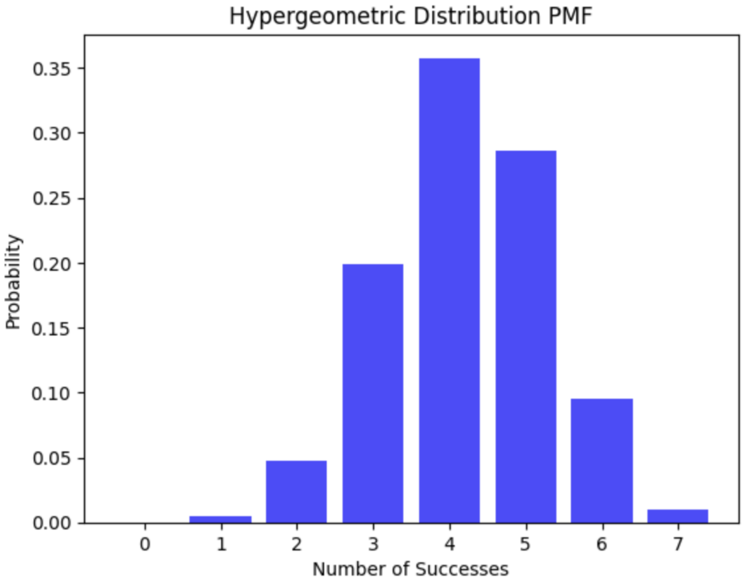
Theorem (Mean and Variance)

If X is a hyper-geometric random variable with parameters N, K, and n, then

$$\mu = E(X) = np, \qquad \sigma^2 = V(X) = np(1-p)\frac{N-n}{N-1},$$

in which $p = \dfrac{K}{N}$.

Example. A committee of size 12 is to be selected at random from 7 chemists and 13 physicists.
a) Find the probability distribution for the number of chemists on the committee.
b) Find the mean and the variance of the number of chemists on the committee.


Hypergeometric Distribution PMF
Probability
0.35
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
1
2
3
4
5
6
7
Number of Successes

Poisson distribution

Given an interval of real numbers, assume events occur at random throughout the interval. If the interval can be partitioned into subintervals of small enough length such that:

1. The probability of more than one event in a subinterval is zero.
2. The probability of one event in a subinterval is the same for all subintervals and proportional to the length of the subinterval,
3. The event in each subinterval is independent of the other subintervals, the random experiment is called the **Poisson Process**.

Definition

The random variable $X =$ the number of events in an interval of time has a **Poisson distribution** with parameter λ, and the probability mass function of X is:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for } x = 0, 1, 2, \ldots$$

Poisson distribution

Theorem (Mean and Variance)

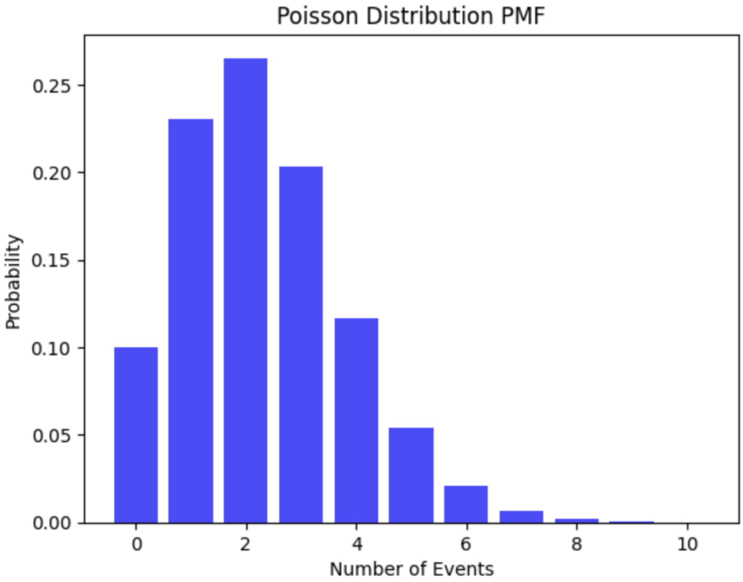
If X is a Poisson random variable with parameter λ, then

$$\mu = E(X) = \lambda, \qquad \sigma^2 = V(X) = \lambda$$

Example. For the case of the thin copper wire, suppose that the number of flaws follows a Poisson distribution with a mean of 2.3 flaws per millimeter.

a. Determine the probability of exactly 2 flaws in 1 millimeter of wire.

b. Determine the probability of at least 1 flaw in 2 millimeters of wire.


Poisson Distribution PMF
Probability
0.25
0.20
0.15
0.10
0.05
0.00
0
2
4
6
8
10
Number of Events

Summary

Distribution	PMF f	Mean μ	Variance σ^2
Uniform distribution	$\frac{1}{b-a+1}$	$\frac{b+a}{2}$	$\frac{(b-a+1)^2-1}{12}$
Binomial distribution	$C_n^x p^x (1-p)^{n-x}$	np	$np(1-p)$
Geometric distribution	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Negative Binomial distribution	$C_{x-1}^{r-1}(1-p)^{x-r}p^r$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Hyper-geometric distribution	$\frac{C_K^x C_{N-K}^{n-x}}{C_N^n}$	np	$np(1-p)\frac{N-n}{N-1}$
Poisson distribution	$\frac{e^{-\lambda}\lambda^x}{x!}$	λ	λ

Thank you!

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