Lecture 10

STATISTICAL INFERENCE FOR TWO SAMPLES

FPT Education

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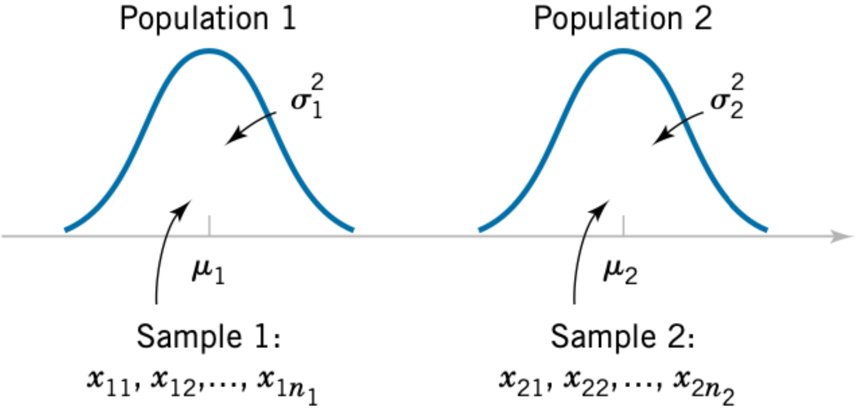
Contents

1. Two independent populations

Two independent populations



Assumptions

- $X_{11}, X_{12}, ..., X_{1n_1}$ is a random sample from population 1.
- $X_{21}, X_{22}, ..., X_{2n_2}$ is a random sample from population 2.
- The two populations represented by X_1 and X_2 are independent.
- Both populations are normal.

Based on the assumptions, we may state the following.

Theorem

The quantitive

$$Z = \frac{\bar{X}_1 - \bar{X}_2 - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

has a $\mathcal{N}(0, 1)$ distribution.

2. Inference on the Difference in Means of Two Normal Distributions

Confidence interval on the difference in means (variances known)

Theorem (σ_1, σ_2 known)

A $100(1-\alpha)\%$ C.I. on the difference in means $\mu_1 - \mu_2$

$$\bar{x}_1 - \bar{x}_2 - z_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_{\alpha/2} \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

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One-sided Confidence interval on the difference in means (variances known)

Theorem (σ_1, σ_2 known)

- *A* $100(1-\alpha)\%$ *upper-confidence bound for* μ *is*

$$\mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + z_\alpha \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

- *A* $100(1-\alpha)\%$ *lower-confidence bound for* μ *is*

$$\mu_1 - \mu_2 \geq \bar{x}_1 - \bar{x}_2 - z_\alpha \cdot \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Example

A product developer aims to reduce the drying time of a primer paint and tests two formulations: formulation 1, the standard chemistry, and formulation 2, which includes a new drying ingredient expected to decrease drying time.

It is known from prior experience that the standard deviation of drying time is 8 minutes, and this inherent variability is assumed to be unaffected by the addition of the new ingredient. Ten specimens are painted with formulation 1, and another 10 specimens with formulation 2; the 20 specimens are painted in random order. The sample average drying times are $\bar{x}_1 = 121$ minutes and $\bar{x}_2 = 112$ minutes, respectively.

Construct a 95% confidence interval based on the difference in means.

Answer. $2 \leq \mu_1 - \mu_2 \leq 16$

Hypothesis Test for Difference in Means (variances known)

Theorem (Traditional Method)

- *Step 1. Construct the two hypotheses*

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 \neq \Delta_0.$$

- *Step 2. Find the test statistic* $z_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$
- *Step 3. Identify acceptance region (use $\mathcal{N}(0,1)$).*
- *Step 4. Make a decision:*
  If z_0 is in critical region, then reject H_0.
  If z_0 is in acceptance region, then we fail to reject H_0

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Hypothesis Test for Difference in Means (variances known)

Theorem (P-value Method)

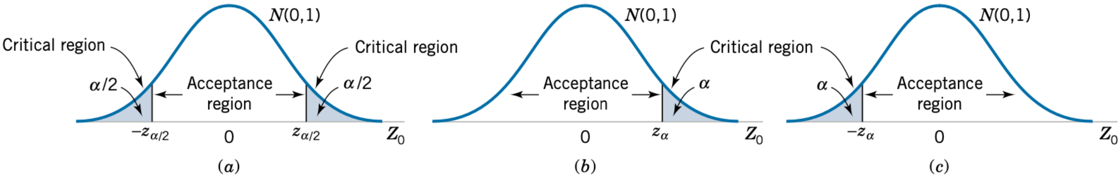
- *Step 1. Construct the two hypotheses*

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 \neq \Delta_0.$$

- *Step 2. Find the test statistic* $z_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$
- *Step 3. Find the P-value.*
- *Step 4. Make a decision:*
  If P-value $\leq \alpha$, *then reject* H_0.
  If P-value $> \alpha$, *then fail to reject* H_0

Critical regions and P-values for one-tailed tests (σ known)

- Critical regions in two-tailed, upper-tailed, and lower-tailed tests (left to right)



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$$\text{P-value} = \begin{cases} 2\Big(1 - \Phi(|z_0|)\Big) & \text{for case} H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 \neq \Delta_0 \\ 1 - \Phi(z_0) & \text{for case } H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 > \Delta_0 \\ \Phi(z_0) & \text{for case} H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 < \Delta_0 \end{cases}$$

Example (cont')

What conclusions can the product developer draw about the effectiveness of the new ingredient, using $\alpha = 0.05$?

Hint. The hypotheses

$$H_0 : \mu_1 - \mu_2 = 0 \text{ vs } H_1 : \mu_1 - \mu_2 > 0.$$

The test statistic

$$z_0 = \frac{121 - 112 - 0}{\sqrt{\frac{8^2}{10} + \frac{8^2}{10}}} \approx 2.52.$$

- Traditional Method: acceptance region is $(-\infty, 1.645]$
- P-value Method: P-value $= 1 - P(Z < 2.52)$

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Inference on the difference in means of two normal distributions, variances unknown (assume equal variances)

Question. What if we do NOT know population variances? (Assume equal variances)

- We need to replace population variances by **pooled variances**

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

- Use t-distribution with degree of freedom

$$df = n_1 + n_2 - 2$$

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Confidence interval on the difference in means (variances unknown)

Theorem (σ_1, σ_2 unknown)

A $100(1-\alpha)\%$ *C.I. on the difference in means* $\mu_1 - \mu_2$

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,df} \cdot \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} \leq \mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,df} \cdot \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}$$

One-sided Confidence interval on the difference in means (variances unknown)

Theorem (σ_1, σ_2 unknown)

- *A* $100(1-\alpha)\%$ *upper-confidence bound for* μ *is*

$$\mu_1 - \mu_2 \leq \bar{x}_1 - \bar{x}_2 + t_{\alpha,df} \cdot \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}$$

- *A* $100(1-\alpha)\%$ *lower-confidence bound for* μ *is*

$$\mu_1 - \mu_2 \geq \bar{x}_1 - \bar{x}_2 - t_{\alpha,df} \cdot \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}$$

Example

Two catalysts are being analyzed to determine how they affect the mean yield of a chemical. Construct a 95% confidence interval for the difference in means.

No.	Catalyst 1	Catalyst 2
1	91.50	89.19
2	94.18	90.95
3	92.18	90.46
4	95.39	93.21
5	91.79	97.19
6	89.07	97.04
7	94.72	91.07
8	89.21	92.75
$\bar{x}$	92.255	92.733
s	2.39	2.98

Hypothesis Test for Difference in Means (variances unknown)

Theorem (Traditional Method)

- *Step 1. Construct the two hypotheses*

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 \neq \Delta_0.$$

- *Step 2. Find the test statistic* $t_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\dfrac{s_p^2}{n_1} + \dfrac{s_p^2}{n_2}}}$
- *Step 3. Identify acceptance region (use t-distribution with* $df = n_1 + n_2 - 2$*).*
- *Step 4. Make a decision:*
  If z_0 *is in critical region, then reject* H_0*.*
  If z_0 *is in acceptance region, then we fail to reject* H_0

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Hypothesis Test for Difference in Means (variances unknown)

Theorem (P-value Method)

- *Step 1. Construct the two hypotheses*

$$H_0 : \mu_1 - \mu_2 = \Delta_0 \text{ vs } H_1 : \mu_1 - \mu_2 \neq \Delta_0.$$

- *Step 2. Find the test statistic* $t_0 = \dfrac{\bar{x}_1 - \bar{x}_2 - \Delta_0}{\sqrt{\dfrac{s_p^2}{n_1} + \dfrac{s_p^2}{n_2}}}$
- *Step 3. Find the P-value (use t-distribution with* $df = n_1 + n_2 - 2$*).*
- *Step 4. Make a decision:*
  If P-value $\leq \alpha$, *then reject* H_0.
  If P-value $> \alpha$, *then fail to reject* H_0

Example (cont')

Use significant level 0.05 and assume equal variances, is there any difference in the mean yields.

Hint. Hypothesis test

$$H_0 : \mu_1 - \mu_2 = 0 \text{ vs } H_1 : \mu_1 - \mu_2 \neq 0.$$

The test statistic

$$t_0 \approx -0.354$$

- Traditional Method: acceptance region $[-2.145, 2.145]$
- P-value Method: P-value 0.728

3. Inference on the Difference of Two Population Proportions

Assumption for two sample inference

Two independent random samples of size n_1 and n_2 (large enough).

Remarks.

- Sample proportion: $\hat{p}_1 = \dfrac{x_1}{n_1}$ and $\hat{p}_2 = \dfrac{x_2}{n_2}$
- $\hat{p}_1 - \hat{p}_2$ is a point estimator of $p_1 - p_2$
- If n_1 and n_2 are large enough, we have

$$\hat{p}_1 - \hat{p}_2 \sim \mathcal{N}\left(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}\right)$$

- Pooled proportion: $\bar{p} = \dfrac{x_1 + x_2}{n_1 + n_2}$

Confidence interval on the difference of two proportions

Theorem (two-sided C.I.)

A $100(1-\alpha)\%$ *confidence interval for* $(p_1 - p_2)$ *is*

$$\hat{p}_1 - \hat{p}_2 - z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} \le p_1 - p_2$$
$$\le \hat{p}_1 - \hat{p}_2 + z_{\alpha/2} \cdot \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

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Example

Extracts of St. Johns Wort are widely used to treat depression. An article in the April 18, 2001, issue of the *Journal of the American Medical Association* ("Effectiveness of St. Johns Wort on Major Depression: A Randomized Controlled Trial") compared the efficacy of a standard extract of St. Johns Wort with a placebo in 200 outpatients diagnosed with major depression. Patients were randomly assigned to two groups; one group received St. Johns Wort, and the other received a placebo. After eight weeks, 19 of the placebo-treated patients showed improvement, and 27 of those treated with St. Johns Wort improved.

Construct a 95% confidence interval for the difference between these two proportions.

Hypothesis Test for Difference in Population proportions

Theorem (Traditional Method)

- *Step 1. Form the two hypotheses*

$$H_0 : p_1 - p_2 = 0 \text{ vs } H_1 : p_1 - p_2 \neq 0.$$

- *Step 2. Find the test statistic* $z_0 = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\dfrac{\bar{p}(1-\bar{p})}{n_1} + \dfrac{\bar{p}(1-\bar{p})}{n_2}}}$
- *Step 3. Identify acceptance region (use $Z = \mathcal{N}(0,1)$).*
- *Step 4. Make a decision:*
  If z_0 is in critical region, then reject H_0.
  If z_0 is in acceptance region, then we fail to reject H_0

Example (cont')

Is there any reason to believe that St. Johns Wort is effective in treating major depression? Use $\alpha = 0.05$.

Remark. We can also use P-value method to solve this problem.

Thank you!

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