Lecture 9

HYPOTHESIS TEST FOR A SINGLE SAMPLE

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Contents

1. Statistical hypothesis

Statistical hypothesis

Definition

A **statistical hypothesis** is a statement (claim) about the parameters of one or more populations.

Example.

- A company claims that the mean weight all product is 150 g.

$\rightarrow$ This is a claim about the population mean: $\mu = 150$ g.

- A university claims that the employment rate of its students after graduation is more than 94%.

$\rightarrow$ This is a claim about the population proportion: $p > 0.94$.

Remark. We will use information from a random sample to decide whether the claim is acceptable.

Test of a hypothesis

- A procedure leading to a decision about a particular hypothesis is called a test of a hypothesis.
- A (two-tailed) hypothesis test about the population mean can be formed as:

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu \neq \mu_0.$$

- We call:

  H_0: Null hypothesis
  H_1: Alternative hypothesis

Example

We wish to test

$$H_0 : \mu = 150 \text{ vs } H_1 : \mu \neq 150.$$

Assume that $H_0 : \mu = 150$ is true, a random sample of $n = 10$ objects is selected and the sample mean $\bar{x}$ is observed.

- If $\bar{x}$ falls close to the hypothesized value of $\mu = 150$, we **fail to reject** H_0; it is evidence in support of the null hypothesis.
- If $\bar{x}$ is considerably different from 150, we **reject** H_0; it is evidence in support of the alternative hypothesis.

Types of error

- **Type I error** (False Positive) This error occurs when you reject a null hypothesis that is actually true.
- **Type II error** (False Negative) This error occurs when you fail to reject a null hypothesis that is actually false.

Decision	H_0 **is True**	H_0 **is False**
Fail to reject H_0	no error	type II error
Reject H_0	type I error	no error

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is True})$$
$$\beta = P(\text{type II error}) = P(\text{fail to reject } H_0 \mid H_0 \text{ is False})$$

- α is called the **significance level**.

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Probability of Type I error

Theorem

$$\alpha = P(\textit{type I error}) = P(\bar{x} \textit{ is in critical region} \mid \mu = \mu_0)$$

Example. Suppose that if $148 \leq \bar{x} \leq 152$, we will not reject the null hypothesis $H_0 : \mu = 150$, and if either $\bar{x} < 148$ or $\bar{x} > 152$, we will reject the null hypothesis in favor of the alternative hypothesis $H_1 : \mu \neq 150$. Thus,

$$\alpha = P(\bar{x} < 148 \cup \bar{x} > 152 | \ \mu = 150).$$

By Central Limit Theorem, we have

$$\alpha = P\left(Z < \frac{148 - 150}{\sigma/\sqrt{n}}\right) + P\left(Z > \frac{152 - 150}{\sigma/\sqrt{n}}\right).$$

2. Hypothesis test for the population mean μ

Hypothesis Test for Population Mean μ (σ known)

Theorem (Traditional Method (two-tailed test))

- *Step 1. Form the two hypotheses*

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu \neq \mu_0.$$

- *Step 2. Find the test statistic* $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$.
- *Step 3. Identify acceptance region* $[-z_{\alpha/2}, z_{\alpha/2}]$.
- *Step 4. Make a decision:*
  If z_0 *is in critical region, then reject* H_0.
  If z_0 *is in acceptance region, then we fail to reject* H_0.

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Hypothesis Test for Population Mean μ (σ known)

Theorem (P-value Method (two-tailed test))

- *Step 1.* *Form the two hypotheses*

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu \neq \mu_0.$$

- *Step 2.* *Find the test statistic* $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$
- *Step 3.* *Find the P-value.*

$$\textit{P-value} = P(Z < -|z_0| \text{ or } Z > |z_0|)$$

- *Step 4.* *Make a decision:*
  If P-value $\leq \alpha$, *then reject* H_0.
  If P-value $> \alpha$, *then fail to reject* H_0.

Example

The heights of all adults in a community are known to have a standard deviation of 0.03m. A random sample of 43 adults is collected, and their average height is 1.68m. Test the hypothesis that the true average height of all adults in the community is 1.7m at a significance level of $\alpha = 0.05$.

Hint. The test statistic

$$z_0 = \frac{1.68 - 1.70}{0.03/\sqrt{43}} \approx -4.37.$$

- Traditional Method: acceptance region is $[-1.96, 1.96]$
- P-value Method: P-value $= 2 * P(Z < -4.37) \approx 0$

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One-tailed test for Population Mean μ (σ known)

In hypothesis testing, when the objective is to determine whether the population mean is significantly greater or smaller than a specified value, a **one-tailed test** can be formulated.

$$H_0 : \mu \leq \mu_0 \text{ vs } H_1 : \mu > \mu_0 \text{ (upper-tailed test)}$$
$$H_0 : \mu \geq \mu_0 \text{ vs } H_1 : \mu < \mu_0 \text{ (lower-tailed test)}$$

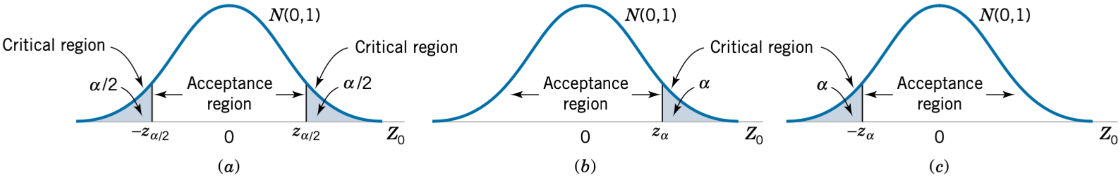
Remark.
- The null hypothesis in these cases can be written as

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu > \mu_0 \text{ (upper-tailed test)}$$
$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu < \mu_0 \text{ (lower-tailed test)}$$

- The test statistic $z_0 = \dfrac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$

Critical regions and P-values for one-tailed tests (σ known)

- Critical regions in two-tailed, upper-tailed, and lower-tailed tests (left to right)



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$$\text{P-value} = \begin{cases} 2\Big(1-\Phi(|z_0|)\Big) & \text{for a two-tailed test: } H_0: \mu = \mu_0, \quad H_1: \mu \neq \mu_0 \\ 1-\Phi(z_0) & \text{for an upper-tailed test: } H_0: \mu = \mu_0, \quad H_1: \mu > \mu_0 \\ \Phi(z_0) & \text{for a lower-tailed test: } H_0: \mu = \mu_0, \quad H_1: \mu < \mu_0 \end{cases}$$

Example

Consider a manufacturing process where the weight of a product is critical. The population standard deviation of the weights is known to be 2.5 grams. A random sample of 30 products is taken, and the average weight is calculated to be 48 grams. The company is concerned that the average weight is less than the specified value of 50 grams.

The hypotheses can be stated as follows:

$$H_0 : \mu \geq 50 \text{ vs } H_1 : \mu < 50$$

where μ is the true average weight.

Using the known population standard deviation $(\sigma = 2.5)$, a one-tailed z-test can be conducted to determine whether there is enough evidence to reject the null hypothesis and conclude that the average weight is less than 50 grams.

Hypothesis Test for Population Mean μ (σ unknown)

Theorem (Traditional Method)

- *Step 1.* *Form the two hypotheses*

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu \neq \mu_0.$$

- *Step 2.* *Find the test statistic* $t_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$
- *Step 3.* *Identify acceptance region (use t-distribution with* $df = n - 1$*).*
- *Step 4.* *Make a decision:*
  If t_0 *is in critical region, then reject* H_0*.*
  If t_0 *is in acceptance region, then we fail to reject* H_0

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Hypothesis Test for Population Mean μ (σ unknown)

Theorem (P-value Method)

- *Step 1.* *Form the two hypotheses*

$$H_0 : \mu = \mu_0 \text{ vs } H_1 : \mu \neq \mu_0.$$

- *Step 2.* *Find the test statistic* $t_0 = \dfrac{\bar{x} - \mu_0}{s/\sqrt{n}}$
- *Step 3.* *Find the P-value (use t-distribution with* $df = n - 1$*).*
- *Step 4.* *Make a decision:*
  If P-value $\leq \alpha$, *then reject* H_0.
  If P-value $> \alpha$, *then fail to reject* H_0

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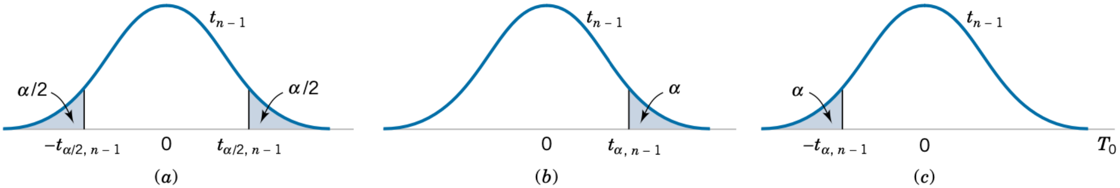
Critical regions and P-value for hypothesis tests μ (σ unknown)

- Critical regions in two-tailed, upper-tailed, and lower-tailed tests (left to right)



- $$\text{P-value} = \begin{cases} 2P(T > |t_0|) & \text{for a two-tailed test: } H_0 : \mu = \mu_0, \quad H_1 : \mu \neq \mu_0 \\ P(T > t_0) & \text{for an upper-tailed test: } H_0 : \mu = \mu_0, \quad H_1 : \mu > \mu_0 \\ P(T < t_0) & \text{for a lower-tailed test: } H_0 : \mu = \mu_0, \quad H_1 : \mu < \mu_0 \end{cases}$$

Example

Example. The heights of all adults in a community are known to have a normal distribution. A random sample are collected and the heights (in meters) are recorded as follows:

$$1.57 \quad 1.60 \quad 1.59 \quad 1.62 \quad 1.65 \quad 1.70 \quad 1.68$$

Test the hypothesis that the average height of all adults in the community is at most 1.65m, at the significance level of 5%.

Hint. Hypothesis test

$$H_0 : \mu = 1.65 \text{ vs } H_1 : \mu \leq 1.65.$$

It is easy to find $n = 7, \bar{x} = 1.63, s = 0.0483$.

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3. Hypothesis test for the population proportion p

Hypothesis Test for Population proportion p

Theorem (Traditional Method)

- *Step 1. Form the two hypotheses*

$$H_0 : p = p_0 \text{ vs } H_1 : p \neq p_0.$$

- *Step 2. Find the test statistic* $z_0 = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$
- *Step 3. Identify acceptance region (use $Z = \mathcal{N}(0,1)$).*
- *Step 4. Make a decision:*
  If z_0 is in critical region, then reject H_0.
  If z_0 is in acceptance region, then we fail to reject H_0.

Hypothesis Test for Population proportion p

Theorem (P-value Method)

- *Step 1. Form the two hypotheses*

$$H_0 : p = p_0 \text{ vs } H_1 : p \neq p_0.$$

- *Step 2. Find the test statistic* $z_0 = \dfrac{\hat{p} - p_0}{\sqrt{\dfrac{p_0(1-p_0)}{n}}}$
- *Step 3. Find the P-value (use* $Z = \mathcal{N}(0,1)$*).*
- *Step 4. Make a decision:*
  If P-value $\leq \alpha$*, then reject* H_0*.*
  If P-value $> \alpha$*, then fail to reject* H_0

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Example

A company claims that the percentage of defective products is kept under control, i.e., less than 3%. In order to test this claim, a random sample of 135 products is taken, and it is found that 6 of them are defective. The hypothesis test is conducted at a significance level of 5%.

Hint. The hypotheses can be stated as follows:

$$H_0 : p \geq 0.03$$

$$H_1 : p < 0.03$$

where p is the true proportion of defective products.

Thank you!

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