Lecture 2

PROBABILITY

FPT Education

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Department of Mathematics

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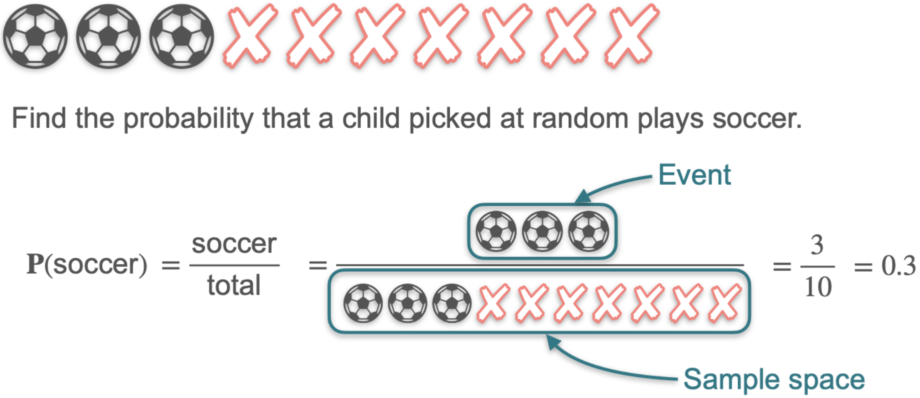
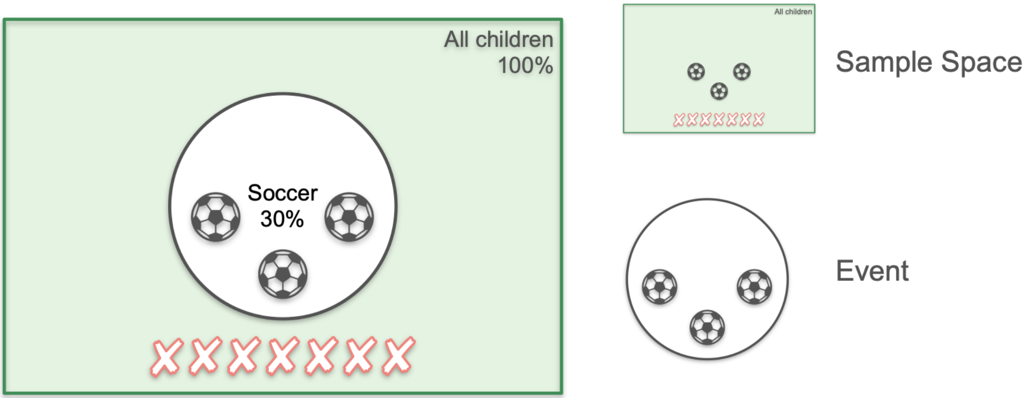
1. Sample Spaces and Events

Introduction to Probability



Introduction to Probability: Venn Diagram

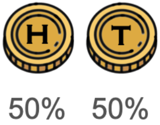


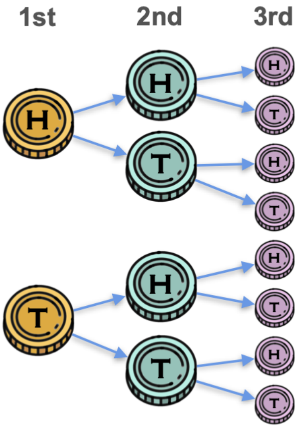
Sample spaces and events

- An **experiment** that can result in different outcomes, even though it is repeated in the same manner every time, is called a random experiment.
- The set of all possible outcomes of a random experiment is called the **sample space** (denoted as S).
- An **event** is a subset of the sample space of a random experiment.

Example. Toss a coin three times. Determine the sample space of that experiment.

Solution.





Quizlets

Quiz 1. A probability experiment consists of tossing a coin and then rolling a six-sided die. Describe the sample space.

Quiz 2. Each message in a digital communication system is classified according to whether it is received within the time specified by the system design. If three messages are classified, use a tree diagram to represent the sample space of possible outcomes.

2. Interpretations of Probability

Interpretations of Probability: Introduction

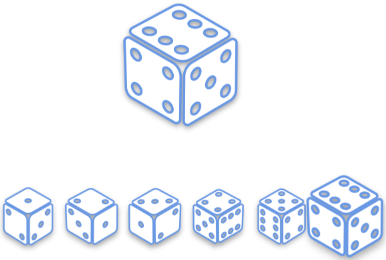
There are different approaches to assessing the probability of an uncertain event:

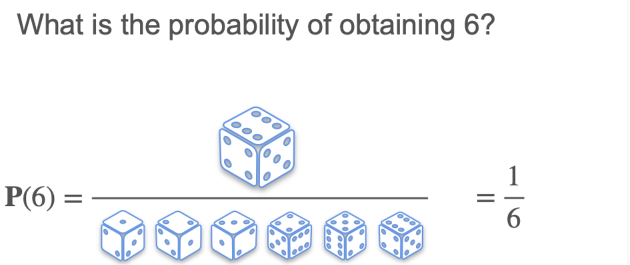
1. **priori classical probability**: the probability of an event is based on prior knowledge of the process involved.
2. **empirical classical probability**: the probability of an event is based on observed data.

Theorem (Equally Likely Outcomes)

Whenever a sample space consists of N possible outcomes that are equally likely, the probability of each outcome is $1/N$.

Example.





Interpretations of Probability

Priori classical probability

$$\text{Probability of Occurrence} = \frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}.$$

Empirical classical probability

$$\text{Probability of Occurrence} = \frac{\text{number of favorable out comes observed}}{\text{total number of outcomes observed}}.$$

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Priori classical probability

Example. Find the probability of selecting a face card (Jack, Queen, or King) from a standard deck of 52 cards.

Solution.

$$\text{Probability of Face Card} = \frac{\text{number of face cards}}{\text{total number of cards}} = \frac{12}{52}.$$

Empirical classical probability

Example. Find the probability of selecting a male using statistics from the population described in the following table:

	Taking Stats	Not Taking Stats	Total
Male	84	145	229
Female	76	134	210
Total	160	279	439

Solution.

$$\text{Probability of males taking stats} = \frac{\text{number of males taking stats}}{\text{total number of people}} = \frac{84}{439}.$$

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Axioms of Probability

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties:

1. $P(S) = 1$,
2. $0 \leq P(E) \leq 1$,
3. For two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2).$$

In which, S is the sample space and E is any event.

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Remark. For a discrete sample space, the probability of an event E, denoted as $P(E)$, equals the sum of the probabilities of the outcomes in E.

Example 1. A random experiment can result in one of the outcomes $S = \{a, b, c, d\}$ with probabilities $0.1, 0.3, 0.5$, and 0.1, respectively. Let $A = \{a, b\}$, $B = \{b, c, d\}$, $A' = S \setminus A$, $B' = S \setminus B$. *Find Probabilities:*

$$P(A), P(B), P(A'), P(B'), P(A \cap B), P(A \cup B).$$

Example 2. A visual inspection of a location on wafers from a semiconductor manufacturing process resulted in the following table. *What is the probability that a wafer contains three or more particles in the inspected location?*

No. of contamination particles	0	1	2	3+
Proportion of wafers	0.4	0.15	0.2	0.25

3. Basic Set Operations

Basic Set Operations

Definitions

- The **union of two events** is the event that consists of all outcomes that are contained in either of the two events. We denote the union as $E_1 \cup E_2$.
- The **intersection of two events** is the event that consists of all outcomes that $E_1 \cap E_2$.
- The **complement of an event** in a sample space is the set of outcomes in the sample space that are not in the event. We denote the component of the event E as E'.

Basic Properties

- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$

- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$

- $A = (A \cap B) \cup (A \cap B')$

Complement Rule

If the **complement** of A, denoted by A' consists of all the outcomes in which the event A does not occur, then we have

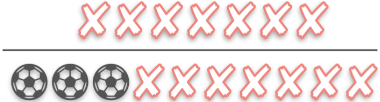
$$P(A) + P(A') = 1.$$

Example.

30%

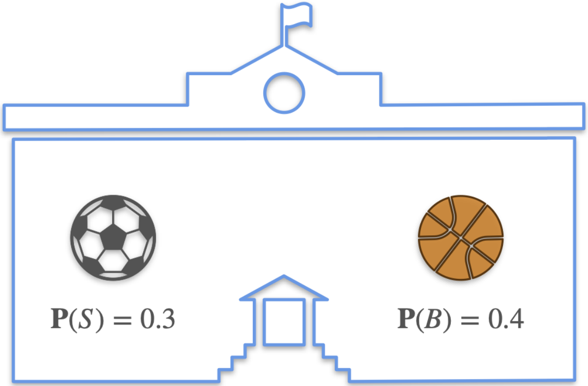
What is the probability of a child NOT playing soccer?

$$\mathbf{P}(\text{not soccer}) = \frac{\text{not soccer}}{\text{total}} =$$



$$= \frac{7}{10}$$

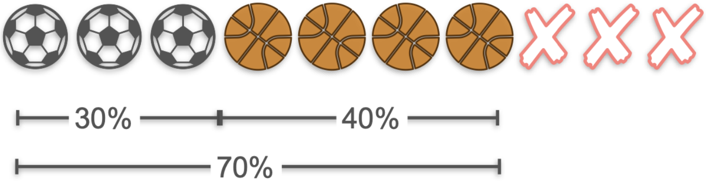
Mutually exclusive



At a school, kids can only play one sport.

What is the probability that a kid plays soccer or basketball?

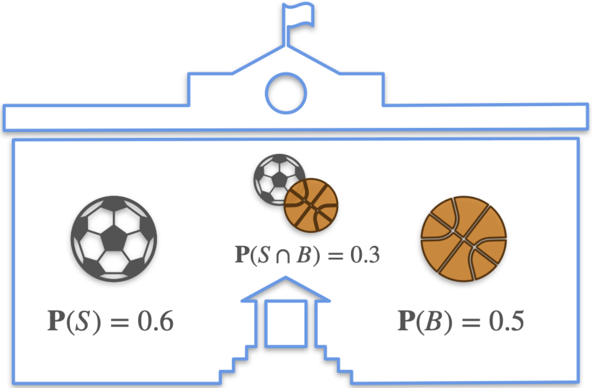
Hint: What if there were only 10 kids?



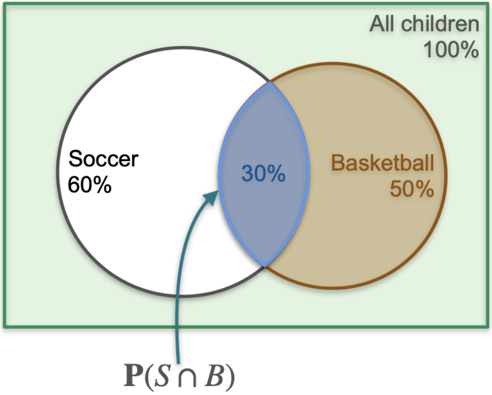
$$\mathbf{P}(\text{soccer or basketball}) = \frac{\text{soccer or basketball}}{\text{total}} = \frac{3+4}{10} = 0.7$$

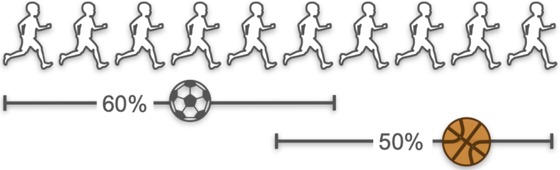
$$\mathbf{P}(\text{soccer or basketball}) = \mathbf{P}(\text{soccer}) + \mathbf{P}(\text{basketball})$$

Non-mutually exclusive



What is the probability that a child plays soccer or basketball?





$$
\begin{aligned}
\mathbf{P}(S \cup B) &= \mathbf{P}(S) + \mathbf{P}(B) - \mathbf{P}(S \cap B) \\
&= 0.6 + 0.5 - 0.3 \\
&= 0.8
\end{aligned}
$$

3. Addition rules

Addition rules

1. If A and B are mutually exclusive events,

$$P(A \cup B) = P(A) + P(B)$$

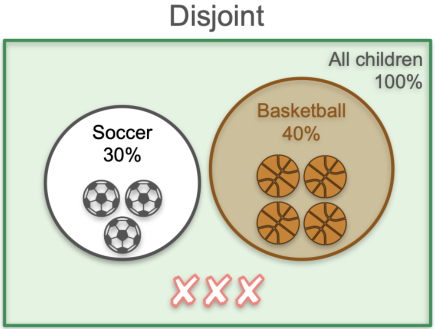
2. A collection of events, $E_1, E_2, ..., E_k$ is said to be mutually exclusive if for all pairs, $E_i \cap E_j = \emptyset$. For a collection of mutually exclusive events,

$$P(E_1 \cup E_2 \cup ... \cup E_k) = P(E_1) + P(E_2) + ... + P(E_k)$$

3. General: If A and B are any events,

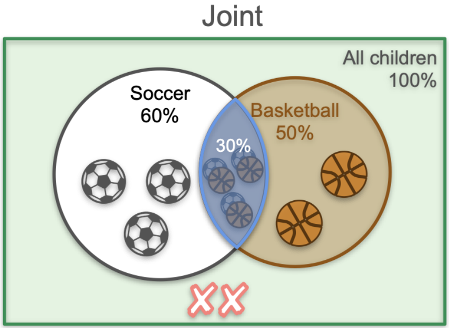
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Disjoint Events Vs Joint Events



Mutually exclusive

$$\mathbf{P}(S \cup B) = \mathbf{P}(S) + \mathbf{P}(B)$$



Non-mutually exclusive

$$\mathbf{P}(S \cup B) = \mathbf{P}(S) + \mathbf{P}(B) - \mathbf{P}(S \cap B)$$

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4. Conditional Probability

Conditional Probability

The **conditional probability** of an event B, given that an event A already occurred, denoted by $P(B|A)$, is computed as

$$P(B|A) = \frac{P(B \cap A)}{P(A)}.$$

Special case: All outcomes are equally likely

$$\frac{P(B \cap A)}{P(A)} = \frac{\text{number of outcomes in } B \cap A}{\text{number of outcomes in } A}.$$

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Conditional Probability: Example 1

Of the cars on a used car lot, 70% have air conditioning (AC) and 40% have a CD player (CD). 20% of the cars have both. *What is the probability that a car has a CD player, given that it has AC ?*

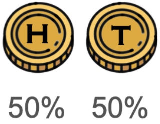
Solution.

	CD	No CD	Total
AC	0.2	0.5	0.7
No AC	0.2	0.1	0.3
Total	0.4	0.6	1.0

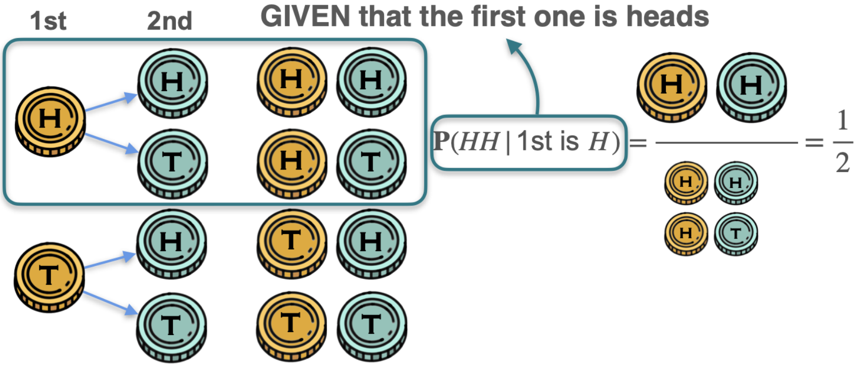
One has

$$P(CD|AC) = \frac{P(CD \text{ and } AC)}{P(AC)} = \frac{0.2}{0.7} = \frac{2}{7}.$$

Conditional Probability: Example 2



What is the probability of landing on heads twice?



5. Multiplication & Total Probability Rules

Multiplication rule

Theorem

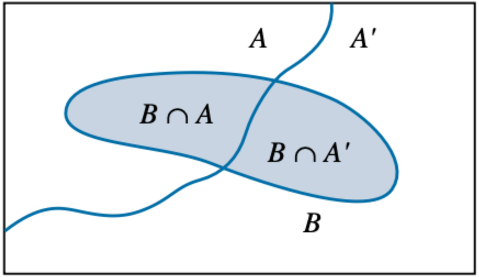
$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A)$$

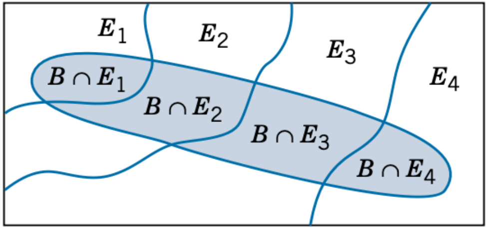
Example. The probability that an automobile battery subjected to high engine compartment temperatures suffers low charging current is 0.7. The probability that a battery is subject to high engine compartment temperatures is 0.05. What is the probability that a battery is subject to low charging current and high engine compartment temperature?

Solution. Let $C = \{$a battery suffers low charging current$\}$ and $T = \{$a battery is subject to high engine compartment temperature$\}$. The probability that a battery is subject to low charging current and high engine compartment temperature is

$$P(C \cap T) = P(C|T)P(T) = 0.7 \times 0.05 = 0.035$$

Partitioning an event





$$B = (B \cap E_1) \cup (B \cap E_2) \cup (B \cap E_3) \cup (B \cap E_4)$$

Total Probability Rule

for two events

$$P(B) = P(B \cap A) + P(B \cap A') = P(B|A)P(A) + P(B|A')P(A')$$

for multiple events

Assume $E_1, E_2, ..., E_k$ are mutually exclusive and exhaustive events. Then,

$$\begin{aligned} P(B) &= P(B \cap E_1) + P(B \cap E_2) + \cdots + P(B \cap E_k) \\ &= P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + \cdots + P(B|E_k)P(E_k) \end{aligned}$$

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Total Probability Rule: Example

Example. The subsequent table provides a comprehensive summary of the data presented in the contamination discussion.

Probability of Failure	Level of Contamination	Probability of Level
0.100	High	0.2
0.005	No high	0.8

Let F denote the event that the product fail. Find $P(F)$.

Solution. Let H denote the event that the chip is exposed to a high level of contamination. We have

$$P(H) = 0.2 \text{ and } P(H') = 0.8$$

Moreover, $P(F|H) = 0.1$ and $P(F|H') = 0.005$. Thus,

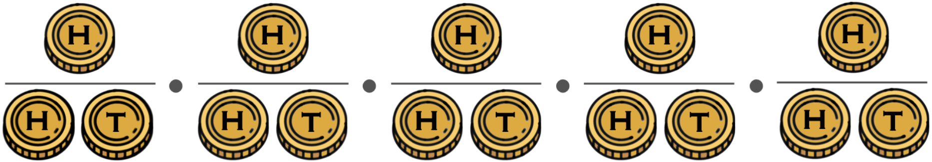
$$P(F) = 0.1 * 0.2 + 0.005 * 0.8 = 0.024$$

6. Independence

50% 50%

What is the probability of landing on heads five times?

P(5 heads) =



$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

Independence

Two events are called **independent** if the occurrence of one event does not change the probability of the other event. Equivalently, any one of the following equivalent statements is true:

1. $P(A|B) = P(A)$
2. $P(A \cap B) = P(A) \cdot P(B)$
3. $P(B|A) = P(B)$

Remark. If A and B are independent events, then so are events A and B', events A' and B, and events A' and B'.

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Example. A day's production of 850 manufactured parts contains 50 parts that do not meet customer requirements. Two parts are selected at random, without replacement, from the batch. Let $A = \{\text{the first part is defective}\}$, and $B = \{\text{the second part is defective}\}$. We suspect that these two events are not independent because knowledge that the first part is defective suggests that it is less likely that the second part selected is defective.

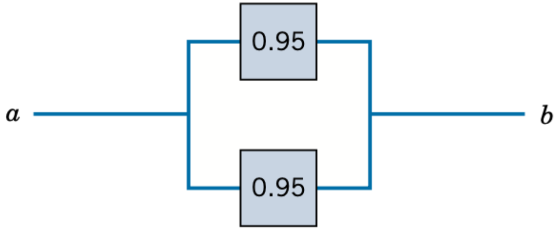
Solution.

$$
\begin{aligned}
P(B|A) &= 49/849 \\
P(B) &= P(B|A)P(A) + P(B|A')P(A') \\
&= (49/849)(50/850) + (50/849)(800/850) = 50/850
\end{aligned}
$$

$\Rightarrow$ Two events are not independent, as we suspected.

Example

The following circuit operates only if there is a path of functional devices from left to right. The probability that each device functions is shown on the graph. Assume that devices fail independently. *What is the probability that the circuit operates?*



Solution. Let T and B denote the events that the top and bottom devices operate, respectively.

$$\begin{aligned} P(T \cup B) &= 1 - P(T' \cap B') \\ &= 1 - (1 - 0.95)^2 = 0.9975 \end{aligned}$$

The events $E_1, E_2, ..., E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, ..., E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap ... \cap E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times ... \times P(E_{i_k})$$

Quiz. Two coins are tossed. Let A denote the event as At most one head on the two tosses. and B denote the event One head and one tail in both tosses. Are A and B independent events?

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The Birthday Problem

You have 30 friends at a party. What do you think is more likely:

- That there exist two people with the same birthday
- That no two of them have the same birthday

(Assume the year has 365 days, nobody has a birthday on Feb 29)

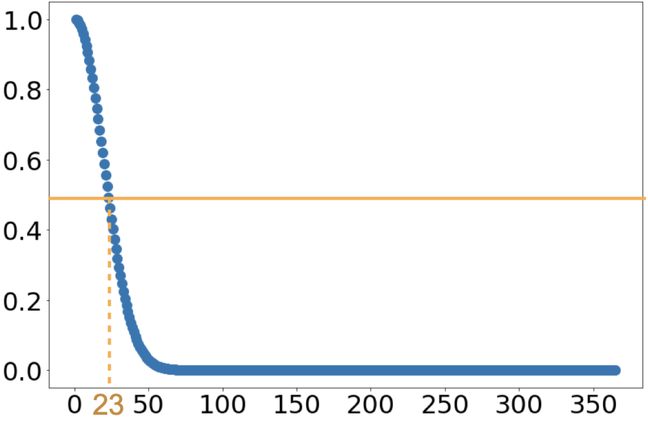
Answer. Its more likely that 2 people have the same birthday. In fact, the probability of no two people having the same birthday is around 0.3.

Probability That Everyone Has a Different Birthday

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdot \frac{362}{365} \cdot \frac{361}{365} \cdot \frac{360}{365} \cdot \frac{359}{365} \cdot \frac{358}{365} \cdot \frac{357}{365} = 0.905$$

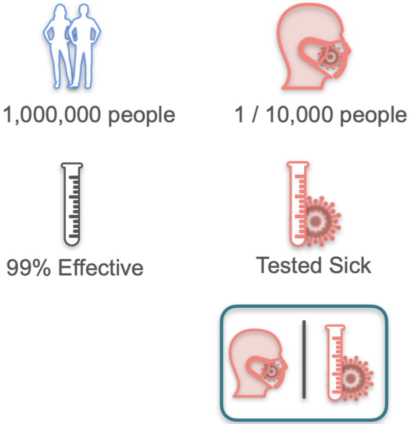
Probability That no Two People Have the Same Birthday

1 person: 1
2 people: 0.997
3 people: 0.992
4 people: 0.984
5 people: 0.973
10 people: 0.883
20 people: 0.589
23 people: 0.493
30 people: 0.294
50 people: 0.030
100 people: 0.0000003
365 people: 0



7. Bayes' Theorem

Motivation



	Diagnosed Sick	Diagnosed Healthy
100 people	99	1
100 people	1	99

What's the probability that **you are sick GIVEN that you tested sick**?

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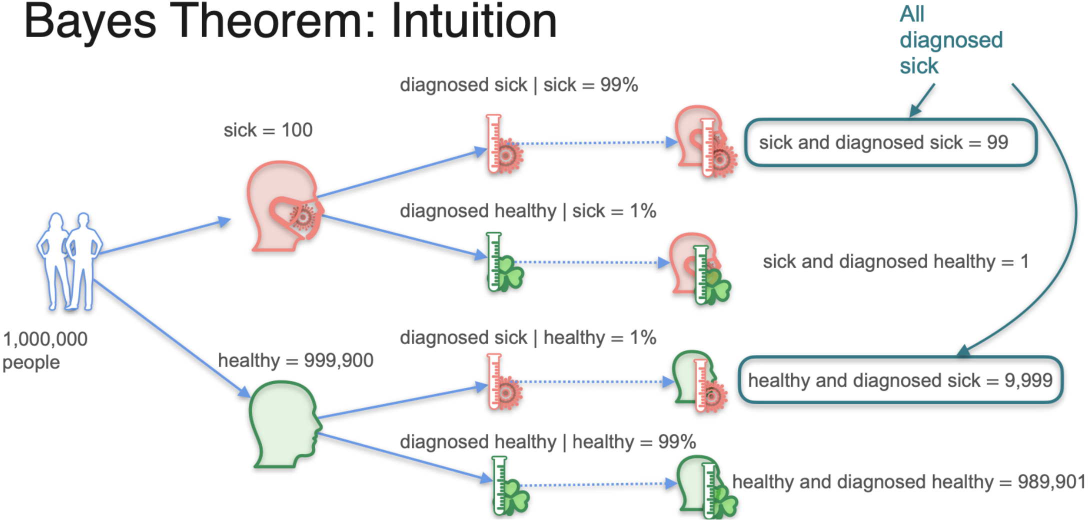
Bayes Theorem: Intuition



Bayes' Theorem

Theorem

If $E_1, E_2, ..., E_k$ *are* k *mutually exclusive and exhaustive events and* B *is any event,*

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{P(B|E_1)P(E_1) + P(B|E_2)P(E_2) + ... + P(B|E_k)P(E_k)}, \text{ for } P(B) > 0.$$

Special case:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \text{ for } P(B) > 0.$$

Quizlets

Quiz 1. Given two events A and B such that $P(A \cap B) = 0.15, P(A \cup B) = 0.65$, and $P(A|B) = 0.5$. Find $P(B|A)$.

Quiz 2. In a state where cars have to be tested for pollution emissions, 25% of all cars emit excessive amounts of pollutants. 99% of all tested cars that emit excessive amounts of pollutants will fail, but 17% of all tested cars that do not emit excessive amounts of pollutants will also fail.
What is the probability that a car that fails the test actually emits excessive amounts of pollutants?

8. Random Variables

From Events to Random Variables

$X =$ Number of heads in 10 coin tosses

$X = 5$

$X = 6$

$X = 5$

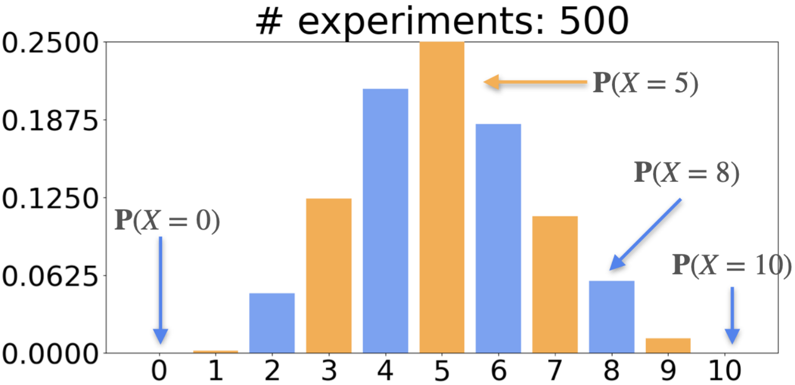
Possible outcomes:

0	4	8
1	5	9
2	6	10
3	7	

$\mathbf{P}(H) = 0.5$

Repeat 500 times

Flipping a Fair Coin 500 Times



$\mathbf{P}(H) = 0.5$

Repeat 500 times

Random Variable

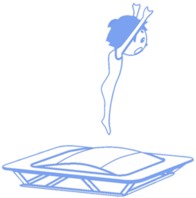
A **random variable** is a mathematical concept used in probability theory and statistics to describe the outcomes of a random experiment or process. It assigns numerical values to each possible outcome of a random phenomenon.

Remark.

1. A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as $x = 70$ milli-amperes.
2. A **discrete random variable** is a random variable with a finite (or countable infinite) range.
3. A **continuous random variable** is a random variable with an interval of real numbers for its range.

Other Random Variables

Wait time until the next bus arrives



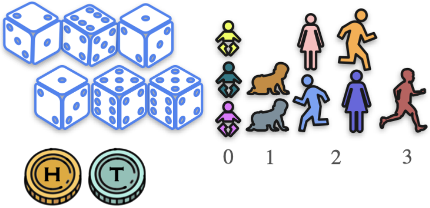
Height of an gymnast's jump

Number of defective products in a shipment

mm. of rain in November

Discrete and Continuous Random Variables

Discrete random variables



~~Finite number of values~~

(Could be infinite too)

Can take only a **countable** number of values

Continuous random variables

Infinite number of values

Takes values on an interval

Thank you!

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