Statmat 2 Kelompok 8

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SLIDE 8

1) Interval kepercayaan untuk $\mu_1 - \mu_2$ dari dua distribusi $N(\mu_1, \sigma_1^2)$ dan $N(\mu_2, \sigma_2^2)$

e) Jika σ_1^2 dan σ_2^2 diketahui tetapi tidak sama

Dengan Teorema Limit Pusat

Misal $X_1, X_2, \ldots, X_n$ adalah random sampel berukuran n yang diambil dr suatu populasi dengan mean $= \mu$ dan variansi $= \sigma^2$. Dimana $\bar{X}$ adalah mean sampel, bentuk limit distribusi : $Z = \dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}$ $n \sim \infty$ adalah distribusi normal

$X_1, X_2, \ldots, X_n \sim N(\mu_1, \sigma_1^2)$

$Y_1, Y_2, \ldots, Y_n \sim N(\mu_2, \sigma_2^2)$

Mean dari $\bar{X} - \bar{Y}$

$$E(\bar{X}-\bar{Y}) = E\left(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i\right) - E\left(\frac{1}{n_2}\sum_{i=1}^{n_2} Y_i\right)$$

$$= \frac{1}{n_1}\sum_{i=1}^{n_1} EX_i - \frac{1}{n_2}\sum_{i=1}^{n_2} EY_i$$

$$= \mu_1 - \mu_2$$

Variansi dari $\bar{X} - \bar{Y}$

$$Var(\bar{X}-\bar{Y}) = Var(\bar{X}) + Var(\bar{Y}) - 2\,cov(\bar{X}, \bar{Y})$$

karena $\bar{X}$ & $\bar{Y}$ saling bebas maka $cov(\bar{X}, \bar{Y}) = 0$

$$Var(\bar{X}-\bar{Y}) = Var\left(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i\right) + Var\left(\frac{1}{n_2}\sum_{i=1}^{n_2} Y_i\right)$$

$$= \frac{1}{n_1^2}\sum_{i=1}^{n_1} Var(X_i) + \frac{1}{n_2^2}\sum_{i=1}^{n_2} Var(Y_i)$$

$$= \frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2}$$

Diambil kembali Teorema Limit Pusat,

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$$

$$= \frac{(\bar{X}-\bar{Y}) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2}}} \sim N(0,1)$$

$$P\left(-z_{\alpha/2} \le \frac{(\bar{X}-\bar{Y}) - (\mu_1-\mu_2)}{\sqrt{\sigma_1^2/n_1^2 + \sigma_2^2/n_2^2}} \le z_{\alpha/2}\right) = 1-\alpha$$

$$P\left(-z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2}} \le (\bar{X}-\bar{Y}) - (\mu_1-\mu_2) \le z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2}}\right) = 1-\alpha$$

$$P\left(\bar{x}-\bar{y} - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2}} \le \mu_1-\mu_2 \le \bar{x}-\bar{y} + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1^2} + \frac{\sigma_2^2}{n_2^2}}\right) = 1-\alpha$$

Date :

maka interval kepercayaan

$$\therefore \bar{x}-\bar{y} - z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1^2}+\frac{\sigma_2^2}{n_2^2}} \le \mu_1-\mu_2 \le \bar{x}-\bar{y} + z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1^2}+\frac{\sigma_1^2}{n_2^2}}$$

b) Jika σ_1^2 dan σ_2^2 tidak diketahui dan tidak sama

Misal $X_1, X_2, \dots, X_n$ adalah sampel random independent berdistribusi $N(\mu_1, \sigma_1^2)$

$Y_1, Y_2, \dots, Y_n$ adalah sampel random independent berdistribusi $N(\mu_2, \sigma_2^2)$

Dari poin 1 (a) didapat $Z = \dfrac{(\bar{x}-\bar{Y}) - (\mu_1-\mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1^2}+\frac{\sigma_2^2}{n_2^2}}} \sim N(0,1)$

$V = \dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2}$ maka statistik uji

$$T = \frac{Z}{\sqrt{V/r}}$$

$$= \frac{\dfrac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{\sigma_1^2/n_1^2+\sigma_2^2/n_2^2}}}{\sqrt{\dfrac{(n_1-1)S_1^2}{(n_1+n_2-2)\sigma_1^2}+\dfrac{(n_2-1)S_2^2}{(n_1+n_2-2)\sigma_2^2}}} \sim t(n_1+n_2-2)$$

$$= \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sigma_2\sqrt{\frac{\sigma_1^2}{n_1\sigma_2^2}+\frac{1}{n_2}}\;\frac{1}{\sigma_1}\sqrt{\frac{(n_1-1)S_1^2}{(n_1+n_2-2)}+\frac{(n_2-1)S_2^2\sigma_1^2}{(n_1+n_2-2)\sigma_2^2}}}$$

karena $\dfrac{\sigma_1^2}{\sigma_2^2} = k$ yang konstan maka $\dfrac{\sigma_1}{\sigma_2} = \sqrt{k}$

$$= \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{k}\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n_1-1)S_1^2}{(n_1+n_2-2)}+k\frac{(n_2-1)S_2^2}{(n_1+n_2-2)}}}$$

$$= \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{k}\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n_1-1)S_1^2+k(n_2-1)S_2^2}{(n_1+n_2-2)}}}$$

$$P\left(-t_{\frac{\alpha}{2};n_1+n_2-2} \le \frac{\bar{X}-\bar{Y}-(\mu_1-\mu_2)}{\sqrt{k}\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n_1-1)S_1^2+k(n_2-1)S_2^2}{(n_1+n_2-2)}}} \le t_{\frac{\alpha}{2};n_1+n_2-2}\right) = 1-\alpha$$

$$P\left(\bar{x}-\bar{y}-t_{\frac{\alpha}{2};n_1+n_2-2}\sqrt{k}\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n_1+1)S_1^2+k(n_2-1)S_2^2}{(n_1+n_2-2)}} \le \mu_1-\mu_2 \le \bar{x}-\bar{y}+t_{\frac{\alpha}{2};n_1+n_2-2}\sqrt{k}\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n_1-1)S_1^2+k(n_2-1)S_2^2}{(n_1+n_2-2)}}\right)$$

1. Interval kepercayaan

$$\bar{x}-\bar{y}-t_{\frac{\alpha}{2};n_1+n_2-2}\sqrt{k}\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n-1)S_1^2+k(n_2-1)S_2^2}{n_1+n_2-2}} \leq \mu_1-\mu_2 \leq \bar{x}-\bar{y}+t_{\frac{\alpha}{2};n_1+n_2-2}\sqrt{k}$$

$$\sqrt{\frac{k}{n_1}+\frac{1}{n_2}}\sqrt{\frac{(n-1)S_1^2+k(n_2-1)S_2^2}{n_1+n_2-2}}$$

2) Interval kepercayaan untuk σ_1^2/σ_2^2 dari dua distribusi $N(\mu_1,\sigma_1^2)$ dan $N(\mu_2,\sigma_2^2)$

a) Jika μ_1 dan μ_2 diketahui

Dengan Teorema Limit Pusat

misal $x \sim N(\mu_1,\sigma_1^2)$ diketahui bahwa $Z_1=\frac{\bar{x}-\mu_1}{\sigma_1/\sqrt{n}} \sim N(0,1)$ maka

$$Z_1^2=\left(\frac{\bar{x}-\mu_1}{\sigma_1/\sqrt{n}}\right)^2 \sim \chi^2_{(1)}$$

misal $Y \sim N(\mu_2,\sigma_2^2)$ diketahui bahwa $Z_2=\frac{\bar{x}-\mu_2}{\sigma_2/\sqrt{n}} \sim N(0,1)$ maka

$$Z_2^2=\left(\frac{\bar{x}-\mu_2}{\sigma_2/\sqrt{n}}\right)^2 \sim \chi^2_{(1)}$$

$$F=\frac{Z_2^2/1}{Z_1^2/1}=\frac{Z_2^2}{Z_1^2} \sim F_{1,1}$$

$$P\left(F_{\frac{\alpha}{2},1,1} < \frac{Z_2^2}{Z_1^2} < F_{1-\frac{\alpha}{2},1,1}\right)=1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},1,1} < \frac{\frac{(\bar{Y}-\mu_2)^2}{\sigma_2^2/m}}{\frac{(\bar{x}-\mu_1)^2}{\sigma_1^2/n}} < F_{1-\frac{\alpha}{2},1,1}\right)=1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},1,1} < \frac{(\bar{Y}-\mu_2)^2}{\sigma_2^2/m}\cdot\frac{\sigma_1^2/n}{(\bar{x}-\mu_1)^2} < F_{1-\frac{\alpha}{2},1,1}\right)=1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},1,1} < \frac{(\bar{Y}-\mu_2)^2/n}{(\bar{x}-\mu_1)^2/m}\;\frac{\sigma_1^2}{\sigma_2^2} < F_{1-\frac{\alpha}{2},1,1}\right)=1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},1,1}\;\frac{(\bar{x}-\mu_1)^2/m}{(\bar{Y}-\mu_2)^2/n} < \frac{\sigma_1^2}{\sigma_2^2} < F_{1-\frac{\alpha}{2},1,1}\cdot\frac{(\bar{x}-\mu_1)^2/m}{(\bar{Y}-\mu_2)^2/n}\right)=1-\alpha$$

∴ Interval kepercayaan

$$F_{\frac{\alpha}{2},1,1}\;\frac{(\bar{x}-\mu_1)^2/m}{(\bar{Y}-\mu_2)^2/n} < \frac{\sigma_1^2}{\sigma_2^2} < F_{1-\frac{\alpha}{2},1,1}\;\frac{(\bar{x}-\mu_1)^2/m}{(\bar{Y}-\mu_2)^2/n}$$

No.

Date :

b) Jika μ_1 dan μ_2 tak diketahui

misal $X \sim N(\mu_1, \sigma_1^2)$ $\qquad Y \sim N(\mu_2, \sigma_2^2)$

$$U = \frac{nS_1^2}{\sigma_1^2} \sim \chi^2_{(n-1)} \qquad V = \frac{mS_2^2}{\sigma_2^2} \sim \chi^2_{(m-1)}$$

$$F = \frac{\chi^2/df}{\chi^2/df}$$

$$= \frac{\frac{mS_2^2}{\sigma_2^2}/m-1}{\frac{nS_1^2}{\sigma_1^2}/n-1} \sim F_{m-1,n-1}$$

$$P\left(F_{\frac{\alpha}{2},m-1,n-1} < \frac{\frac{mS_2^2}{\sigma_2^2}/m-1}{\frac{nS_1^2}{\sigma_1^2}/n-1} < F_{1-\frac{\alpha}{2},m-1,n-1}\right) = 1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},m-1,n-1} < \frac{n-1}{m-1}\,\frac{mS_2^2/\sigma_2^2}{nS_1^2/\sigma_1^2} < F_{1-\frac{\alpha}{2},m-1,n-1}\right) = 1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},m-1,n-1} < \frac{n-1}{m-1}\,\frac{\sigma_1^2}{\sigma_2^2}\,\frac{mS_2^2}{nS_1^2} < F_{1-\frac{\alpha}{2},m-1,n-1}\right) = 1-\alpha$$

$$P\left(F_{\frac{\alpha}{2},m-1,n-1}\,\frac{(m-1)nS_1^2}{(n-1)mS_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < F_{1-\frac{\alpha}{2},m-1,n-1}\,\frac{(m-1)nS_1^2}{(n-1)mS_2^2}\right) = 1-\alpha$$

$\therefore$ Interval kepercayaan

$$F_{\frac{\alpha}{2},m-1,n-1}\,\frac{(m-1)nS_1^2}{(n-1)mS_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < F_{1-\frac{\alpha}{2},m-1,n-1}\,\frac{(m-1)nS_1^2}{(n-1)mS_2^2}$$

3) Interval kepercayaan untuk $\frac{\sigma_1^2}{\sigma_2^2}$ dari dua distribusi (bukan normal) yang mempunyai mean μ_1 dan μ_2 diketahui

Dengan Teorema Limit Pusat

X bukan distribusi normal mean $= \mu$ variansi $= \sigma_1^2$

Y bukan distribusi normal mean $= \mu$ variansi $= \sigma_2^2$

$$\frac{\bar{X}-\mu_1}{\sigma_1} \approx N(0,1) \qquad \frac{\bar{Y}-\mu_2}{\sigma_2} \approx N(0,1)$$

misal terdapat $W = \frac{\bar{X}-\mu}{\sigma} \approx N(0,1)$ maka $V = W^2 = \left(\frac{\bar{X}-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$

No

Date :

misal $U = \frac{\bar{X} - \mu_1}{\sigma_1} \sim N(0,1)$ maka $V = \left(\frac{\bar{X} - \mu_1}{\sigma_1}\right)^2 \sim \chi^2_{(1)}$

$R = \frac{\bar{Y} - \mu_2}{\sigma_2} \sim N(0,1)$ maka $W = \left(\frac{\bar{Y} - \mu_2}{\sigma_2}\right)^2 \sim \chi^2_{(1)}$

karena V dan W saling bebas maka

$$F = \frac{V/1}{W/1} = \frac{V}{W}$$

$$= \frac{\left(\frac{\bar{X} - \mu_1}{\sigma_1}\right)^2}{\left(\frac{\bar{Y} - \mu_2}{\sigma_2}\right)^2}$$

$$= \left(\frac{\bar{X} - \mu_1}{Y - \mu_2}\right)^2 \left(\frac{\sigma_2^{\,2}}{\sigma_1^{\,2}}\right) \sim F_{1,1}$$

$$P\left(F_{\alpha/2,1,1} \le \left(\frac{\bar{X} - \mu_1}{\bar{Y} - \mu_2}\right)^2 \frac{\sigma_2^2}{\sigma_1^2} \le F_{1-\frac{\alpha}{2},1,1}\right) = 1-\alpha$$

$$P\left(\frac{1}{F_{\alpha/2,1,1}} \le \frac{\sigma_1^2}{\sigma_2^2}\left(\frac{\bar{Y} - \mu_2}{\bar{X} - \mu_1}\right)^2 \le \frac{1}{F_{1-\alpha/2,1,1}}\right) = 1-\alpha$$

$$P\left(\frac{1}{F_{\alpha/2,1,1}}\left(\frac{\bar{X} - \mu_1}{\bar{Y} - \mu_2}\right)^2 \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{1}{F_{1-\alpha/2,1,1}}\left(\frac{\bar{X} - \mu_1}{\bar{Y} - \mu_2}\right)^2\right) = 1-\alpha$$

∴ interval kepercayaan

$$\frac{1}{F_{\alpha/2,1,1}}\left(\frac{\bar{X} - \mu_1}{\bar{Y} - \mu_2}\right)^2 \le \frac{\sigma_1^2}{\sigma_2^2} \le F_{\alpha/2,1,1}\left(\frac{\bar{X} - \mu_1}{\bar{Y} - \mu_2}\right)^2$$

Slide 11

Misalkan $Y_1 \sim b(n_1, p_1)$ dan $Y_2 \sim b(n_2, p_2)$ dimana Y_1, dan Y_2 saling bebas, Misalkan $c = \frac{n_1}{n_2}$, c konstanta positif yang fixed maka Tunjukkan bahwa

$$V = \sqrt{\frac{\frac{\frac{Y_1}{n_1}\left[1-\frac{Y_1}{n_1}\right]}{n_1} + \frac{\frac{Y_2}{n_2}\left[1-\frac{Y_2}{n_2}\right]}{n_2}}{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}}$$

konvergen dalam probabilitas ke 1 untuk $n_2 \to \infty$ dan untuk $n_1 \to \infty$ karena $c = \frac{n_1}{n_2}$, $c > 0$.

Jawab

Untuk membuktikan V konvergen dalam probabilitas ke 1 untuk $n_1, n_2 \to \infty$, kita memerlukan teorema-teorema berikut.

teorema 1

Misalkan $F_n(u)$ menyatakan fungsi distribusi dari variabel acak U_n yang distribusinya bergantung pada bilangan bulat positif n. Misalkan U_n konvergen dalam probabilitas ke konstanta $c \neq 0$, maka variabel acak $\frac{U_n}{c}$ konvergen dalam probabilitas ke 1.

teorema 2

Misalkan $F_n(u)$ menyatakan fungsi distribusi dari variabel acak U_n yang distribusinya bergantung pada bilangan bulat positif n. Misalkan U_n konvergen dalam probabilitas ke konstanta positif c dan misalkan $Pr(U_n < 0) = 0 \ \forall n$. Maka variabel acak $\sqrt{U_n}$ konvergen dalam probabilitas ke $\sqrt{c}$.

teorema 3

Misalkan $F_n(u)$ menyatakan fungsi distribusi dari variabel acak U_n yang distribusinya bergantung pada bilangan bulat positif n. Misalkan U_n mempunyai suatu distribusi pendekatan dengan fungsi distribusi $F(u)$. Misalkan pula variabel acak V_n konvergen secara probabilitas ke 1, Maka distribusi pendekatan dari variabel acak $W_n = \frac{U_n}{V_n}$ sama dengan distribusi pendekatan dari U_n, atau W_n mempunyai fungsi distribusi pendekatan dengan fungsi distribusi $F(w)$.

Dari teorema-teorema berikut. Akan ditunjukkan kebenaran dari pernyataan klaim akibat teorema di atas.

Misalkan b suatu konstanta positif, U_n konvergen dalam probabilitas ke konstanta $c > 0$ dan V_n konvergen dalam probabilitas ke konstanta $d > 0$. Maka

1) $U_n V_n$ konvergen dalam probabilitas ke cd

2) $\frac{U_n}{V_n}$ konvergen dalam probabilitas ke $\frac{c}{d}$, $d \neq 0$

3) bU_n konvergen dalam probabilitas ke bc.

Akan dibuktikan pernyataan 1) $U_n V_n$ konvergen dalam probabilitas ke cd.

Diketahui

- U_n konvergen dalam probabilitas ke $c>0$ artinya $\forall \varepsilon_1 > 0$ berlaku

$\lim_{n\to\infty} Pr(|U_n - c| < \varepsilon_1) = 1 \iff \lim_{n\to\infty} Pr(-\varepsilon_1 < U_n - c < \varepsilon_1) = 1$

- V_n konvergen dalam probabilitas ke $d>0$ artinya $\forall \varepsilon_2 > 0$ berlaku

$\lim_{n\to\infty} Pr(|V_n - d| < \varepsilon_2) = 1 \iff \lim_{n\to\infty} Pr(-\varepsilon_2 < V_n - d < \varepsilon_2) = 1.$

- Perhatikan bahwa operasi biner pada perkalian bilangan real berlaku

$U_n V_n$ artinya $\forall \varepsilon_1 > 0$ dan $\forall \varepsilon_2 > 0$ berlaku

$(\lim_{n\to\infty} Pr(|U_n - c| < \varepsilon_1) = 1)(\cdot \lim_{n\to\infty} Pr(|V_n - d| < \varepsilon_2) = 1) \iff$

$\lim_{n\to\infty} \left(Pr(|U_n - c| < \varepsilon_1) \cdot Pr(|V_n - d| < \varepsilon_2) \right) = 1 \iff$

$\lim_{n\to\infty} \left(Pr(|U_n V_n - dU_n - cV_n + cd|) < Pr(|U_n V_n + cd|) \right) = 1 \iff$

$\lim_{n\to\infty} (Pr(|U_n V_n + cd|) < Pr(||U_n V_n| + |cd||) = 1 \iff$

$\lim_{n\to\infty} (Pr(||U_n V_n| + |cd||) < Pr(|U_n V_n - cd|) = 1 \iff$

$\lim_{n\to\infty} (Pr(|U_n V_n - cd|) = 1$

Terbukti $U_n V_n$ konvergen ke cd.

Akan dibuktikan pernyantaan 2) $\frac{U_n}{V_n}$ convergent dalam probabilitas ke $\frac{c}{d}$, $d \neq 0$

$U_n \xrightarrow{P} c$ dan $V_n \xrightarrow{P} d \iff U_n \xrightarrow{P} c$ dan $\frac{1}{V_n} \xrightarrow{P} \frac{1}{d}$

$\iff \frac{U_n}{V_n} \xrightarrow{P} \frac{c}{d}$

$\therefore \frac{U_n}{V_n}$ konvergen dalam probabilitas ke $\frac{c}{d}$

Akan dibuktikan pernyataan 3) bU_n konvergen dalam probabilitas ke bc.

Diketahui $U_n \xrightarrow{P} c \neq 0$ artinya $\forall \varepsilon > 0$ berlaku

$\lim_{n\to\infty} Pr(|U_n - c| < \varepsilon) = 1 \iff \lim_{n\to\infty} Pr(|bU_n - bc| < b\varepsilon) = 1$, misal $\varepsilon' = b\varepsilon$

$\iff \lim_{n\to\infty} Pr(|bU_n - bc| < \varepsilon') = 1$

Terbukti $bU_n \xrightarrow{P} bc$.

- Perhatikan bahwa $\frac{Y_1}{n_1}$ konvergen dalam probabilitas ke p_1

Bukti

$$E\left[\frac{Y_1}{n_1}\right] = \frac{E[Y_1]}{n_1} = \frac{n_1 p_1}{n_1} = p_1$$

$$Var\left[\frac{Y_1}{n_1}\right] = \frac{var[Y_1]}{n_1^2} = \frac{n_1 p_1(1-p_1)}{n_1^2} = \frac{p_1(1-p_1)}{n_1}$$

Maka $\frac{Y_1}{n_1}$ mempunyai mean p_1 dan variansi $\frac{p_1(1-p_1)}{n_1}$

$\forall \varepsilon > 0$ berlaku

$$Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \varepsilon\right) = Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \frac{k\sqrt{p_1(1-p_1)}}{\sqrt{n_1}}\right)$$

dimana $\varepsilon = \frac{k\sqrt{p_1(1-p_1)}}{\sqrt{n_1}} \iff k = \frac{\varepsilon\sqrt{n_1}}{\sqrt{p_1(1-p_1)}}$

berdasarkan pertidaksamaan Chebyshev, maka

$$Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \varepsilon\right) = Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \frac{k\sqrt{p_1(1-p_1)}}{\sqrt{n_1}}\right) \leq \frac{1}{k^2} = \frac{p_1(1-p_1)}{\varepsilon^2 n_1}$$

Jika σ^2 berhingga, maka $\forall \varepsilon > 0$ berlaku

$$\lim_{n_1\to\infty} Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \varepsilon\right) = \lim_{n_1\to\infty} Pr\left[\left|\frac{Y_1}{n_1} - p_1\right| \geq \frac{k\sqrt{p_1(1-p_1)}}{\sqrt{n_1}}\right) \leq \lim_{n_1\to\infty} \frac{p_1(1-p_1)}{\varepsilon^2 n_1} = 0$$

Karena probabilitas itu positif maka $\forall \varepsilon > 0$ berlaku

$$\lim_{n\to\infty} Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \varepsilon\right) = 0$$

$\therefore$ Jadi, $\frac{Y_1}{n_1}$ konvergen dalam probabilitas ke p_1 jika σ^2 berhingga.

- Perhatikan bahwa $1 - \frac{Y_1}{n_1}$ konvergen dalam probabilitas ke $1 - p_1$

Bukti

$$E\left[1 - \frac{Y_1}{n_1}\right] = E[1] - E\left[\frac{Y_1}{n_1}\right] = 1 - p_1$$

$$Var\left[1 - \frac{Y_1}{n_1}\right] = Var[1] + Var\left[\frac{Y_1}{n_1}\right] = + \frac{p_1(1-p_1)}{n_1}$$

Maka $1 - \frac{Y_1}{n_1}$ mempunyai mean $1 - p_1$ dan variansi $+ \frac{p_1(1-p_1)}{n_1}$

$\forall \varepsilon > 0$ berlaku

$$Pr\left(\left|1 - \frac{Y_1}{n_1} - (1-p_1)\right| \geq \varepsilon\right) = Pr\left(\left|1 - \frac{Y_1}{n_1} - 1 + p_1\right| \geq \varepsilon\right) = Pr\left(\left|-\frac{Y_1}{n_1} + p_1\right| \geq \varepsilon\right)$$

$$= Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \varepsilon\right)$$

Berdasarkan pembuktian sebelumnya, maka

$$\lim_{n_1 \to \infty} \Pr\left(\left|1 - \frac{Y_1}{n_1} - (1-p_1)\right| \geq \varepsilon\right) = \lim_{n_1 \to \infty} \Pr\left(\left|\frac{Y_1}{n_1} - p_1\right| \geq \varepsilon\right) = 0$$

$\therefore$ Jadi $1 - \frac{Y_1}{n_1}$ konvergen dalam probabilitas ke $1 - p_1$

- Berdasarkan teorema sebelumnya jika $\frac{Y_1}{n_1}$ konvergen dalam probabilitas ke p_1 dan $1 - \frac{Y_1}{n_1}$ konvergen dalam probabilitas ke $1 - p_1$ maka,

  $\frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)$ konvergen dalam probabilitas ke $p_1(1-p_1)$

- Untuk $c = \frac{n_1}{n_2} \Leftrightarrow n_1 = cn_2$ berdasarkan teorema sebelumnya jika konstanta positif cn_2 dan $\frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)$ berlaku

  $\frac{1}{cn_2} \frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)$ konvergen dalam probabilitas ke $\frac{1}{cn_2} p_1(1-p_1)$

$$\Leftrightarrow \quad \frac{\frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)}{n_1} \xrightarrow{p} \frac{p_1(1-p_1)}{n_1}$$

- WLOG pada kasus $\frac{Y_2}{n_2}$ sedemikian sehingga $\frac{\frac{Y_2}{n_2}\left(1 - \frac{Y_2}{n_2}\right)}{n_2} \xrightarrow{p} \frac{p_2(1-p_2)}{n_2}$

- Tambahkan kedua ruas untuk kasus $\frac{Y_1}{n_1}$ & $\frac{Y_2}{n_2}$

$$\frac{\frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)}{n_1} + \frac{\frac{Y_2}{n_2}\left(1 - \frac{Y_2}{n_2}\right)}{n_2} \xrightarrow{p} \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

- Berdasarkan teorema 1 berlaku

$$\frac{\frac{\frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)}{n_1} + \frac{\frac{Y_2}{n_2}\left(1 - \frac{Y_2}{n_2}\right)}{n_2}}{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \xrightarrow{p} 1$$

- Berdasarkan teorema 2 berlaku

$$\sqrt{\frac{\frac{\frac{Y_1}{n_1}\left(1 - \frac{Y_1}{n_1}\right)}{n_1} + \frac{\frac{Y_2}{n_2}\left(1 - \frac{Y_2}{n_2}\right)}{n_2}}{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}} \xrightarrow{p} \sqrt{1} = 1$$

Karena telah ditunjukkan $V = \sqrt{\dfrac{\dfrac{\frac{Y_1}{n_1}\left(1-\frac{Y_1}{n_1}\right)}{n_1} + \dfrac{\frac{Y_2}{n_2}\left(1-\frac{Y_2}{n_2}\right)}{n_2}}{\dfrac{p_1(1-p_1)}{n_1} + \dfrac{p_2(1-p_2)}{n_2}}}$ konvergen dalam

probabilitas ke 1 maka terbukti $V \xrightarrow{P} 1$, artinya probabilitas ke 1 ▨

No.

Date

6.34 $X_1, X_2, \ldots, X_n$ dan $y_1, y_2, \ldots, y_n$ merepresentasikan sample random distribusi $X \sim N(\mu_1, \sigma_1^2)$ dan $y \sim N(\mu_2, \sigma_2^2)$ dimana $n_1 = 9$ $n_2 = 12$ dan variansi tidak diketahui dgn $\sigma_1^2 = 3\sigma_2^2$ akan didefinisikan var random berdistribusi t untuk menentukan interval 95% utk $\mu_1 - \mu_2$

→ Dengan rasio $k = 3$

$$T = \frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{k}\sqrt{k/n_1 + 1/n_2}\sqrt{\dfrac{(n_1-1)s_1^2 + k(n_2-1)s_2^2}{n_1+n_2-2}}}$$

$$= \frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{\sqrt{3}\sqrt{3/9 + 1/12}\sqrt{\dfrac{(9-1)s_1^2 + 3(12-1)s_2^2}{9+12-2}}}$$

$$= \frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{1.118033989\sqrt{\dfrac{8s_1^2 + 33s_2^2}{19}}}$$

$$= \frac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{0.2565\sqrt{8s_1^2 + 33s_2^2}}$$

→ Interval kepercayaan 95% utk T

$\Leftrightarrow P\left[-t_{\frac{\alpha}{2}} < T < t_{\frac{\alpha}{2}}\right] = 1 - \alpha$

$\Leftrightarrow P\left[-t_{\frac{\alpha}{2}} < \dfrac{\bar{x} - \bar{y} - (\mu_1 - \mu_2)}{0.2565\sqrt{8s_1^2 + 33s_2^2}} < t_{\frac{\alpha}{2}}\right] = 0.95$

$\Leftrightarrow P\left[-t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s^2}\right) < \bar{x} - \bar{y} - (\mu_1 - \mu_2) < t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right)\right]$
$= 0.95$

$\Leftrightarrow P\left[-t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right) - (\bar{x}-\bar{y}) < -(\mu_1 - \mu_2) < t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right) + (\bar{x}-\bar{y})\right]$
$= 0.95$

$\Leftrightarrow P\left[-t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right) + (\bar{x}-\bar{y}) < \mu_1 - \mu_2 < t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right) - (\bar{x}-\bar{y})\right]$
$= 0.95$

Maka interval kepercayaannya $\left(-t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right) + (\bar{x}-\bar{y}),\ t_{\frac{\alpha}{2}}\left(0.2565\sqrt{8s_1^2+33s_2^2}\right) - (\bar{x}-\bar{y})\right)$ //

No.

Date . .

dengan diketahui $t(0,025;19) = 2.093$

maka interval kepercayaannya

$$\left[-2.093\left(0.2565\sqrt{8s_1^2+33s_2^2}\right)+(\bar{x}-\bar{y}),\ 2.093\left(0.2565\sqrt{8s_1^2+33s_2^2}\right)-(\bar{x}-\bar{y})\right]$$

6.35.

Let $\bar{X}$ and $\bar{Y}$ be the means of two independent random samples, each of size n from the respective distributions $N(\mu_1, \sigma^2)$, $N(\mu_2, \sigma^2)$, where the common variance is known. Find n such that

$Pr(\bar{X}-\bar{Y}-\sigma/5 < \mu_1-\mu_2 < \bar{X}-\bar{Y}+\sigma/5) = 0,90.$

Jwb:

Masuk ke dalam "2 populasi berdistribusi normal dengan varians σ^2 sama dan diketahui)

Misalkan $X_1, X_2, \ldots, X_n$ sampel random berukuran n dengan mean $= \bar{x}$, variansi $= \frac{\sigma^2}{n}$

$Y_1, Y_2, \ldots, Y_n$ sampel random berukuran n dengan mean $= \bar{Y}$, variansi $= \frac{\sigma^2}{n}$

$$E[\bar{X}-\bar{Y}] = E[\bar{X}] - E[\bar{Y}] = \frac{X_1+X_2+\ldots+X_n}{n} - \frac{Y_1+Y_2+\ldots+Y_n}{n} = \frac{n\,E[\bar{X}]}{n} - \frac{n\,E[\bar{Y}]}{n} = \mu_1 - \mu_2$$

$$Var[\bar{X}-\bar{Y}] = 1^2 Var[\bar{X}] + (-1)^2 Var[\bar{Y}] = Var[\bar{X}] + Var[\bar{Y}] = Var\left[\frac{X_1+X_2+\ldots+X_n}{n}\right] + Var\left[\frac{Y_1+Y_2+\ldots+Y_n}{n}\right]$$

$$= \frac{1}{n^2} Var[X_1+X_2+\ldots+X_n] + \frac{1}{n^2} Var[Y_1+Y_2+\ldots+Y_n]$$

$$= \frac{1}{n^2} \Sigma Var(X_i) + \frac{1}{n^2} \Sigma Var(Y_i)$$

$$= \frac{1}{n^2} \cdot n \cdot Var[X] + \frac{1}{n^2} \cdot n \cdot Var[Y]$$

$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} = \frac{2\sigma^2}{n}$$

$$\text{CLT}: Z = \frac{\bar{x}-\mu}{\sigma/\sqrt{n}} = \frac{(\bar{X}-\bar{Y}) - \mu_{\bar{X}-\bar{Y}}}{\sigma_{\bar{X}-\bar{Y}}/\sqrt{n}}$$

$$Z = \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{2\sigma^2}/\sqrt{n}} \sim N(0,1)$$

C.I untuk $\mu_1 - \mu_2$:

$$Pr(-Z_{\alpha/2} < Z < Z_{\alpha/2}) = 1-\alpha$$

$$\Leftrightarrow Pr\left(-Z_{\alpha/2} < \frac{(\bar{X}-\bar{Y})-(\mu_1-\mu_2)}{\sqrt{2\sigma^2}/\sqrt{n}} < Z_{\alpha/2}\right) = 1-\alpha$$

$$\Leftrightarrow Pr\left(-Z_{\alpha/2}\frac{\sqrt{2\sigma^2}}{\sqrt{n}} < (\bar{x}-\bar{y})-(\mu_1-\mu_2) < Z_{\alpha/2}\frac{\sqrt{2\sigma^2}}{\sqrt{n}}\right) = 1-\alpha$$

$$\Leftrightarrow Pr\left((\bar{x}-\bar{y}) - Z_{\alpha/2}\sqrt{\frac{2\sigma^2}{n}} < \mu_1-\mu_2 < (\bar{x}-\bar{Y}) + Z_{\alpha/2}\sqrt{\frac{2\sigma^2}{n}}\right) = 1-\alpha$$

$1-\alpha = 0,90$

$\alpha = 0,1 \quad \Rightarrow Z_{\alpha/2} = Z_{0,05} = 1,645$

$\alpha/2 = 0,05$

$$Z_{\alpha/2}\sqrt{\frac{2\sigma^2}{n}} = \frac{\sigma}{5} \Leftrightarrow Z_{0,05}\sqrt{\frac{2}{n}}\,\sigma = \frac{\sigma}{5}$$

$$\Leftrightarrow 1,645\sqrt{\frac{2}{n}} = \frac{1}{5}$$

$$\Leftrightarrow \sqrt{\frac{2}{n}} = \frac{40}{329}$$

$$\Leftrightarrow \frac{2}{n} \approx 0,0148$$

$$\Leftrightarrow n \approx 135$$

6.36

Under the conditions given, show that the random variable defined by ratio (1) of the text converges in probability to 1.

Jwb:

Membuktikan : $$\frac{\frac{\frac{Y_1}{n_1}\left(1-\frac{Y_1}{n_1}\right)}{n_1}+\frac{\frac{Y_2}{n_2}\left(1-\frac{Y_2}{n_2}\right)}{n_2}}{\frac{p_1(1-p_1)}{n_1}+\frac{p_2(1-p_2)}{n_2}} \xrightarrow{P} 1$$

Misalkan $x_1, x_2, \ldots, x_n$, sampel acak dari populasi berdistribusi $b(1,p_1)$, $Y_1 = \sum X_i \sim b(n_1, p_1)$

$Y_1, Y_2, \ldots, Y_n$, sampel acak dr populasi berdistribusi $b(1,p_2)$, $Y_2 = \sum Y_i \sim b(n_2, p_2)$

dimana $\hat{p}_1 = \frac{Y_1}{n_1}$; $\hat{p}_2 = \frac{Y_2}{n_2}$

$E[\hat{p}_1] = E\left[\frac{Y_1}{n_1}\right] = \frac{1}{n_1} E[Y_1] = \frac{1}{n_1} \cdot n_1 \cdot p_1 = p_1$

$E[\hat{p}_2] = E\left[\frac{Y_2}{n_2}\right] = \frac{1}{n_2} E[Y_2] = \frac{1}{n_2} \cdot n_2 \cdot p_2 = p_2$

$Var[\hat{p}_1] = Var\left[\frac{Y_1}{n_1}\right] = \frac{1}{n_1^2} Var[Y_1] = \frac{1}{n_1^2} n_1 p_1 (1-p_1) = \frac{p_1(1-p_1)}{n_1}$

$Var[\hat{p}_2] = Var\left[\frac{Y_2}{n_2}\right] = \frac{1}{n_2^2} Var[Y_2] = \frac{1}{n_2^2} n_2 p_2 (1-p_2) = \frac{p_2(1-p_2)}{n_2}$.

karena $\frac{Y_1}{n_1} \xrightarrow{P} p_1$

$(\Rightarrow) \; 1-\frac{Y_1}{n_1} \xrightarrow{P} p_1(1-p_1)$

$\Leftrightarrow \frac{Y_1}{n_1}\left(1-\frac{Y_1}{n_1}\right) \xrightarrow{P} p_1(1-p_1)$

$\Leftrightarrow \frac{\frac{Y_1}{n_1}\left(1-\frac{Y_1}{n_1}\right)}{n_1} \xrightarrow{P} \frac{p_1(1-p_1)}{n_1}$

enggunakan teorema; $U_n \xrightarrow{P} c$, $V_n \xrightarrow{P} d$

a. $(\Rightarrow)$ $U_n + V_n \xrightarrow{P} c+d$

b. $(\Rightarrow)$ $U_n V_n \xrightarrow{P} cd$

c. $(\Rightarrow)$ $\frac{U_n}{c} \xrightarrow{P} 1$, $c \neq 0$

dari

eorema tsb; $\frac{\left(\frac{Y_1}{n_1}\right)\left(1-\frac{Y_1}{n_1}\right)}{n_1} \xrightarrow{P} \frac{p_1(1-p_1)}{n_1}$ dan $\frac{\left(\frac{Y_2}{n_2}\right)\left(1-\frac{Y_2}{n_2}\right)}{n_2} \xrightarrow{P} \frac{p_2(1-p_2)}{n_2}$

$$= \frac{\left(\frac{Y_1}{n_1}\right)\left(1-\frac{Y_1}{n_1}\right)}{n_1} + \frac{\left(\frac{Y_2}{n_2}\right)\left(1-\frac{Y_2}{n_2}\right)}{n_2} \xrightarrow{P} \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

$$= \frac{\frac{\left(\frac{Y_1}{n_1}\right)\left(1-\frac{Y_1}{n_1}\right)}{n_1} + \frac{\left(\frac{Y_2}{n_2}\right)\left(1-\frac{Y_2}{n_2}\right)}{n_2}}{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \xrightarrow{P} 1$$

(terbukti) //