UOA MSDS C2

18/06/2022

Agenda:

1. Calculus
2. Limits
3. Derivatives – Some basic formulaes

   Rules of Differentiation

   Chain Rule

   Partial Derivatives
4. Matrix – Determinant

   Trace

   Eigen Vector & Eigen Value

——— X ———

Calculus.

- comes from Latin $\longrightarrow$ Small Stone

↓ why?

Bcz it is like understanding something by looking at small pieces

Two Types:

Differential Calculus

- It cuts something into small pieces to find out how it changes

Integral Calculus

It joins (integrates) the small pieces together to find out how much there is.

Limits:

Sometimes we can't work something out directly...
but we can see what it should be as we get closer & closer

eg: $\dfrac{x^2-1}{x-1}$

If $x=1$,

$$\frac{1^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0} \quad (\text{Indeterminate})$$

ex. Contd:

x	$\frac{x^2-1}{x-1}$
0.5	1.5
0.9	1.9
0.99	1.99
0.999	1.999
0.9999	1.9999
0.99999	1.99999

Pl Observe: As x gets close to 1, then $\dfrac{x^2-1}{x-1}$ gets close to 2.

When $x=1 \rightarrow$ Indeterminate o/p
But we can certainly see that it is going to be 2.
||

$\Downarrow$

LIMIT

i.e. The limit of $\frac{x^2-1}{x-1}$ as x approaches 1 is 2.

$\Big\downarrow$ Mathematical Representation

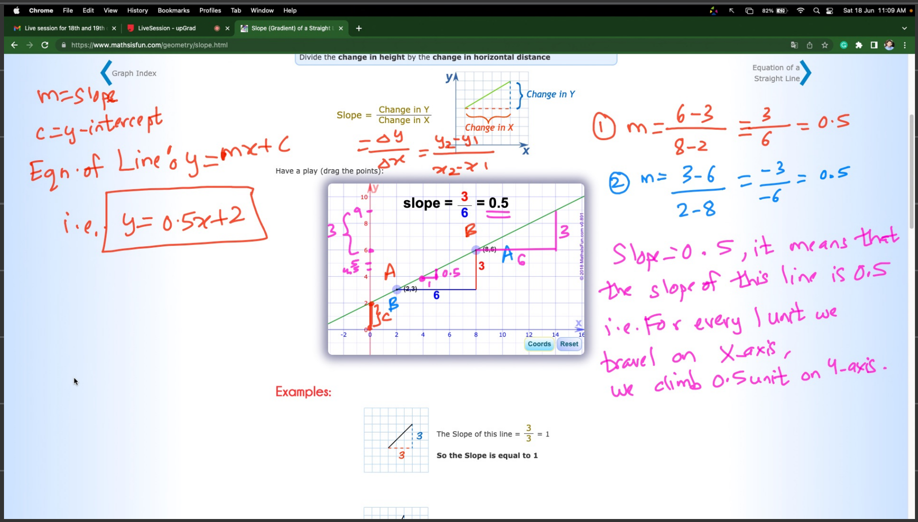
$$\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

eg Contd:

x	$\frac{x^2-1}{x-1}$
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001
1.00001	2.00001

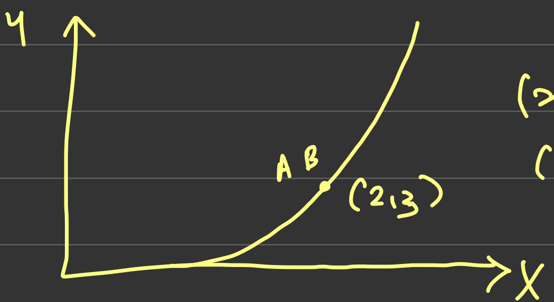
Again, its heading for 2.

Derivatives : It is all about slope.



Finding Slope of a line is very easy as the slope is constant i.e. 0.5 throughout.

What if we want to calculate slope of a curve?



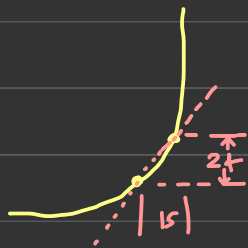
$(x_1, y_1) = (2,3)$

$(x_2, y_2) = (2,3)$

$$m = \frac{3-3}{2-2} = 0$$

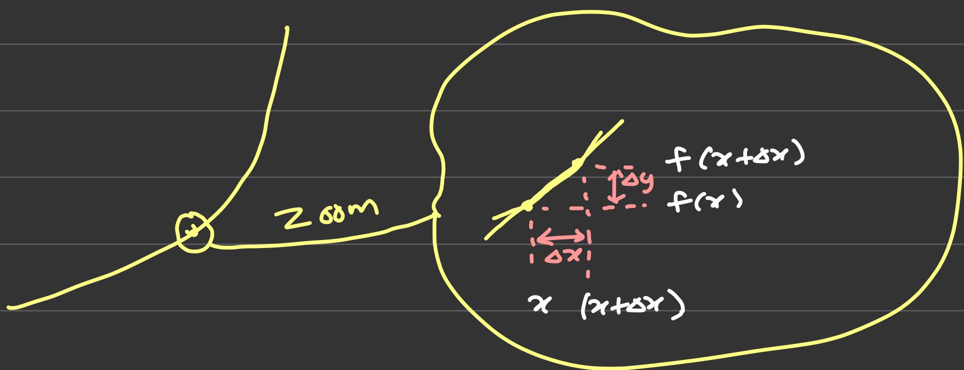
↓

Calculus : Small Pebbles.



$m = \frac{2t}{15}$

Derivatives : We use a small difference.



$m = \frac{\Delta y}{\Delta x}$

x changes from x to $x+\Delta x$

y changes from $f(x)$ to $f(x+\Delta x)$

$$m = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

& make Δx shrink toward zero.

eg: $f(x) = x^2$

Soln: $f(x+\Delta x) = (x+\Delta x)^2$

$$= x^2 + 2x\Delta x + (\Delta x)^2$$

$$m = \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x^2}}{\Delta x}$$

$$= \frac{2x\Delta x + (\Delta x)^2}{\Delta x}$$

$$= \frac{\cancel{\Delta x}(2x+\Delta x)}{\cancel{\Delta x}}$$

$$= 2x + \Delta x$$

$\because$ Δx is heading towards 0.

$\therefore$ $m = 2x + \Delta x$

$= 2x + (0)$

$= 2x$

Derivative of x^2 is $2x$

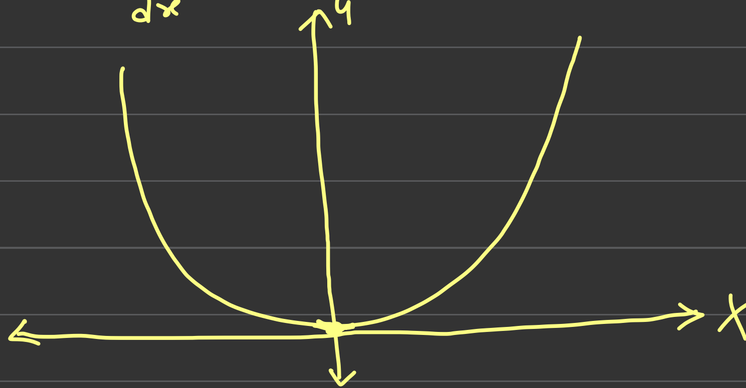
or

Slope at x is $2x$.

or

$\frac{d}{dx}(x^2) = 2x$.

What does $\frac{d}{dx}(x^2) = 2x$ mean?



It means that the rate of change at any point is $2x$.

i.e. when $x=2$, Slope is $2x = 2(2) = 4$

when $x=3$, Slope is $2x = 2(3) = 6$

⋮

when $x=25$, —" ——— $=2(25)=50$

when $x=-3$, —" ——— $=2(-3)=-6$

Common Functions	Function	Derivatives
Constant	c	0
Line	x	1
	ax	a
Square	x^2	$2x \left\} = 2 \cdot x^{2-1} = 2x^1 = 2x \right.$
	x^n	$n \cdot x^{n-1}$
Sq. Root	$\sqrt{x} = x^{\frac{1}{2}}$	$\left(\frac{1}{2}\right) x^{-\frac{1}{2}}$
Exponential	e^x	e^x
	a^x	$\ln(a) \cdot a^x$
Logarithms	$\ln(x)$	$\frac{1}{x}$
	$\log_a(x)$	$\frac{1}{x \cdot \ln(a)}$
Trigonometry	$\sin(x)$	$\cos(x)$
(x is radians)	$\cos(x)$	$-\sin(x)$
	$\tan(x)$	$\sec^2(x)$
Inverse Trigonometry	$\sin^{-1}(x)$	$1 / \sqrt{(1-x^2)}$
	$\cos^{-1}(x)$	$-1 / \sqrt{(1-x^2)}$
	$\tan^{-1}(x)$	$1 / (1+x^2)$

Rules for Derivatives:

Rules	Function	Derivative
1. Multiplication by a constant	cf	cf'
2. Power Rule	x^n	$n \cdot x^{n-1}$
3. Sum Rule	$f+g$	$f'+g'$
4. Difference Rule	$f-g$	$f'-g'$
5. Product Rule	$f \cdot g$	$f \cdot g' + f' \cdot g$
6. Quotient Rule	f/g	$\frac{f' \cdot g - g' \cdot f}{g^2}$
7. Reciprocal Rule	$\frac{1}{f}$	$\frac{-f'}{f^2}$

eg: $\frac{d}{dx}(\sin x) = ?$

$\sin(x)' = \cos(x)$

eg: $\frac{d}{dx} x^3 = ?$

Power Rule. $\frac{d}{dx} x^3 = 3 \cdot x^{3-1}$

$= 3x^2$

eg: $\frac{d}{dx}\left(\frac{1}{x}\right) = ?$

$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}\left(x^{-1}\right)$ Power Rule

$= -1 \cdot x^{-1-1}$

$= -1 \cdot x^{-2}$

$= -x^{-2}$

$= -\frac{1}{x^2}$.

eg: $\frac{d}{dx}(5x^3) = ?$

$\frac{d}{df}(cf) = cf'$

$\frac{d}{dx}(5x^3) = 5 \cdot \frac{d}{dx}(x^3)$

$= 5 \cdot \left[3 \cdot x^{3-1}\right] = \underline{\underline{15x^2}}$.

eg: $\frac{d}{dx}(x^2+x^3) = ?$

Sum Rule: $\frac{d}{dx}(x^2) + \frac{d}{dx}(x^3)$

$= 2x + 3x^2$ (Power rule).

Diff Rule = $\frac{d}{dx}(x^2) - \frac{d}{dx}(x^3) = 2x - 3x^2.$

eg: $\frac{d}{dz}(5z^2 + z^3 - 7z^4) = ?$

$= \frac{d}{dz}(5z^2) + \frac{d}{dz}(z^3) - \frac{d}{dz}(7z^4)$ [Add Diff. rule]

$= 10z + 3z^2 - 28z^3$ [Power Rule].

Product Rule:

eg: $\frac{d}{dx}[\underline{\cos(x)} \cdot \underline{\sin(x)}] = ?$

Der. of $f \cdot g = f \cdot g' + f' \cdot g$

Here, $f = \cos$, $f' = -\sin$

$g = \sin$, $g' = \cos$

$$\frac{d}{dx}(\cos(x) \cdot \sin(x)) = \cos x \cdot \sin(x)' + \cos(x)' \cdot \sin(x)$$
$$= \cos x \cdot \cos x + (-\sin(x)) \cdot \sin(x)$$
$$= \cos^2 x - \sin^2 x.$$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{g \cdot f' - f \cdot g'}{g^2}$

eg:

$$\frac{d}{dx}\left[\cos(x)/x\right] = ?$$

$f = \cos$, $f' = -\sin$

$g = x$, $g' = 1$

$$= \frac{x \cdot (-\sin(x)) - \cos(x) \cdot 1}{x^2}$$

$$= -\frac{x \cdot \sin(x) - \cos(x)}{x^2} .$$

Composition of Functions:

It is applying one function to the results of another.

$$\longrightarrow \boxed{f()} \longrightarrow \boxed{g()} \longrightarrow$$

i.e. $g(f(x))$

i.e. $(g \circ f)(x)$

eg: $f(x) = 2x+3$ & $g(x) = x^2$

Soln: x is just a placeholder.
Lets call it as "input" to avoid confusion.

$$f(\text{input}) = 2(\text{input})+3$$
$$g(\text{input}) = (\text{input})^2$$

$$(g \circ f)(x) = g(f(x))$$

$$\xrightarrow{x} \boxed{2(\text{input})+3} \xrightarrow{2x+3} \boxed{(\text{input})^2} \xrightarrow{(2x+3)^2}$$

$$(g \circ f)(x) = (2x+3)^2$$

What is $(f \circ g)(x)$?

$$(f \circ g)(x) = f(g(x))$$

$$\xrightarrow{x} \boxed{(\text{input})^2} \xrightarrow{x^2} \boxed{2(\text{input})+3} \xrightarrow{2(x^2)+3}$$

Note: We get a different result.

Chain Rule of Derivatives:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

eg: $\frac{d}{dx} \sin(x^2) = ?$

Mtd1:

Let $u = x^2$, $\therefore y = \sin(u)$

$$\frac{d}{dx} \sin(x^2) = \frac{d}{du}(\sin u) \cdot \frac{d}{dx} \cdot x^2$$

$$= \cos(u) \cdot (2x)$$

Substitute back $u = x^2$.

$$\frac{d}{dx} \sin(x^2) = 2x \cdot \cos(x^2).$$

Mtd2:

$\sin(x^2)$ is made of $\sin()$ & x^2.

$\therefore f(g) = \sin(g)$

$g(x) = x^2$

ChainRule Says: Derivative of $f(g(x)) = f'(g(x)) \cdot g'(x)$

$f'(g) = \cos(g)$

$g'(x) = 2x$ $\Rightarrow \therefore \frac{d}{dx} \sin(x^2) = \cos(g(x)) \cdot (2x)$

$= 2x \cdot \cos(x^2).$

eg: $\frac{d}{dx}(5x-2)^3 = ?$

Chain Rule: Der. of $f(g(x)) = f'(g(x)) \cdot g'(x)$

$$(5x-2)^3 \begin{cases} g^3 \xrightarrow{\text{Der.}} f'(g) = 3g^2 \\ 5x-2 \xrightarrow{\text{Der.}} g'(x) = 5 \end{cases}$$

$$\therefore \frac{d}{dx}(5x-2)^3 = (3\,g(x)^2)(5) = \underline{\underline{15\,(5x-2)^2}}$$

Partial Derivatives: It is a derivative where we hold some variable constant.

Function of One Variable

$$f(x) = x^2$$
$$f'(x) = 2x \qquad \text{(Power Rule)}$$

Function of two variables:

$$f(x,y) = x^2 + y^3$$

$$f'_x = \frac{d}{dx}(x^2 + y^3) = 2x + 0 \qquad = 2x$$

$$f'_y = \frac{d}{dy}(x^2 + y^3) = 0 + 3y^2 \qquad = 3y^2.$$

eg: $f(x,y) = y^3 \cdot \sin(x) + x^2 \cdot \tan(y)$

Wrt x, we can change y to k.

$$f(x,y) = k^3 \sin(x) + 2x \cdot \tan(k)$$

$$f'_x = k^3 \cdot \cos(x) + 2x \cdot \tan(k)$$

$$= y^3 \cdot \cos(x) + 2x \cdot \tan(y).$$

Likewise wrt y, we turn x into k,

$$f(x,y) = y^3 \cdot \sin(k) + k^2 \cdot \tan(y)$$

$$f'_y = 3y^2 \cdot \sin(k) + k^2 \cdot \sec^2(y)$$

$$= 3y^2 \cdot \sin(x) + x^2 \cdot \sec^2(y).$$

eg: Find Partial Derivatives of

$$f(x,y,z) = x^4 - 3xyz$$

Soln:

$$\frac{\partial f}{\partial x} = 4x^3 - 3yz$$

$$\frac{\partial f}{\partial y} = -3xz$$

$$\frac{\partial f}{\partial z} = -3xy$$

——————— X ———————

Matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

Addn: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ $B = \begin{bmatrix} 2 & 2 \\ 3 & 7 \end{bmatrix}_{2\times 2}$

$(m\times n) + (m\times n) = (m\times n)$

Subt: $(m\times n) - (m\times n) = (m\times n)$

Multiply a Matrix by a constant.

$$2 \times \begin{bmatrix} 1 & 2 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 18 & 6 \end{bmatrix}$$

Matrix Multiplication:

$$A_{m\times p} * B_{p\times n} = C_{m\times n}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1\cdot 7+2\cdot 9+3\cdot 11 & 1\ 8+2\cdot 10+3\cdot 12 \\ 4\ 7+5\cdot 9+6\cdot 11 & 4\cdot 8+5\cdot 10+6\cdot 12 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}_{2\times 2}.$$

Why we do Mat · Mult this way?

eg: There is a local shop who sells 3 types of icecreams.

a. Apricot costs \$3 each

b. Custard Apple costs \$4 each

c. Butterscotch costs \$2 each.

Their sales for last 4 days are as below:

		Mon	Tue	Wed	Thurs
\$3	Apricot	13	9	7	15
\$4	Cu. Apple	8	7	4	6
\$2	Buttersc.	6	4	0	3

Soln: Value of sales for Monday:

$= \$3 \times 13 + \$4 \times 8 + \$2 \cdot 6$

$= \$83.$

$$\underset{\underline{\underline{1\times 3}}}{\begin{bmatrix} \$3 & \$4 & \$2 \end{bmatrix}} \cdot \underset{\underline{\underline{3\times 4}}}{\begin{bmatrix} 13 & 9 & 7 & 15 \\ 8 & 7 & 4 & 6 \\ 6 & 4 & 0 & 3 \end{bmatrix}}$$

$= [3\cdot 13+4\cdot 8+2\cdot 6 \quad 3\cdot 9+4\cdot 7+2\cdot 4 \quad 3\cdot 7+4\cdot 4+2\cdot 0 \quad 3\cdot 15+4\cdot 6+2\cdot 3]$

$= [83 \quad 63 \quad 75]$ //

They sold \$83 worth icecreams on Monday, \$63 ... Tue, & \$75 ... Wed.

Transpose:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$$

$$A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times 2}$$

Determinant: - It is a special no that can be calculated from a matrix.
- Matrix has to be square.

2X2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

eg: $A = \begin{bmatrix} 4 & 6 \\ 3 & 8 \end{bmatrix}$

$|A| = ad - bc$

$|A| = 4 \cdot 8 - 6 \cdot 3 = 32 - 18$
$= 14.$

3x3

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$|A| = \left[+a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} \right] - \left[b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} \right] + \left[\times c \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right]$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

4x4 or Higher

$$A = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$= +a \times \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + \cdots - \cdots$$

Inverse of a Matrix:

$$8 \underset{\text{Reciprocal}}{\overset{\text{Reciprocal}}{\rightleftarrows}} \frac{1}{8}$$

$$A \underset{\text{Inverse}}{\overset{\text{Inverse}}{\rightleftarrows}} A^{-1}$$

$$8 \times \frac{1}{8} = 1$$

$$A \times A^{-1} = I$$

Property: $A \cdot A^{-1} = A^{-1} \cdot A = I$.

2x2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Take Home: Given $A = \begin{bmatrix} 4 & 2 \\ 7 & 6 \end{bmatrix}$,

find A^{-1}. Confirm that $A \cdot A^{-1}$ & $A^{-1} \cdot A$ equals Identity matrix.

Eigen Vector & Eigen Value: [PCA]

Eigen vector is a special vector that will <u>not</u> <u>change direction</u> in transform.

For a square Matrix, an E.vector & E.value make this eqn true:

$$\boxed{Av = \lambda v}$$

Matrix

E. Vector

E. Value

eg: $A = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix}$

E. Vector $= \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

with a matching E. value of 6.

Soln:

$$A \cdot v = \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}$$

λv:

$$6 \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix}.$$

$$\therefore \boxed{Av = \lambda v}.$$

How to find these Eigen things?

Lets start with Eigen value.

$$Av = \lambda v$$

$$Av = \lambda . I . v$$

$$\boxed{Av - \lambda I v = 0}$$

If v is non-zero then we can solve for λ using just the determinant:

$$|A - \lambda I| = 0$$

eg: Solve for λ:

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} -6-\lambda & 3 \\ 4 & 5-\lambda \end{vmatrix} = 0$$

$$(-6-\lambda)(5-\lambda) - 3 \times 4 = 0$$

$$\lambda^2 + \lambda - 42 = 0$$

$\therefore$ $\boxed{\lambda = -7}$ or $\boxed{\lambda = +6}$

$\therefore$ It says that there are 2 possible eigen values.

Now we know E. values, so lets find the E. vector for E. value $\lambda = 6$.

$$\boxed{Av = \lambda v}$$

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 6 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$-6x + 3y = 6x$$
$$4x + 5y = 6y$$

$$\left.\begin{aligned} -12x + 3y &= 0 \\ 4x - 1y &= 0 \end{aligned}\right\} \quad \therefore \boxed{y = 4x}$$

i.e. if $x = 1$, $y = 4$

$$\therefore \text{E.vector} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

III'ly, solve for $\boxed{\lambda = -7}$ (Home Task).

X