Chapter 18: B-trees

Algorithms and Data Structures
(based on book "Introduction to Algorithms" of Cormen et al.)

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Academic year 2024–2025

Outline

1. Definition of a B-tree
2. Use-cases
3. Height of a B-tree
4. Operations on B-trees

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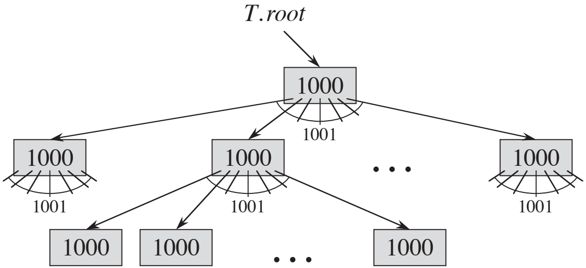
- Main concept $\rightarrow$ remember well

Example of a B-tree



1 node,
1000 keys

1001 nodes,
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1,002,001 nodes,
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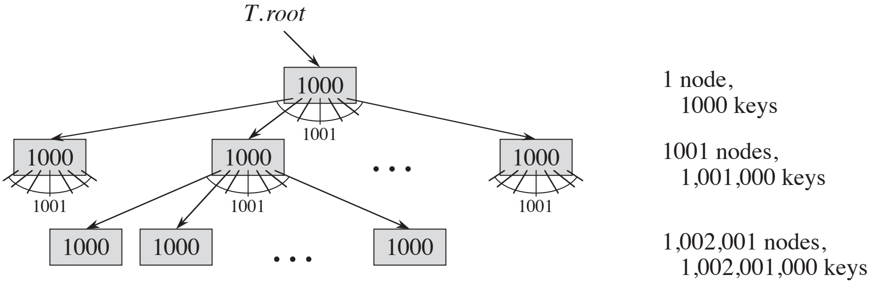

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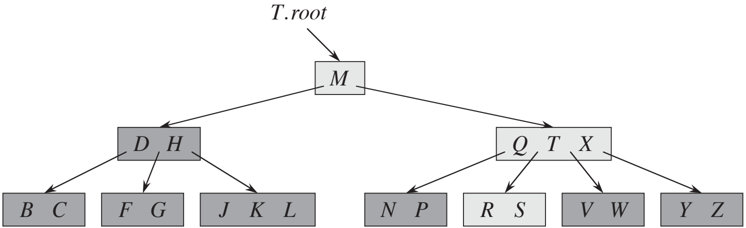

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  - We say that a node is *full* if it contains exactly $2t - 1$ keys.

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  - E.g. SSD's and HDD's.

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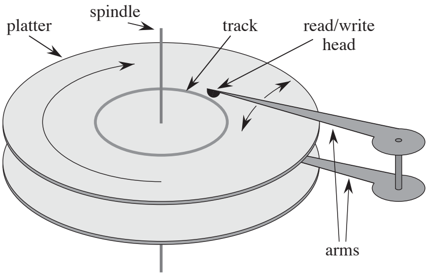

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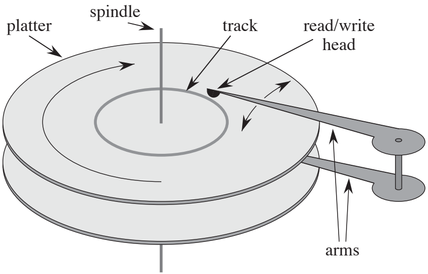

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  → latency!

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  - So we want the height to be as small as possible.

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Height of a B-tree

Theorem

If $n \geq 1$, then for any n-key B-tree T of height h and minimum degree $t \geq 2$

$$h \leq \log_t \left(\frac{n+1}{2}\right)$$

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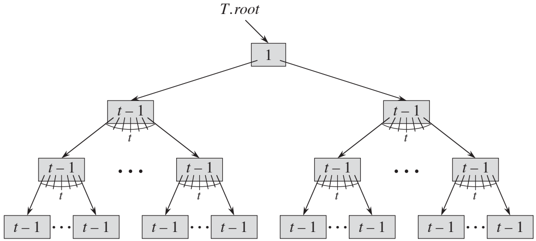

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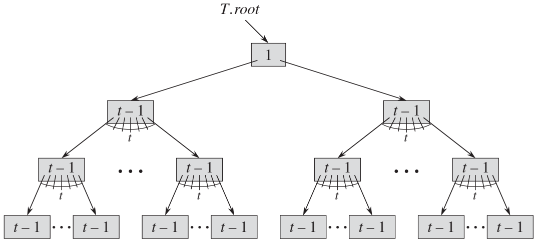

depth	0	1	2	3	...	h
number of nodes	1	2	$2t$	$2t^2$	...	$2t^{h-1}$

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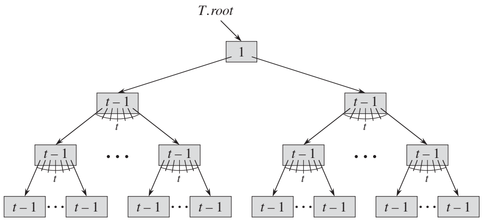

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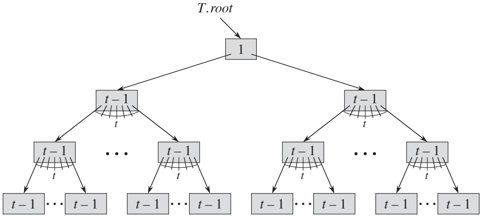

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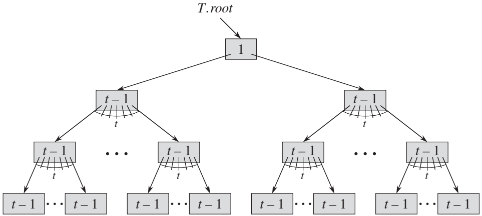

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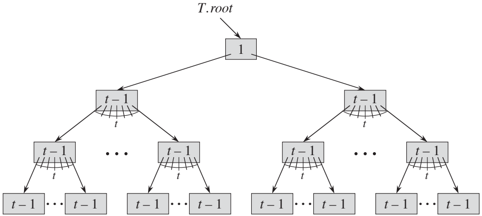

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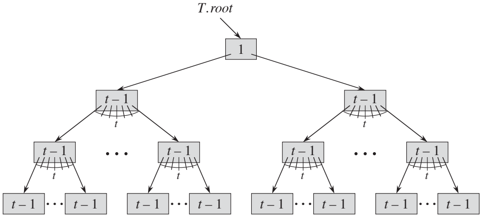

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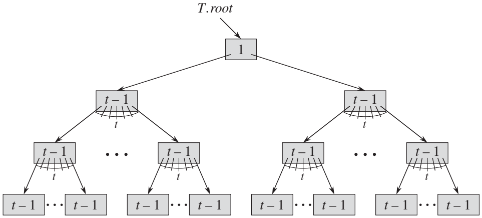
$$t^h \leq \frac{n+1}{2} \iff \boxed{h \leq \log_t\left(\frac{n+1}{2}\right)}$$

$\square$



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$\rightarrow$ Few memory operations, so minimal memory latency.

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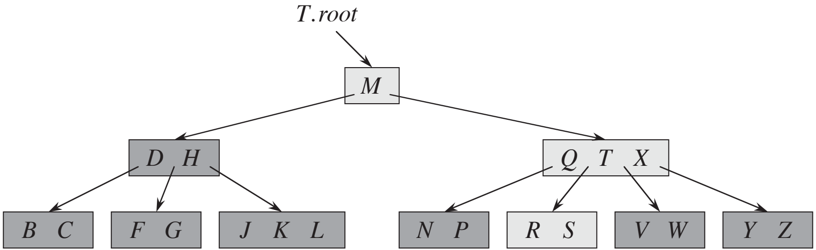
- Example: searching for the letter S using lexicographic order.



Searching a B-tree: pseudocode

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B-TREE-SEARCH(x, k)
i = 1
while i ≤ x.n and k > x.key_i
    i = i + 1
if i ≤ x.n and k == x.key_i
    return (x, i)
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- If k is in the B-tree, B-TREE-SEARCH returns the node-index pair (y, i) such that $y.key_i = k$. Otherwise, the procedure returns NIL.

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- Thus, B-TREE-SEARCH executes $O(h) = O(\log_t n)$ memory accesses.
  - Recall that $h \leq \log_t \left(\frac{n+1}{2}\right)$.
  - Constants are left out in O-notation.

Searching a B-tree: runtime analysis

```
B-TREE-SEARCH(x, k)
i = 1
while i ≤ x.n and k > x.key_i
    i = i + 1
if i ≤ x.n and k == x.key_i
    return (x, i)
elseif x.leaf
    return NIL
else DISK-READ(x.c_i)
    return B-TREE-SEARCH(x.c_i, k)
```

- For every node x, $x.n < 2t$, so the while loop takes $O(t)$ time for each encountered node.

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- For every node x, $x.n < 2t$, so the while loop takes $O(t)$ time for each encountered node.
- The total runtime is $O(th) = O(t \log_t n)$.

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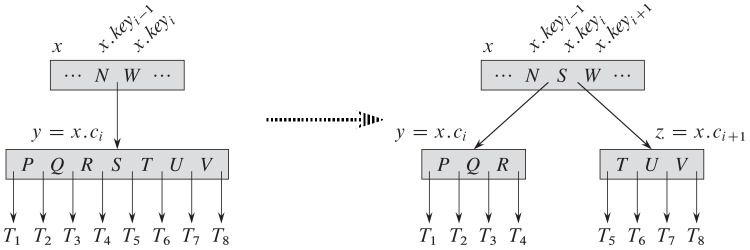
- What if a node y is *full*, i.e. what if $y.n = 2t - 1$?

Splitting a full node y with $y.n = 2t - 1$



B-TREE-SPLIT-CHILD(x, i)

- x is a non-full node and $y = x.c_i$ is a full child of x.
- Split y about its median key S and move S up into y's parent node x.
- Every key $y.key_i$ that is greater than the median S, is placed in a new node z, which is a new child of x.

Inserting a key (continued)

Complexity of $\textsc{B-Tree-Insert}(T, k)$?

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Inserting a key (continued)

Complexity of $\textsc{B-Tree-Insert}(T, k)$?

- Again $O(h) = O(\log_t n)$ disk accesses.
- Again $O(th) = O(t \log_t n)$ time required.

Questions?