Problem 1: Vectors

Consider the following three-dimensional real vectors:

$$u = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad v = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \qquad w = \begin{bmatrix} 9 \\ -3 \\ 9 \end{bmatrix}$$

a) Find the length of each vector.

b) Find the inner product between each pair of vectors and state whether or not they are orthogonal.

c) Add each pair of vectors.

SOLUTION:

a) The length of each vector is as follows:

$$\begin{aligned} \|u\| &= \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14} \\ \|v\| &= \sqrt{(5)^2 + (-1)^2 + (3)^2} = \sqrt{35} \\ \|w\| &= \sqrt{(9)^2 + (-3)^2 + (9)^2} = \sqrt{171} \end{aligned}$$

b) The inner product between each pair of vectors is as follows:

$$\begin{aligned} \langle u, v \rangle &= (2)(5) + (3)(-1) + (-1)(3) = 4 \\ \langle u, w \rangle &= (2)(9) + (3)(-3) + (-1)(9) = 0 \\ \langle v, w \rangle &= (5)(9) + (-1)(-3) + (3)(9) = 75 \end{aligned}$$

Note that the inner product is symmetric, so $\langle v, u \rangle = \langle u, v \rangle$ and so on, so we do not need to compute these separately. The vectors u and w are orthogonal since their inner product is zero, while all other pairs are not orthogonal.

c) The sum of each pair of vectors is as follows:

$$\begin{aligned} u + v &= \begin{bmatrix} (2) + (5) \\ (3) + (-1) \\ (-1) + (3) \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 2 \end{bmatrix} \\ u + w &= \begin{bmatrix} (2) + (9) \\ (3) + (-3) \\ (-1) + (9) \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 8 \end{bmatrix} \\ v + w &= \begin{bmatrix} (5) + (9) \\ (-1) + (-3) \\ (3) + (9) \end{bmatrix} = \begin{bmatrix} 14 \\ -4 \\ 12 \end{bmatrix} \end{aligned}$$

Note that vector addition is *commutative*, meaning that $v + u = u + v$ and so on, so we do not need to compute these separately.

Problem 2: Matrices

For each matrix A, find the eigenvalues and eigenvectors. If it exists, find A^{-1} and compute the eigenvalues and eigenvectors of A^{-1}. How are the eigenvalues and eigenvectors of A^{-1} related to those of A?

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix}$

SOLUTION:

a) The eigenvalues are the solution to the characteristic equation

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{bmatrix}\right) = (\lambda - 1)(\lambda - 2) - (-1)(0) = (\lambda - 1)(\lambda - 2)$$

which are $\lambda_1 = 1$ and $\lambda_2 = 2$. To find the corresponding eigenvectors, we solve the equation

$$(\lambda I - A)\, v = 0$$

for each value eigenvalue λ.

- Case: $\lambda_1 = 1$

$$0 = (\lambda_1 I - A)\, v_1 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -b \end{bmatrix}$$

  This implies that $b = 0$ and a is arbitrary, so we can just set it to $a = 1$.

- Case: $\lambda_2 = 2$

$$0 = (\lambda_2 I - A)\, v_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ 0 \end{bmatrix}$$

  This implies that $b = a$, where a is again arbitrary, so we can set it to $a = 1$.

Therefore, the matrix has the two eigenvalues and eigenvectors

$$\lambda_1 = 1,\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 2,\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The determinant of the matrix is

$$\det(A) = \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) = (1)(2) - (1)(0) = 2$$

(the determinant is also the product of the eigenvalues: $\det(A) = \lambda_1\, \lambda_2 = 2$). The matrix is invertible since the determinant is not zero. Using the adjugate, the inverse of A is

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

We can compute the eigenvalues and eigenvectors of A^{-1} the same as before. Doing so, we find that the eigenvectors of A^{-1} are also v_1 and v_2 (the same as before) with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0.5$ (the inverses of before).

b) The matrix has two eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$, respectively. The matrix is *not* invertible since its determinant is zero.

c) The matrix has three eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = -2$, respectively. The matrix is *not* invertible since its determinant is zero.

Comment: Whenever a matrix A is invertible, its inverse A^{-1} has the same eigenvectors as A, and the eigenvalues are the inverses of the corresponding eigenvalues of A. In particular, suppose that $Av = \lambda v$ where A is invertible. Note that λ must be nonzero since A is invertible. Then we can multiply on the left by $(1/\lambda)\, A^{-1}$ to obtain

$$(1/\lambda)\, v = A^{-1} v$$

Therefore, v is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

Problem 3: Eigenvalues and eigenvectors

For each question, give a short answer. No proof or explanation is required.

a) If a square matrix is invertible, what can you say about its eigenvalues?

SOLUTION: None of the eigenvalues are zero.

Proof: A is not invertible iff its columns are not linearly independent, which is true iff some nonzero linear combination of the columns is zero, i.e., $Av = 0$ for some $v \neq 0$. But this is equivalent to zero being an eigenvalue of A.

b) If a square matrix is diagonalizable, what can you say about its eigenvectors?

SOLUTION: The matrix has a set of eigenvectors that are linearly independent.

Proof: For each eigenvalue λ_i, we can write $Ax_i = \lambda_i x_i$ where x_i is the associated eigenvector. Assembling these equations, we get:

$$A \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \vdots & \\ & & \lambda_n \end{bmatrix}$$

Or, in other words, $AT = T\Lambda$. We can diagonalize A and write $A = T\Lambda T^{-1}$ if and only if T is invertible, which is equivalent to its columns being linearly independent, i.e., A has n linearly independent eigenvectors.

c) If a square matrix with real entries is symmetric, what can you say about its eigenvalues?

SOLUTION: All eigenvalues of the matrix are real.

Proof: Let (x, λ) be a (possibly complex) eigenpair for A, i.e., $Ax = \lambda x$. Since A is real-valued and symmetric, we have $A^\mathsf{T} = A$. Now compute $x^* A x$ in two different ways:

$$\begin{aligned} x^* A x &= x^*(Ax) = \lambda\, x^* x = \lambda\, |x|^2 \\ &= x^* A^\mathsf{T} x = (Ax)^* x = (\lambda x)^* x = \lambda^*\, |x|^2 \end{aligned}$$

Therefore, $\lambda\, |x|^2 = \lambda^*\, |x|^2$. Since $x \neq 0$, this implies that $\lambda = \lambda^*$, i.e., λ is real.

d) If a square and symmetric matrix is indefinite (neither positive definite nor negative definite), what can you say about its eigenvalues?

SOLUTION: The matrix must have at least one positive eigenvalue and at least one negative eigenvalue.

A matrix is positive definite (by definition) if all eigenvalues are positive. A matrix is negative definite (by definition) if all eigenvalues are negative. If neither of these things are true, then either at least one eigenvalue is zero, or there are both positive and negative eigenvalues. As a side note, usually people mean the latter when they say "indefinite". I accepted both answers.

Problem 4: State-space models

Find state-space realizations for each of the following linear systems.

a) The transfer function:

$$\frac{Y(s)}{U(s)} = \frac{s^3+s-1}{3s^3+2s^2-s+2}$$

b) The system of differential equations:

$$\begin{aligned}\ddot{y}_1(t) + 5\dot{y}_1(t) - 10\left(y_2(t)-y_1(t)\right) &= u_1(t)\\ 2\ddot{y}_2(t) + \dot{y}_2(t) + 10\left(y_2(t)-y_1(t)\right) &= u_2(t)\end{aligned}$$

Note that this system has two inputs (u_1, u_2) and two outputs (y_1, y_2).

c) The Fibonacci sequence, which is an autonomous discrete-time system defined by

$$F_k = F_{k-1} + F_{k-2}$$

SOLUTION:

a) Normalizing so that $a_n = 1$ and performing polynomial division,

$$\frac{s^3+s-1}{3s^3+2s^2-s+2} = \frac{\frac{1}{3}s^3+\frac{1}{3}s-\frac{1}{3}}{s^3+\frac{2}{3}s^2-\frac{1}{3}s+\frac{2}{3}} = \frac{1}{3} + \frac{-\frac{2}{9}s^2+\frac{4}{9}s-\frac{5}{9}}{s^3+\frac{2}{3}s^2-\frac{1}{3}s+\frac{2}{3}}$$

From here, we can read off the coefficients, and we obtain the state-space realization:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} & 1 \\ \hline -\frac{5}{9} & \frac{4}{9} & -\frac{2}{9} & \frac{1}{3} \end{array}\right)$$

b) Define the state to be $x = \begin{bmatrix} y_1 & y_2 & \dot{y}_1 & \dot{y}_2 \end{bmatrix}^\mathsf{T}$. Then we can write the equations as

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_4\\ \dot{x}_3 &= -5x_3 + 10\left(x_2 - x_1\right) + u_1\\ \dot{x}_4 &= \tfrac{1}{2}(-x_4 - 10\left(x_2 - x_1\right) + u_2)\end{aligned}$$

In state-space form, this becomes:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -10 & 10 & -5 & 0 & 1 & 0 \\ 5 & -5 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right)$$

c) Define the state to be $x(k) = \begin{bmatrix} F_{k-1} & F_k \end{bmatrix}^\mathsf{T}$. The equations are then

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_1(k) + x_2(k) \end{aligned}$$

so the discrete-time state-space equations are simply:

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x(k)$$

Problem 5: Linearization

When a rigid cylinder is freely rotating in space, it is subject to the Euler equations of motion. If we fix a coordinate frame to the cylinder's center of mass with the z-axis aligned with the axis of rotational symmetry, the equations of motions in three dimensions are:

$$\begin{aligned} I_p\dot{x}_1(t) &= (I_p - I_q)x_2(t)x_3(t) + u_1(t) \\ I_p\dot{x}_2(t) &= (I_q - I_p)x_1(t)x_3(t) + u_2(t) \\ I_q\dot{x}_3(t) &= u_3(t) \end{aligned}$$

Here, $(x_1(t), x_2(t), x_3(t))$ and $(u_1(t), u_2(t), u_3(t))$ are the angular velocities and applied torques in the fixed coordinate frame, respectively. You can think of the applied torques as inputs (e.g. from gyroscopes) and the angular velocities as state variables. The constants $I_p > 0$ and $I_q > 0$ are the moments of inertia of the cylinder and you may assume they are known constants.

a) Consider the nominal state values $\tilde{x}_1 = \tilde{x}_2 = 0$ and $\tilde{x}_3 = \omega_0$ for some fixed $\omega_0 > 0$ and nominal input values $\tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = 0$. Verify that $\tilde{x}$ and $\tilde{u}$ satisfy the equations of motion. Find linearized state-space equations about the nominal values $(\tilde{x}, \tilde{u})$.

b) We will now consider a *time-varying* nominal trajectory. Show that the trajectory

$$\tilde{x}_1(t) = \sin\Big(\Big(1 - \tfrac{I_q}{I_p}\Big)\,\omega_0 t\Big), \qquad \tilde{x}_2(t) = \cos\Big(\Big(1 - \tfrac{I_q}{I_p}\Big)\,\omega_0 t\Big), \qquad \tilde{x}_3(t) = \omega_0$$

satisfies the equations of motion when $\tilde{u}_1(t) = \tilde{u}_2(t) = \tilde{u}_3(t) = 0$ for all t (no input). As in part **(a)**, $\omega_0 > 0$ is a fixed constant.

c) Linearize the equations of motion about the time-varying nominal trajectory $\tilde{x}(t)$ from the previous part. Express your answer as state-space equations where the inputs are $(\delta u_1(t), \delta u_2(t), \delta u_3(t))$ and the states are the perturbations of angular momentum from the nominal trajectory $(\delta x_1(t), \delta x_2(t), \delta x_3(t))$.
Hint: the solution will be a linear time-*varying* system.

SOLUTION:

a) When substituting, the time-derivatives are zero because $\tilde{x}$ is a constant. The only nonzero variable is $\tilde{x}_3$, but it gets multiplied by $\tilde{x}_1$ or $\tilde{x}_2$, both of which are zero. The first equation is of the form:

$$I_p\dot{x}_1 = f(x_1, x_2, x_3, u_1, u_2, u_3)$$

Linearizing yields:

$$\begin{aligned} I_p\,\delta\dot{x}_1 &= \sum_{i=1}^{3}\left[\frac{\partial f}{\partial x_i}(\tilde{x},\tilde{u})\right]\delta x_i + \sum_{j=1}^{3}\left[\frac{\partial f}{\partial u_j}(\tilde{x},\tilde{u})\right]\delta u_j \\ &= [0]\,\delta x_1 + [(I_p - I_q)\,\omega_0]\,\delta x_2 + [0]\,\delta x_3 + [1]\,\delta u_1 + [0]\,\delta u_2 + [0]\,\delta u_3 \end{aligned}$$

Doing something similar for the other equations, we obtain:

$$\begin{bmatrix} I_p\,\delta\dot{x}_1 \\ I_p\,\delta\dot{x}_2 \\ I_p\,\delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & (I_p - I_q)\,\omega_0 & 0 \\ (I_q - I_p)\,\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

Dividing through by I_p and I_q (multiplying by the inverse), we obtain:

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & I_p^{-1}(I_p - I_q)\,\omega_0 & 0 \\ I_p^{-1}(I_q - I_p)\,\omega_0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} I_p^{-1} & 0 & 0 \\ 0 & I_p^{-1} & 0 \\ 0 & 0 & I_q^{-1} \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

b) Substituting the nominal trajectory into the first equation, we obtain:

$$I_p\,\dot{\tilde{x}}_1(t) = I_p\left(1 - \tfrac{I_q}{I_p}\right)\omega_0 \cos\left(\left(1 - \tfrac{I_q}{I_p}\right)\omega_0 t\right) = (I_p - I_q)\,\tilde{x}_2(t)\,\tilde{x}_3(t) + \tilde{u}_1(t)$$

and similarly for the other two equations.

c) The method is exactly analogous to part **(a)**, except this time when we substitute $\tilde{x}$ and $\tilde{u}$, these are functions of time. The result is:

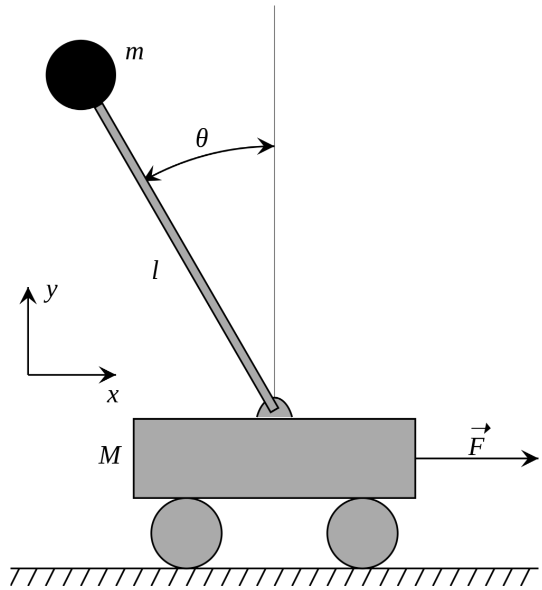
$$\begin{bmatrix} I_p\,\delta\dot{x}_1 \\ I_p\,\delta\dot{x}_2 \\ I_p\,\delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & (I_p - I_q)\,\omega_0 & (I_p - I_q)\cos\big((1 - \tfrac{I_q}{I_p})\,\omega_0 t\big) \\ (I_q - I_p)\,\omega_0 & 0 & (I_p - I_q)\sin\big((1 - \tfrac{I_q}{I_p})\,\omega_0 t\big) \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{bmatrix}$$

As in part **(a)**, we can divide through by I_p and I_q and we obtain a linear time-varying system of the form:

$$\delta\dot{x}(t) = A(t)\,\delta x(t) + Bu(t)$$

Problem 6: Cart-pole: Modeling

The cart-pole is a dynamical system consisting of a pole with a mass on one end with the other end connected by a revolute joint to a cart as shown below.



The rod has length ℓ and is assumed to be massless, while the mass on the end of the rod has mass m. The cart has mass M and has wheels with no friction that are connected to a fixed surface. The input to the system is the force f applied to the cart in the x direction. The variable θ measures the angle of the rod from vertical. The acceleration due to gravity has magnitude g in the negative y direction.

a) Derive the equations of motion for the system.

b) Linearize the equations of motion about the fixed point $\theta = 0$ with no input force.

SOLUTION:

a) While we could use Newtonian mechanics to find the equations of motion for the system, this would require using two orthogonal coordinates, such as (x, y), and balancing the forces in each direction for both the cart and the pole. A simpler approach, however, is to express the dynamics in the *generalized coordinates* (x, θ), which are independent but not orthogonal. To construct the equations of motion in these generalized coordinates, we will use *variational* (i.e., Lagrangian and Hamiltonian) mechanics.

First, we need the kinetic and potential energy of the system. The kinetic energy is the sum of the kinetic energies of the cart and the pendulum, which is

$$T = \underbrace{\frac{M}{2}\dot{x}^2}_{\text{cart}} + \underbrace{\frac{m}{2}\left(\dot{x}_{\text{pend}}^2 + \dot{y}_{\text{pend}}^2\right)}_{\text{pendulum}}$$

where $(x_{\textsf{pend}}, y_{\textsf{pend}})$ is the position of the pendulum. Using trigonometry, we can express the position of the pendulum as

$$x_{\textsf{pend}} = x - \ell \sin\theta \qquad \text{and} \qquad y_{\textsf{pend}} = \ell\cos\theta$$

Taking the time derivative, the velocity of the pendulum is

$$\dot{x}_{\textsf{pend}} = \dot{x} - \ell\dot{\theta}\cos\theta \qquad \text{and} \qquad \dot{y}_{\textsf{pend}} = -\ell\dot{\theta}\sin\theta$$

Therefore, the kinetic energy is

$$T = \frac{M}{2}\dot{x}^2 + \frac{m}{2}\Big((\dot{x} - \ell\dot{\theta}\cos\theta)^2 + (-\ell\dot{\theta}\sin\theta)^2\Big)$$

The potential energy of the system is due to the height of the pendulum and is given by

$$V = mgy_{\textsf{pend}} = mg\ell\cos\theta$$

The Lagrangian is then the difference between the kinetic and potential energies, $L = T - V$. Substituting quantities, expanding, and using that $\cos^2\theta + \sin^2\theta = 1$, we can write the Lagrangian in terms of the generalized coorinates (x, θ) and their derivatives as follows:

$$L(x, \dot{x}, \theta, \dot{\theta}) = \frac{M+m}{2}\,\dot{x}^2 - m\ell\dot{x}\dot{\theta}\cos\theta + \frac{m\ell^2}{2}\dot{\theta}^2 - mg\ell\cos\theta$$

We can now obtain the equations of motion using Lagrange's equations. To do so, we need the partial derivatives of the Lagrangian with respect to each generalized coordinate along with their derivatives, which are as follows:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= 0 \\
\frac{\partial L}{\partial \dot{x}} &= (M+m)\,\dot{x} - m\ell\dot{\theta}\cos\theta \\
\frac{\partial L}{\partial \theta} &= m\ell\sin\theta\,(\dot{x}\dot{\theta} + g) \\
\frac{\partial L}{\partial \dot{\theta}} &= m\ell^2\dot{\theta} - m\ell\dot{x}\cos\theta
\end{aligned}$$

We also need the time derivative of the partial derivative of the Lagrangian with respect to the derivatives of the generalized coordinates. Using the chain rule, these are given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} &= (M+m)\,\ddot{x} - m\ell\,(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} &= m\ell^2\ddot{\theta} - m\ell\ddot{x}\cos\theta + m\ell\dot{x}\dot{\theta}\sin\theta
\end{aligned}$$

We can now substitute these expressions into Lagrange's equations to obtain the equations of motion:

$$\begin{aligned}
f &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = (M+m)\,\ddot{x} - m\ell\,(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta) \\
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = m\ell^2\ddot{\theta} - m\ell\ddot{x}\cos\theta - mg\ell\sin\theta
\end{aligned}$$

This is a set of two coupled second-order nonlinear differential equations, so the state is $(x, \dot{x}, \theta, \dot{\theta})$ which has dimension four. Alternatively, we can use Hamiltonian mechanics to derive the equations of motion. The benefit of this approach, as we will see, is that it naturally identifies an invariant of the system that allows us to reduce the state dimension to three.

Instead of the derivatives $(\dot{x}, \dot{\theta})$, the Hamiltonian is expressed in terms of the generalized momenta (p_x, p_θ). The generalized momenta with respect to the generalized coordinates are given by

$$\begin{bmatrix} p_x \\ p_\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial L}{\partial \dot{x}} \\ \frac{\partial L}{\partial \dot{\theta}} \end{bmatrix} = \begin{bmatrix} M+m & -m\ell\cos\theta \\ -m\ell\cos\theta & m\ell^2 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix}$$

where we had already computed the partial derivatives above. Inverting this linear system of equations, we can express the derivatives of the generalized coordinates in terms of the generalized momenta as

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} = \frac{1}{M+m-m\cos^2\theta} \begin{bmatrix} 1 & \frac{\cos\theta}{\ell} \\ \frac{\cos\theta}{\ell} & \frac{M+m}{m\ell^2} \end{bmatrix} \begin{bmatrix} p_x \\ p_\theta \end{bmatrix}$$

The Hamiltonian is then defined as

$$H(x, p_x, \theta, p_\theta) = p_x\,\dot{x} + p_\theta\,\dot{\theta} - L(x, \dot{x}, \theta, \dot{\theta})$$

Substituting the expressions for $(\dot{x}, \dot{\theta})$ in terms of the generalized momenta (p_x, p_θ) found above and simplifying, we obtain

$$H = \underbrace{\frac{(M+m)\,p_\theta^2 + 2m\ell\cos\theta\,p_x\,p_\theta + m\ell^2 p_x^2}{m\ell^2\,(2M+m-m\cos 2\theta)}}_{T} + \underbrace{mg\ell\cos\theta}_{V}$$

In this case, the Hamiltonian is equal to the total energy in the system (this is not true in general, but it does hold when the potential energy does not depend on the velocities of the generalized coordinates and the constraint equations do not depend explicitly on time).

The partial derivatives of the Hamiltonian with respect to the generalized coordinates and momenta are

$$\begin{aligned}
\frac{\partial H}{\partial x} &= 0 \\
\frac{\partial H}{\partial \theta} &= -\left(\frac{2p_x p_\theta}{\ell\,(2M+m-m\cos 2\theta)} + mg\ell\right)\sin\theta \\
&\qquad - 2\,\frac{(M+m)p_\theta^2 + m\ell^2 p_x^2 + 2m\ell p_x p_\theta\cos\theta}{\ell^2\,(2M+m-m\cos 2\theta)^2}\sin 2\theta \\
\frac{\partial H}{\partial p_x} &= \frac{2\,(\ell p_x + p_\theta\cos\theta)}{\ell\,(2M+m-m\cos 2\theta)} \\
\frac{\partial H}{\partial p_\theta} &= \frac{2\,(M+m)p_\theta + 2m\ell p_x\cos\theta}{m\ell^2\,(2M+m-m\cos 2\theta)}
\end{aligned}$$

Hamilton's equations are then the following set of four coupled first-order nonlinear differential equations:

$$\dot{x} = \frac{\partial H}{\partial p_x} \qquad \dot{\theta} = \frac{\partial H}{\partial p_\theta} \qquad \dot{p}_x = -\frac{\partial H}{\partial x} + f \qquad \dot{p}_\theta = -\frac{\partial H}{\partial \theta}$$

Since the Hamiltonian does not explicitly depend on x, the corresponding generalized momentum p_x satisfies $\dot{p}_x = f$. In particular, if the input force is zero, then the generalized momentum is conserved (that is, p_x is constant in time). In this case, the equations of motion are a set of three coupled first-order differential equations.

b) We now linearize the equations of motion about the fixed point corresponding to the pendulum in the vertical position. Using the Hamiltonian formulation, the state $(x, \theta, p_x, p_\theta)$ satisfies the equations

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{p}_x \\ \dot{p}_\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial H}{\partial p_x} \\ \frac{\partial H}{\partial p_\theta} \\ -\frac{\partial H}{\partial x} \\ -\frac{\partial H}{\partial \theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} f$$

The Jacobian of the state transition function with respect to the state is

$$J(x, \theta, p_x, p_\theta) = \begin{bmatrix} \frac{\partial^2 H}{\partial p_x \partial x} & \frac{\partial^2 H}{\partial p_x \partial \theta} & \frac{\partial^2 H}{\partial p_x \partial p_x} & \frac{\partial^2 H}{\partial p_x \partial p_\theta} \\ \frac{\partial^2 H}{\partial p_\theta \partial x} & \frac{\partial^2 H}{\partial p_\theta \partial \theta} & \frac{\partial^2 H}{\partial p_\theta \partial p_x} & \frac{\partial^2 H}{\partial p_\theta \partial p_\theta} \\ -\frac{\partial^2 H}{\partial x \partial x} & -\frac{\partial^2 H}{\partial x \partial \theta} & -\frac{\partial^2 H}{\partial x \partial p_x} & -\frac{\partial^2 H}{\partial x \partial p_\theta} \\ -\frac{\partial^2 H}{\partial \theta \partial x} & -\frac{\partial^2 H}{\partial \theta \partial \theta} & -\frac{\partial^2 H}{\partial \theta \partial p_x} & -\frac{\partial^2 H}{\partial \theta \partial p_\theta} \end{bmatrix}$$

While this is a complicated function of the state, it simplifies considerably when we substitute the fixed point $(x_0, 0, 0, 0)$ where x_0 is arbitrary (the system has a fixed point where the pendulum is balanced for any horizontal position of the cart). With this fixed point, the Jacobian is

$$J(x_0, 0, 0, 0) = \begin{bmatrix} 0 & 0 & \frac{1}{M} & \frac{1}{M\ell} \\ 0 & 0 & \frac{1}{M\ell} & \frac{M+m}{Mm\ell^2} \\ 0 & 0 & 0 & 0 \\ 0 & mg\ell & 0 & 0 \end{bmatrix}$$

Therefore, the linearization of the system about the balanced vertical position is

$$\begin{bmatrix} \dot{x} \\ \dot{\theta} \\ \dot{p}_x \\ \dot{p}_\theta \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{M} & \frac{1}{M\ell} \\ 0 & 0 & \frac{1}{M\ell} & \frac{M+m}{Mm\ell^2} \\ 0 & 0 & 0 & 0 \\ 0 & mg\ell & 0 & 0 \end{bmatrix} \begin{bmatrix} x - x_0 \\ \theta \\ p_x \\ p_\theta \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} f$$

Note that the linearization does not actually depend on the horizontal position x_0.

Problem 7: State-space simulation

In this problem, we will use MATLAB to simulate a state-space system. Consider the spring-mass-damper model:

$$\ddot{y} + c\dot{y} + ky = u$$

a) Let's pick values of $c = 0.2$ and $k = 1$. Create a transfer function model for this system in MATLAB using the command `tf`. This can be done by specifying numerator and denominator coefficients or by using `s` directly. You can read the documentation and see examples by running `doc tf`. Plot the impulse response for $0 \le t \le 60$ using the `impulse` command, and repeat the task using $c = 0.01$ (less damping) and $c = 0.4$ (more damping). Describe what you see.

b) This time, create a state-space model for this system in MATLAB using the command `ss`. Start by finding the (A, B, C, D) matrices by hand. Again, refer to `doc ss` for guidelines. Then, plot the impulse response as in part **(a)** and verify that you get the same results.

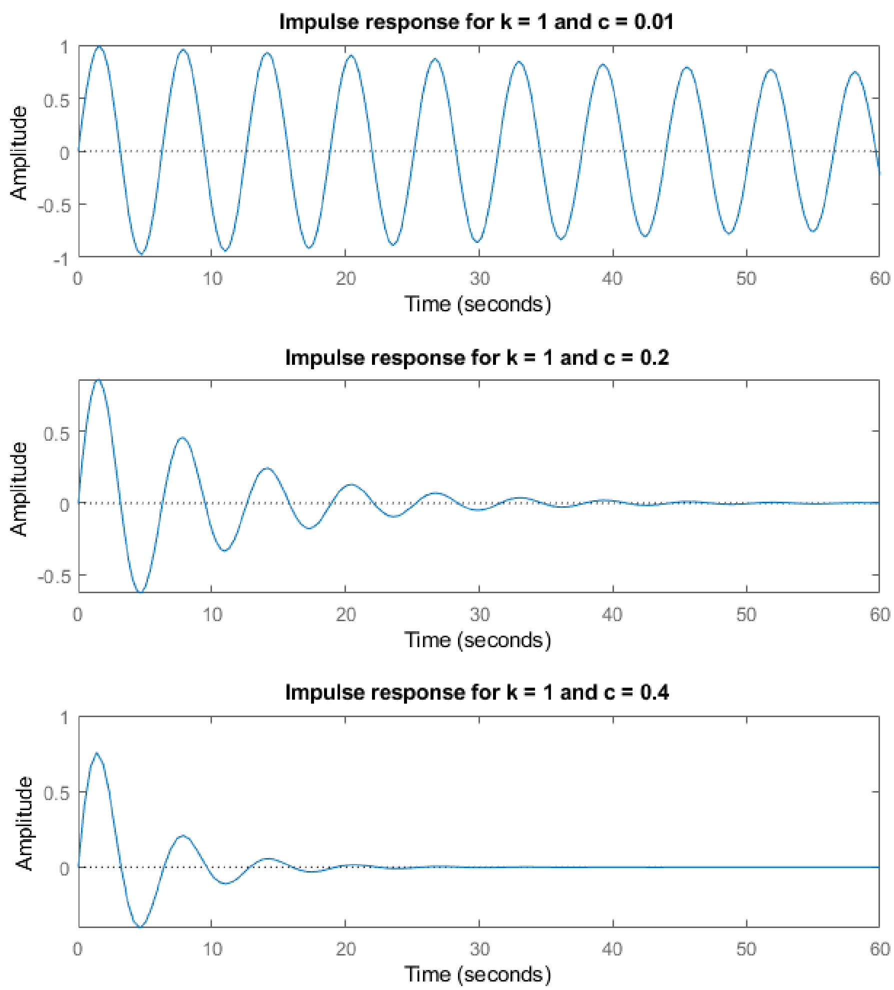
c) MATLAB can convert between state-space and transfer function for you. If `H` is the state-space object from part **(b)**, run `tf(H)` and verify that you recover the transfer function from part **(a)**. Likewise, if `G` is the transfer function object from part **(a)**, run `ss(G)` to to find a state-space realization. Is this realization the same as `H`? Explain.

SOLUTION: Here is MATLAB code that generates the plots:

```
figure;
t_final = 60;
cvalues = [0.01, 0.2, 0.4];  % possible c values to test

for i = 1:length(cvalues)

    c = cvalues(i);  % set value of c
    k = 1;           % set value of k

    % transfer function using num/den coefficients
    H1 = tf([1], [1 c k]);

    % transfer function using direct specification
    s = tf('s');
    H2 = 1/(s^2 + c*s + k);

    % transfer function from state-space model
    A = [0 1; -k -c];
    B = [0; 1];
    C = [1 0];
    D = [0];

    H3 = ss(A,B,C,D);

    % plot impulse response
    subplot(3,1,i)
    impulse(H1,t_final)
    title(['Impulse response for k = ' num2str(k)
```

```
        ' and c = ' num2str(c)])
end
```

The system oscillates due to the spring, and the oscillations dampen over time due to the damper. Larger values of the damping coefficient result in faster dampening of the response.



The state-space matrices look different because they use a different representation of the state, but they have the same transfer function and therefore represent the same system.

Problem 8: Discretizing a continuous-time system

In practice it is difficult to work with continuous-time signals because of storage and computational considerations. In this problem, we will discretize continuous state-space equations in a way that preserves the continuous-time behavior.

a) Consider the continuous-time state-space system: $\dot{x}(t) = Ax(t)$ with initial condition $x(0) = x_0$. We would like to find an exact discretization of this system. Specifically, we will sample $x(t)$ at times $t = 0, T, 2T, \ldots$. To this effect, define the discrete-time signal

$$x_d[k] = x(kT) \quad \text{for } k = 0, 1, 2, \ldots$$

Find a matrix A_d such that $x_d[k+1] = A_d x_d[k]$. The matrix A_d should depend on both A and T.

b) Let's add an input signal. Suppose we have the discrete input $u_d[k]$, and define the continuous-time input signal that is constant between sampling times, that is,

$$u(t) = u_d[k] \quad \text{for } kT \le t < (k+1)\,T$$

Now consider the continuous-time system: $\dot{x}(t) = Ax(t) + Bu(t)$ using the piecewise-constant input defined above. Find matrices A_d and B_d such that $x_d[k+1] = A_d x_d[k] + B_d u_d[k]$. Here, $x_d[k]$ is defined the same as in part **(a)**. You may express B_d as an integral.

c) Use MATLAB to simulate $x(t)$ for part **(a)** for the case where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -9 & -5 \end{bmatrix}$$

Plot all three components of $x(t)$ for $0 \le t \le 5$ with $x_0 = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\mathsf{T}$. Then, simulate the discretized system using $T = 0.2$. Verify that the two simulations agree (the second one should be a sampled version of the first one).
Note: the MATLAB command for computing matrix exponentials is `expm`.

SOLUTION:

a) The solution to the autonomous system is $x(t) = e^{At}x_0$. Substituting $t = kT$ and using the definition for $x_d[k]$, we obtain

$$x_d[k+1] = e^{A(k+1)T}\,x_0 = e^{AT}e^{AkT}\,x_0 = e^{AT}\,x_d[k]$$

So the discretized version has $A_d = e^{AT}$.

b) The solution to the system at time t_2 as a function of the initial time t_1 is given by

$$x(t_2) = e^{A(t_2-t_1)}\,x(t_1) + \int_{t_1}^{t_2} e^{A(t_2-\tau)}\,Bu(\tau)\,\mathrm{d}\tau$$

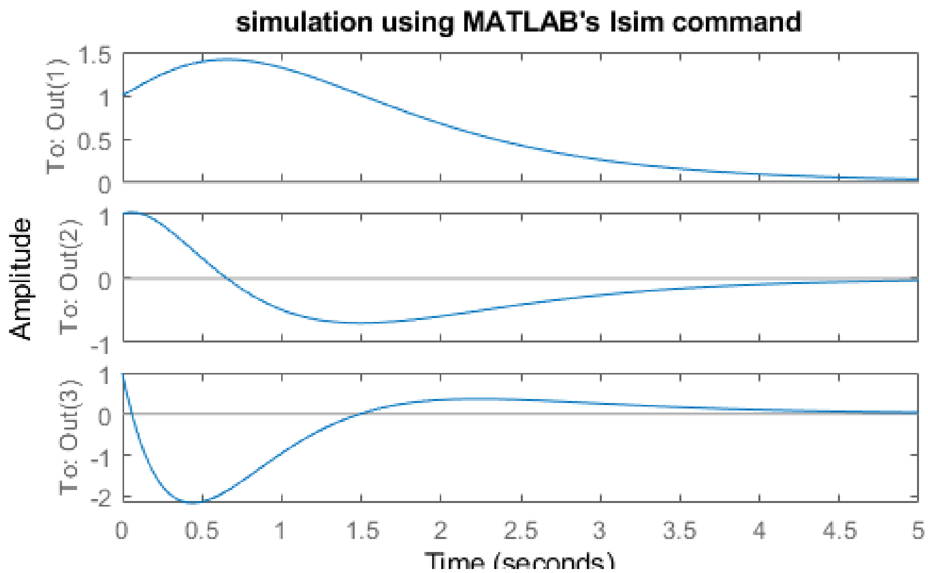
Substituting $t_1 = kT$ and $t_2 = (k+1)T$ and using the definition for $x_d[k]$, we obtain

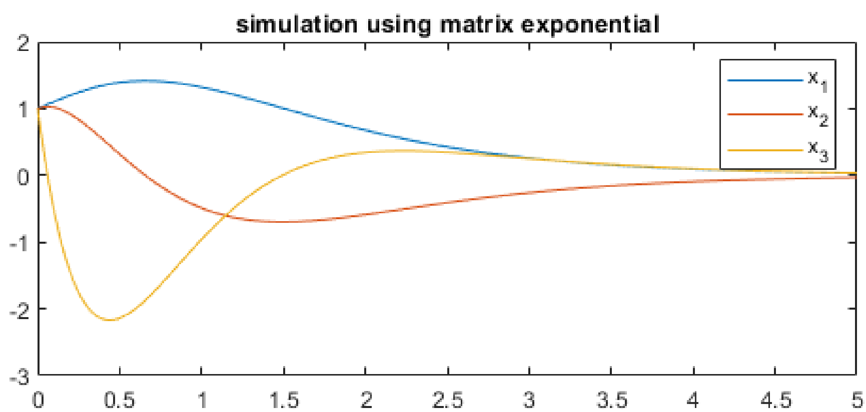
$$\begin{aligned}
x_d[k+1] &= e^{AT}x_d[k] + \int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}Bu(\tau)\,\mathrm{d}\tau \\
&= e^{AT}x_d[k] + \left(\int_{kT}^{(k+1)T} e^{A((k+1)T-\tau)}B\,\mathrm{d}\tau\right)u_d[k] \\
&= e^{AT}x_d[k] + \left(\int_0^T e^{A\tau}B\,\mathrm{d}\tau\right)u_d[k]
\end{aligned}$$

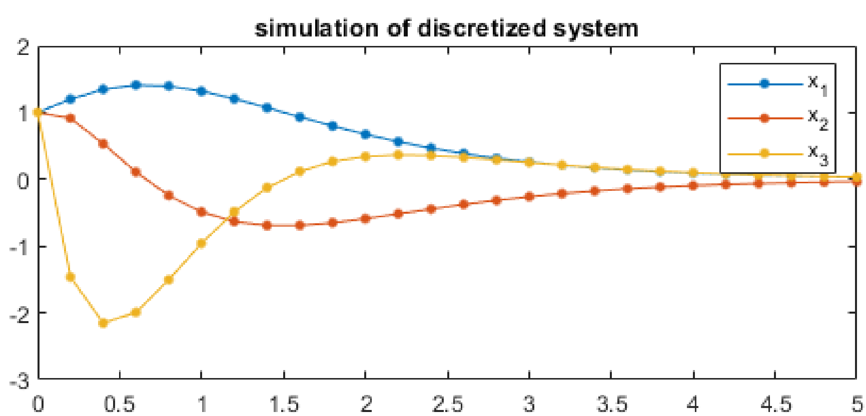
So the discretized version has $A_d = e^{AT}$ and $B_d = \int_0^T e^{A\tau}B\,\mathrm{d}\tau$.

c) Here is code that simulates using MATLAB's built-in `lsim` function, then manually using the matrix exponential (continuous-time solution), and finally using the discretized system. We can see that the three agree.

```
A  = [0 1 0; 0 0 1; -5 -9 -5];  % system matrix
x0 = [1; 1; 1];                 % initial condition
t  = linspace(0,5,300);         % time vector

% a) simulation using LSIM command
%     - set B to zero since no inputs
%     - set C to the identity to return the state as the output
sys = ss(A,zeros(3,1),eye(3),0);
u = zeros(size(t));  % zero input

figure(1);
lsim(sys,u,t,x0);
title('simulation using MATLAB''s lsim command');

% b) simulation using matrix exponential
x = zeros(3,300);
for k = 1:300
    x(:,k) = expm(A*t(k))*x0;
end

figure(2);
plot(t,x);
title('simulation using matrix exponential');
legend('x_1','x_2','x_3');

% c) simulation of discretized system
T  = 0.2;
Ad = expm(A*T);
td = 0:T:5;
n  = numel(td);
xd = zeros(3,n);
xd(:,1) = x0;
for j = 1:n-1
    xd(:,j+1) = Ad*xd(:,j);
end

figure(3);
plot(td,xd,'.-','MarkerSize',12);
title('simulation of discretized system');
legend('x_1','x_2','x_3');
```


simulation using MATLAB's lsim command
Amplitude
To: Out(1)
To: Out(2)
To: Out(3)
Time (seconds)


simulation using matrix exponential
x_1
x_2
x_3


simulation of discretized system
x_1
x_2
x_3

Problem 9: Discretizing a continuous-time system

Discretize the continuous-time system

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 10 & -15 & -20 \\ -4 & 6 & 8 \\ 8 & -12 & -16 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x(t) + u(t)\end{aligned}$$

assuming a piecewise constant input $u(t)$ and sampling interval $h = 0.1$. In other words, find the state-space matrices (A_d, B_d, C_d, D_d) of a discrete-time system

$$\begin{aligned}x_d[k+1] &= A_d\, x_d[k] + B_d\, u_d[k] \\ y_d[k] &= C_d\, x_d[k] + D_d\, u_d[k]\end{aligned}$$

such that the $x_d[k] = x(kh)$ and $y_d[k] = y(kh)$ for $k = 0, 1, 2, \ldots$ when the input signal satisfies

$$u(t) = u_d[k] \quad \text{for } kh \leq t < (k+1)h$$

SOLUTION: The discrete-time state-space matrices are given by

$$A_d = e^{Ah} \qquad \text{and} \qquad B_d = \int_0^h e^{At}\, B\, dt$$

Therefore, we need to compute the matrix exponential of A. We first convert the system to Jordan form using the state transformation $x(t) = Tz(t)$, where

$$T = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 2 & 1 \\ 1 & -4 & 1 \end{bmatrix} \qquad \text{and} \qquad T^{-1} = \begin{bmatrix} 6 & -7 & -11 \\ 1 & -1 & -2 \\ -2 & 3 & 4 \end{bmatrix}$$

The transformed state-space matrices are

$$\hat{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \qquad \text{and} \qquad \hat{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then the matrix exponential in the transformed coordinates is

$$e^{\hat{A}t} = \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & t\,e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

Converting back to the original coordinates,

$$e^{At} = Te^{\hat{A}t}T^{-1} = \begin{bmatrix} 2 & -5 & 3 \\ 0 & 2 & 1 \\ 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & t\,e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 6 & -7 & -11 \\ 1 & -1 & -2 \\ -2 & 3 & 4 \end{bmatrix}$$

The discrete-time input matrix is then

$$
\begin{aligned}
B_d &= \int_0^h e^{At}\, B\, dt \\
&= \int_0^h \left(T e^{\hat{A}t}\, T^{-1}\right)\left(T\hat{B}\right) dt \\
&= T \int_0^h \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & t\,e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} dt \\
&= T \int_0^h \begin{bmatrix} e^{-t} \\ t\,e^{-2t} \\ e^{-2t} \end{bmatrix} \mathrm{d}t \\
&= T \begin{bmatrix} -e^{-t} \\ -\frac{1}{4}(2t+1)\,e^{-2t} \\ -\frac{1}{2}e^{-2t} \end{bmatrix} \Bigg|_{t=0}^{h} \\
&= \begin{bmatrix} 2 & -5 & 3 \\ 0 & 2 & 1 \\ 1 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 - e^{-h} \\ \frac{1}{4} - \frac{1}{4}(2h+1)\,e^{-2h} \\ \frac{1}{2}(1 - e^{-2h}) \end{bmatrix}
\end{aligned}
$$

The output matrices are the same, $C_d = C$ and $D_d = D$.

Problem 10: Linear time-varying system

Consider the linear time-*varying* dynamical system

$$
\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad x(0) = x_0 \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{aligned}
$$

a) Suppose we can find a matrix function $P(t)$ with the property that

$$\dot{P}(t) = A(t)P(t) \quad \text{and} \quad \det(P(t)) \neq 0 \quad \text{for all } t$$

Such a $P(t)$ is called a *fundamental matrix*. Note that in the case where $A(t)$ is constant, we can use $P(t) = e^{At}$, as seen in class. Define $\Phi(t,\tau) = P(t)\,P(\tau)^{-1}$. Prove that the output of the time-varying system is given by

$$y(t) = C(t)\,\Phi(t,0)\,x_0 + \int_0^t C(t)\,\Phi(t,\tau)\,B(\tau)\,u(\tau)\,\mathrm{d}\tau + D(t)\,u(t)$$

Hint: Use the transformation $x(t) = P(t)\,z(t)$.

b) Based on the result from part **(a)**, it's enough to find a fundamental matrix $P(t)$ and we automatically obtain a solution to the time-varying system. Consider the special case where $x(t)$ is a scalar. In this case, $A(t) = a(t)$ is also a scalar, and the fundamental matrix $P(t) = p(t)$ is a scalar as well. Find $p(t)$.

c) Consider the sequence of functions $\{M_k\}$ defined recursively by

$$M_{k+1}(t) = I + \int_0^t A(\tau)\,M_k(\tau)\,\mathrm{d}\tau \qquad \text{for } k = 0, 1, 2, \dots$$

with initial condition $M_0(t) = I$, and let $P(t) = \lim_{k\to\infty} M_k(t)$. In other words,

$$
\begin{aligned}
P(t) = I + \int_0^t A(\tau_1)\,\mathrm{d}\tau_1 &+ \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2)\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 \\
&+ \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \int_0^{\tau_2} A(\tau_3)\,\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 + \dots
\end{aligned}
$$

It turns out this is a well-defined limit (the sequence $\{M_k\}$ converges uniformly). Prove that $P(t)$ is a fundamental matrix.

SOLUTION:

a) Using the state transformation $x(t) = P(t)z(t)$, we have

$$\dot{x}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(P(t)z(t)\Big) = \dot{P}(t)z(t) + P(t)\dot{z}(t) = A(t)P(t)z(t) + P(t)\dot{z}(t)$$

Substituting the system dynamics,

$$\begin{aligned} A(t)P(t)z(t) + P(t)\dot{z}(t) &= A(t)P(t)z(t) + B(t)u(t) \\ y(t) &= C(t)P(t)z(t) + D(t)u(t) \end{aligned}$$

The first equation has cancellations and simplifies to $\dot{z}(t) = P(t)^{-1}B(t)u(t)$. Therefore, we can integrate and conclude that

$$z(t) = z(0) + \int_0^t P(\tau)^{-1}B(\tau)u(\tau)\,\mathrm{d}\tau$$

Substituting this into the output equation and letting $\Phi(t,\tau) = P(t)\,P(\tau)^{-1}$ gives the result.

b) If $A(t) = a(t)$ is a scalar, the fundamental matrix is also a scalar $P(t) = p(t)$. We seek to solve the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = a(t)\,p(t)$$

We saw in class how to solve this using an integrating factor. Another way is by writing it in differential form and integrating:

$$\begin{aligned} \frac{\mathrm{d}p}{p} = a\,\mathrm{d}t \implies & \int_{p(0)}^{p(t)} \frac{1}{p}\mathrm{d}p = \int_0^t a(\tau)\,\mathrm{d}\tau \\ \implies & \log\big(p(t)\big) - \log\big(p(0)\big) = \int_0^t a(\tau)\,\mathrm{d}\tau \\ \implies & p(t) = p(0)\exp\left(\int_0^t a(\tau)\,\mathrm{d}\tau\right) \end{aligned}$$

We can use any $p(0) \neq 0$. Ultimately, it doesn't matter since it will cancel out when we compute

$$\Phi(t_1, t_2) = \frac{p(t_1)}{p(t_2)} = \exp\left(\int_{t_1}^{t_2} a(\tau)\,\mathrm{d}\tau\right)$$

c) The function $P(t)$ is a series: $P(t) = \sum_{k=0}^{\infty} P_k(t)$. The terms in this series are:

$$\begin{aligned} &P_0(t) = I \\ &P_1(t) = \int_0^t A(\tau_1)\,\mathrm{d}\tau_1 \\ &P_k(t) = \int_0^t A(\tau_1)\int_0^{\tau_1} A(\tau_2)\int_0^{\tau_2} A(\tau_3)\cdots\int_0^{\tau_k} A(\tau_k)\,\mathrm{d}\tau_k\cdots\mathrm{d}\tau_3\,\mathrm{d}\tau_2\,\mathrm{d}\tau_1 \end{aligned}$$

Differentiating a general term with respect to t, we obtain $\frac{\mathrm{d}}{\mathrm{d}t} P_0(t) = 0$ and for $k \geq 1$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} P_k(t) &= A(t) \int_0^t A(\tau_2) \int_0^{\tau_2} A(\tau_3) \cdots \int_0^{\tau_{k-1}} A(\tau_k)\, \mathrm{d}\tau_k \cdots \mathrm{d}\tau_3\, \mathrm{d}\tau_2 \\
&= A(t) \int_0^t A(\tau_1) \int_0^{\tau_1} A(\tau_2) \cdots \int_0^{\tau_{k-2}} A(\tau_{k-1})\, \mathrm{d}\tau_{k-1} \cdots \mathrm{d}\tau_2\, \mathrm{d}\tau_1 \\
&= A(t)\, P_{k-1}(t)
\end{aligned}$$

where we relabeled the dummy integration variables in the second step. Substituting this in to the series definition, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} P(t) &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{k=0}^{\infty} P_k(t) \\
&= \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t} P_k(t) \\
&= \sum_{k=1}^{\infty} A(t) P_{k-1}(t) \\
&= A(t) \sum_{k=0}^{\infty} P_k(t) \\
&= A(t) P(t)
\end{aligned}$$

Problem 11: System representation

Consider a continuous-time system with transfer function

$$H(s) = \frac{1}{(s+1)(s+2)}$$

a) What is the impulse response of the system? Note: the Laplace transform of e^{-at} is $\frac{1}{s+a}$.

SOLUTION: The impulse response is the inverse Laplace transform of the transfer function, which can be decomposed as

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Therefore, the impulse response is $h(t) = e^{-t} - e^{-2t}$.

ALT. SOLUTION: We can also use convolution. Since $H(s) = \left(\frac{1}{s+1}\right)\left(\frac{1}{s+2}\right)$, the impulse response is

$$\begin{aligned} h(t) &= e^{-t} * e^{-2t} \\ &= \int_0^t e^{-\tau}\, e^{-2(t-\tau)}\, \mathrm{d}\tau \\ &= e^{-2t} \int_0^t e^{\tau}\, \mathrm{d}\tau \\ &= e^{-2t}\left(e^t - 1\right) \\ &= e^{-t} - e^{-2t} \end{aligned}$$

b) Find a state-space realization (A, B, C, D) for this system.

SOLUTION: The transfer function is

$$H(s) = \frac{1}{s^2 + 3s + 2}$$

so one possible realization is the controllable canonical form:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 1 & 0 & 0 \end{array}\right)$$

ALT. SOLUTION: We can also write this as the differential equation

$$\ddot{y} + 3\dot{y} + 2y = u$$

Setting the state as $x = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$, this is equivalent to the state-space system

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \end{aligned}$$

which is the same result as before.

Problem 12: System representation

Consider a continuous-time system with transfer function

$$H(s) = \frac{s^2}{(s+1)(s+2)}$$

a) What is the impulse response of the system?

Note: the Laplace transform of e^{-at} is $\frac{1}{s+a}$, and the Laplace transform of 1 is $\delta(t)$.

SOLUTION: The impulse response is the inverse Laplace transform of the transfer function, which can be decomposed as

$$H(s) = \frac{s^2}{(s+1)(s+2)} = 1 + \frac{1}{s+1} - \frac{4}{s+2}$$

Therefore, the impulse response is $h(t) = \delta(t) + e^{-t} - 4\,e^{-2t}$.

b) Find a state-space realization (A, B, C, D) for this system.

SOLUTION: The transfer function is

$$H(s) = 1 + \frac{-3s-2}{s^2+3s+2}$$

so one possible realization is the controllable canonical form:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline -2 & -3 & 1 \end{array}\right)$$

Problem 13: Diagonal form

Consider the state-space equation $\dot{x}(t) = Ax(t) + Bu(t)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$$

a) Use a state transformation to convert this system into diagonal canonical form.

b) Is this system controllable? Explain your answer by analyzing your solution to part **(a)**.

c) Compute closed-form expressions for A^k and for e^{At}.

SOLUTION:

a) Based on the form of the A matrix, its characteristic polynomial is

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = (\lambda+1)(\lambda+2)(\lambda+3)$$

so the eigenvalues of A are $\{-1, -2, -3\}$. One possible diagonalization is

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Computing the inverse, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

The diagonal for of the system is $(T^{-1}AT, T^{-1}B)$, which we can compute to be

$$\hat{A} = T^{-1}AT = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \quad \text{and} \quad \hat{B} = T^{-1}B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

b) The system is not controllable. The controllability matrix (which we may compute in the transformed coordinates) is

$$[\hat{B} \quad \hat{A}\hat{B} \quad \hat{A}^2\hat{B}] = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

which is clearly rank-deficient because it has a row of zeros.

c) We can also compute powers and exponentials using the diagonalized form:

$$A^k = T\hat{A}^kT^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & (-3)^k \end{bmatrix} \begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

$$e^{At} = Te^{\hat{A}t}T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 6 & 5 & 1 \\ -6 & -8 & -2 \\ 2 & 3 & 1 \end{bmatrix}$$

Problem 14: Jordan normal form

Any square matrix $A \in \mathbb{R}^{n\times n}$ can be transformed (using a similarity transform) into Jordan normal form, which is a block-diagonal matrix that looks like:

$$T^{-1}AT = \begin{bmatrix} J_{-1} & & 0 \\ & \ddots & \\ 0 & & J_r \end{bmatrix} \qquad \text{with} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda_i & 1 \\ 0 & 0 & 0 & 0 & \lambda_i \end{bmatrix}$$

Each Jordan block J_i has an eigenvalue λ_i of A on its main diagonal with 1's immediately above the main diagonal. The size of each Jordan block depends on the geometric multiplicity of the corresponding eigenvalue. If A is diagonalizable, then each Jordan block is 1×1 and $T^{-1}AT$ is a diagonal matrix, recovering the standard eigenvalue decomposition in this case.

Suppose that a SISO LTI system (A, B, C, D) has an A matrix that is a single Jordan block:

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccccc|c} \lambda & 1 & 0 & \dots & 0 & b_1 \\ 0 & \lambda & 1 & \dots & 0 & b_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \lambda & 1 & b_{n-1} \\ 0 & 0 & 0 & 0 & \lambda & b_n \\ \hline c_1 & c_2 & c_3 & \dots & c_n & d \end{array}\right]$$

a) Show that (A, B) is controllable if and only if $b_n \neq 0$.

b) Show that (C, A) is observable if and only if $c_1 \neq 0$.

SOLUTION:

a) We will use the PBH test for controllability, which states that (A, B) is controllable if and only if $B^\mathsf{T} x = 0$ for all $x \in \mathbb{C}^n$ that are left-eigenvectors of A. Suppose that $A^\mathsf{T} x = \mu x$. Letting $x = \begin{bmatrix} x_1^* & \dots & x_n^* \end{bmatrix}^*$, we have:

$$\begin{array}{ccccc} \lambda x_1 = \mu x_1 & & (\lambda-\mu)x_1 = 0 & & (\lambda-\mu)x_1 = 0 \\ \lambda x_2 + x_1 = \mu x_2 & & (\lambda-\mu)x_2 + x_1 = 0 & & (\lambda-\mu)^2 x_2 = 0 \\ \lambda x_3 + x_2 = \mu x_3 & \iff & (\lambda-\mu)x_3 + x_2 = 0 & \iff & (\lambda-\mu)^3 x_3 = 0 \\ \vdots & & \vdots & & \vdots \\ \lambda x_n + x_{n-1} = \mu x_n & & (\lambda-\mu)x_n + x_{n-1} = 0 & & (\lambda-\mu)^n x_n = 0 \end{array}$$

Since $x \neq 0$, the only way to satisfy these equations is if $\mu = \lambda$. Making this substitution back into the original equations, we conclude that $x_1 = x_2 = \ldots = x_{n-1} = 0$ and $x_n \neq 0$ is arbitrary. But now $B^\mathsf{T} x = b_n x_n$. So $B^\mathsf{T} x \neq 0$ if and only if $b_n \neq 0$, as required.

b) We can prove this part similar to part **(a)**. The PBH test says that (C, A) is observable if and only if every eigenpair (x, μ) of A satisfies $Cx \neq 0$. This is essentially the same as part **(a)** except we are working with A rather than A^T. Once again, we conclude that $\mu = \lambda$, except this time $x_1 \neq 0$ is arbitrary and $x_2 = x_3 = \ldots = x_n = 0$. Then $Cx = c_1 x_1$, so (C, A) is observable if and only if $c_1 \neq 0$, as required.

Problem 15: Matrix functions

Let $A = \begin{bmatrix} -4 & 1 \\ -6 & 1 \end{bmatrix}$.

a) Compute the matrix power A^k.

SOLUTION: The matrix is diagonalizable with eigendecomposition

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

Therefore, the matrix power is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} (-1)^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

b) Compute the matrix exponential e^{At}.

SOLUTION: Using the eigendecomposition above, the matrix exponential is

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}$$

Problem 16: Controllability

For which values of α is the following system controllable?

$$\dot{x}(t) = \begin{bmatrix} 1 & \alpha & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

SOLUTION: The controllability matrix is

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2+\alpha & 2+2\alpha \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The system is controllable if and only if the controllability matrix is full rank. We can determine this in numerous ways such as computing the determinant, looking at the columns, looking at the rows, etc. Let's look at columns. The first column is always independent from the other two because of the zeros in the third entry. So we conclude that the matrix drops rank only when the second column is a multiple of the third column. This happens when $2 + \alpha = 2 + 2\alpha$, i.e. when $\alpha = 0$. So the system is controllable if and only if $\alpha \neq 0$.

Problem 17: Controllability

For which values of α is the following system controllable?

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$

SOLUTION: The controllability matrix is

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & \alpha & 3\alpha \\ 1 & 0 & 0 \\ 1 & 3 & 3+\alpha \end{bmatrix}$$

The system is controllable if and only if the controllability matrix is full rank. We can determine this in numerous ways such as computing the determinant, looking at the columns, looking at the rows, etc. Let's compute the determinant:

$$\begin{aligned} \det(P) &= \begin{vmatrix} 0 & 0 \\ 3 & 3+\alpha \end{vmatrix} - \alpha \begin{vmatrix} 1 & 0 \\ 1 & 3+\alpha \end{vmatrix} + 3\alpha \begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} \\ &= 0 - \alpha\,(3+\alpha) + 3\alpha\,(3) \\ &= \alpha\,(6-\alpha) \end{aligned}$$

So the system is controllable if and only if $\alpha \neq 0$ and $\alpha \neq 6$.

Problem 18: Controllable subspace

Consider the continuous-time LTI system

$$\dot{x}(t) = \begin{bmatrix} -5 & -1 & -4 & 5 \\ 12 & 0 & 5 & -13 \\ -6 & -1 & -3 & 5 \\ -6 & -1 & -4 & 6 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 5 \\ -2 \\ -2 \end{bmatrix} u(t)$$

a) State whether or not the system is controllable, and explain your reasoning.

b) Find the controllable subspace, that is, the set of states that the system can be driven to from the origin.

c) Consider the following state vectors:

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \qquad x_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad x_5 = \begin{bmatrix} -2 \\ 3 \\ -2 \\ -2 \end{bmatrix}$$

For each combination of the state vectors, determine whether or not an input $u(t)$ exists that moves the state from one state vector to the other in one second.

SOLUTION:

a) The controllability matrix is

$$P = \begin{bmatrix} -1 & -2 & 3 & -13 \\ 5 & 4 & 0 & 20 \\ -2 & -3 & 2 & -14 \\ -2 & -3 & 2 & -14 \end{bmatrix}$$

The matrix P is not full rank since its last two rows are identical, so the system is *not* controllable.

b) The controllable subspace is the range of the controllability matrix.

$$\text{range}(P) = \text{range}\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right)$$

c) The system can be driven between any two states in the controllable subspace in arbitrary finite time. Only the state x_4 is not in the controllable subspace. Therefore, the state can be driven between any combination of states that do not include x_4.

Problem 19: Controllable decomposition

If (A, B) is not controllable and its controllability matrix has rank $q < n$, then there exists a state transformation T such that

$$(A, B) \to \left(\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix} \right)$$

where $\hat{A}_{11} \in \mathbb{R}^{q \times q}$.

Prove that $(\hat{A}_{11}, \hat{B}_1)$ is controllable.

SOLUTION: The controllability matrix of the transformed system is

$$\hat{P} = \begin{bmatrix} \hat{B}_1 & \hat{A}_{11} & \dots & \hat{A}_{11}^{n-1} \hat{B}_1 \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

Since linear state transformations do not change the rank of the controllability matrix,

$$q = \text{rank}(P) = \text{rank}(\hat{P})$$

But the rows of zeros in $\hat{P}$ do not contribute to the rank, so we can remove them and conclude that

$$\text{rank}\big(\begin{bmatrix} \hat{B}_1 & \hat{A}_{11} & \dots & \hat{A}_{11}^{n-1} \hat{B}_1 \end{bmatrix} \big) = q$$

Since $\hat{A}_{11} \in \mathbb{R}^{q \times q}$, this matrix has only q rows and therefore has full row rank. But this is also the controllability matrix for $(\hat{A}_{11}, \hat{B}_1)$, so we conclude that $(\hat{A}_{11}, \hat{B}_1)$ is controllable.

Problem 20: Observability

Consider two linear time-invariant systems (A, B, C, D) and $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ that have the same transfer function and are both observable. Suppose that the states of the two systems are related by a similarity transformation T. Find an explicit expression for this transformation matrix T.

SOLUTION: The observability matrix of the transformed system is

$$\hat{Q} = \begin{bmatrix} \hat{C} \\ \hat{C}\hat{A} \\ \vdots \\ \hat{C}\hat{A}^{n-1} \end{bmatrix} = \begin{bmatrix} CT \\ (CT)(T^{-1}AT) \\ \vdots \\ (CT)(T^{-1}AT)^{n-1} \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} T = QT$$

where Q is the observability matrix of the original system. Since the system is observable, Q has full column rank. Then its pseudoinverse

$$Q^{\dagger} = \left(Q^{\mathsf{T}}Q\right)^{-1}Q^{\mathsf{T}}$$

is a right inverse, so the transformation matrix is

$$T = Q^{\dagger}\hat{Q}$$

Problem 21: Observability

a) If possible, find the initial state $x(0)$ of the system

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x(k)$$
$$y(k) = \begin{bmatrix} 1 & 2 \end{bmatrix} x(k)$$

when the output sequence is $y(k) = \{1, 1, -2, 1, 1, -2, \ldots\}$.

SOLUTION: The initial state is the solution to the set of linear equations

$$\begin{bmatrix} y(0) \\ y(1) \end{bmatrix} = \begin{bmatrix} C \\ CA \end{bmatrix} x(0)$$

Substituting the specific values,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix} x(0)$$

which has the solution $x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

b) Find the unobservable subspace of the system

$$x(k+1) = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} x(k) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) + u(k)$$

(that is, the set of initial conditions $x(0)$ that are not observable).

SOLUTION: The observability matrix of the system is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

The unobservable subspace is the nullspace of the observability matrix, which is

$$\text{null}(Q) = \left\{ \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \mid \alpha \in \mathbb{R} \right\}$$

Problem 22: Realizing a transfer function

For the transfer function $H(s) = \frac{s+1}{s^2+2}$, find:

a) an uncontrollable and observable realization,

b) a controllable and unobservable realization,

c) an uncontrollable and unobservable realization,

d) a controllable and observable (minimal) realization.

SOLUTION: Recall the Kalman decomposition:

$$H(s) = \left[\begin{array}{cccc|c} A_{c\bar{o}} & A_{12} & A_{13} & A_{14} & B_{c\bar{o}} \\ 0 & A_{co} & 0 & A_{24} & B_{co} \\ 0 & 0 & A_{\bar{c}\bar{o}} & A_{34} & 0 \\ 0 & 0 & 0 & A_{\bar{c}o} & 0 \\ \hline 0 & C_{co} & 0 & C_{\bar{c}o} & D \end{array}\right]$$

The controllable canonical form for the system is $H(s) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}\right]$. This realization has two states, which is the same as the degree of $H(s)$ (and there are no pole-zero cancellations). Therefore, this realization is minimal, and we can use it for (A_{co}, B_{co}, C_{co}). We can get solutions to the other parts by adding a component to the appropriate part of the Kalman decomposition. Possible solutions are below, where red indicates the states that are removable.

a) uncontrollable and observable: $H(s) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 1 & 0 \end{array}\right]$

b) controllable and unobservable: $H(s) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ \hline 1 & 1 & 0 & 0 \end{array}\right]$

c) uncontrollable and unobservable: $H(s) = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 \end{array}\right]$

d) controllable and observable: $H(s) = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & 0 & 1 \\ \hline 1 & 1 & 0 \end{array}\right]$

Problem 23: Kalman decomposition

In this problem, you will write a MATLAB function that performs the Kalman decomposition. The following MATLAB commands might be useful as you write your code:

- `P = ctrb(A,B)` returns the controllability matrix.
- `Q = obsv(A,C)` returns the observability matrix.
- `T = orth(M)` returns a matrix whose columns are a basis for $\mathrm{col}(M)$.
- `T = null(M)` returns a matrix whose columns are a basis for (M).

Hint: For any matrix A, the *orthogonal complement* is a matrix $A^\perp$ whose columns are a basis for the set of all vectors that are orthogonal to the columns of A, in other words, the matrix $\begin{bmatrix} A & A^\perp \end{bmatrix}$ is nonsingular. Some useful facts are that $\left(A^\mathsf{T}\right) = \mathrm{col}(A)^\perp$ and $(A^\perp)^\perp = A$.

a) Given two subspaces of $\mathbb{R}^n$ whose bases are given by the columns of T_1 and T_2 respectively, the MATLAB function below returns a matrix T whose columns form a basis for the intersection of the two subspaces. Explain how it works.

```
function T = intersect_subspaces(T1,T2)
    [~,r] = size(T1);
    S = null( [T1 T2] );
    T = T1 * S(1:r,:);
end
```

SOLUTION: We first note that the following three statements are equivalent:

(i) x is in the intersection of the subspaces with bases given by the columns of T_1 and T_2

(ii) $x = T_1 z_1$ and $x = -T_2 z_2$ for some z_1 and z_2

(iii) $x = T_1 z_1$ and $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \in \left(\begin{bmatrix} T_1 & T_2 \end{bmatrix}\right)$

The code first computes r, the number of columns of T_1, which is also the size of z_1. It then computes $S = \left(\begin{bmatrix} T_1 & T_2 \end{bmatrix}\right)$ which is the set of all possible z's. Then from item (iii), the basis for the intersection is the product of T_1 and the first r rows of S since $x = T_1 z_1$.

b) Given two subspaces of $\mathbb{R}^n$ whose bases are given by the columns of T_1 and T_2 respectively, with $\mathrm{col}(T_1) \subseteq \mathrm{col}(T_2)$, the MATLAB function below returns a matrix T with columns that complete the basis in T_2. That is, $\mathrm{col}\left(\begin{bmatrix} T_1 & T \end{bmatrix}\right) = \mathrm{col}(T_2)$. Explain how it works.

```
function T = complete_basis(T1,T2)
    T1bar = null(T1');
    T = intersect_subspaces(T2,T1bar);
end
```

SOLUTION: From the first line of code, the columns of $\bar{T}_1$ are a basis for the nullspace of T_1^T, which is the orthogonal complement of the columnspace of T_1. In other words, $\begin{bmatrix} T_1 & \bar{T}_1 \end{bmatrix}$ is square and invertible. The second line of code then computes the intersection of T_2 and $\bar{T}_1$, which is the set of vectors that are both in T_2 and orthogonal to T_1, that is, the completion of T_1 in T_2.

c) Write a MATLAB function `[T1,T2,T3,T4] = kcf(A,B,C)` that takes as input the matrices for a state-space realization and returns the blocks of the matrix T that transforms the system into its Kalan decomposition. Use the convention that $T = \begin{bmatrix} T_{c\bar{o}} & T_{co} & T_{\bar{c}\bar{o}} & T_{\bar{c}o} \end{bmatrix}$.

SOLUTION: We can now put everything together and follow the steps laid out in class for constructing the T_i matrices for the Kalman decomposition. The code is below.

```
function [T1,T2,T3,T4] = kcf(A,B,C)
% computes the state transformation T = [T1 T2 T3 T4] that
% transforms a system with state-space matrices (A,B,C,D)
% to its Kalman decomposition

n = size(A,1);  % number of states

P = ctrb(A,B);  % controllability matrix
Q = obsv(A,C);  % observability matrix

controllable = orth(P);  % basis for controllable subspace
unobservable = null(Q);  % basis for unobservable subspace

% controllable and unobservable
T1 = intersect_subspaces(controllable,unobservable);

% controllable and observable
T2 = complete_basis(T1,controllable);

% uncontrollable and unobservable
T3 = complete_basis(T1,unobservable);

% uncontrollable and observable
T4 = complete_basis([T1 T2 T3],eye(n));

end
```

d) Write a MATLAB function `Am,Bm,Cm = reduce(A,B,C)` that returns a minimal realization via the Kalman decomposition. To test your function, find a minimal realization for

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} 5 & 9 & 2 & 1 & 2 \\ -5 & -6 & -1 & 0 & -1 \\ -7 & -13 & -5 & -2 & -3 \\ 14 & 5 & 2 & -4 & 4 \\ \hline 7 & 16 & 3 & 3 & 0 \end{array}\right]$$

Compare your result to the MATLAB command `minreal(ss(A,B,C,D))`. You can see if the realizations are the same by comparing their transfer function computed using `tf(...)`.

SOLUTION: To find a minimal realization, we first compute the Kalman decomposition and then extract the controllable and observable parts of the realization. Here is the code.

```
function [Am,Bm,Cm,Dm] = reduce(A,B,C,D)
% returns a minimal realization (Am,Bm,Cm,Dm) of the LTI
% system with state-space matrices (A,B,C,D)

% compute the Kalman decomposition transformation matrix
[T1,T2,T3,T4] = kcf(A,B,C);
T = [T1 T2 T3 T4];

% apply the state transformation
At = T\A*T;
Bt = T\B;
Ct = C*T;
Dt = D;

% isolate controllable + observable states (the T2 block)
[~,n1] = size(T1);
[~,n2] = size(T2);
ix = n1+1:n1+n2;

% extract corresponding blocks of the realization
Am = At(ix,ix);
Bm = Bt(ix,:);
Cm = Ct(:,ix);
Dm = Dt;

end
```

We can test our function on the example using the following code.

```
% state-space matrices
A = [5 9 2 1; -5 -6 -1 0; -7 -13 -5 -2; 14 5 2 -4];
B = [2; -1; -3; 4];
C = [7 16 3 3];
D = 0;

[Am,Bm,Cm,Dm] = reduce(A,B,C,D);

Hmin1 = ss(Am,Bm,Cm,Dm)        % using our function
Hmin2 = minreal(ss(A,B,C,D))  % using MATLAB commands

% compare transfer functions
tf(Hmin1)
tf(Hmin2)

% compute Kalman canonical form (round to 4 decimal digits)
[T1,T2,T3,T4] = kcf(A,B,C);
T = [T1 T2 T3 T4];
At = round(T\A*T,4);
Bt = round(T\B,4);
Ct = round(C*T,4);
Dt = round(D,4);

```

```
% print out the state-space realization
ss(At,Bt,Ct,Dt)
```

The Kalman decomposition of the system is

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{cccc|c} -1 & 1.757 & -4.715 & 13.65 & -6.396 \\ 0 & -2 & 0 & 24.97 & -4.369 \\ 0 & 0 & -3 & -10.35 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ \hline 0 & -0.2289 & 0 & -12.7 & 0 \end{array}\right]$$

where the controllable and observable part is shown in red. The minimal realization of this system computed using our function is

$$\left[\begin{array}{c|c} -2 & -4.369 \\ \hline -0.2289 & 0 \end{array}\right]$$

while MATLAB's `minreal` function returns

$$\left[\begin{array}{c|c} -2 & -3.09 \\ \hline -0.3237 & 0 \end{array}\right]$$

While these realization look different, they are actually equivalent since both correspond to the same transfer function

$$H(s) = \frac{1}{s+2}$$

Problem 24: Minimal realizations

a) Consider a system with the following state-space realization:

$$\left[\begin{array}{ccccc|cc} 3 & 0 & 0 & 3 & 0 & 0 & 7 \\ 3 & 2 & 7 & 2 & 8 & 1 & 0 \\ 0 & 3 & 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 2 & 0 & 0 \\ \hline 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Find a minimal realization for the system.

SOLUTION: The last two states are not affected by any other states, nor are they affected by the input. Therefore, these states are uncontrollable. Removing them, we are left with

$$\left[\begin{array}{ccc|cc} 3 & 0 & 0 & 0 & 7 \\ 3 & 2 & 7 & 1 & 0 \\ 0 & 3 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last two states in this realization do not affect any other states, nor do they affect the output. Therefore, these states are unobservable. Removing them, we are left with

$$\left[\begin{array}{c|cc} 3 & 0 & 7 \\ \hline 0 & 0 & 0 \\ 2 & 0 & 0 \end{array}\right]$$

which has only has one state and $B \neq 0$ and $C \neq 0$, so it is minimal.

b) Find a minimal realization for a system whose impulse response is $h(t) = 2e^{-t} - 3e^{-2t}$.

SOLUTION: We can use linearity. A system with impulse response e^{at} has realization

$$\left[\begin{array}{c|c} a & 1 \\ \hline 1 & 0 \end{array}\right]$$

Therefore, a system with impulse response $2e^{-t} - 3e^{-2t}$ has the minimal realization

$$\left[\begin{array}{c|c} -1 & 2 \\ \hline 1 & 0 \end{array}\right] + \left[\begin{array}{c|c} -2 & -3 \\ \hline 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & -2 & -3 \\ \hline 1 & 1 & 0 \end{array}\right]$$

ALT. SOLUTION: An alternate approach is to compute the transfer function

$$G(s) = \frac{2}{s+1} - \frac{3}{s+2} = \frac{-s+1}{(s+1)(s+2)} = \frac{-s+1}{s^2+3s+2}$$

and then use the controllable canonical form,

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ -2 & -3 & 1 \\ \hline 1 & -1 & 0 \end{array}\right]$$

Problem 25: Minimality of diagonal canonical form

Consider the two-dimensional system

$$\dot{x}(t) = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x(t) + Du(t)$$

where $\lambda_1 \neq \lambda_2$. Find conditions under which the system is minimal.

SOLUTION: The state-space realization is minimal if and only if it is both controllable and observable.

- **Controllability:** The controllability matrix is

  $$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} b_1 & \lambda_1 b_1 \\ b_2 & \lambda_2 b_2 \end{bmatrix}$$

  which has the determinant
  $$\det(P) = (\lambda_1 - \lambda_2)\, b_1\, b_2$$
  Since $\lambda_1 \neq \lambda_2$, this is nonzero if and only if both $b_1 \neq 0$ and $b_2 \neq 0$.

- **Observability:** The observability matrix is

  $$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ \lambda_1 c_1 & \lambda_2 c_2 \end{bmatrix}$$

  which has the determinant
  $$\det(Q) = (\lambda_1 - \lambda_2)\, c_1\, c_2$$
  Since $\lambda_1 \neq \lambda_2$, this is nonzero if and only if both $c_1 \neq 0$ and $c_2 \neq 0$.

Therefore, the system is minimal if and only if

$$b_1 \neq 0 \qquad b_2 \neq 0 \qquad c_1 \neq 0 \qquad c_2 \neq 0$$

Problem 26: Minimal realization

Consider the following state-space realization:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cccc|c} -3 & 7 & 4 & 0 & 1 \\ 0 & 5 & 0 & 0 & 0 \\ -1 & 9 & 2 & 0 & 0 \\ -2 & 3 & 6 & 1 & 4 \\ \hline 1 & 2 & -1 & 0 & 0 \end{array}\right)$$

Find a minimal realization for the system.

SOLUTION: The last state does not affect any other states or the output, so it is unobservable. Removing it, we are left with

$$\left(\begin{array}{ccc|c} -3 & 7 & 4 & 1 \\ 0 & 5 & 0 & 0 \\ -1 & 9 & 2 & 0 \\ \hline 1 & 2 & -1 & 0 \end{array}\right)$$

The second state in this realization is not affected by any other states or the input, so it is uncontrollable. Removing it, we are left with

$$\left(\begin{array}{cc|c} -3 & 4 & 1 \\ -1 & 2 & 0 \\ \hline 1 & -1 & 0 \end{array}\right)$$

The observability matrix of this realization is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

which has rank one, so the system is *not* observable (and therefore not minimal). One way to find a minimal realization is to compute the transfer function

$$\begin{aligned} H(s) &= C\,(sI - A)^{-1}B + D \\ &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s+3 & -4 \\ 1 & s-2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(s-1)(s+2)} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s-2 & 4 \\ -1 & s+3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{s-1}{(s-1)(s+2)} \\ &= \frac{1}{s+2} \end{aligned}$$

which, after a pole-zero cancellation, has the minimal realization

$$\left(\begin{array}{c|c} -2 & 1 \\ \hline 1 & 0 \end{array}\right)$$

Problem 27: Stability of LTI systems

Consider the linear system $\dot{x} = Ax$. For each case, examine the eigenvalues to determine whether the system is asymptotically stable, marginally stable, or unstable.

a) $A = \begin{bmatrix} 0 & 1 \\ -14 & -4 \end{bmatrix}$ **b)** $A = \begin{bmatrix} 0 & 1 \\ -14 & 4 \end{bmatrix}$ **c)** $A = \begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$ **d)** $A = \begin{bmatrix} 0 & 1 \\ -14 & 0 \end{bmatrix}$

SOLUTION:

a) Eigenvalues are $-2 \pm 3.16228i$, so the system is *asymptotically stable*.

b) Eigenvalues are $2 \pm 3.16228i$, so the system is *unstable*.

c) Eigenvalues are 0 and -4, so the system is *marginally stable*.

d) Eigenvalues are $\pm 3.74166i$, so the system is *marginally stable*.

Problem 28: Discrete-time Lyapunov equation

Prove the discrete-time version of the Lyapunov theorem:

For any $Q = Q^\mathsf{T} \succ 0$, the equation $A^\mathsf{T} P A - P + Q = 0$ has a solution $P = P^\mathsf{T} \succ 0$ if and only if A is Schur (that is, all eigenvalues of A have magnitude less than one).

Hint: To prove the *if* part, P will involve an infinite sum rather than an infinite integral. Be sure to prove both directions (*if* and *only if*).

SOLUTION:
($\Longrightarrow$) Suppose $P = P^\mathsf{T} \succ 0$ satisfies $A^\mathsf{T} P A - P + Q = 0$. Let (x, λ) be an eigenpair of A, that is, $Ax = \lambda x$. Multiplying the equation on the right and left by x and its conjugate transpose, respectively, gives

$$\begin{aligned} 0 &= x^* \left(A^\mathsf{T} P A - P + Q\right) x \\ &= (\lambda\bar{\lambda})\, x^* P x + x^* Q x \\ &= \left(|\lambda|^2 - 1\right)(x^* P x) + (x^* Q x) \end{aligned}$$

Since $P \succ 0$ and $Q \succ 0$, we have that $x^* P x > 0$ and $x^* Q x > 0$. These imply that $(|\lambda|^2 - 1) < 0$, or equivalently, $|\lambda| < 1$. Therefore, A is Schur.

($\Longleftarrow$) Suppose A is Schur. Then $A^k \to 0$ as $k \to \infty$. Define the matrix

$$P = \sum_{k=0}^{\infty} \left(A^k\right)^\mathsf{T} Q \left(A^k\right)$$

which is symmetric and satisfies the matrix equation since

$$\begin{aligned} A^\mathsf{T} P A - P &= \sum_{k=0}^{\infty} \left(A^{k+1}\right)^\mathsf{T} Q \left(A^{k+1}\right) - \left(A^k\right)^\mathsf{T} Q \left(A^k\right) \\ &= \lim_{k\to\infty} \left(A^k\right)^\mathsf{T} Q \left(A^k\right) - Q \\ &= -Q \end{aligned}$$

To prove that $P \succ 0$, multiply on the right and left by an arbitrary nonzero vector $v \in \mathbb{R}^n$ and its transpose, respectively, to obtain

$$v^\mathsf{T} P v = \sum_{k=0}^{\infty} \left(A^k v\right)^\mathsf{T} Q \left(A^k v\right)$$

Since $Q \succ 0$, each term of the sum is nonnegative. Moreover, the first term in the sum $(k = 0)$ is simply $v^\mathsf{T} Q v$, which is stricly positive if $v \neq 0$. Therefore, $P \succ 0$. ■

Problem 29: Transient behavior

An asymptotically stable linear dynamical system with state $x(t)$ satisfies $\lim_{t\to\infty} \|x(t)\| = 0$. However, this convergence may not be monotonic. In other words, $\|x(t)\|$ can get very large before it settles down to zero.

a) As an example, consider the stable linear system

$$\dot{x}(t) = \begin{bmatrix} -0.1 & 100 \\ 0 & -0.2 \end{bmatrix} x(t) \qquad \text{with} \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Plot $\|x(t)\|$ as a function of t for $0 \le t \le 100$.

b) Suppose $\dot{x} = Ax$ is an asymptotically stable linear system. Let $P \succ 0$ be the solution to the Lyapunov equation $A^{\mathsf{T}}P + PA + I = 0$. Prove that if we define $z(t) = P^{1/2}x(t)$, then the transformed state $z(t)$ has the property that $\|z(t)\|$ converges monotonically to zero.

Note: $P^{1/2}$ is the matrix square root, which is the unique symmetric positive definite matrix such that $P^{1/2}P^{1/2} = P$.

c) We will numerically verify the result of part **(b)** on the example system of part **(a)**. To do this, make use MATLAB's `lyap` function to solve the Lyapunov equation and `sqrtm` to find the matrix square root. Then, plot $\|z(t)\|$ as a function of t and verify that it's a decreasing function.

Note: Be aware that MATLAB uses a different convention (transposes!) than what we covered in class. Type `help lyap` to learn more about the syntax.

SOLUTION:

a) (see part **(c)**)

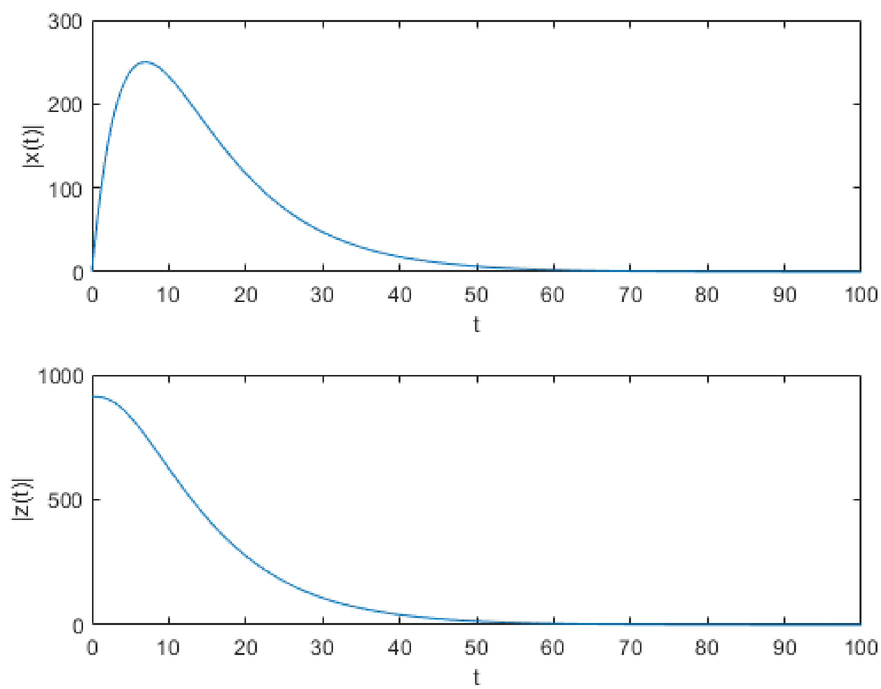
b) Note that $\|z\|^2 = \|P^{1/2}x\|^2 = x^{\mathsf{T}}Px$. Therefore, the rate of change of this quantity over time is

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 &= \frac{\mathrm{d}}{\mathrm{d}t} x(t)^{\mathsf{T}} P\, x(t) \\ &= \dot{x}(t)^{\mathsf{T}} P x(t) + x(t)^{\mathsf{T}} P \dot{x}(t) \\ &= x(t)^{\mathsf{T}} \left(A^{\mathsf{T}}P + PA\right) x(t) \\ &= -\|x(t)\|^2 \\ &\le 0 \end{aligned}$$

where we used that $A^{\mathsf{T}}P + PA = -I$. Therefore, $\|z(t)\|$ is decreasing as long as $x(t) \neq 0$.

c) The code and plots are shown below. As expected, $\|x(t)\|$ shoots up before settling down while $\|z(t)\|$ is monotonically decreasing.

```
A   = [-0.1 100; 0 -0.2];   % system matrix
x0  = [0; 1];               % initial state
P   = lyap(A',eye(2));      % solution to Lyapunov equation
N   = 1000;                 % number of samples
t   = linspace(0,100,N);    % time vector

xnorm = zeros(1,N);         % pre-allocate arrays
znorm = zeros(1,N);

for k = 1:N
    x = expm(A*t(k))*x0;    % state at time t(k)
    z = sqrtm(P)*x;         % transform to z-coordinates
    xnorm(k) = norm(x);     % norm of x(t)
    znorm(k) = norm(z);     % norm of z(t)
end

figure;
subplot(211); plot(t,xnorm); ylabel('|x(t)|'); xlabel('t');
subplot(212); plot(t,znorm); ylabel('|z(t)|'); xlabel('t');
```



Problem 30: Stability via the controllability Gramian

Consider the LTI system $\dot{x} = Ax + Bu$ where (A, B) is controllable.

a) Prove that A is Hurwitz if and only if there is a solution $V \succ 0$ to the Lyapunov equation

$$AV + VA^\mathsf{T} + BB^\mathsf{T} = 0$$

Hint: This problem is similar to Problem 3 in Homework 4.

b) Prove that A is Hurwitz if and only if there is a solution $V \succ 0$ to the equation

$$AV + VA^\mathsf{T} + BB^\mathsf{T} \preceq 0$$

In other words, we don't have to solve the Lyapunov equation exactly (equals zero) in order to prove stability. It's enough to solve so the left-hand side is negative semidefinite.
Hint: Rearrange the inequality so it becomes an equality, and use part **(a)**.

c) Recall that the controllability Gramian at time t is

$$W(t) = \int_0^t e^{A\tau} BB^\mathsf{T} e^{A^\mathsf{T}\tau}\,\mathrm{d}\tau$$

Prove that the feedback law

$$u(t) = -B^\mathsf{T} W(-T)^{-1} x(t)$$

stabilizes the system for any $T > 0$. (Since (A, B) is controllable, we have that $W(-T) \succ 0$ for all $T > 0$, so the inverse exists.)
Hint: What is the feedback gain K? You need to prove that $A + BK$ is Hurwitz. Make use of the result from part **(b)** with $V = W(-T)$.

SOLUTION:

a) ($\Longleftarrow$) Suppose $V = V^\mathsf{T} \succ 0$ is a solution to the Lyapunov equation. Let (w, λ) be an left-eigenpair of A, that is, $w^*A = \lambda w^*$. Multiplying the Lyapunov equation on the right and left by w and its conjugate transpose, respectively, gives

$$\begin{aligned} 0 &= w^*\left(AV + VA^\mathsf{T} + BB^\mathsf{T}\right) w \\ &= (\lambda + \bar{\lambda})\, w^*Vw + (B^\mathsf{T}w)^*(B^\mathsf{T}w) \\ &= 2\,\mathrm{Re}(\lambda)\,(w^*Vw) + \|B^\mathsf{T}w\|^2 \end{aligned}$$

By the PBH test for observability, we must have $B^\mathsf{T}w \neq 0$ since $w^*A = \lambda w^*$. Therefore, $\|B^\mathsf{T}x\|^2 > 0$. Since $V \succ 0$ and $w \neq 0$, we must also have $w^*Vw > 0$. Therefore, $\mathrm{Re}(\lambda) < 0$ so A is Hurwitz.
($\Longrightarrow$) Suppose A is Hurwitz. Then $e^{At} \to 0$ as $t \to \infty$. Define the matrix

$$V = \int_0^\infty e^{At} BB^\mathsf{T} e^{A^\mathsf{T}t}\,\mathrm{d}t$$

which is symmetric and satisfies the matrix equation. To prove that $V \succ 0$, multiply on the right and left by an arbitrary nonzero vector $w \in \mathbb{R}^n$ and its transpose, respectively, to obtain

$$w^\mathsf{T} V w = \int_0^\infty \|B^\mathsf{T} e^{A^\mathsf{T} t} w\|^2 \, \mathrm{d}t$$

We need to show that this quantity is strictly positive. Suppose (by contradiction) that it is not. Then $\|B^\mathsf{T} e^{A^\mathsf{T} t} w\| = 0$ for all t. Taking derivatives and setting $t = 0$, we conclude that $B^\mathsf{T}(A^k)^\mathsf{T} w = 0$ for all k. This implies that $w^\mathsf{T} P = 0$, where P is the controllability matrix. Since w is a nonzero vector in the left-nullspace of P, the pair (A, B) is uncontrollable, which is a contradiction. Therefore, $\|B^\mathsf{T} e^{A^\mathsf{T} t} w\|$ is not zero for all t and so the integral is strictly positive, that is, $w^\mathsf{T} V w > 0$. ■

b) ($\Longrightarrow$) This follows directly from part **(a)** since we can use the same solution V.

($\Longleftarrow$) Suppose $V \succ 0$ is a solution to the Lyapunov inequality. In other words, we have $R = AV + VA^\mathsf{T} + BB^\mathsf{T} \succeq 0$. This means we can find a factorization $R = -SS^\mathsf{T}$ for some choice of S (for example, by taking an eigenvalue decomposition). Rearranging the equation, we obtain

$$AV + VA^\mathsf{T} + \begin{bmatrix} B & S \end{bmatrix} \begin{bmatrix} B & S \end{bmatrix}^\mathsf{T} = 0$$

We have turned the Lyapunov *inequality* into an *equality*. We can apply the result from part **(a)** once again; note that (A, B) is controllable, so $\left(A, \begin{bmatrix} B & S \end{bmatrix}\right)$ is also controllable by the PBH test. Therefore, A is Hurwitz. ■

c) Since (A, B) is controllable, so is $(A + BK, B)$ for any K. We need to verify that $A + BK$ is Hurwitz when $K = -B^\mathsf{T} W(-T)^{-1}$. By applying the result from part **(b)**, it suffices to find a solution to the Lyapunov inequality

$$(A + BK)V + V(A + BK)^\mathsf{T} + BB^\mathsf{T} \preceq 0$$

Substituting our definition for K, we obtain

$$\left(A - BB^\mathsf{T} W(-T)^{-1}\right) V + V \left(A - BB^\mathsf{T} W(-T)^{-1}\right)^\mathsf{T} + BB^\mathsf{T} \preceq 0$$

Let's try setting $V = W(-T)$ so that the inverse cancels and we get

$$AW(-T) + W(-T)A^\mathsf{T} - BB^\mathsf{T} \preceq 0$$

Rearranging, we obtain

$$(-A)W(-T) + W(-T)(-A)^\mathsf{T} + BB^\mathsf{T} \succeq 0$$

We need to prove that $W(-T)$ does in fact satisfy this equation. Substituting its definition,

$$
\begin{aligned}
&(-A)W(-T) + W(-T)(-A)^\mathsf{T} + BB^\mathsf{T} \\
&\qquad = \int_0^T \left((-A)e^{-At}BB^\mathsf{T}e^{-A^\mathsf{T}t} + e^{-At}BB^\mathsf{T}e^{-A^\mathsf{T}t}(-A)^\mathsf{T}\right)\mathrm{d}t + BB^\mathsf{T} \\
&\qquad = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-At}BB^\mathsf{T}e^{-A^\mathsf{T}t}\right)\mathrm{d}t + BB^\mathsf{T} \\
&\qquad = \left(e^{-AT}BB^\mathsf{T}e^{-A^\mathsf{T}T} - BB^\mathsf{T}\right) + BB^\mathsf{T} \\
&\qquad = e^{-AT}BB^\mathsf{T}e^{-A^\mathsf{T}T} \\
&\qquad \succeq 0
\end{aligned}
$$

The last line follows from the fact that

$$
x^\mathsf{T}\left(e^{-AT}BB^\mathsf{T}e^{-A^\mathsf{T}T}\right)x = \left\|B^\mathsf{T}e^{-A^\mathsf{T}T}x\right\|^2 \geq 0
$$

Therefore, $V = W(-T)$ does in fact satisfy the Lyapunov inequality, so $A+BK$ is Hurwitz from part **(b)**.

Note that we must be careful here. The equation $AW(-T)+W(-T)A^\mathsf{T} - BB^\mathsf{T} \preceq 0$ *looks* like a Lyapunov inquality, but remember that A is not Hurwitz, so none of our existing results apply! Indeed, we can't even let $T \to \infty$ because e^{-At} does not go to zero as $t \to \infty$.

Problem 31: Continuous-time Lyapunov equation

The Lyapunov stability result states that for any $Q = Q^{\mathsf{T}} \succ 0$, the equation $A^{\mathsf{T}}P + PA + Q = 0$ has a solution $P = P^{\mathsf{T}} \succ 0$ if and only if A is Hurwitz (that is, all eigenvalues of A have negative real part).

Suppose (C, A) is observable. Prove that the equation $A^{\mathsf{T}}W + WA + C^{\mathsf{T}}C = 0$ has a solution $W = W^{\mathsf{T}} \succ 0$ if and only if A is Hurwitz.

Note: This result is more general than the one we saw in class because typically we will have $C^{\mathsf{T}}C \succeq 0$ but not $C^{\mathsf{T}}C \succ 0$. Be sure to prove both directions (if and only if).

SOLUTION:
($\Longrightarrow$) Suppose (C, A) is observable and $W = W^{\mathsf{T}} \succ 0$ is such that $A^{\mathsf{T}}W + WA + C^{\mathsf{T}}C = 0$. Let (x, λ) be an eigenpair of A, that is, $Ax = \lambda x$. Multiplying the equation on the right and left by x and its conjugate transpose, respectively, gives

$$\begin{aligned} 0 &= x^* \left(A^{\mathsf{T}}W + WA + C^{\mathsf{T}}C\right) x \\ &= (\lambda + \bar{\lambda})\, x^*Wx + (Cx)^*(Cx) \\ &= 2\,\mathrm{Re}(\lambda)\,(x^*Wx) + \|Cx\|^2 \end{aligned}$$

By the PBH test for observability, we must have $Cx \neq 0$ since $Ax = \lambda x$. Therefore, $\|Cx\|^2 > 0$. Since $W \succ 0$ and $x \neq 0$, we must also have $x^*Wx > 0$. Therefore, $\mathrm{Re}(\lambda) < 0$ so A is Hurwitz.

($\Longleftarrow$) Suppose A is Hurwitz. Then $e^{At} \to 0$ as $t \to \infty$. Define the matrix

$$W = \int_0^\infty e^{A^{\mathsf{T}}t} C^{\mathsf{T}} C e^{At} \, \mathrm{d}t$$

which is symmetric and satisfies the matrix equation. To prove that $W \succ 0$, multiply on the right and left by an arbitrary nonzero vector $v \in \mathbb{R}^n$ and its transpose, respectively, to obtain

$$v^{\mathsf{T}}Wv = \int_0^\infty \|Ce^{At}v\|^2 \, \mathrm{d}t$$

We need to show that this quantity is strictly positive. Suppose (by contradiction) that it is not. Then $\|Ce^{At}v\| = 0$ for all t. Taking derivatives and setting $t = 0$, we conclude that $CA^k v = 0$ for all k. This implies that $Qv = 0$, where Q is the observability matrix. Since v is a nonzero vector in the nullspace of Q, the pair (C, A) is unobservable, which is a contradiction. Therefore, $\|Ce^{At}v\|$ is not zero for all t and so the integral is strictly positive, that is, $v^{\mathsf{T}}Wv > 0$. ■

Problem 32: Lyapunov function for nonlinear system

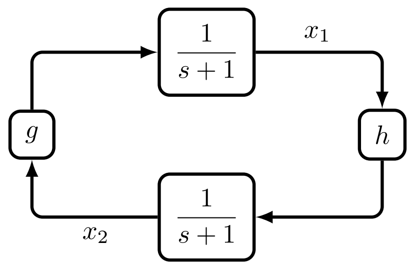
Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + g(x_2)\\ \dot{x}_2 &= -x_2 + h(x_1)\end{aligned}$$

where the nonlinear functions g and h satisfy

$$|g(u)| \le |u|/2 \qquad \text{and} \qquad |h(u)| \le |u|/2$$

for all u. This can be viewed as the interconnection of two linear systems with cross-coupling:



Prove that the system is globally asymptotically stable.

Hint: use the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$.

SOLUTION: It is clear that V is positive definite, so we just need to show that it is decreasing along system trajectories. Computing the time derivative,

$$\begin{aligned}\dot{V}(x) &= \big(\nabla V(x)\big)^{\mathsf{T}} f(x)\\
&= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -x_1 + g(x_2) \\ -x_2 + h(x_1) \end{bmatrix}\\
&= x_1\left(-x_1 + g(x_2)\right) + x_2\left(-x_2 + h(x_1)\right)\\
&= -x_1^2 - x_2^2 + x_1\, g(x_2) + x_2\, h(x_1)\\
&\le -x_1^2 - x_2^2 + |x_1 x_2|\end{aligned}$$

Rearranging the inequality $(|x_1| - |x_2|)^2 \ge 0$, we have that $|x_1 x_2| \le \frac{1}{2}(x_1^2 + x_2^2)$. Therefore,

$$\dot{V}(x) \le -V(x) \le 0$$

so the system is globally exponentially stable.

Problem 33: Lyapunov stability for a nonlinear system

Consider the scalar nonlinear dynamical system

$$\dot{x}(t) = -x(t) + \frac{1}{4}x(t)^3 \qquad x(0) = x_0$$

Prove that the point $\tilde{x} = 0$ is a locally stable equilibrium point.
Hint: use the Lyapunov function $V(x) = x^2$.

SOLUTION: We will use the Lyapunov function $V(x) = x^2$ to prove stability. Since V is obviously positive definite, we just need to show that $V\big(x(t)\big)$ is a decreasing function of t. We can ascertain this by evaluating the time derivative:

$$\begin{aligned}\dot{V}(x) &= \big(\nabla V(x)\big)^{\mathsf{T}} f(x) \\ &= 2x\left(-x + \frac{1}{4}x^3\right) \\ &= -\frac{1}{2}x^2\left(4 - x^2\right)\end{aligned}$$

Whenever $-2 < x(t) < 2$, the expression above is negative, i.e. V is decreasing. But if V is decreasing, then so is $|x(t)|$. So if $|x_0| < 2$, then $|x(t)| < 2$ for all $t \geq 0$ as well. In other words, the point $x = 0$ is a locally stable equilibrium point.

Note that this Lyapunov function does not prove global stability, because if $|x(t)| > 2$, then V is not decreasing.

ALT. SOLUTION: The solution above is based on Lyapunov's direct method, which is to pick a Lyapunov candidate and directly show that it satisfies the definition of a Lyapunov function in some local neighborhood of the equilibrium point.

Another way to solve the problem is to use Lyapunov's indirect method, which states that if the linearized dynamics about the equilibrium point are stable, then the nonlinear system is locally asymptotically stable and we can find a quadratic Lyapunov function that certifies it.

In this case, the Jacobian of the system is

$$\frac{\partial f}{\partial x} = -1 + \frac{3}{4}x(t)^2$$

so linearized dynamics about $\tilde{x} = 0$ are

$$\dot{x}(t) = -x(t)$$

whichi is clearly stable (the A matrix is -1, which has a negative real eigenvalue). Therefore, the nonlinear system is locally asymptotically stable as well.

Problem 34: Lyapunov stability for a nonlinear system

Consider the nonlinear dynamical system

$$\dot{x}_1 = -x_1 + 4\,x_2$$
$$\dot{x}_2 = -x_1 - x_2^3$$

Prove that the origin $(x_1, x_2) = (0, 0)$ is a *globally* stable equilibrium point.

Hint: Choose the constant $a > 0$ such that $V(x_1, x_2) = x_1^2 + a\,x_2^2$ is a Lyapunov function.

SOLUTION: For any $a > 0$, the function $V(x_1, x_2)$ is clearly positive definite since $V(0, 0) = 0$ and $V(x_1, x_2) > 0$ whenever $x_1 \neq 0$ or $x_2 \neq 0$. The time derivative of V along trajectories of the system is given by

$$\begin{aligned}\dot{V}(x_1, x_2) &= \big(\nabla V(x_1, x_2)\big)^{\mathsf{T}} f(x_1, x_2) \\ &= \begin{bmatrix} 2x_1 \\ 2ax_2 \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} -x_1 + 4x_2 \\ -x_1 - x_2^3 \end{bmatrix} \\ &= 2x_1\,(-x_1 + 4x_2) + 2ax_2\,(-x_1 - x_2^3) \\ &= -2x_1^2 + (8 - 2a)x_1x_2 - 2a\,x_2^4\end{aligned}$$

If we choose $a = 4$, then the cross term x_1x_2 is eliminated and we obtain

$$\dot{V}(x_1, x_2) = -2x_1^2 - 8x_2^4 \leq 0$$

which is clearly negative semidefinite for all x_1 and x_2. Therefore, the origin is a globally stable equilibrium point.

Comment: We could use Lyapunov's indirect method (that is, check stability of the linearization about the equilibrium point) to conclude that the origin is a *locally* stable equilibrium. However, Lyapunov's indirect method cannot be used to conclude *global* stability.

Problem 35: Evaluating quadratic integrals

Suppose $\dot{x}(t) = Ax(t)$ and A is Hurwitz (all eigenvalues of A have a strictly negative real part). Suppose $Q = Q^{\mathsf{T}}$ is a symmetric matrix. We are interested in evaluating the integral

$$J(x_0) = \int_0^\infty x(t)^{\mathsf{T}} Q\, x(t)\, \mathrm{d}t$$

where $x(t)$ is the state of the system at time t, assuming we start at $x(0) = x_0$. Prove that the value of this integral is given by

$$J(x_0) = x_0^{\mathsf{T}} P x_0$$

where P satisfies the Lyapunov equation $A^{\mathsf{T}} P + PA + Q = 0$.

SOLUTION: Supposed P satisfies the Lyapunov equation. Substitute the expression for Q into the integral:

$$\begin{aligned}
J(x_0) &= \int_0^\infty x(t)^{\mathsf{T}} Q\, x(t)\, \mathrm{d}t \\
&= \int_0^\infty x(t)^{\mathsf{T}} \Big(-A^{\mathsf{T}} P - PA\Big)\, x(t)\, \mathrm{d}t \\
&= -\int_0^\infty \Big(\dot{x}(t)^{\mathsf{T}} P x(t) + x(t) P \dot{x}(t)\Big)\, \mathrm{d}t \\
&= -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}t}\Big(x(t)^{\mathsf{T}} P\, x(t)\Big)\, \mathrm{d}t \\
&= -\Big[x(t)^{\mathsf{T}} P\, x(t)\Big]_{t=0}^{\infty} \\
&= x_0^{\mathsf{T}} P\, x_0
\end{aligned}$$

where we used that A is Hurwitz in the last step, which implies that the system is asymptotically stable, so $\lim_{t\to\infty} x(t) = 0$.

Problem 36: Closed-loop stability using transfer functions

Consider a system with transfer function

$$H(s) = \frac{(s-1)(s+2)}{(s+1)(s-2)(s+3)}$$

Is it possible to make the closed-loop transfer function

$$\hat{H}(s) = \frac{1}{s+3}$$

using state feedback? Is the resulting system BIBO stable? Asymptotically stable?

SOLUTION: Yes, it is possible to make the closed-loop transfer function $\hat{H}(s) = \frac{1}{s+3}$.

- For example, if we place $K(s)$ in the feedforward path, the closed-loop transfer function is $\frac{KH}{1+KH}$. In this case, the desired closed-loop transfer function is obtained by setting

$$K(s) = \frac{(s+1)(s+3)(s-2)}{(s-1)(s+2)(s+4)}$$

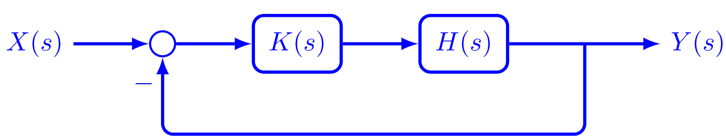


Figure 1: controller in feedforward path

- If, on the other hand, we place $K(s)$ in the feedback path, the closed-loop transfer function is $\frac{H}{1+KH}$. In this case, the desired closed-loop transfer function is obtained by setting

$$K(s) = \frac{2s(s+3)}{(s+2)(s-1)}$$

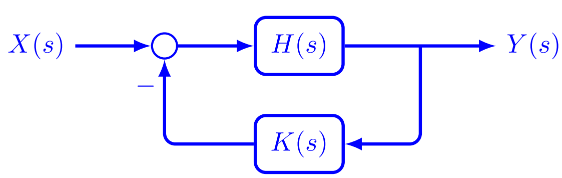


Figure 2: controller in feedback path

In either case (putting the controller in the feedforward or feedback back), the closed-loop transfer function is $\frac{1}{s+3}$, so the system is BIBO (input-output) stable by definition.

To see whether either possibility leads to an internally stable system, we can compute the realization of the feedback interconnection. We'll demonstrate the second case (controller in feedback path), but the feedforward case is similar and leads to the same conclusion.

First, we use MATLAB to compute minimal realizations for G and K.

$$G(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right] = \left[\begin{array}{ccc|c} -2 & 2.5 & 1.5 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ \hline 0.5 & 0.25 & -0.25 & 0 \end{array}\right]$$

$$K(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 0 & 0 \\ \hline -1 & -1 & -2 \end{array}\right]$$

The only thing that matters when considering internal stability is the eigenvalues of the closed-loop A-matrix. From the class notes (with $D_1 = 0$), this is

$$A_{\mathsf{cl}} = \begin{bmatrix} A_1 + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 \end{bmatrix} = \begin{bmatrix} -4 & 1.5 & 2.5 & -2 & -2 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 2 & 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The eigenvalues of this matrix are $\{-3, -2, -2, 1, 1\}$. Since at least one eigenvalue is unstable, the system is not asymptotically stable. Now, of course, since the closed-loop transfer function is $\frac{1}{s+3}$, only the -3 eigenvalue is controllable and observable. We could, in principle, reduce the realization. But the controller and plant are physically separate; these states are physical states that cannot simply be canceled out!

Problem 37: Internal and external stability

Consider the system

$$\dot{x}(t) = \begin{pmatrix} -3 & 4 \\ -1 & 2 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & -1 \end{pmatrix} x(t)$$

a) When the input is zero, is the origin an asymptotically stable equilibrium?

b) Is the system BIBO stable?

c) Find the state-space realization of a system with the same transfer function that is both asymptotically stable and BIBO stable.

SOLUTION:

a) The characteristic equation of the A matrix is

$$0 = \det(sI - A) = \det \begin{pmatrix} s+3 & -4 \\ 1 & s-2 \end{pmatrix} = (s+3)(s-2)+4 = s^2+s-2 = (s-1)(s+2)$$

so the eigenvalues of A are $s = 1$ and $s = -2$. Since one of the eigenvalues is not in the left-half plane, the system is NOT asymptotically stable.

b) The transfer function of the system is

$$\begin{aligned} H(s) &= C\,(sI-A)^{-1}B + D \\ &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s+3 & -4 \\ 1 & s-2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(s-1)(s+2)} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s-2 & 4 \\ -1 & s+3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{s-1}{(s-1)(s+2)} \\ &= \frac{1}{s+2} \end{aligned}$$

which has a single pole at $s = -2$. Since all of the poles are in the left-half plane, the system is BIBO stable.

c) Since the system is BIBO stable, a minimal realization will also be asymptotically stable. Since the transfer function $H(s) = 1/(s+2)$ has no pole-zero cancellations, *any* one-state realization is minimal. One such realization is the controllable canonical form,

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} -2 & 1 \\ \hline 1 & 0 \end{array}\right)$$

This realization is asymptotically stable, BIBO stable, and has the same transfer function as the original system.

Problem 38: BIBO stability

a) Determine whether or not the following SISO system is BIBO stable:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 7 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 3 \\ -4 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 9 & 2 \end{bmatrix} x(t) + 5\, u(t)$$

SOLUTION: The A matrix is lower-triangular, so its eigenvalues are the diagonal elements which are -2 and -1. All of the eigenvalues are in the left-half plane, so the system is asymptotically stable and therefore BIBO stable as well.

b) For the MIMO system

$$\dot{x}(t) = -x(t) + Bu(t)$$
$$y(t) = Cx(t)$$

determine the smallest constant $a > 0$ such that $\|y\|_\infty \le a\,\|u\|_\infty$ for all input signals u. Note that this is a MIMO system, so $B \in \mathbb{R}^{1\times m}$ and $C \in \mathbb{R}^{p\times 1}$.

SOLUTION: The constant a is the induced L_∞-norm of the system, which is given by

$$a = \int_0^\infty \|h(t)\|_2 \,\mathrm{d}t$$

where $h(t)$ is the impulse response of the system, given by

$$h(t) = C\, e^{At} B = e^{-t}\,(CB)$$

Since the exponential is always nonnegative, the norm of the impulse response is

$$\|h(t)\|_2 = e^{-t}\,\|CB\|_2$$

Then integrating,

$$a = \int_0^\infty \|h(t)\|_2 \,\mathrm{d}t = \|CB\|_2 \int_0^\infty e^{-t}\,\mathrm{d}t = \|CB\|_2$$

Therefore, the smallest constant is $a = \|CB\|_2$.

Problem 39: Internal and external stability

Consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & -1 \end{pmatrix} x(t)\end{aligned}$$

a) Is the system asymptotically stable?

b) Is the system BIBO stable?

c) Find the state-space realization of a system with the same transfer function that is both asymptotically stable and BIBO stable.

SOLUTION:

a) The characteristic equation of the A matrix is

$$0 = \det(sI - A) = \det \begin{pmatrix} s & 0 \\ -1 & s+1 \end{pmatrix} = s\,(s+1)$$

so the eigenvalues of A are $s = 0$ and $s = -1$. Since one of the eigenvalues is not in the left-half plane, the system is *not* asymptotically stable.

b) The transfer function of the system is

$$\begin{aligned}H(s) &= C\,(sI - A)^{-1} B + D \\ &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s & 0 \\ -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{s\,(s+1)} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} s+1 & 0 \\ 1 & s \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{s}{s\,(s+1)} \\ &= \frac{1}{s+1}\end{aligned}$$

which has a single pole at $s = -1$. Since all of the poles are in the left-half plane, the system is BIBO stable.

c) Since the system is BIBO stable, a minimal realization will also be asymptotically stable. Since the transfer function $H(s) = 1/(s+1)$ has no pole-zero cancellations, *any* one-state realization is minimal. One such realization is the controllable canonical form,

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} -1 & 1 \\ \hline 1 & 0 \end{array}\right)$$

This realization is asymptotically stable, BIBO stable, and has the same transfer function as the original system.

Problem 40: Internal and external stability

Consider the system

$$x(k+1) = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k)$$
$$y(k) = \begin{pmatrix} 1 & -1 \end{pmatrix} x(k)$$

a) Is the system asymptotically stable?

SOLUTION: The characteristic polynomial of the A matrix is

$$\det(\lambda I - A) = \det \begin{pmatrix} \lambda & 0 \\ -1 & \lambda + 1 \end{pmatrix} = \lambda\,(\lambda + 1)$$

so the eigenvalues of A are $\lambda = 0$ and $\lambda = -1$. Since one of the eigenvalues is on the unit circle, the system is *not* asymptotically stable.

b) Is the system BIBO stable?

SOLUTION: The transfer function of the system is

$$\begin{aligned} H(z) &= C\,(zI - A)^{-1} B + D \\ &= \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} z & 0 \\ -1 & z+1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{s\,(z+1)} \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} z+1 & 0 \\ 1 & z \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{z}{z\,(z+1)} \\ &= \frac{1}{z+1} \end{aligned}$$

which has a single pole at $z = -1$. Since the transfer function has a pole on the unit circle, the system is *not* BIBO stable.

c) If possible, find the state-space realization of a system with the same transfer function that is both asymptotically stable and BIBO stable. Otherwise, explain why this is not possible.

SOLUTION: The system is not BIBO stable, so any state-space realization of the system will not be asymptotically stable. Therefore, this is not possible.

Problem 41: Internal and external stability

Consider the system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix}0 & 0\\ 1 & -1\end{pmatrix} x(t) + \begin{pmatrix}1\\ 0\end{pmatrix} u(t)\\ y(t) &= \begin{pmatrix}1 & -1\end{pmatrix} x(t)\end{aligned}$$

a) Is the system asymptotically stable?

b) Is the system BIBO stable?

c) If possible, find the state-space realization of a system with the same transfer function that is both asymptotically stable and BIBO stable. Otherwise, explain why this is not possible.

SOLUTION:

a) The characteristic equation of the A matrix is

$$0 = \det(sI - A) = \det\begin{pmatrix}s & 0\\ -1 & s+1\end{pmatrix} = s\,(s+1)$$

so the eigenvalues of A are $s = 0$ and $s = -1$. Since one of the eigenvalues is not in the left-half plane, the system is *not* asymptotically stable.

b) The transfer function of the system is

$$\begin{aligned}H(s) &= C\,(sI - A)^{-1}B + D\\ &= \begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}s & 0\\ -1 & s+1\end{pmatrix}^{-1}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{1}{s\,(s+1)}\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}s+1 & 0\\ 1 & s\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix}\\ &= \frac{s}{s\,(s+1)}\\ &= \frac{1}{s+1}\end{aligned}$$

which has a single pole at $s = -1$. Since all of the poles are in the left-half plane, the system is BIBO stable.

c) Since the system is BIBO stable, a minimal realization will also be asymptotically stable. Since the transfer function $H(s) = 1/(s+1)$ has no pole-zero cancellations, *any* one-state realization is minimal. One such realization is the controllable canonical form,

$$\left(\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right) = \left(\begin{array}{c|c} -1 & 1\\ \hline 1 & 0\end{array}\right)$$

This realization is asymptotically stable, BIBO stable, and has the same transfer function as the original system.

Problem 42: $\mathcal{H}_2$ norm

a) Show that the $\mathcal{H}_2$ norm of a strictly proper stable continuous-time system with transfer function $G(s)$ and state-space realization $(A, B, C, 0)$ is

$$\|G\|_{\mathcal{H}_2} = \sqrt{\operatorname{tr}(CW_cC^\mathsf{T})}$$

where W_c is the controllability Gramian, which satisfies the Lyapunov equation

$$AW_c + W_cA^\mathsf{T} + BB^\mathsf{T} = 0$$

SOLUTION: Let $h(t) = C\, e^{At} B$ be the impulse response of the system. Then the squared $\mathcal{H}_2$ norm is

$$\begin{aligned}
\|G\|^2_{\mathcal{H}_2} &= \int_0^\infty \|h(t)\|_F^2 \,\mathrm{d}t \\
&= \int_0^\infty \operatorname{tr}\Big(h(t)\, h(t)^\mathsf{T}\Big)\,\mathrm{d}t \\
&= \int_0^\infty \operatorname{tr}\Big(C\, e^{At} BB^\mathsf{T} e^{A^\mathsf{T} t} C^\mathsf{T}\Big)\,\mathrm{d}t \\
&= \operatorname{tr}\left[C\left(\int_0^\infty e^{At} BB^\mathsf{T} e^{A^\mathsf{T} t}\,\mathrm{d}t\right) C^\mathsf{T}\right] \\
&= \operatorname{tr}\big(CW_cC^\mathsf{T}\big)
\end{aligned}$$

b) Compute the $\mathcal{H}_2$ norm of the system with transfer function $G(s) = 1/(s + a)$.

SOLUTION: A realization of the system is

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} -a & 1 \\ \hline 1 & 0 \end{array}\right)$$

The Lyapunov equation for the controllability Gramian is then

$$(-a)\, W_c + W_c\,(-a) + 1 = 0$$

which implies that $W_c = 1/(2a)$. Then the $\mathcal{H}_2$ norm is

$$\|G\|_{\mathcal{H}_2} = \sqrt{\operatorname{tr}(CW_cC^\mathsf{T})} = \frac{1}{\sqrt{2a}}$$

Problem 43: Second-order system

Consider the second-order system

$$G(s) = \frac{1}{s^2 + 0.5s + 1}$$

a) Compute the percent overshoot of the step response.

b) Compute the rise time of the step response.

c) Compute the (approximate) 5% settling time of the step response.

d) Compute the $\mathcal{H}_2$ norm of the system.

SOLUTION: This system is in standard form with $\zeta = 0.25$ and $\omega_n = 1$. The parameters of the transient response are then

$$PO = 100 \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 44.4\%$$

$$t_p = \frac{\pi}{\omega_d} = 3.24\, s$$

$$t_s \leq \frac{1}{\zeta\omega_n} \log\left(\frac{1}{0.05\sqrt{1-\zeta^2}}\right) = 12.11\, s$$

To compute the $\mathcal{H}_2$ norm, we first obtain a state-space realization

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & -0.5 & 1 \\ \hline 1 & 0 & 0 \end{array}\right)$$

Then the observability Gramian $W_o = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is the solution to the Lyapunov equation

$$0 = A^\mathsf{T} W_o + W_o A + C^\mathsf{T} C = \begin{bmatrix} 1 - 2b & a - \frac{1}{2}b - c \\ a - \frac{1}{2}b - c & 2b - c \end{bmatrix}$$

Solving this system of linear equations gives

$$W_o = \begin{pmatrix} \frac{5}{4} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$$

The $\mathcal{H}_2$ norm of the system is then

$$\|G\|_2 = \sqrt{\operatorname{tr}(B^\mathsf{T} W_o B)} = 1$$

Problem 44: Static state feedback

Consider the LTI system

$$\dot{x} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

a) Find a state feedback gain matrix K such that $A + BK$ has eigenvalues at -1 and -2. Do this by directly computing eigenvalues.

b) Repeat the previous exercise, but this time solve it by transforming the system to controllable canonical form, finding K in those coordinates, and then transforming back to the original coordinates.

SOLUTION:

a) Suppose $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Then the closed-loop system matrix is

$$A + BK = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 2 + k_1 & 1 + k_2 \\ -1 + 2k_1 & 1 + 2k_2 \end{bmatrix}$$

Since we want the eigenvalues at -1 and -2, the characteristic polynomial should be equal to $(\lambda + 1)(\lambda + 2) = \lambda^2 + 3\lambda + 2$. Setting this equal to the characteristic polynomial for the above matrix, we have

$$\begin{aligned} \lambda^2 + 3\lambda + 2 &= (\lambda - 2 - k_1)(\lambda - 1 - 2k_2) - (1 + k_2)(-1 + 2k_1) \\ &= \lambda^2 + (-3 - k_1 - 2k_2)\,\lambda + (3 - k_1 + 5k_2) \end{aligned}$$

Equating coefficients, we obtain the linear system of equations

$$\begin{aligned} 3 &= -3 - k_1 - 2k_2 \\ 2 &= 3 - k_1 + 5k_2 \end{aligned}$$

which has the solution $k_1 = -4$ and $k_2 = -1$. Therefore, $K = \begin{bmatrix} -4 & -1 \end{bmatrix}$.

b) The transformation to CCF is

$$T_{\mathsf{CCF}} = P P_{\mathsf{CCF}}^{-1} = \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}$$

Using this to transform the system, we obtain

$$\left[\begin{array}{c|c} A_{\mathsf{CCF}} & B_{\mathsf{CCF}} \\ \hline C_{\mathsf{CCF}} & D_{\mathsf{CCF}} \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -3 & 3 & 1 \\ \hline -4 & 3 & 0 \end{array}\right]$$

Note that $a_0 = 3$ and $a_1 = -3$. Our desired characteristic polynomial is $\lambda^2 + 3\lambda + 2$, so $\alpha_0 = 2$ and $\alpha_1 = 3$. Therefore, we should use the controller

$$K_{\mathsf{CCF}} = \begin{bmatrix} (a_0 - \alpha_0) & (a_1 - \alpha_1) \end{bmatrix} = \begin{bmatrix} 1 & -6 \end{bmatrix}$$

Transforming back to the original coordinates,

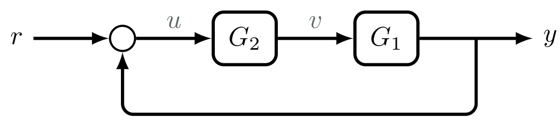
$$K = K_{\mathsf{CCF}} T_{\mathsf{CCF}}^{-1} = \begin{bmatrix} 1 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -5 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} -4 & -1 \end{bmatrix}$$

Problem 45: Feedback interconnection

Consider two LTI systems with state-space descriptions

$$G_1 = \left(\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right) \qquad \text{and} \qquad G_2 = \left(\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right)$$

Find the state-space matrices for the interconnected system from the reference r to the output y in the following feedback interconnection.



What condition do the state-space matrices need to satisfy for the feedback interconnection to be well-posed?

SOLUTION: The state equations for the individual systems are

$$\begin{cases} \dot{x}_1 = A_1 x_1 + B_1 v \\ \;y = C_1 x_1 + D_1 v \end{cases} \quad \text{and} \quad \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u \\ \;v = C_2 x_2 + D_2 u \end{cases} \quad \text{and} \quad u = r + y$$

Defining the matrix $\Delta = I - D_1 D_2$, the output is

$$y = C_1 x_1 + D_1\big(C_2 x_2 + D_2(r+y)\big) = \Delta^{-1} C_1 x_1 + \Delta^{-1} D_1 C_2 x_2 + \Delta^{-1} D_1 D_2 r$$

Now expanding the state equations,

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + B_1\big(C_2 x_2 + D_2(r + \Delta^{-1} C_1 x_1 + \Delta^{-1} D_1 C_2 x_2 + \Delta^{-1} D_1 D_2 r)\big) \\ &= \big(A_1 + B_1 D_2 \Delta^{-1} C_1\big)\, x_1 + B_1\big(C_2 + D_2 \Delta^{-1} D_1 C_2\big)\, x_2 + B_1 D_2\big(I + \Delta^{-1} D_1 D_2\big)\, r \end{aligned}$$

and

$$\begin{aligned} \dot{x}_2 &= A_2 x_2 + B_2(r + \Delta^{-1} C_1 x_1 + \Delta^{-1} D_1 C_2 x_2 + \Delta^{-1} D_1 D_2 r) \\ &= \big(B_2 \Delta^{-1} C_1\big)\, x_1 + \big(A_2 + B_2 \Delta^{-1} D_1 C_2\big)\, x_2 + B_2\big(I + \Delta^{-1} D_1 D_2\big)\, r \end{aligned}$$

We can simplify the formulas a bit by noticing that

$$I + \Delta^{-1} D_1 D_2 = \Delta^{-1}$$

Therefore, the state-space matrices for the feedback interconnection are

$$\left(\begin{array}{cc|c} A_1 + B_1 D_2 \Delta^{-1} C_1 & B_1\big(C_2 + D_2 \Delta^{-1} D_1 C_2\big) & B_1 D_2 \Delta^{-1} \\ B_2 \Delta^{-1} C_1 & A_2 + B_2 \Delta^{-1} D_1 C_2 & B_2 \Delta^{-1} \\ \hline \Delta^{-1} C_1 & \Delta^{-1} D_1 C_2 & \Delta^{-1} D_1 D_2 \end{array}\right)$$

where $\Delta = I - D_1 D_2$. For the feedback interconnection to be well-posed, the state-space matrices must satisfy $\det(\Delta) \neq 0$.

ALT. SOLUTION: We could have also obtained the state-space matrices for the interconnection by first combining G_1 and G_2 using the formula for a series connection, and then applying the formula for a feedback interconnection derived in class where the system in the feedback path is a unit gain.

Problem 46: Shaping the transient response

Consider the LTI system

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -680 & -176 & -86 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 100 & 20 & 10 & 0 \end{bmatrix} x$$

a) Determine the desired eigenvalues for a generic second-order system to obtain 2% overshoot and 2-s settling time (within a tolerance of 2%).

b) Design a state feedback control law for the given system to achieve the transient requirements in part **(a)**. Compare open- and closed-loop responses to a step input.

c) Design a state feedback control law using integral control for the given system to achieve the transient requirements in part **(a)** and a steady-state output value of 1. Place the poles using the 5th order ITAE polynomial: $s^5 + 2.8\omega_n s^4 + 5\omega_n^2 s^3 + 5.5\omega_n^3 s^2 + 3.4\omega_n^4 s + \omega_n^5$. Compare the open- and closed-loop responses to a step input.

SOLUTION:

a) For a generic second-order system $H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, the percent overshoot is given by $\exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right)$ and the settling time is approximately $\frac{4}{\zeta\omega_n}$. Using the percent overshoot equation, we have

$$0.02 = \exp\left(\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}\right) \qquad \Longrightarrow \qquad \zeta \approx 0.7797$$

Substituting into the settling time equation (equal to 2 seconds) and solving for ω_n yields $\omega_n \approx 2.565$. The poles of the transfer function are therefore located at

$$\lambda = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = -2 \pm 1.606i$$

b) The current characteristic polynomial is $s^4 + 6s^3 + 86s^2 + 176s + 680$, which has roots at $-1 \pm 3i$ and $-2 \pm 8i$. Let's place two of the poles at $-2 \pm 1.606i$ as in part **(a)**, and the other poles around time times futher left, so let's say at -20. Therefore we want our characteristic polynomial to be

$$(s+2+1.606i)(s+2-1.606i)(s+20)^2 = s^4 + 44s^3 + 566.58s^2 + 1863.2s + 2631.7$$

Since the systme is already in controllable canonical form, we need to use feedback gain matrix

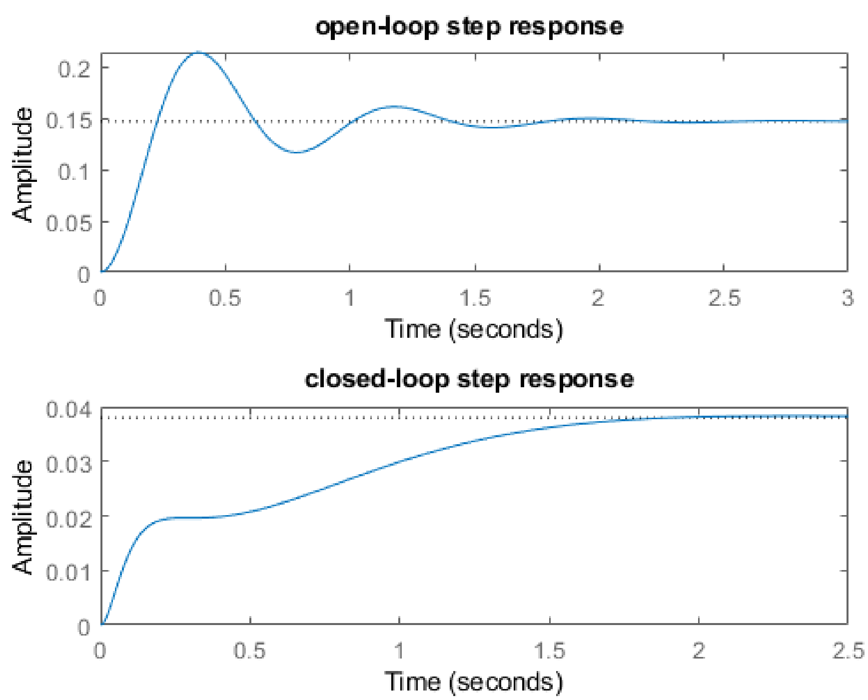
$$\begin{aligned} K &= \begin{bmatrix} (a_0 - \alpha_0) & (a_1 - \alpha_1) & (a_2 - \alpha_2) & (a_3 - \alpha_3) \end{bmatrix} \\ &= \begin{bmatrix} 680 - 2631.7 & 176 - 1863.2 & 86 - 566.58 & 6 - 44 \end{bmatrix} \\ &= \begin{bmatrix} -1951.7 & -1687.2 & -480.58 & -38 \end{bmatrix} \end{aligned}$$

The resulting closed-loop matrix is given by

$$A + BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2631.7 & -1863.2 & -566.58 & -44 \end{bmatrix}$$

Here is the code and plots comparing the step responses of the two systems.

```
% state-space matrices
A = [0 1 0 0; 0 0 1 0; 0 0 0 1; -680 -176 -86 -6];
B = [0; 0; 0; 1];
C = [100 20 10 0];
D = 0;

% feedback gain matrix
K = [-1951.7 -1687.2 -480.58 -38];

G1 = ss(A,B,C,D);       % open-loop system
G2 = ss(A+B*K,B,C,D);   % closed-loop system

figure;
subplot(211); step(G1); title('open-loop step response');
subplot(212); step(G2); title('closed-loop step response');
```



c) The 5th-order ITAE polynomial with the desired natural frequency $\omega_n = 2.565$ is

$$s^5 + 7.182s^4 + 32.90s^3 + 92.81s^2 + 147.17s + 111.03$$

The closed-loop map using integral action with some (augmented) controller K is

$$\left[\begin{array}{cc|c} A+BK & Bk_i & 0 \\ -C & 0 & 1 \\ \hline C & 0 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ k_1-680 & k_2-176 & k_3-86 & k_4-6 & k_i & 0 \\ -100 & -20 & -10 & 0 & 0 & 1 \\ \hline 100 & 20 & 10 & 0 & 0 & 0 \end{array}\right]$$

Comparing coefficients with the ITAE polynomial above, we conclude that

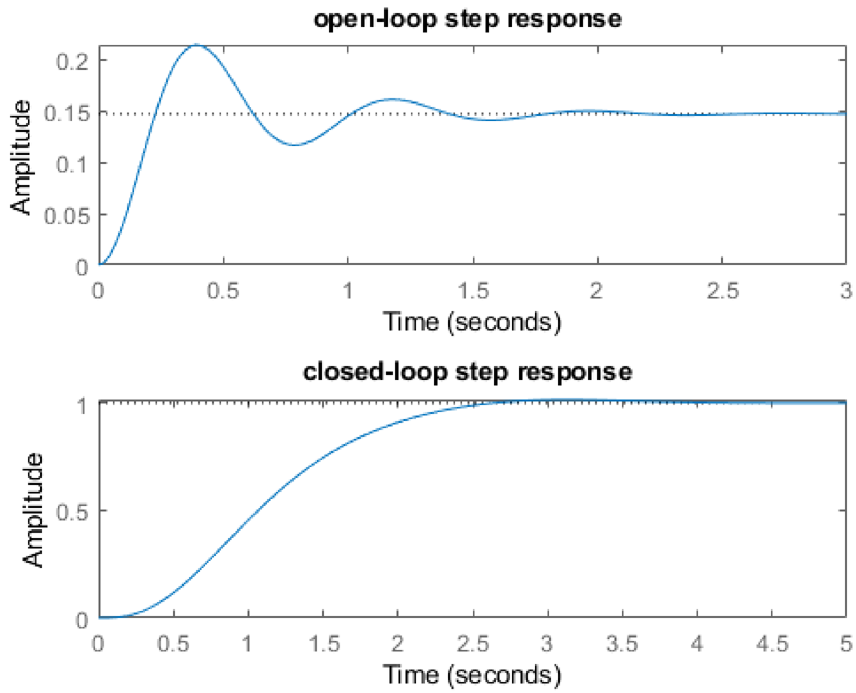
$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 & k_i \end{bmatrix} = \begin{bmatrix} 555 & 94.29 & 53.1 & -1.182 & 1.11 \end{bmatrix}$$

More specifically, the control law is given by

$$u(t) = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} x(t) + k_i \int_0^t \big(r(\tau) - y(\tau)\big)\, \mathrm{d}\tau$$

where the reference input $r(t)$ is the input to the closed-loop system. The corresponding code and plot are shown below.

```
% state-space matrices
A = [0 1 0 0; 0 0 1 0; 0 0 0 1; -680 -176 -86 -6];
B = [0; 0; 0; 1];
C = [100 20 10 0];
D = 0;

% feedback gain matrix
K = [555 94.29 53.1 -1.182];

% integral gain
ki = 1.11;

% closed-loop system
Acl = [A+B*K B*ki; -C 0];
Bcl = [zeros(4,1); 1];
Ccl = [C 0];
Dcl = 0;

G1 = ss(A,B,C,D);           % open-loop system
G2 = ss(Acl,Bcl,Ccl,Dcl);   % closed-loop system

figure;
subplot(211); step(G1); title('open-loop step response');
subplot(212); step(G2); title('closed-loop step response');
```


open-loop step response
Amplitude
0.2
0.15
0.1
0.05
0
0
0.5
1
1.5
2
2.5
3
Time (seconds)
closed-loop step response
Amplitude
1
0.5
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
Time (seconds)

Problem 47: Transient response

Consider the system with transfer function

$$G(s) = \frac{1}{(s^2+s+1)(s+20)(s^2+50s+1025)}$$

a) What are the dominant poles of the system?

SOLUTION: The poles of the system are at $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$, -20, and $-25 \pm j20$. The dominant poles of the system are the complex conjugate poles closest to the imaginary axis, which are

$$s = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

b) Find the percent overshoot, peak time, and (approximate) 5% settling time of the system.

SOLUTION: For the dominant poles, the natural frequency is $\omega_n = 1$ and the damping ratio is $\zeta = \frac{1}{2}$. Then the damped frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{3}}{2}$$

so the percent overshoot, peak time, and (approximate) 5% settling time of the system are

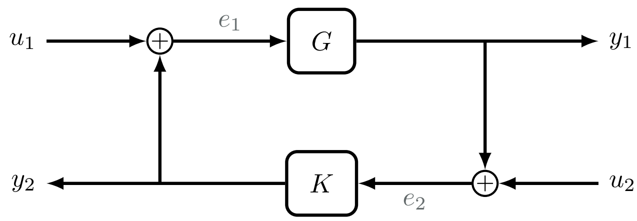
$$\text{overshoot} = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) = 16.3\%$$

$$t_p = \frac{\pi}{\omega_d} \approx 3.63\,s$$

$$t_s \geq \frac{1}{\zeta\omega_n}\log\left(\frac{1}{0.05\sqrt{1-\zeta^2}}\right) = 6.28\,s$$

Problem 48: MIMO system

Consider the following interconnection of LTI systems $G(s)$ and $K(s)$.



Find the closed-loop transfer matrix $H(s)$ such that

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Note: treat both $G(s)$ and $K(s)$ as matrices.

SOLUTION: From the block diagram, the signals in the system satisfy the set of equations

$$\begin{aligned} e_1 &= u_1 + y_2 \\ e_2 &= u_2 + y_1 \\ y_1 &= Ge_1 \\ y_2 &= Ke_2 \end{aligned}$$

We can express y_1 in terms of u_1 and u_2 as

$$y_1 = Ge_1 = G\,(u_1 + y_2) = G\,\big(u_1 + K\,(u_2 + y_1)\big)$$

Solving for y_1 gives

$$y_1 = (I - GK)^{-1}G\,(u_1 + Ku_2)$$

Similarly, substituting into the last equation gives

$$y_2 = Ke_2 = K\,(u_2 + y_1) = K\,\big(u_2 + G\,(u_1 + y_2)\big)$$

Solving for y_2 gives

$$y_2 = (I - KG)^{-1}K\,(Gu_1 + u_2)$$

Therefore, the signals satisfy

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} (I-GK)^{-1}G & (I-GK)^{-1}GK \\ (I-KG)^{-1}KG & (I-KG)^{-1}K \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Note that many other (equivalent) expressions for the solution are possible depending on the order and manner in which the variables are eliminated. For example, the H_{11} term has the following equivalent expressions:

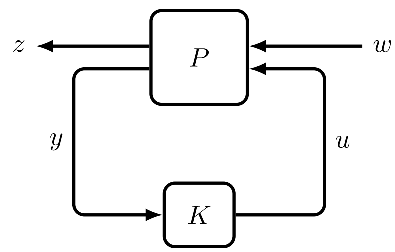
$$H_{11} = (I-GK)^{-1}G = G\,(I-KG)^{-1} = G + GK\,(I-GK)^{-1}G = G + G\,(I-KG)^{-1}KG$$

Problem 49: MIMO system

Consider the interconnection of the multi-input multi-output (MIMO) system

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}$$

in feedback with the controller $K(s)$ as shown below.



Find the closed-loop transfer function $H(s)$ from the input w to the output z.
Note: treat all transfer functions as matrices.

SOLUTION: From the block diagram, the signals in the system satisfy the set of equations

$$\begin{aligned} z &= P_{11}w + P_{12}u \\ y &= P_{21}w + P_{22}u \\ u &= Ky \end{aligned}$$

Substituting the third equation into the second and rearranging yields

$$(I - P_{22}K)\, y = P_{21}w$$

Multiplying on the left by the inverse to solve for y yields

$$y = (I - P_{22}K)^{-1}P_{21}w$$

Substituting this into the first equation, we obtain the regulated output z in terms of the exogenous input w as

$$z = \left(P_{11} + P_{12}\, K\, (I - P_{22}K)^{-1}P_{21}\right) w$$

Therefore, the transfer function from z to w is

$$H(s) = P_{11} + P_{12}\, K\, (I - P_{22}K)^{-1}P_{21}$$

Problem 50: State observer

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

Compute a state observer for this system such that the estimation error decays at a rate e^{-10t}. Write out the dynamics of the observer and draw a block diagram.

SOLUTION: To get a decay rate of e^{-10t}, we should place one of the eigenvalues at $\lambda_1 = -10$ and the other eigenvalue at $\lambda_2 \leq -10$. For simplicity, let's place them both at -10. Therefore, we seek $L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}$ such that the closed-loop system matrix

$$A - LC = \begin{bmatrix} -\ell_1 & -1-\ell_1 \\ 1-\ell_2 & -2-\ell_2 \end{bmatrix}$$

has its eigenvalues at $\{-10, -10\}$. The characteristic polynomial of $A - LC$ is

$$\det\Big(\lambda I - (A - LC)\Big) = \det \begin{bmatrix} \lambda + \ell_1 & 1 + \ell_1 \\ -1 + \ell_2 & \lambda + 2 + \ell_2 \end{bmatrix} = \lambda^2 + (\ell_1 + \ell_2 + 2)\,\lambda + (3\ell_1 - \ell_2 + 1)$$

For the closed-loop system to have both eigenvalues at -10, the characteristic polynomial should be

$$(\lambda + 10)^2 = \lambda^2 + 20\lambda + 100$$

Equating coefficients leads to the system of linear equations

$$20 = \ell_1 + \ell_2 + 2$$
$$100 = 3\ell_1 - \ell_2 + 1$$

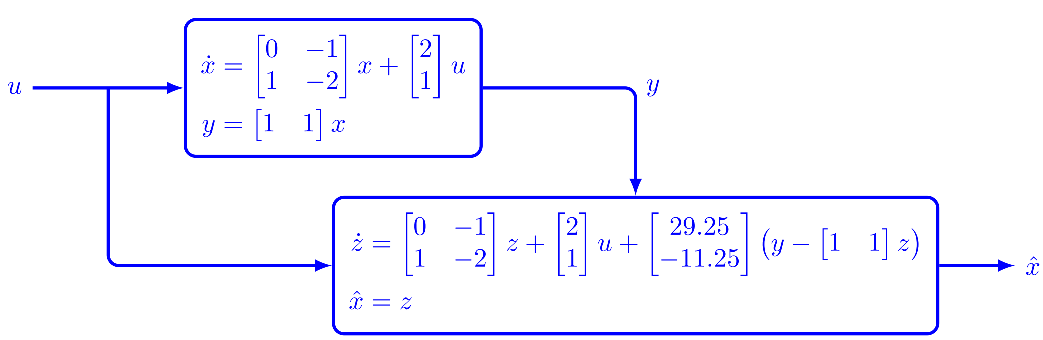
which has the solution

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} = \begin{bmatrix} 29.25 \\ -11.25 \end{bmatrix}$$

Therefore, the observer dynamics are

$$\dot{z} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} z + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u + \begin{bmatrix} 29.25 \\ -11.25 \end{bmatrix} \left(y - \begin{bmatrix} 1 & 1 \end{bmatrix} z\right)$$

and the block diagram is as follows:



Problem 51: Pole placement

Consider the system

$$\dot{x}(t) = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} -1 & 1 \end{pmatrix} x(t)$$

a) If possible, design a static state feedback controller $u(t) = Kx(t)$ such that the closed-loop system has poles at $s = -1$ and $s = -2$. Otherwise, explain why this is not possible.

SOLUTION: The controllability matrix is

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

which has rank one, so the system is *not* controllable. The characteristic polynomial of the system matrix is

$$\det(sI - A) = \det \begin{bmatrix} s-2 & 3 \\ 0 & s+1 \end{bmatrix} = (s+1)(s-2)$$

so A has eigenvalues at -1 and 2. If the eigenvalue at 2 is detectable, then we can still place the closed-loop eigenvalues as desired. Applying the PBH test to this eigenvalue,

$$\operatorname{rank} \begin{bmatrix} B & A - 2I \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & 0 & -3 \\ 1 & 0 & -3 \end{bmatrix} = 1$$

so the eigenvalue is not detectable and cannot be moved. Therefore, it is *not* possible to place the closed-loop eigenvalues at $s = -1$ and $s = -2$ using static state feedback.

b) If possible, design an observer such that the estimation error decays at a rate of e^{-t}. Otherwise, explain why this is not possible.

SOLUTION: The observability matrix is

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

which has rank one, so the system is *not* observable. However, applying the PBH test to the eigenvalue at 2, we obtain

$$\operatorname{rank} \begin{bmatrix} C \\ A - 2I \end{bmatrix} = \operatorname{rank} \begin{bmatrix} 1 & -1 \\ 0 & -3 \\ 0 & -3 \end{bmatrix} = 2$$

This eigenvalue can be moved, so we can design the observer. The characteristic polynomial of the observer is

$$\det\big(sI - (A + LC)\big) = \det \begin{bmatrix} s - 2 + \ell_1 & 3 - \ell_1 \\ \ell_2 & s + 1 - \ell_2 \end{bmatrix} = s^2 + (\ell_1 - \ell_2 - 1)s + (\ell_1 - \ell_2 - 2)$$

To place both poles at $s = -1$, the desired characteristic polynomial is

$$(s+1)^2 = s^2 + 2s + 1$$

Matching coefficients, the observer gain is $L = \begin{bmatrix} 1.5 \\ -1.5 \end{bmatrix}$.

Problem 52: LQR with cross-terms

Find the optimal control policy under the dynamics $\dot{x} = Ax + Bu$ with initial condition $x(0) = x_0$ that optimizes the functional

$$J(x_0) = \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \mathrm{d}t$$

where $R \succ 0$ and $\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \succeq 0$ (so the cost J is always nonnegative).

Note that the version we solved in class was with $S = 0$. Following a similar derivation, you should obtain a similar solution but with a slightly different Algebraic Riccati Equation.

SOLUTION: Assuming that $\lim_{t\to\infty} x(t) = 0$ and introducing a symmetric matrix $P = P^\mathsf{T}$, the cost is

$$\begin{aligned} J(x_0) &= \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \mathrm{d}t \\ &= x_0^\mathsf{T} P x_0 + \int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}t} x^\mathsf{T} P x + \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \right) \mathrm{d}t \end{aligned}$$

The derivative of the quadratic form can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(x^\mathsf{T} P x) &= \dot{x}^\mathsf{T} P x + x^\mathsf{T} P x \\ &= (Ax + Bu)^\mathsf{T} P x + x^\mathsf{T} P (Ax + Bu) \\ &= \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & 0 \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \end{aligned}$$

Substituting this into the expression for the cost, we have

$$J(x_0) = x_0^\mathsf{T} P x_0 + \int_0^\infty \begin{bmatrix} x \\ u \end{bmatrix}^\mathsf{T} \left(\begin{bmatrix} A^\mathsf{T} P + PA & PB \\ B^\mathsf{T} P & 0 \end{bmatrix} + \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \right) \begin{bmatrix} x \\ u \end{bmatrix} \mathrm{d}t$$

The first term does not depend on the control u, so we cannot do anything about it. However, we can choose the control to minimize the term inside the integral. Doing so gives $u(t) = Kx(t)$ where the feedback gain K is given by

$$K = -R^{-1}(B^\mathsf{T} P + S^\mathsf{T})$$

and P is the solution to the ARE

$$A^\mathsf{T} P + PA + Q - (PB + S)R^{-1}(PB + S)^\mathsf{T} = 0$$

Using this control law, the cost is $J(x_0) = x_0^\mathsf{T} P x_0$.

Problem 53: Mini LQR

Consider the dynamical system $\dot{x}(t) = -x(t) + u(t)$, where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. Consider the state-feedback control law that minimizes the cost

$$\int_0^\infty \left(x(t)^2 + \mu\, u(t)^2\right) \mathrm{d}t$$

where $\mu > 0$ is a parameter. Compute the optimal controller (which depends on μ), and verify that the closed-loop dynamics are stable for all values of $\mu > 0$.

Hint: If (A, B) is stabilizable and (Q, A) is detectable, then the algebraic Riccati equation $A^\mathsf{T} P + PA + Q - PBR^{-1}B^\mathsf{T} P = 0$ has a unique solution P that is symmetric and positive definite. Moreover, this P is *stabilizing* in that $A + BK$ is stable, where $K = -R^{-1}B^\mathsf{T} P$. Also, $u(t) = Kx(t)$ is the solution to the LQR problem with cost $\int_0^\infty (x^\mathsf{T} Qx + u^\mathsf{T} Ru)\,\mathrm{d}t$.

SOLUTION: We can directly apply the general result to this mini example. In this case,

$$A = -1 \qquad B = 1 \qquad Q = 1 \qquad R = \mu$$

The ARE becomes the following scalar equation in the scalar variable $P = p \in \mathbb{R}$,

$$-2p + 1 - \frac{1}{\mu} p^2 = 0$$

This is equivalent to the quadratic equation $p^2 + 2\mu p - \mu = 0$ which has the two solutions

$$p = -\mu \pm \sqrt{\mu^2 + \mu}$$

Notice that when $\mu > 0$, one of these roots is positive and the other is negative. Since we want the positive definite solution (i.e., $p > 0$), we choose the positive root. The closed-loop eigenvalues are

$$\begin{aligned} A + BK &= A - BR^{-1}B^\mathsf{T} P \\ &= (-1) - \frac{1}{\mu}\left(-\mu + \sqrt{\mu^2 + \mu}\right) \\ &= -\frac{\sqrt{\mu^2 + \mu}}{\mu} \\ &= -\sqrt{1 + \frac{1}{\mu}} \end{aligned}$$

Therefore, the closed-loop dynamics are

$$\dot{x}(t) = -\left(\sqrt{1 + \frac{1}{\mu}}\right) x(t)$$

The coefficient multiplying $x(t)$ is negative for all $\mu > 0$, so the closed-loop dynamics are stable.

- As $\mu \to 0$, we place relatively more weight on the state being small. The closed-loop system matrix tends to negative infinity, so the dynamics get faster and drive the state to zero more quickly.
- As $\mu \to \infty$, we place relatively more weight on the input being small. The closed-loop system matrix tends to -1, resulting in $K = 0$, that is, we use no control effort at all.

Problem 54: Algebraic Riccati Equation

We will study the simplified ARE

$$A^\mathsf{T} X + XA + Q + XRX = 0$$

where $Q, R \in \mathbb{R}^{n\times n}$ are symmetric matrices, and we want to find a solution $X = X^\mathsf{T} \in \mathbb{R}^{n\times n}$. Associated with this equation is the *Hamiltonian* matrix, defined as

$$H = \begin{bmatrix} A & R \\ -Q & -A^\mathsf{T} \end{bmatrix} \in \mathbb{R}^{2n\times 2n}$$

a) Prove that we can rewrite the ARE as $\begin{bmatrix} X & -I \end{bmatrix} H \begin{bmatrix} I \\ X \end{bmatrix} = 0$.

b) Define the matrix $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$ where the blocks of J are the same sizes as the blocks of H. Show that $J^{-1}HJ = -H^\mathsf{T}$, and use this fact to prove that if λ is an eigenvalue of H, then so it $-\bar{\lambda}$. In other words, if H has no eigenvalues on the imaginary axis, then exactly n of them are stable and the other n are unstable.

c) Suppose we can find three matrices $X_1, X_2, M \in \mathbb{R}^{n\times n}$ such that X_1 is invertible and the matrices satisfy $H\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} M$. Prove that $X = X_2 X_1^{-1}$ is a solution to the ARE.

SOLUTION:

a) Substituting the expression for H and simplifying, we have

$$0 = \begin{bmatrix} X & -I \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^\mathsf{T} \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = A^\mathsf{T} X + XA + Q + XRX$$

which is the ARE.

b) Since $J^2 = -I$, its inverse is simply $J^{-1} = -J$. Then

$$\begin{aligned} J^{-1}HJ = -JHJ &= \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & R \\ -Q & -A^\mathsf{T} \end{bmatrix} \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} R & -A \\ -A^\mathsf{T} & Q \end{bmatrix} \\ &= \begin{bmatrix} -A^\mathsf{T} & Q \\ -R & A \end{bmatrix} = -H^\mathsf{T} \end{aligned}$$

Suppose (λ, v) is an eigenpair of H. Then

$$\lambda v = Hv = \left(-JH^\mathsf{T} J^{-1}\right) v = J\left(-H^\mathsf{T}\right) J^{-1} v$$

Rearranging, we obtain $H^\mathsf{T}(J^{-1}v) = -\lambda(J^{-1}v)$. Taking the transpose, we notice that $-\lambda$ must also be an eigenvalue of H. We could take the conjugate-transpose and conclude that $-\bar{\lambda}$ is an eigenvalue of H. This means that all eigenvalues of H occur in quadruplets: if λ is an eigenvalue, then so are $\pm\lambda$ and $\pm\bar{\lambda}$.

c) Start with the given expression

$$H\begin{bmatrix}X_1\\X_2\end{bmatrix} = \begin{bmatrix}X_1\\X_2\end{bmatrix}M$$

Multiply both sides on the right by X_1^{-1} to obtain

$$H\begin{bmatrix}I\\X_2X_1^{-1}\end{bmatrix} = \begin{bmatrix}X_1\\X_2\end{bmatrix}MX_1^{-1} = \begin{bmatrix}I\\X_2X_1^{-1}\end{bmatrix}(X_1MX_1^{-1})$$

Setting $X = X_2X_1^{-1}$, we have

$$H\begin{bmatrix}I\\X\end{bmatrix} = \begin{bmatrix}I\\X\end{bmatrix}(X_1MX_1^{-1})$$

Multiply both sides on the left by $\begin{bmatrix}X & -I\end{bmatrix}$. The right-hand side becomes zero and we obtain

$$\begin{bmatrix}X & -I\end{bmatrix}H\begin{bmatrix}I\\X\end{bmatrix} = 0$$

By the result in part **(a)**, we conclude that $X = X_2X_1^{-1}$ is a solution to the ARE.

Problem 55: Dual state-space system

Consider the MIMO system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

Suppose the transfer function of this system (from u to y) is given by $H(s)$. Now consider the dual system with state $z(t)$ and state-space equations

$$\begin{aligned}\dot{z}(t) &= -A^{\mathsf{T}}z(t) - C^{\mathsf{T}}v(t)\\ w(t) &= B^{\mathsf{T}}z(t) + D^{\mathsf{T}}v(t)\end{aligned}$$

Prove that the transfer function for this new system (from v to w) is given by $\big(H(-s)\big)^{\mathsf{T}}$.

SOLUTION: The transfer function for (A, B, C, D) is

$$H(s) = C\,(sI - A)^{-1}B + D$$

so the transfer function for the dual system is $B^{\mathsf{T}}(sI + A^{\mathsf{T}})^{-1}(-C^{\mathsf{T}}) + D^{\mathsf{T}}$. We can now check that

$$\begin{aligned}\big(H(-s)\big)^{\mathsf{T}} &= \big(C\,(-sI - A)^{-1}B + D\big)^{\mathsf{T}}\\ &= B^{\mathsf{T}}(-sI - A)^{-\mathsf{T}}C^{\mathsf{T}} + D^{\mathsf{T}}\\ &= B^{\mathsf{T}}(sI + A^{\mathsf{T}})^{-1}(-C^{\mathsf{T}}) + D^{\mathsf{T}}\end{aligned}$$

which is the same as in the previous expression.

Problem 56: True/False

For each question, circle TRUE or FALSE. No explanation is required.

a) If two realizations for the same transfer function have the same number of states (A is the same size for both), then there must exist a state transformation matrix T that transforms one realization into the other (and vice versa), even if the realizations are not minimal. TRUE $\boxed{\text{FALSE}}$

SOLUTION: (False) This is only true for minimal realizations.

b) If a linear time-invariant system is BIBO stable, then it is always Lyapunov stable. TRUE $\boxed{\text{FALSE}}$

SOLUTION: (False) This is only true for minimal realizations.

c) If a linear time-invariant system is asymptotically stable, then it is always exponentially stable as well. $\boxed{\text{TRUE}}$ FALSE

SOLUTION: (True) For linear systems, asymptotic stability implies exponential stability.

d) When using the input $u(t) = \sin(t)$, the output $y(t)$ of a system grows unbounded with time. We can conclude that the system is definitely *not* BIBO stable. $\boxed{\text{TRUE}}$ FALSE

SOLUTION: (True) BIBO stability means all bounded inputs produce bounded outputs. One example of this failing to occur means that the system is not BIBO stable

e) When using the input $u(t) = \sin(t)$, the output $y(t)$ of the system is bounded. We can conclude that system is definitely BIBO stable. TRUE $\boxed{\text{FALSE}}$

SOLUTION: (False) Showing that one bounded input produces a bounded output does not necessarily mean that *all* bounded inputs produce bounded outputs.

f) If $\dot{x}(t) = Ax(t)$ is asymptotically stable, then so is $x[k+1] = Ax[k]$ (same A matrix for both systems). TRUE $\boxed{\text{FALSE}}$

SOLUTION: (False) The stability requirements are different in continuous and discrete time. In continuous time, we require $\mathrm{Re}(\lambda) < 0$, while in discrete time we require $|\lambda| < 1$.

Problem 57: True/False

For each statement, circle TRUE or FALSE. No explanation is required.

a) Integral control achieves zero steady-state error for *any* time-varying reference input signal. TRUE $\boxed{\text{FALSE}}$

SOLUTION: (False) Integral control achieves zero steady-state error for any *constant* reference input signal.

b) If the continuous-time system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable, then so is the discrete-time system $x(k+1) = Ax(k) + Bu(k)$ (same A and B matrices for both systems). $\boxed{\text{TRUE}}$ FALSE

SOLUTION: (True) The controllability requirements are the same in continuous and discrete time (for example, both use the same controllability matrix P).

c) If an LTI system is stabilizable, then it is possible to arbitrarily place the closed-loop poles (in complex conjugate pairs) using static state feedback. TRUE $\boxed{\text{FALSE}}$

SOLUTION: (False) The closed-loop system can be stabilized, but the poles cannot be placed arbitrarily unless the system is controllable.

d) If an LTI system is detectable, then it is possible to design an observer that asymptotically estimates the state. $\boxed{\text{TRUE}}$ FALSE

SOLUTION: (True) For a detectable system, the eigenvalues of the observer can be made stable.

e) Consider an LTI system whose state is estimated by an observer and whose input is a static multiple of the state estimate. The closed-loop eigenvalues of this system are the same as those of the observer and the system when using static *state* feedback. $\boxed{\text{TRUE}}$ FALSE

SOLUTION: (True) This is the separation property.

f) Balanced truncation is a method of model reduction that removes states that are not very controllable and observable. $\boxed{\text{TRUE}}$ FALSE

SOLUTION: (True) Balanced truncation removes states with small Hankel singular values, which are a measure of the controllability and observability of each state in a balanced realization.

Problem 58: Linearization

For each nonlinear system, find all the equilibrium points and construct the Jacobian linearization of the system about each equilibrium.

a)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -4x_2 \\ 4x_1 - x_1^2 - 0.5x_2 \end{bmatrix}$$

b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 9x_1 + x_2^2 \\ x_1 - x_2 \end{bmatrix}$$

c)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - x_1 x_2 \\ -x_2 + x_2^2 x_1 \end{bmatrix}$$

SOLUTION:

a) This is a continuous-time nonlinear system with state transition function

$$f(x_1, x_2) = \begin{bmatrix} -4x_2 \\ 4x_1 - x_1^2 - 0.5x_2 \end{bmatrix}$$

The equilibrium points are the solutions to $f(\tilde{x}_1, \tilde{x}_2) = 0$. The first component of this equation implies that $\tilde{x}_2 = 0$, and the second component implies that $\tilde{x}_1 = 0$ or 4. Therefore, the system has two equilibriums at $(0, 0)$ and $(4, 0)$. The Jacobian of the state transition function is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & -4 \\ 4 - 2x_1 & -0.5 \end{bmatrix}$$

The Jacobian linearization of the system about the equilibrium $(0, 0)$ is

$$\begin{bmatrix} \dot{\delta}_{x_1} \\ \dot{\delta}_{x_2} \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 4 & -0.5 \end{bmatrix} \begin{bmatrix} \delta_{x_1} \\ \delta_{x_2} \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} \delta_{x_1} &= x_1 \\ \delta_{x_2} &= x_2 \end{aligned}$$

and the linearization about the equilibrium $(4, 0)$ is

$$\begin{bmatrix} \dot{\delta}_{x_1} \\ \dot{\delta}_{x_2} \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ -4 & -0.5 \end{bmatrix} \begin{bmatrix} \delta_{x_1} \\ \delta_{x_2} \end{bmatrix} \qquad \text{where} \qquad \begin{aligned} \delta_{x_1} &= x_1 - 4 \\ \delta_{x_2} &= x_2 \end{aligned}$$

b) This is a continuous-time nonlinear system with state transition function

$$f(x_1, x_2) = \begin{bmatrix} 9x_1 + x_2^2 \\ x_1 - x_2 \end{bmatrix}$$

The equilibrium points are the solutions to $f(\tilde{x}_1, \tilde{x}_2) = 0$. The second component of this equation implies that $\tilde{x}_1 = \tilde{x}_2$, and the first component implies that $\tilde{x}_1 = 0$ or -9. Therefore, the system has two equilibriums at $(0, 0)$ and $(-9, -9)$. The Jacobian of the state transition function is

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 9 & 2x_2 \\ 1 & -1 \end{bmatrix}$$

The Jacobian linearization of the system about the equilibrium $(0,0)$ is

$$\begin{bmatrix}\dot{\delta}_{x_1}\\ \dot{\delta}_{x_2}\end{bmatrix} = \begin{bmatrix}9 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}\delta_{x_1}\\ \delta_{x_2}\end{bmatrix} \qquad \text{where} \qquad \begin{aligned}\delta_{x_1} &= x_1\\ \delta_{x_2} &= x_2\end{aligned}$$

and the linearization about the equilibrium $(4,0)$ is

$$\begin{bmatrix}\dot{\delta}_{x_1}\\ \dot{\delta}_{x_2}\end{bmatrix} = \begin{bmatrix}9 & -18\\ 1 & -1\end{bmatrix}\begin{bmatrix}\delta_{x_1}\\ \delta_{x_2}\end{bmatrix} \qquad \text{where} \qquad \begin{aligned}\delta_{x_1} &= x_1 + 9\\ \delta_{x_2} &= x_2 + 9\end{aligned}$$

c) This is a continuous-time nonlinear system with state transition function

$$f(x_1,x_2) = \begin{bmatrix}2x_1 - x_1x_2\\ -x_2 + x_2^2x_1\end{bmatrix}$$

The equilibrium points are the solutions to $f(\tilde{x}_1,\tilde{x}_2) = 0$. The first component of this equation implies that $\tilde{x}_1 = 0$ or $\tilde{x}_2 = 2$. For the case $\tilde{x}_1 = 0$, the second component implies that $\tilde{x}_2 = 0$. For the case $\tilde{x}_2 = 2$, the second component implies that $\tilde{x}_1 = 1/2$. Therefore, the system has two equilibriums at $(0,0)$ and $(1/2,2)$. The Jacobian of the state transition function is

$$\frac{\partial f}{\partial x} = \begin{bmatrix}2 - x_2 & -x_1\\ x_2^2 & -1 + 2x_2x_1\end{bmatrix}$$

The Jacobian linearization of the system about the equilibrium $(0,0)$ is

$$\begin{bmatrix}\dot{\delta}_{x_1}\\ \dot{\delta}_{x_2}\end{bmatrix} = \begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\delta_{x_1}\\ \delta_{x_2}\end{bmatrix} \qquad \text{where} \qquad \begin{aligned}\delta_{x_1} &= x_1\\ \delta_{x_2} &= x_2\end{aligned}$$

and the linearization about the equilibrium $(1/2,2)$ is

$$\begin{bmatrix}\dot{\delta}_{x_1}\\ \dot{\delta}_{x_2}\end{bmatrix} = \begin{bmatrix}0 & -1/2\\ 4 & 1\end{bmatrix}\begin{bmatrix}\delta_{x_1}\\ \delta_{x_2}\end{bmatrix} \qquad \text{where} \qquad \begin{aligned}\delta_{x_1} &= x_1 - 1/2\\ \delta_{x_2} &= x_2 - 2\end{aligned}$$

Problem 59: Controllable subspace

Consider the continuous-time LTI system

$$\dot{x}(t) = \begin{bmatrix} -5 & -1 & -4 & 5 \\ 12 & 0 & 5 & -13 \\ -6 & -1 & -3 & 5 \\ -6 & -1 & -4 & 6 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 5 \\ -2 \\ -2 \end{bmatrix} u(t)$$

a) State whether or not the system is controllable, and explain your reasoning.

b) Find the controllable subspace, that is, the set of states that the system can be driven to from the origin.

c) Consider the following possible state vectors:

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad x_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \qquad x_4 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \qquad x_5 = \begin{bmatrix} -2 \\ 3 \\ -2 \\ -2 \end{bmatrix}$$

For each combination of the state vectors, determine whether or not an input $u(t)$ exists that moves the state from one state vector to the other in one second.

d) If possible, find the input signal $u(t)$ for $0 \leq t \leq 1$ that drives the system from the initial state $x(0) = x_1$ to the final state $x(1) = x_5$ in one second.

SOLUTION:

a) The controllability matrix is

$$P = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} -1 & -2 & 3 & -13 \\ 5 & 4 & 0 & 20 \\ -2 & -3 & 2 & -14 \\ -2 & -3 & 2 & -14 \end{bmatrix}$$

which has rank three, so the system is not controllable.

b) The controllable subspace is the range of the controllability matrix, which is

$$\operatorname{range}(P) = \operatorname{range} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \mid x_3 = x_4 \right\}$$

c) There exists an input that moves the system from one state to another if and only if both states are in the controllable subspace. The states x_1, x_2, x_3, and x_5 are in the controllable subspace, so the system can move between any combination of these states in arbitrary finite time. The state x_4 is not in the controllable subspace, so the system cannot be driven to this state from any of the other states, and it cannot be driven to any other state from this state in any amount of time.