Problem 1: Complex numbers

Consider the following complex numbers, where $j = \sqrt{-1}$ is the imaginary unit:

$$z_1 = 3 - j4 \qquad z_2 = -2 + j4 \qquad z_3 = 2\,e^{j90^\circ} \qquad z_4 = e^{j\pi}$$

For each complex number:

a) Plot the number in the complex plane.

b) Find the real part, imaginary part, magnitude, and angle.

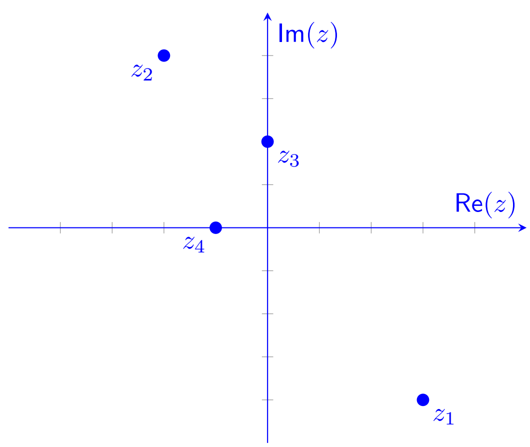
Also, compute the following quantities:

c) $z_1 + z_3$

d) $z_1\, z_4$

SOLUTION:

a) We can visualize each complex number in the complex plane as follows.



b) The following table shows the real part, imaginary part, magnitude, and angle of each complex number.

	real part	imaginary part	magnitude	angle (degrees)
z_1	3	-4	5	-53°
z_2	-2	4	$\sqrt{20}$	116.6°
z_3	0	2	2	90°
z_4	-1	0	1	180°

c) To sum two complex numbers, we use their rectangular form.

$$z_1 + z_3 = (3 - j4) + (0 + j2) = (3 + 0) + j(-4 + 2) = 3 - j2$$

d) To multiply two complex numbers, we use their polar form.

$$z_1\, z_4 = \left(5\, e^{-j53^\circ}\right)\left(e^{j180^\circ}\right) = (5 \cdot 1)\, e^{j(180^\circ - 53^\circ)} = 5\, e^{j127^\circ}$$

Problem 2: Vectors

Consider the following three-dimensional real vectors:

$$u = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \qquad v = \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \qquad w = \begin{bmatrix} 9 \\ -3 \\ 9 \end{bmatrix}$$

a) Find the length of each vector.

b) Find the inner product between each pair of vectors and state whether or not they are orthogonal.

c) Add each pair of vectors.

SOLUTION:

a) The length of each vector is as follows:

$$\begin{aligned} \|u\| &= \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14} \\ \|v\| &= \sqrt{(5)^2 + (-1)^2 + (3)^2} = \sqrt{35} \\ \|w\| &= \sqrt{(9)^2 + (-3)^2 + (9)^2} = \sqrt{171} \end{aligned}$$

b) The inner product between each pair of vectors is as follows:

$$\begin{aligned} \langle u, v\rangle &= (2)(5) + (3)(-1) + (-1)(3) = 2 \\ \langle u, w\rangle &= (2)(9) + (3)(-3) + (-1)(9) = 0 \\ \langle v, w\rangle &= (5)(9) + (-1)(-3) + (3)(9) = 75 \end{aligned}$$

Note that the inner product is symmetric, so $\langle v, u\rangle = \langle u, v\rangle$ and so on, so we do not need to compute these separately. The vectors u and w are orthogonal since their inner product is zero, while all other pairs are not orthogonal.

c) The sum of each pair of vectors is as follows:

$$\begin{aligned} u + v &= \begin{bmatrix} (2) + (5) \\ (3) + (-1) \\ (-1) + (3) \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 2 \end{bmatrix} \\ u + w &= \begin{bmatrix} (2) + (9) \\ (3) + (-3) \\ (-1) + (9) \end{bmatrix} = \begin{bmatrix} 11 \\ 0 \\ 8 \end{bmatrix} \\ v + w &= \begin{bmatrix} (5) + (9) \\ (-1) + (-3) \\ (3) + (9) \end{bmatrix} = \begin{bmatrix} 14 \\ -4 \\ 12 \end{bmatrix} \end{aligned}$$

Note that vector addition is *commutative*, meaning that $v + u = u + v$ and so on, so we do not need to compute these separately.

Problem 3: Matrices

For each matrix A, find the eigenvalues and eigenvectors. If it exists, find A^{-1} and compute the eigenvalues and eigenvectors of A^{-1}. How are the eigenvalues and eigenvectors of A^{-1} related to those of A?

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$

b) $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$

c) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{pmatrix}$

SOLUTION:

a) The eigenvalues are the solution to the characteristic equation

$$0 = \det(\lambda I - A) = \det\left(\begin{bmatrix} \lambda - 1 & -1 \\ 0 & \lambda - 2 \end{bmatrix}\right) = (\lambda - 1)(\lambda - 2) - (-1)(0) = (\lambda - 1)(\lambda - 2)$$

which are $\lambda_1 = 1$ and $\lambda_2 = 2$. To find the corresponding eigenvectors, we solve the equation

$$(\lambda I - A)\, v = 0$$

for each value eigenvalue λ.

- Case: $\lambda_1 = 1$
$$0 = (\lambda_1 I - A)\, v_1 = \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ -b \end{bmatrix}$$
This implies that $b = 0$ and a is arbitrary, so we can just set it to $a = 1$.
- Case: $\lambda_2 = 2$
$$0 = (\lambda_2 I - A)\, v_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ 0 \end{bmatrix}$$
This implies that $b = a$, where a is again arbitrary, so we can set it to $a = 1$.

Therefore, the matrix has the two eigenvalues and eigenvectors

$$\lambda_1 = 1,\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \lambda_2 = 2,\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The determinant of the matrix is

$$\det(A) = \det\left(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) = (1)(2) - (1)(0) = 2$$

(the determinant is also the product of the eigenvalues: $\det(A) = \lambda_1\, \lambda_2 = 2$). The matrix is invertible since the determinant is not zero. Using the adjugate, the inverse of A is

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A) = \frac{1}{2} \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -0.5 \\ 0 & 0.5 \end{bmatrix}$$

We can compute the eigenvalues and eigenvectors of A^{-1} the same as before. Doing so, we find that the eigenvectors of A^{-1} are also v_1 and v_2 (the same as before) with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0.5$ (the inverses of before).

b) The matrix has two eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 5$, respectively. The matrix is *not* invertible since its determinant is zero.

c) The matrix has three eigenvectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad v_3 = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$$

with corresponding eigenvalues $\lambda_1 = 0$, $\lambda_2 = -1$, and $\lambda_3 = -2$, respectively. The matrix is *not* invertible since its determinant is zero.

Comment: Whenever a matrix A is invertible, its inverse A^{-1} has the same eigenvectors as A, and the eigenvalues are the inverses of the corresponding eigenvalues of A. In particular, suppose that $Av = \lambda v$ where A is invertible. Note that λ must be nonzero since A is invertible. Then we can multiply on the left by $(1/\lambda)\, A^{-1}$ to obtain

$$(1/\lambda)\, v = A^{-1} v$$

Therefore, v is an eigenvector of A^{-1} with eigenvalue $1/\lambda$.

Problem 4: Programming

Use MATLAB to do the following.

a) Find the determinant, inverse, eigenvalues, and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & -0.5 & -0.5 \\ 1 & 2.5 & 1.5 \\ -1 & 0.5 & 1.5 \end{bmatrix}$$

b) Define a function that evaluates $f(t) = \sin\left(\frac{\pi}{2}(t-1)\right)$.

c) Create a vector with 100 evenly-spaced elements in the interval $[-\pi, \pi]$.

d) Write a for loop to iterate from 0 to 10 in increments of 2.

e) Plot the function $f(t)$ defined previously over the interval $[-\pi, \pi]$. Label both axes.

SOLUTION:

a) We can use the following code to compute the desired quantities:

```
A = [2 -0.5 -0.5; 1 2.5 1.5; -1 0.5 1.5]
det(A)
inv(A)
[V,D] = eig(A)
```

This code gives the answers using floating point numbers. If we want the corresponding rational numbers, we can use the `rats` function, such as `rats(inv(A))`. Doing so, we find that $\det(A) = 6$ and the inverse is

$$A^{-1} = \begin{bmatrix} 0.5 & 0.0833 & 0.08333 \\ -0.5 & 0.4167 & -0.5833 \\ 0.5 & -0.0833 & 0.9167 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 6 & 1 & 1 \\ -6 & 5 & -7 \\ 6 & -1 & 11 \end{bmatrix}$$

The eigenvectors are the columns of `V`, and the eigenvalues are the elements on the diagonal of `D`. In particular, the matrix has the following three eigenpairs.

$$\lambda_1 = 1 \qquad v_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 2 \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\lambda_3 = 3 \qquad v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Note that we scaled the eigenvectors so that they are easier to read. MATLAB scales them so that they have unit norm, that is, $\|v\| = 1$.

b) We can define the function in two ways. An anonymous function is simply defined using the single command `f = @(t) sin((pi/2)*(t-1))`. We can also define this function inside a separate m-file called `f.m` as follows:

```
function y = f(t)

y = sin((pi/2)*(t-1));

end
```

Note that the name of the file has to match the name of the function inside the file.

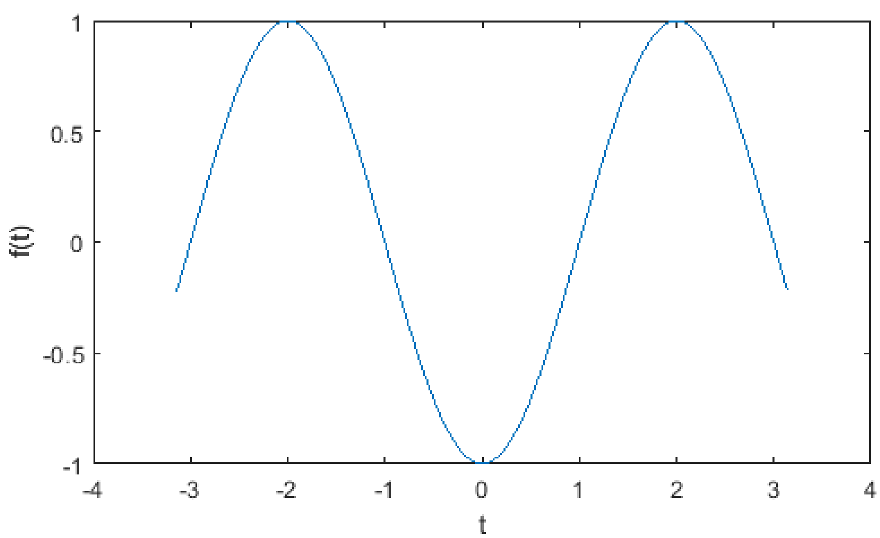
c) `linspace(-pi,pi,100)`

d) We can use the notation `start:increment:stop` to construct the for loop.

```
for k = 0:2:10
    disp(k)
end
```

e) We can plot the function as follows.

```
t = linspace(-pi,pi,100);
f = @(t) sin((pi/2)*(t-1));

plot(t,f(t));
xlabel('t');
ylabel('f(t)');
```



Problem 5: Plotting discrete-time signals

Use a computer to plot each of the following discrete-time signals.

a) $x(k) = 2\,u_s(k) - u_s(k-2) - u_s(k-5)$

b) $x(k) = \delta(k) + \delta(k-1) + u_s(k) - u_s(k-5)$

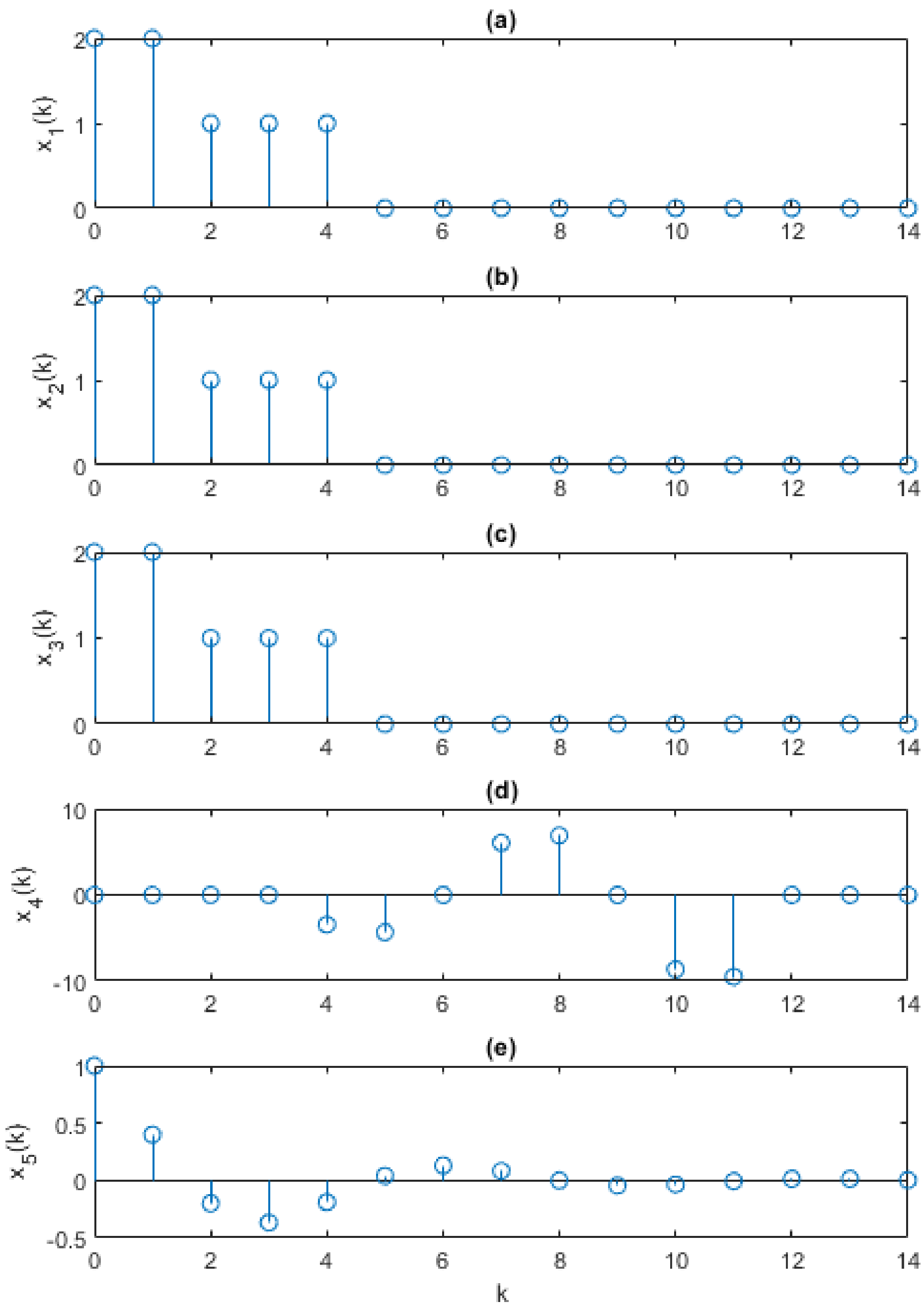
c) $x(k) = 2\,[u_s(k) - u_s(k-2)] + [u_s(k-2) - u_s(k-5)]$

d) $x(k) = k\sin(k\pi/3)\,[u_s(k-3) - u_s(k-12)]$

e) $x(k) = \mathrm{Re}\{(0.4 + j0.6)^k\,u_s(k)\}$

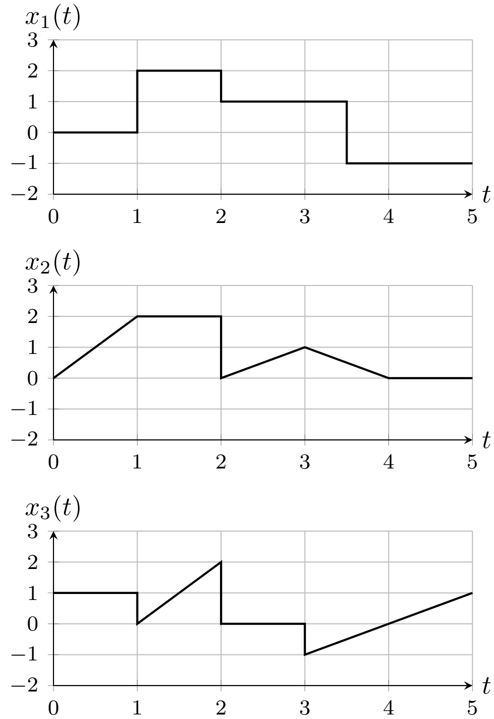
SOLUTION:

```
% number of iterations
N = 15;

% discrete-time index (vector)
k = 0:N-1;

u_step = @(x) (x >= 0);  % unit step sequence
delta  = @(x) (x == 0);  % unit impulse sequence

% signals
x1 = 2*u_step(k) - u_step(k-2) - u_step(k-5);
x2 = delta(k) + delta(k-1) + u_step(k) - u_step(k-5);
x3 = 2*(u_step(k) - u_step(k-2)) + (u_step(k-2) - u_step(k-5));
x4 = k.*sin(k*pi/3).*(u_step(k-3) - u_step(k-12));
x5 = real( ((0.4 + 0.6*1i).^k) .* u_step(k) );

% plot
figure;
subplot(5,1,1); stem(k,x1); title('(a)'); ylabel('x_1(k)');
subplot(5,1,2); stem(k,x2); title('(b)'); ylabel('x_2(k)');
subplot(5,1,3); stem(k,x3); title('(c)'); ylabel('x_3(k)');
subplot(5,1,4); stem(k,x4); title('(d)'); ylabel('x_4(k)');
subplot(5,1,5); stem(k,x5); title('(e)'); ylabel('x_5(k)');
xlabel('k');
```


(a)
2
1
0
$x_1(k)$
0
2
4
6
8
10
12
14
(b)
2
1
0
$x_2(k)$
0
2
4
6
8
10
12
14
(c)
2
1
0
$x_3(k)$
0
2
4
6
8
10
12
14
(d)
10
0
-10
$x_4(k)$
0
2
4
6
8
10
12
14
(e)
1
0.5
0
-0.5
$x_5(k)$
0
2
4
6
8
10
12
14
k

Problem 6: Plotting continuous-time signals

Write an expression for each of the following signals using shifted and scaled versions of the unit step signal $u_s(t)$ and unit ramp signal $u_r(t)$.



SOLUTION: The first signal $x_1(t)$ is zero until time $t = 1$, at which point it has a discontinuity and "jumps" to a value of two. We can represent this as a unit step signal that is shifted to the right by one and scaled by two, $2\,u_s(t-1)$. This signal has a value of two for all times $t \geq 1$. Since $x_1(t)$ jumps down by one at time $t = 2$, we need to subtract a unit step starting at time two, which gives $2\,u_s(t-1) - u_s(t-2)$. For all times $t \geq 2$, both unit steps are "on" so the value of the signal is $2 - 1 = 1$. Since $x_1(t)$ jumps down by two at time $t = 3.5$, we need to subtract a scaled unit step shifted to this time: $-2\,u_s(t-3.5)$. Combining all these together gives the expression

$$x_1(t) = 2\,u_s(t-1) - u_s(t-2) - 2\,u_s(t-3.5)$$

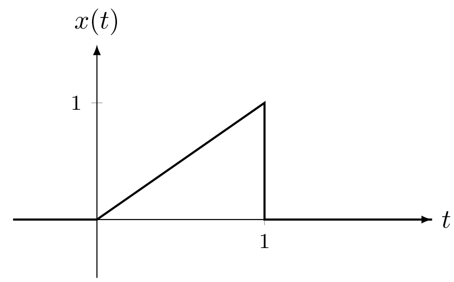
The process of finding the expressions for the other two signals is similar, and the resulting expressions are:

$$x_2(t) = 2\,u_r(t) - 2\,u_r(t-1) - 2\,u_s(t-2) + u_r(t-2) - 2\,u_r(t-3) + u_r(t-4)$$

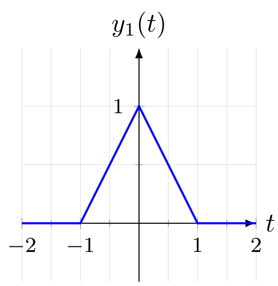
$$x_3(t) = u_s(t) - u_s(t-1) + 2\,u_r(t-1) - 2\,u_s(t-2) - u_s(t-3) + u_r(t-3)$$

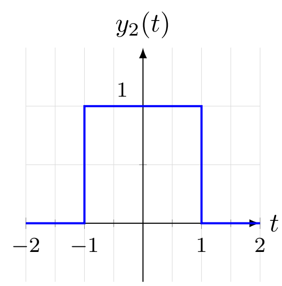
Problem 7: Signal transformations

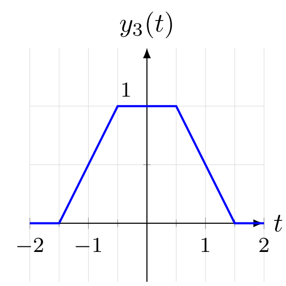
Consider the following continuous-time signal $x(t)$.



Express each of the following signals as a linear combination of time-shifted and/or time-scaled versions of $x(t)$.







SOLUTION: We can decompose each signal in terms of $x(t)$ as follows. The left triangle of $y_1(t)$ is given by $x(t)$ shifted to the left by one, which is $x(t+1)$. Likewise, the right triangle of $y_1(t)$ is given by $x(t+1)$ reflected about the vertical axis, which is $x(-t+1)$. Therefore, we can express $y_1(t)$ in terms of $x(t)$ as

$$y_1(t) = x(t+1) + x(-t+1)$$

Similarly, we can use $y_1(t)$ to construct

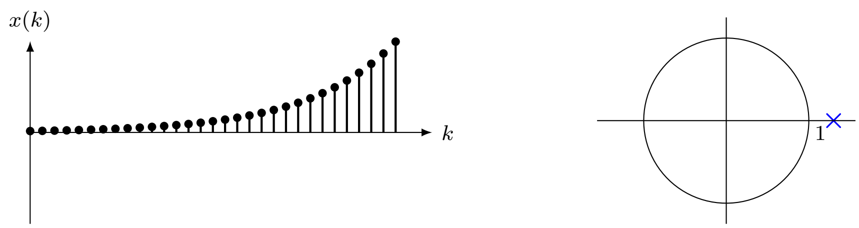
$$y_2(t) = y_1(t) + x(-t) + x(t)$$

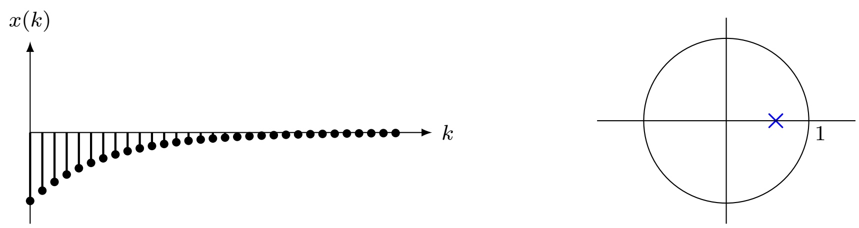
and then use $y_2(t)$ to construct

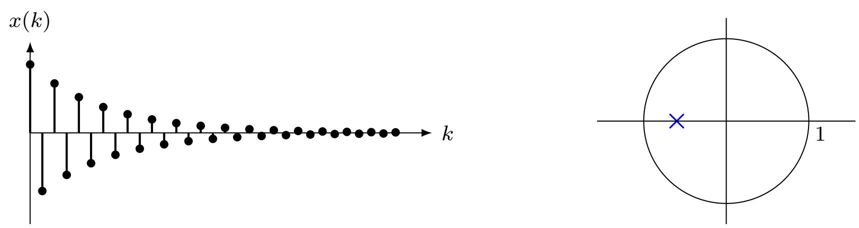
$$y_3(t) = y_2(2t) + x(t+1.5) + x(-t+1.5)$$

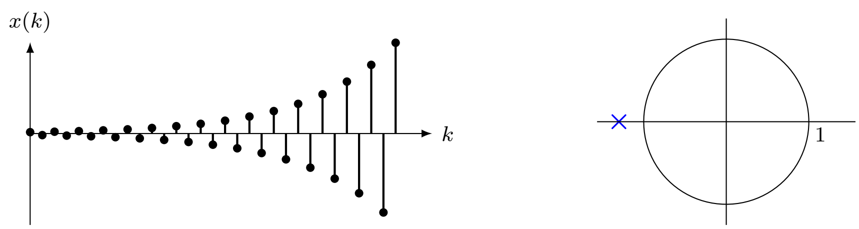
Problem 8: Discrete-time complex exponential signals

Each discrete-time signal shown on the left is the real part of a complex exponential signal of the form $x(k) = c\,z^k$ for some scalar c and complex number z. On the corresponding axis on the right, mark the approximate complex-plane location(s) of z.







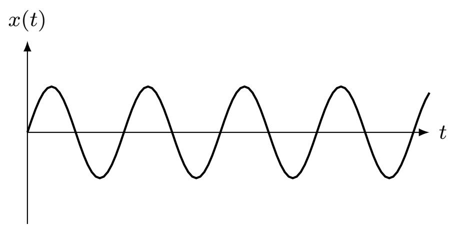


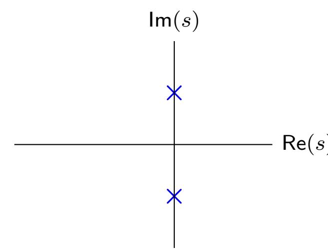
SOLUTION:

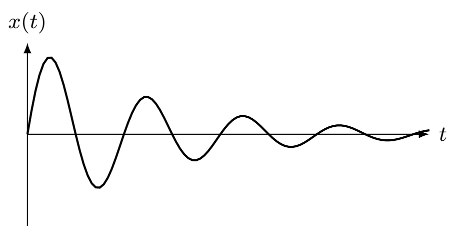
- The first signal grows exponentially without bound, so z must be outside of the unit circle in the complex plane. The signal is monotonic (not oscillating or changing sign), so z must be on the positive real axis.
- This signal is also monotonic, but is decaying exponentially so z should be on the positive real axis but inside the unit circle.
- The size of this signal is exponentially decaying, so z should be inside the unit circle. But now the signal is "bouncing" back and forth between positive and negative values, so z should be on the negative real axis.
- This signal also changes sign at each iteration, but is now growing without bound, so z should be on the negative real axis outside of the unit circle.

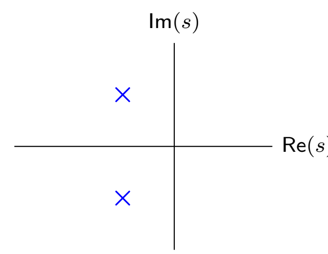
Problem 9: Continuous-time complex exponential signals

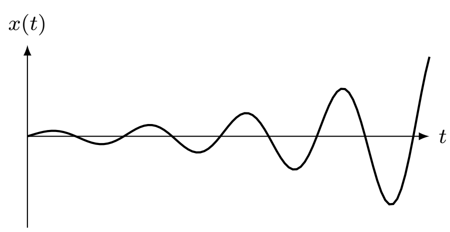
Each continuous-time signal shown on the left is the real part of a complex exponential signal of the form $x(t) = c\,e^{st}$ for some constant c and complex number s. On the corresponding axis on the right, mark the approximate complex-plane location(s) of s.

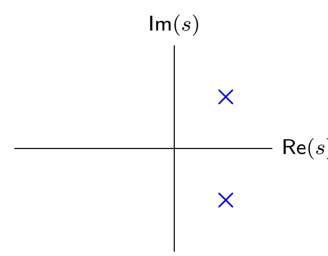


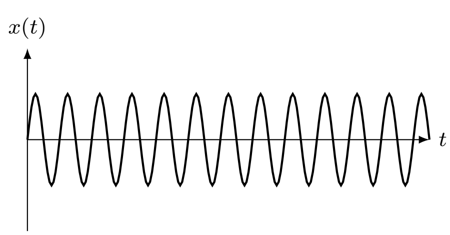


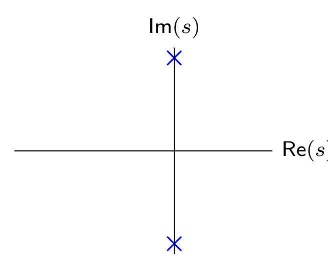






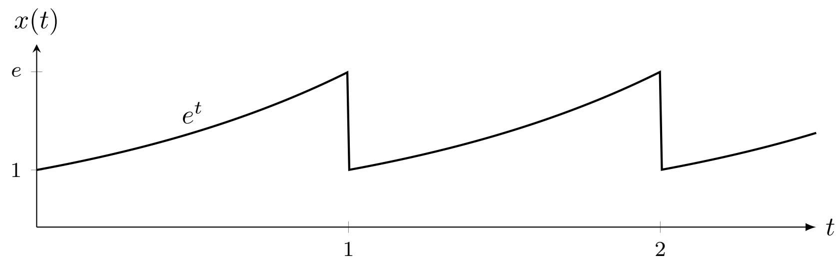






Problem 10: Fourier series

Consider the following signal $x(t)$.



a) Determine the fundamental period T.

b) Determine the fundamental frequency ω_0.

c) Compute the (exponential) Fourier series coefficients c_k.

Note: The general form of the exponential Fourier series is

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \, e^{jk\omega_0 t} \qquad \text{where} \qquad c_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,\mathrm{d}t$$

SOLUTION:

a) The fundamental period is the smallest time before the signal repeats, which is $T = 1$.

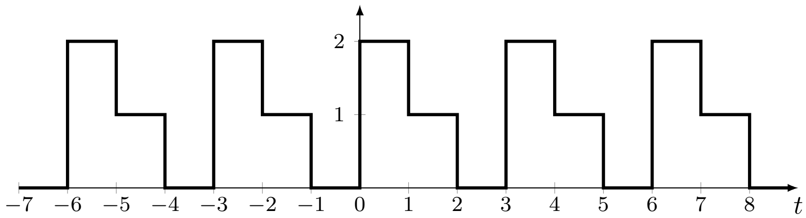
b) The fundamental frequency is

$$\omega_0 = \frac{2\pi}{T} = 2\pi$$

c) The Fourier series coefficients are

$$\begin{aligned} c_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,\mathrm{d}t \\ &= \int_0^1 e^{(1-jk2\pi)t}\,\mathrm{d}t \\ &= \frac{1}{1-jk2\pi}\left(e^{1-jk2\pi} - 1\right) \end{aligned}$$

Problem 11: Fourier series



For the periodic signal shown above, do the following:

a) find the fundamental period T and fundamental frequency ω_0

b) find the Fourier series coefficients in exponential form (c_k)

c) find the Fourier series coefficients in trigonometric form (A_k and θ_k)

d) use MATLAB to plot the original signal and its 10-term approximation

$$\hat{x}(t) = A_0 + 2\sum_{k=1}^{10} A_k \cos(k\omega_0 t + \theta_k)$$

on the same axis (remember to label the axes, and include your code)

SOLUTION:

a) The signal has fundamental period $T = 3$ seconds and fundamental frequency $\omega_0 = 2\pi/T = 2\pi/3$ radians per second.

b) The Fourier series coefficients in exponential form are given by

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\,\mathrm{d}t \\
&= \frac{1}{3}\int_0^1 2\, e^{-jk2\pi/3t}\,\mathrm{d}t + \frac{1}{3}\int_1^2 e^{-jk2\pi/3t}\,\mathrm{d}t \\
&= -\frac{1}{j\pi k} e^{-jk2\pi/3t}\Big|_{t=0}^{1} - \frac{1}{j2\pi k} e^{-jk2\pi/3t}\Big|_{t=1}^{2} \\
&= \frac{1}{j2\pi k}\Big(2 - 2\, e^{-jk2\pi/3} - e^{-jk4\pi/3} + e^{-jk2\pi/3}\Big) \\
&= \frac{1}{j2\pi k}\Big(2 - e^{-jk2\pi/3} - e^{jk2\pi/3}\Big) \\
&= \frac{1}{j\pi k}\Big(1 - \cos\big(\tfrac{2\pi}{3}k\big)\Big)
\end{aligned}$$

for $k \neq 0$, and the average value of the signal is $c_0 = 1$.

c) In trigonometric form, the magnitude of each Fourier series coefficient is

$$A_k = |c_k| = \frac{1}{\pi k}\Big(1 - \cos\big(\tfrac{2\pi}{3}k\big)\Big)$$

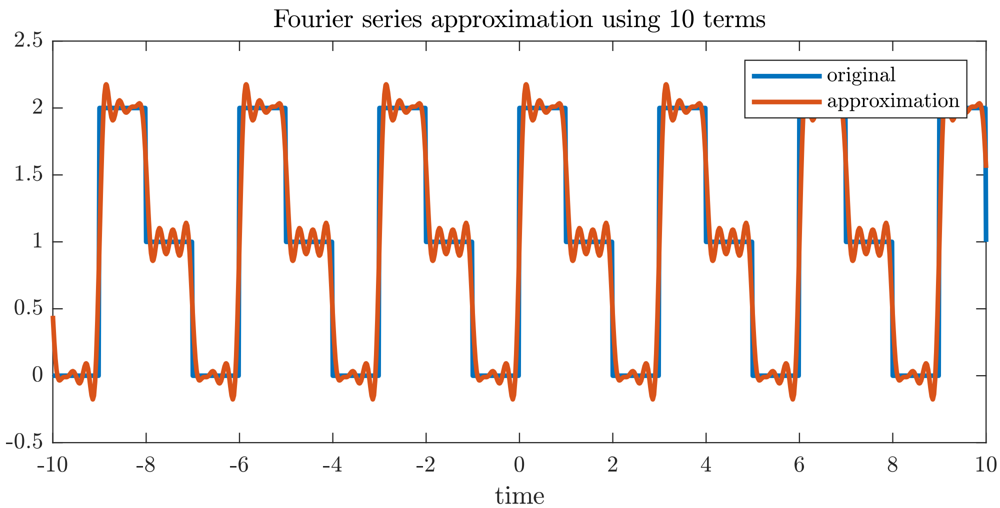
with $A_0 = 1$, and the phase of each coefficient is

$$\theta_k = \angle c_k = -\frac{\pi}{2}$$

Therefore, the trigonometric Fourier series representation of the signal is

$$x(t) = 1 + 2\sum_{k=1}^{\infty} \frac{1}{\pi k}\Big(1 - \cos\big(\tfrac{2\pi}{3}k\big)\Big)\cos\Big(\tfrac{2\pi k}{3}t - \tfrac{\pi}{2}\Big)$$

d) The plot of the original signal $x(t)$ and its Fourier series approximation $\hat{x}(t)$ is as follows:



```
%% Problem 1: Fourier series
clc; clear; close all;

% time vector
t = -10:0.0001:10;

% number of terms in Fourier series approximation
N = 10;

% fundamental period
T = 3;

% fundamental frequency
w = 2*pi/T;

% original periodic signal
x = zeros(size(t));
for m = 1:length(t)
    if mod(t(m),T) < 1
        x(m) = 2;
    elseif mod(t(m),T) < 2
        x(m) = 1;
    end
end

% Fourier series coefficients (exponential form)
c0 = 1;
c  = @(k) 1/(1i*pi*k)*(1 - cos(2*pi*k/3));

% Fourier series coefficients (trigonometric form)
A0    = 1;
A     = @(k) (1-cos(2*pi*k/3))/(pi*k);
theta = @(k) -pi/2;

% Fourier series approximation
xhat = A0*ones(size(t));
for k = 1:N
    xhat = xhat + 2*A(k)*cos(k*w*t + theta(k));
end

% plot the original signal and Fourier series approximation
plot(t,x,t,xhat);
xlabel('time');
legend('original','approximation');
title('Fourier series approximation using 10 terms');
```

Problem 12: Discrete Fourier transform

a) Write a MATLAB function that computes the $N \times N$ DFT matrix Q whose $(k,n)^{\text{th}}$ entry is given by

$$Q_{kn} = \exp(-j\,(k-1)(n-1)\,2\pi/N) \qquad k,n = 1,\ldots,N$$

b) We now verify that we can compute the forward and inverse DFT using the matrix Q. To do so, we will use a normally distributed random discrete-time signal of length $N = 10$, which can be generated using the MATLAB command `x = randn(N,1)`.

- **Analysis:** Compute the forward DFT by multiplying the vector `x` by the DFT matrix, that is, `y = Q*x`. Verify that you get the same result using the MATLAB command `fft`.
- **Synthesis:** Compute the inverse DFT by multiplying the vector `y` by the transpose of the DFT matrix and dividing by N, that is, `x = (1/N)*Q'*y`. Verify that you get the same result using the MATLAB command `ifft`.

c) Compute the DFT of the discrete-time sinusoidal signals

$$x_1(k) = \cos\big(3\tfrac{2\pi}{N}k\big) \qquad \text{and} \qquad x_2(k) = \cos\big(7\tfrac{2\pi}{N}k\big) \qquad \text{for } k = 0,\ldots,N-1$$

using $N = 10$. What do you notice? Does anything change if you increase N?

d) Use MATLAB to find the absolute values of the eigenvalues of $\frac{1}{\sqrt{N}}Q$. What do you notice?

SOLUTION:

a) We can construct the DFT matrix Q using the following function in the file `DFT.m`.

```
function Q = DFT(N)
% Constructs the Discrete Fourier Transform matrix. Given a
% discrete-time signal x of length N, the coefficients of the
% DFT are y = Q*x, and the signal can be reconstructed from
% its coefficients using x = (1/N)*Q'*y.

Q = zeros(N);
for k = 1:N
    for n = 1:N
        Q(k,n) = exp(-1i*(k-1)*(n-1)*2*pi/N);
    end
end

end
```

Alternatively, we can use an anonymous function to make the same function in the single command:

```
DFT = @(N) exp(-1i*(0:N-1)'*(0:N-1)*2*pi/N);
```

b) We can verify the analysis and synthesis equations with the MATLAB commands `fft` and `ifft` as follows.

```
N  = 10;             % number of samples
x  = randn(N,1);     % random signal
Q  = DFT(N);         % DFT matrix
y1 = Q*x;            % analysis using Q
y2 = fft(x);         % analysis using fft
x1 = (1/N)*Q'*y1;    % synthesis using Q
x2 = ifft(y2);       % synthesis using ifft
disp(norm(y1-y2));   % this should be close to zero
disp(norm(x1-x2));   % this should be close to zero
```

c) We can compute the DFT of the two sinusoidal signals as follows:

```
N  = 10;                % number of samples
k  = (0:N-1)';          % index (column vector)
x1 = cos(3*2*pi/N*k);   % x1
x2 = cos(7*2*pi/N*k);   % x2
Q  = DFT(N);            % DFT matrix

disp([Q*x1, Q*x2])      % show the DFT's side-by-side
```

Notice that the two DFT's are the same, even though the sinusoids oscillate at different frequencies! (In fact, x_1 and x_2 are identical.) This is referred to as *aliasing*, which is when high frequencies appear the same as low frequencies. If we increase the length N, then the fundamental frequency $2\pi/N$ decreases and the DFT's are no longer the same.

d) To find the absolute values of the eigenvalues of $\frac{1}{\sqrt{N}}Q$, we can use the MATLAB command `abs(eig(1/sqrt(N)*Q))`. While the eigenvalues are complex, they all have magnitude one. Such a matrix is called *orthogonal* and has the special property that its inverse is equal to its transpose, that is,

$$\left(\tfrac{1}{\sqrt{N}}Q\right)^{-1} = \left(\tfrac{1}{\sqrt{N}}Q\right)^{\mathsf{T}}$$

This is why we can compute the inverse DFT using the (scaled) transpose of Q (which is easy) instead of needing to compute its inverse (which is more difficult).

Problem 13: System properties

For each of the following continuous-time systems with input $x(t)$ and output $y(t)$, determine whether the system is static or dynamic, time invariant or time varying, linear or nonlinear, and causal or noncausal.

a) $y(t) = x(t-2) + x(2-t)$

b) $y(t) = \cos(3t)\, x(t)$

c) $y(t) = \int_{-\infty}^{2t} x(\tau)\, \mathrm{d}\tau$

d) $y(t) = \begin{cases} x(t) + x(t-2) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$

e) $y(t) = \begin{cases} x(t) + x(t-2) & \text{if } x(t) \geq 0 \\ 0 & \text{if } x(t) < 0 \end{cases}$

f) $y(t) = x(t/3)$

SOLUTION:

	a)	b)	c)	d)	e)	f)
static	✗	✓	✗	✗	✗	✗
time invariant	✗	✗	✗	✗	✓	✗
linear	✓	✓	✓	✓	✗	✓
causal	✗	✓	✗	✓	✓	✗

Problem 14: System properties

The output $y(t)$ of a continuous-time system is given by

$$y(t) = \cos(2t + 1)\, u(t - 1)$$

where $u(t)$ is the input signal. For each part, circle the property that describes the system.

a)	$\boxed{\text{linear}}$	or	nonlinear
b)	$\boxed{\text{causal}}$	or	noncausal
c)	time-invariant	or	$\boxed{\text{time-varying}}$
d)	static	or	$\boxed{\text{dynamic}}$
e)	$\boxed{\text{BIBO stable}}$	or	not BIBO stable

Problem 15: System properties

The output $y(k)$ of a discrete-time system is given by the recursion

$$y(k+1) = 0.8\,y(k) + \sin(k+2)\,x(k)$$

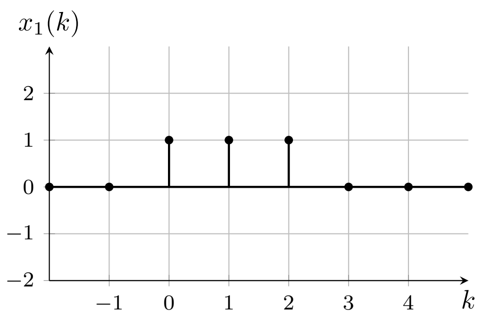
for $k \geq 0$, where $x(k)$ is the input sequence and $y(0)$ is the initial condition. For each part, circle the property that describes the system.

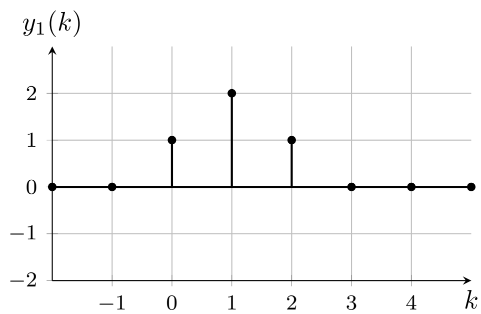
a)	$\boxed{\text{linear}}$	or	nonlinear
b)	$\boxed{\text{causal}}$	or	noncausal
c)	time-invariant	or	$\boxed{\text{time-varying}}$
d)	static	or	$\boxed{\text{dynamic}}$

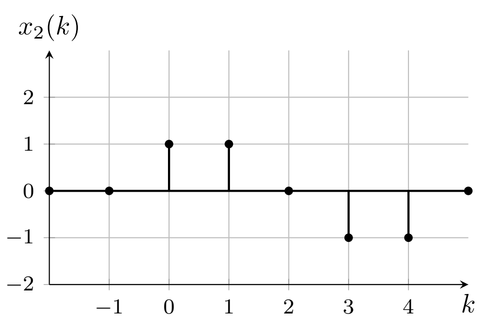
Problem 16: Discrete-time LTI systems

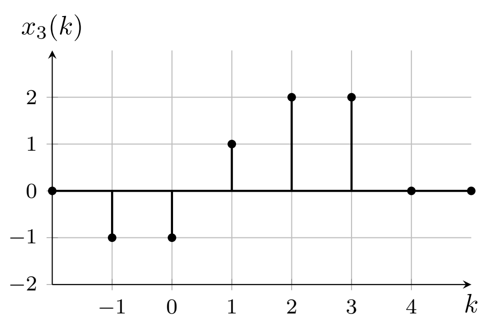
In this problem, we illustrate one of the most important consequences of the properties of linearity and time invariance. Specifically, once we know the response of a linear system or a linear time-invariant (LTI) system to a single input or the response to several inputs, we can directly compute the responses to many other input signals. Much of the remainder of this course deals with a thorough exploitation of this fact in order to develop results and techniques for analyzing and synthesizing LTI systems.

Consider an LTI system whose response to the signal $x_1(k)$ is the signal $y_1(k)$ below.









a) Carefully sketch the response of the system to the input $x_2(k)$.

b) Carefully sketch the response of the system to the input $x_3(k)$.

SOLUTION:

a) The input $x_2(k)$ can be written in terms of the input $x_1(k)$ as

$$x_2(k) = x_1(k) - x_1(k-2)$$

so the output of the system due to the input $x_2(k)$ is

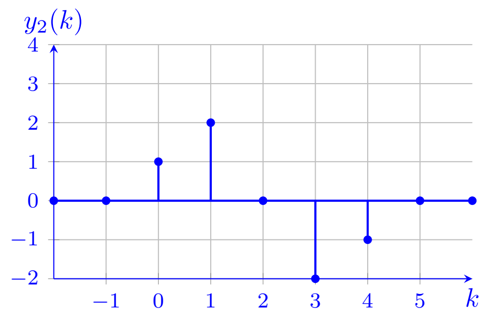
$$y_2(k) = y_1(k) - y_1(k-2)$$

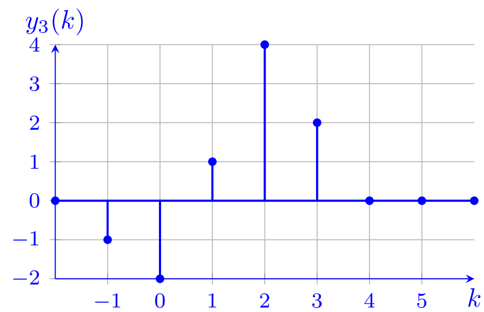
b) The input $x_3(k)$ can be written in terms of the input $x_1(k)$ as

$$x_3(k) = -x_1(k+1) + 2\,x_1(k-1)$$

so the output of the system due to the input $x_3(k)$ is

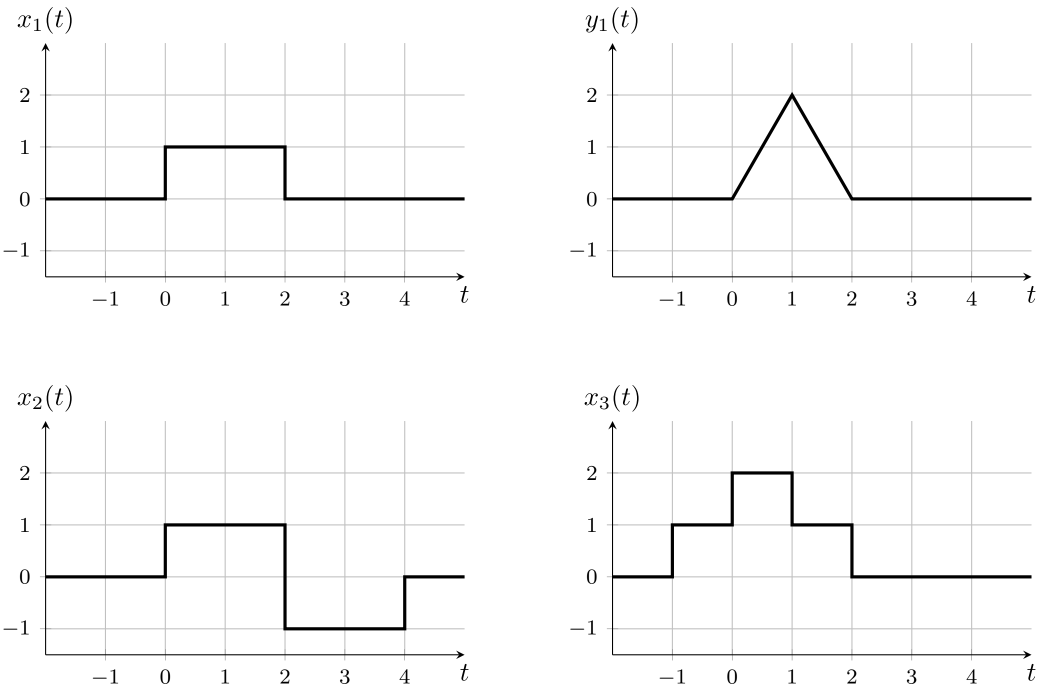
$$y_3(k) = -y_1(k+1) + 2\,y_1(k-1)$$





Problem 17: Continuous-time LTI systems

Consider an LTI system whose response to the signal $x_1(t)$ is the signal $y_1(t)$ below.



a) Determine and sketch carefully the response of the system to the input $x_2(t)$.

b) Determine and sketch carefully the response of the system to the input $x_3(t)$.

SOLUTION:

a) The input $x_2(t)$ can be written in terms of the input $x_1(t)$ as

$$x_2(t) = x_1(t) - x_1(t-2)$$

so the output of the system due to the input $x_2(t)$ is

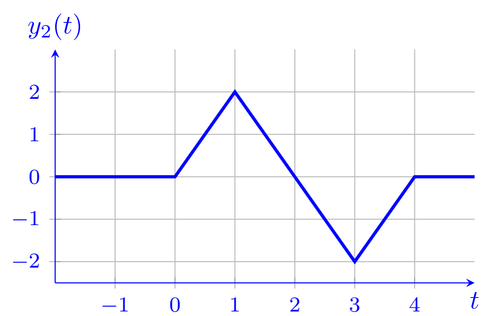
$$y_2(t) = y_1(t) - y_1(t-2)$$

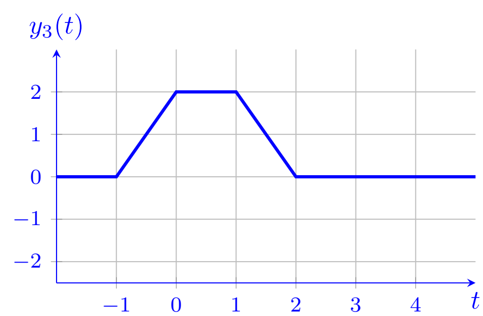
b) The input $x_3(t)$ can be written in terms of the input $x_1(t)$ as

$$x_3(t) = x_1(t) + x_1(t+1)$$

so the output of the system due to the input $x_3(t)$ is

$$y_3(t) = y_1(t) + y_1(t+1)$$





Comments:

- For LTI systems, the time-scaled input $x(at)$ does *not* in general produce the time-scaled output $y(at)$.

Problem 18: Iterating a difference equation

Iterate by hand to find the first few values of $y(k)$ for the following difference equations.

a) $y(k+1) + y(k) = x(k)$, $y(0) = 0$, $x(k) = u_s(k)$

b) $y(k+1) + y(k) = 0$, $y(0) = 1$

c) $y(k+2) - y(k+1) - 2y(k) = x(k+1) + x(k)$, $y(0) = y(1) = 0$, $x(k) = \delta(k)$

d) $y(k+2) - y(k+1) - 2y(k) = 0$, $y(0) = 1$, $y(1) = 0$

SOLUTION:

a) $y(k) = \{0, 0, 1, 0, 1, 0, \dots\}$

b) $y(k) = \{1, -1, 1, -1, 1, -1, \dots\}$

c) $y(k) = \{0, 0, 1, 1, 3, 5, 11, 21, \dots\}$

d) $y(k) = \{1, 0, 2, 2, 6, 10, 22, 42, \dots\}$

Problem 19: Drawing a simulation diagram

For each difference equation or operational relation, draw a corresponding simulation diagram having the minimum possible number of delay blocks.

a) $y(k+1) + y(k) = x(k)$

b) $y(k+2) - y(k+1) = 1.5\,x(k+1) - 0.5\,x(k)$

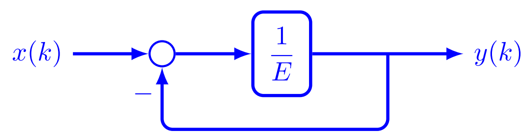
c) $(E^2 - 1)\,\{y(k)\} = (E+2)\,\{x(k)\}$

d) $y(k) = \dfrac{6E^3 + 2E}{E^3 + 5E + 4}\,\{x(k)\}$

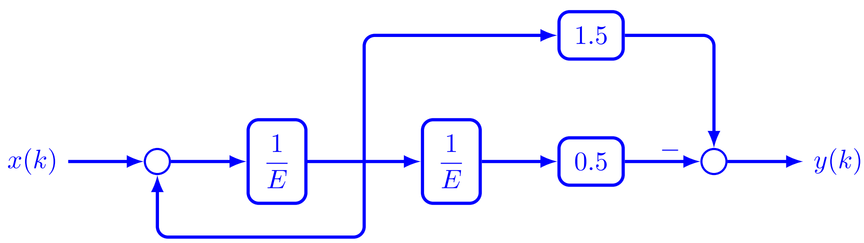
e) $y(k) = \dfrac{6E^3 + 2}{E^3}\,\{x(k)\}$

SOLUTION:

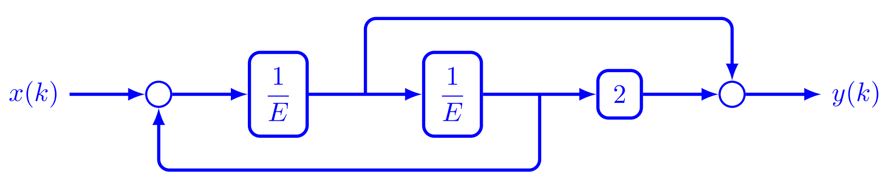
a) The operational transfer function is $H(E) = \dfrac{1}{E+1}$, and a simulation diagram is



b) The operational transfer function is $H(E) = \dfrac{1.5E - 0.5}{E^2 - E}$, and a simulation diagram is



c) The operational transfer function is $H(E) = \dfrac{E+2}{E^2 - 1}$, and a simulation diagram is



d) Since this is a third-order system, we will derive the canonical form for the system. The output can be written as

$$y(k) = (6E^3 + 2E)\{w(k)\}$$

where the intermediate signal $w(k)$ is defined as

$$w(k) = \frac{1}{E^3 + 5E + 4}\,\{x(k)\}$$

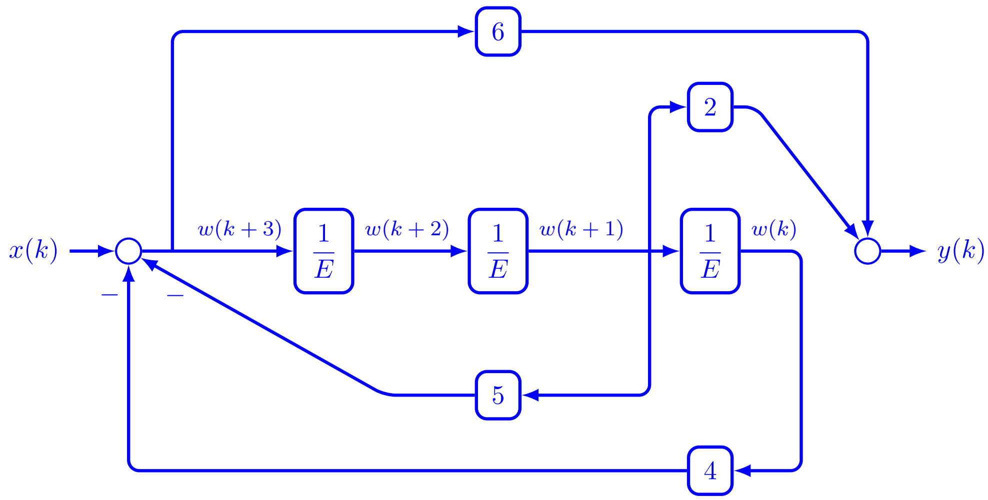
The corresponding difference equations are

$$y(k) = 6w(k+3) + 2w(k+1)$$
$$x(k) = w(k+3) + 5w(k+1) + 4w(k)$$

Solving the second equation for $w(k+3)$ gives

$$w(k+3) = x(k) - 5w(k+1) - 4w(k)$$

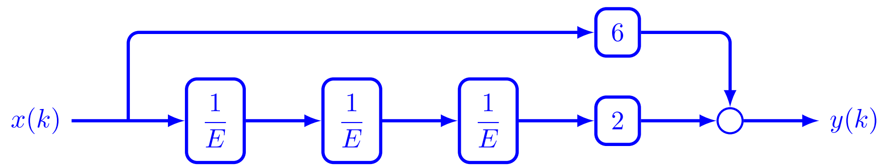
Therefore, a simulation diagram for the system is



e) The output of the system is

$$y(k) = \frac{6E^3+2}{E^3}\{x(k)\} = (6+2E^{-3})\{x(k)\} = 6x(k) + 2x(k-3)$$

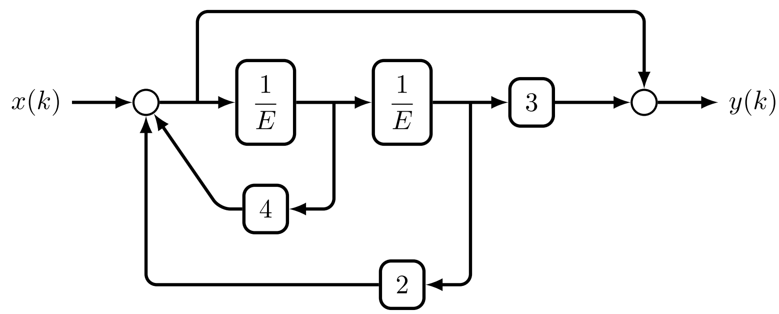
so a simulation diagram is



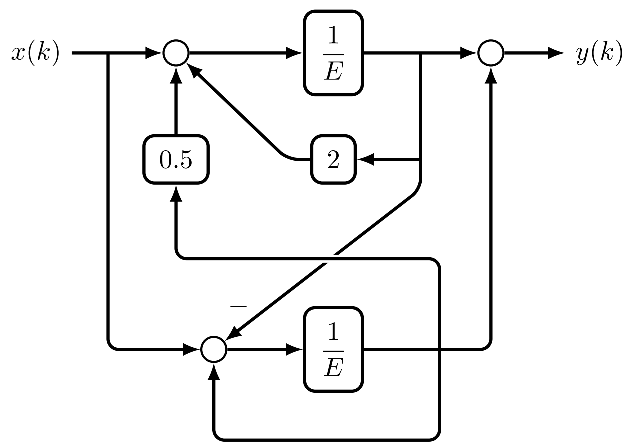
Problem 20: Computing the operational transfer function

Find the operational transfer function for each of the following simulation diagrams.

a)



b)



SOLUTION:

a) Labeling the outputs of the delay blocks as $q_1(k)$ and $q_2(k)$, the state satisfies the coupled difference equations

$$\begin{aligned} q_1(k+1) &= 4q_1(k) + 2q_2(k) + x(k) \\ q_2(k+1) &= q_1(k) \\ y(k) &= 3q_2(k) + q_1(k+1) \end{aligned}$$

Converting these to operator form and substituting the expression for $q_1(k+1)$ into $y(k)$

gives

$$\begin{aligned} E\{q_1(k)\} &= 4q_1(k) + 2q_2(k) + x(k) \\ E\{q_2(k)\} &= q_1(k) \\ y(k) &= 5q_2(k) + 4q_1(k) + x(k) \end{aligned}$$

We can now eliminate the state variables q_1 and q_2 to solve for y in terms of x. Doing so gives

$$\begin{aligned} q_1(k) &= \frac{E}{E^2 - 4E - 2}\{x(k)\} \\ q_2(k) &= \frac{1}{E^2 - 4E - 2}\{x(k)\} \end{aligned}$$

Substituting these into the expression for the output gives

$$y(k) = 5q_2(k) + 4q_1(k) + x(k) = \frac{E^2 + 3}{E^2 - 4E - 2}\{x(k)\}$$

so the operational transfer function is

$$H(E) = \frac{E^2 + 3}{E^2 - 4E - 2}$$

b) Labeling the outputs of the delay blocks as $q_1(k)$ and $q_2(k)$, the state satisfies the coupled difference equations

$$\begin{aligned} q_1(k+1) &= 2q_1(k) + 0.5q_2(k) + x(k) \\ q_2(k+1) &= -q_1(k) + q_2(k) + x(k) \\ y(k) &= q_1(k) + q_2(k) \end{aligned}$$

Converting these to operator form gives

$$\begin{aligned} E\{q_1(k)\} &= 2q_1(k) + 0.5q_2(k) + x(k) \\ E\{q_2(k)\} &= -q_1(k) + q_2(k) + x(k) \\ y(k) &= q_1(k) + q_2(k) \end{aligned}$$

We can now eliminate the state variables q_1 and q_2 to solve for y in terms of x. Doing so gives

$$\begin{aligned} q_1(k) &= \frac{E - 0.5}{E^2 - 3E + 2.5}\{x(k)\} \\ q_2(k) &= \frac{E - 3}{E^2 - 3E + 2.5}\{x(k)\} \end{aligned}$$

Substituting these into the expression for the output gives

$$y(k) = q_1(k) + q_2(k) = \frac{2E - 3.5}{E^2 - 3E + 2.5}\{x(k)\}$$

so the operational transfer function is

$$H(E) = \frac{2E - 3.5}{E^2 - 3E + 2.5}$$

Problem 21: Simulation diagram/transfer function from impulse response

Find the simulation diagram and operational transfer function for the system with the following impulse response.

a) $h(k) = \begin{cases} 1, & k = 2 \\ 0, & k \neq 2 \end{cases}$

b) $h(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

c) $h(k) = \begin{cases} 1, & k \geq 1 \\ 0, & k < 1 \end{cases}$

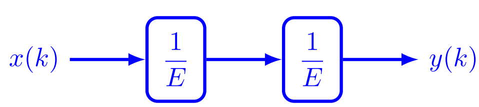
d) $h(k) = \begin{cases} 1, & k \geq 2 \\ 0, & k < 2 \end{cases}$

e) $h(k) = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \\ 3, & k = 2 \\ 0, & \text{otherwise} \end{cases}$

SOLUTION: Recall that the output of the system is the convolution of the input with the impulse response:

$$y(k) = \sum_{k=-\infty}^{\infty} h(m)\, x(k-m)$$

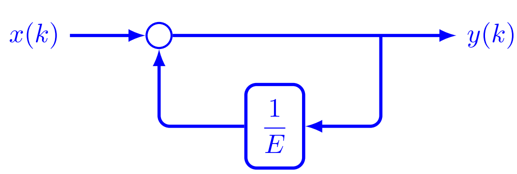
a) The convolution only has one nonzero value (when $m = 2$), so the output is $y(k) = x(k-2)$. This is simply a delay of two time steps, so the operational transfer function is $H(E) = \frac{1}{E^2}$ and the simulation diagram is



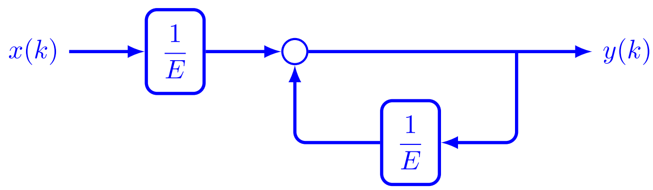
b) In this case, the output of the system is the cumulative summation of the output:

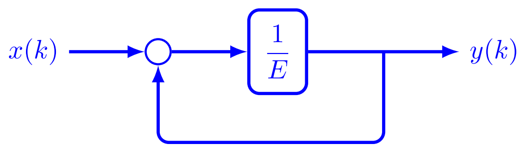
$$y(k) = \sum_{m=0}^{\infty} x(m)$$

The difference equation that produces this output is $y(k) = y(k-1) + x(k)$, and the corresponding operational transfer function is $H(E) = \frac{E}{E-1}$. The simulation diagram is

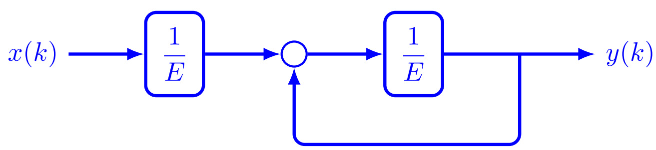


c) The impulse response is the same as that in part **(b)**, except delayed by one time step. Therefore, the operational transfer function is that in part **(b)** multiplied by $\frac{1}{E}$, that is, $H(E) = \frac{1}{E-1}$. The corresponding simulation diagram is either of the following:





d) The impulse response is delayed by one more time step, so the operational transfer function is $H(E) = \frac{1}{E(E-1)}$ and the simulation diagram is



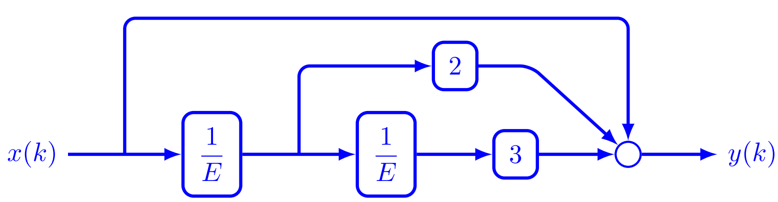
e) The impulse response has only three nonzero values (FIR), so the output is

$$y(k) = x(k) + 2\,x(k-1) + 3\,x(k-2)$$

The operational transfer function is

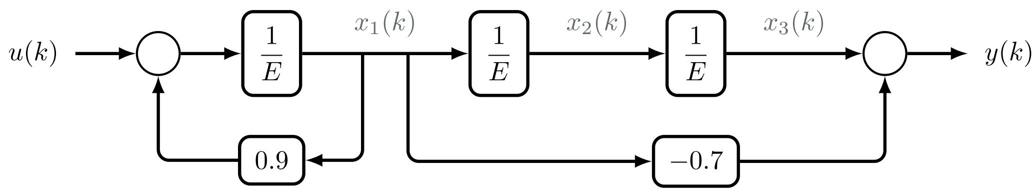
$$H(E) = 1 + \frac{2}{E} + \frac{3}{E^2} = \frac{E^2 + 2E + 3}{E^2}$$

and the simulation diagram is



Problem 22: Programming

Consider the discrete-time system described by the following simulation diagram.

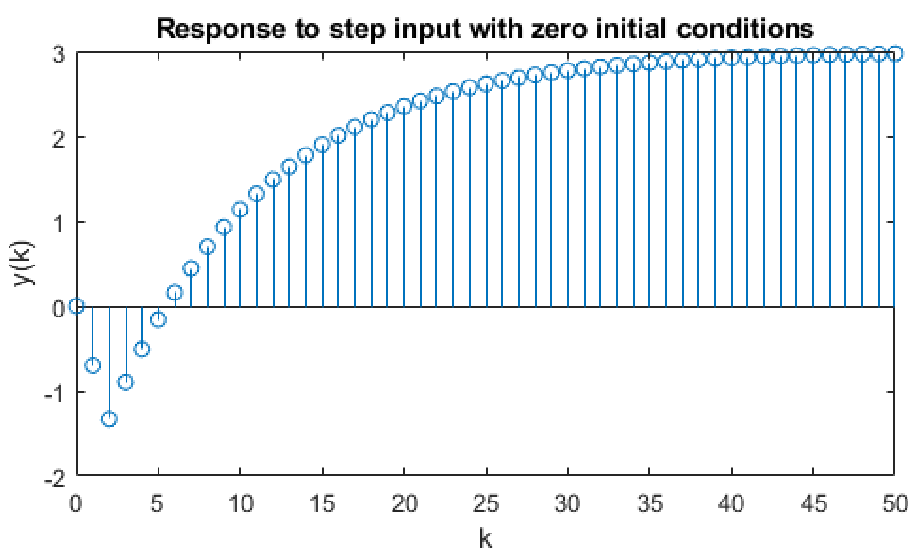


a) Write a computer program to simulate the system and plot its output. Make your program correspond to the given simulation diagram; that is, program coupled difference equations in variables that correspond to the outputs of the delay blocks.

b) Run your program to obtain the sequence $y(k)$, assuming $u(k)$ is the unit step sequence and given zero initial conditions in all delay blocks.

SOLUTION:

```
% number of iterations
N = 51;

% unit step input (shifted by one time step)
u = @(k) (k-1 >= 0);

% pre-allocate arrays
x1 = zeros(1,N+1);
x2 = zeros(1,N+1);
x3 = zeros(1,N+1);
y  = zeros(1,N);

% iterate difference equations
for k = 1:N
    x1(k+1) = 0.9*x1(k) + u(k);
    x2(k+1) = x1(k);
    x3(k+1) = x2(k);
    y(k)    = -0.7*x1(k) + x3(k);
end

% plot the output
stem(0:N-1,y);
xlabel('k');
ylabel('y(k)');
title('Response to step input with zero initial conditions');
```



Problem 23: System response

Find the complete closed-form solution of the following difference equations.

a) $(E - 0.5)\{y(k)\} = 0$, $y(0) = 7$

b) $(E - 1)\{y(k)\} = 3\,(0.5)^k$, $y(0) = 0$

c) $(E^2 + 3E + 2)\{y(k)\} = 0$, $y(0) = 1$, $y(1) = 0$

d) $(E^2 + 1)\{y(k)\} = 3\,(2)^k$, $y(0) = y(1) = 0$

e) $(E^3 - E^2 - E + 1)\{y(k)\} = 2$, $y(0) = 0$, $y(1) = 1$, $y(2) = 2$

SOLUTION:

a) The total response is just the homogeneous solution $y_h(k) = m\,(0.5)^k$ since the input is zero. Applying the initial condition, the coefficient is $m = 7$, so

$$y(k) = 7\,(0.5)^k \quad \text{for } k \geq 0.$$

b) The homogeneous solution has the form $y_h(k) = m\,(1)^k = m$, and the particular solution has the form $y_p(k) = A\,(0.5)^k$. Substituting the particular solution into the difference equation, we find that $A = -6$. Then applying the initial condition gives that $m = 6$. Therefore, the total response is

$$y(k) = 6 - 6\,(0.5)^k \quad \text{for } k \geq 0.$$

c) The total response is just the homogeneous solution $y_h(k) = m_1\,(-1)^k + m_2\,(-2)^k$ since the characteristic polynomial factors as $(E+1)(E+2)$ and the input is zero. Applying the initial conditions gives that $m_1 = 2$ and $m_2 = -1$. Therefore, the total response is

$$y(k) = 2\,(-1)^k - (-2)^k \quad \text{for } k \geq 0.$$

d) The homogeneous solution is of the form $y_h(k) = m_1 \cos(\frac{\pi}{2}k) + m_2 \sin(\frac{\pi}{2}k)$ since the roots of the characteristic polynomial are $\pm j$, which have magnitude one and angle $\pi/2 = 90°$. The particular solution is of the form $y_p(k) = A\,(2)^k$. Substituting the particular solution into the difference equation, we find that $A = 3/5$. Then applying the initial condition gives that $m_1 = -3/5$ and $m_2 = -6/5$. Therefore, the total response is

$$y(k) = -\tfrac{3}{5}\cos\left(\tfrac{\pi}{2}k\right) - \tfrac{6}{5}\sin\left(\tfrac{\pi}{2}k\right) + \tfrac{3}{5}\,(2)^k \quad \text{for } k \geq 0.$$

e) The characteristic polynomial factors as $D(E) = (E-1)^2(E+1)$, so the homogeneous solution is of the form $y_h(k) = m_1\,(-1)^k + m_2 + m_3\,k$. The particular solution is then of the form $y_p(k) = A\,k^2$. Substituting the particular solution into the difference equation, we find that $A = 1/2$. Then applying the initial condition gives that $m_1 = 1/4$, $m_2 = -1/4$, and $m_3 = 0$. Therefore, the total response is

$$y(k) = \tfrac{1}{4} - \tfrac{1}{4}\,(-1)^k + \tfrac{1}{2}\,k^2 \quad \text{for } k \geq 0.$$

Problem 24: Operational transfer function

A certain discrete-time LTI system is described by the operational transfer function

$$H(E) = \frac{1}{E^2 + E + 0.5}$$

a) Write the difference equation that describes this system, using $x(k)$ to denote the input signal and $y(k)$ to denote the output signal, where k is an integer time index.

b) Find the form of the natural response of the system (you do not need to solve for the coefficients).

c) Find the form of the forced response if the input is a constant sequence (you do not need to solve for the coefficients).

d) State whether or not the system is BIBO stable, and give a reason for your statement.

e) Draw a simulation diagram for the system using the minimal number of delay blocks.

SOLUTION:

a) The difference equation describing the system is

$$y(k+2) + y(k+1) - 0.5\,y(k) = x(k)$$

b) The roots of the characteristic polynomial are complex and given by $0.5 \pm j0.5 = \sqrt{2}e^{j\pi/4}$, so the form of the natural response is

$$y(k) = m\left(\sqrt{2}\right)^k \cos\left(k\tfrac{\pi}{4} + \phi\right) = \left(\sqrt{2}\right)^k \left(m_1 \cos\left(k\tfrac{\pi}{4}\right) + m_2 \sin\left(k\tfrac{\pi}{4}\right)\right)$$

c) Since there is no constant term in the natural response, the form of the forced response due to a constant input is

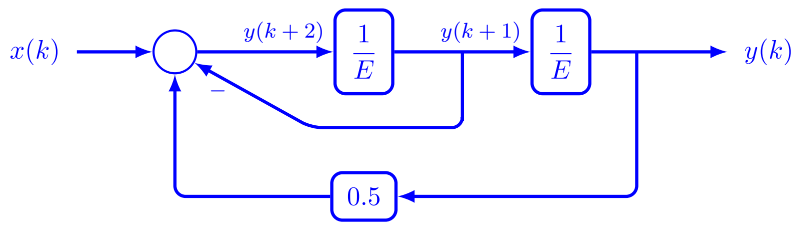
$$y(k) = A$$

d) The system *is* BIBO stable since both roots of the characteristic polynomial are strictly inside the unit circle in the complex plane.

e) From the difference equation, we have that

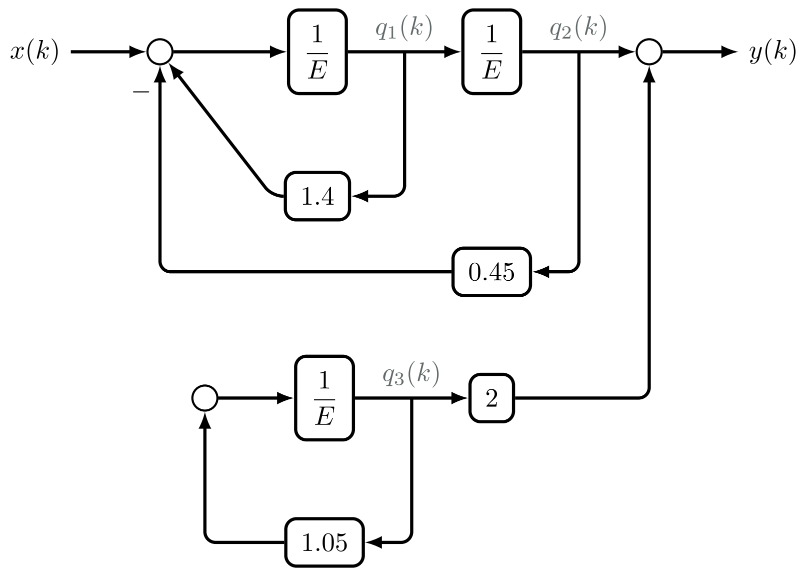
$$y(k+2) = x(k) - y(k+1) + 0.5\,y(k)$$

Therefore, a simulation diagram using the minimal number of delay blocks is



Problem 25: Simulating the ZSR and ZIR

Consider the discrete-time LTI system with input $x(k)$ and output $y(k)$ that is described by the following simulation diagram.



a) Find the operational transfer function of the system.

b) Write a computer program to simulate the system and plot the output. Make your program correspond to the given simulation diagram; that is, program coupled difference equations in variables that correspond to the outputs of the delay blocks.

c) Run the program to determine the zero-input response of $y(k)$ if the initial conditions are $q_1(0) = q_2(0) = q_3(0) = 1$.

d) Run the program to determine the zero-state response of $y(k)$ if the input $x(k)$ is the unit step sequence.

SOLUTION:

a) From the simulation diagram, the sequences satisfy the following difference equations:

$$
\begin{aligned}
q_1(k+1) &= 1.4\, q_1(k) - 0.45\, q_2(k) + x(k) \\
q_2(k+1) &= q_1(k) \\
q_3(k+1) &= 1.05\, q_3(k) \\
y(k) &= q_2(k) + 2\, q_3(k)
\end{aligned}
$$

The first two equations can be used to solve for

$$
q_2(k) = \frac{1}{E^2 - 1.4\, E + 0.45}\{x(k)\}
$$

and the third equation can be used to solve for

$$q_3(k) = \frac{0}{E - 1.05} = 0$$

The input is then related to the output by

$$y(k) = \frac{1}{E^2 - 1.4\,E + 0.45}\{x(k)\}$$

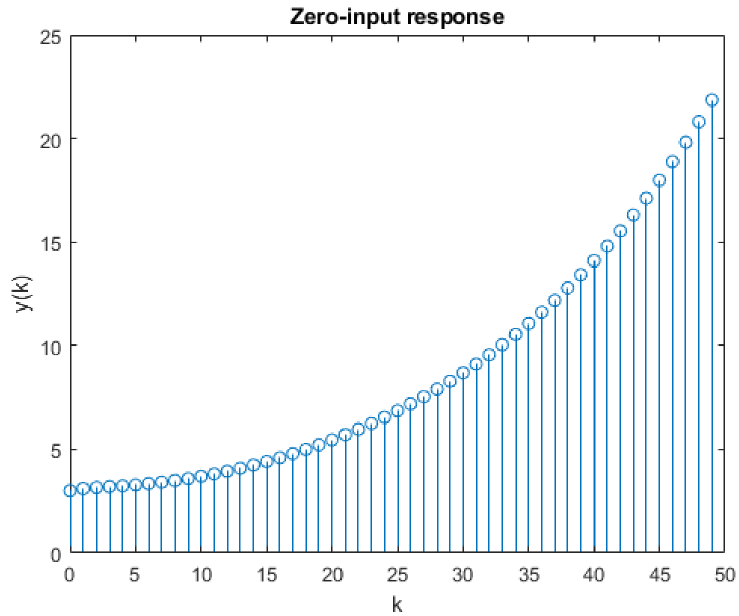
so the operational transfer function is

$$H(E) = \frac{1}{E^2 - 1.4\,E + 0.45}$$

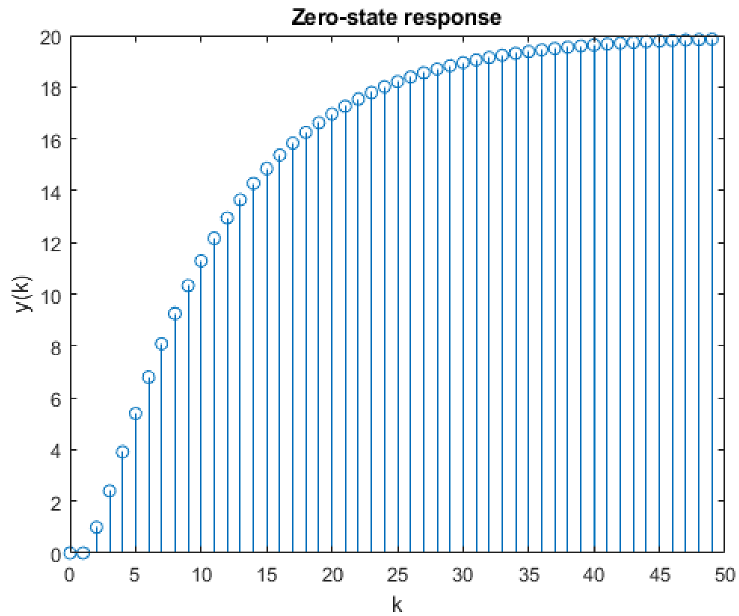
b) The system can be simulated using the following function in MATLAB.

```
function [y] = P1(x,init,description)

% number of iterations
N = length(x);

% pre-allocate arrays
q1 = zeros(1,N);
q2 = zeros(1,N);
q3 = zeros(1,N);
y  = zeros(1,N);

% initial conditions
q1(1) = init(1);
q2(1) = init(2);
q3(1) = init(3);

% iterate difference equations
for k = 1:N
    q1(k+1) = 1.4*q1(k) - 0.45*q2(k) + x(k);
    q2(k+1) = q1(k);
    q3(k+1) = 1.05*q3(k);
    y(k)    = q2(k) + 2*q3(k);
end

% plot the output
stem(0:N-1,y);
xlabel('k');
ylabel('y(k)');
title(description);

end
```

c) The command "P1(zeros(1,50),[1 1 1],'Zero-input response')" produces the following plot of the zero-input response.



d) The command "P1(ones(1,50),[0 0 0],'Zero-state response')" produces the following plot of the zero-state response.



Problem 26: Computing the impulse response

For each of the following systems, find an expression for the impulse response $h(k)$ by iterating until you see a pattern.

a) $y(k) = \dfrac{1}{E - 0.25}\{x(k)\}$ [Ans: $h(k) = (0.25)^{k-1}\, u_s(k-1)$]

b) $y(k) = \dfrac{1}{E\,(E - 0.25)}\{x(k)\}$

c) $y(k) = \dfrac{1}{E^2 + 2E + 1}\{x(k)\}$

SOLUTION:

a) Iterating the difference equation, the impulse response is $h(k) = \{0, 1, \frac{1}{4}, \frac{1}{16}, \frac{1}{32}, \ldots\}$, which has the general form

$$h(k) = (0.25)^{k-1}\, u_s(k-1)$$

b) Iterating the difference equation, the impulse response is $h(k) = \{0, 0, 1, \frac{1}{4}, \frac{1}{16}, \ldots\}$, which has the general form

$$h(k) = (0.25)^{k-2}\, u_s(k-2)$$

Note that the output of this system is simply the output of the system in part **(a)** delayed by one time step.

c) Iterating the difference equation, the impulse response is $h(k) = \{0, 0, 1, -2, 3, -4, 5, \ldots\}$, which has the general form

$$h(k) = (-1)^k\,(k-1)\, u_s(k-2)$$

Problem 27: Computing the ZSR using convolution

Use a convolutional sum to find the zero-state response of an LTI system to a unit step sequence if the system has the following impulse response.

a) $h(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

b) $h(k) = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \\ 3, & k = 2 \\ 0, & \text{otherwise} \end{cases}$

c) $h(k) = \begin{cases} (0.9)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$ [Ans: $y(k) = (10 - 9\,(0.9)^k)\,u_s(k)$]

SOLUTION: The zero-state response due to a unit step input is the cumulative summation of the impulse response:

$$y(k) = \sum_{m=-\infty}^{\infty} h(m)\,x(k-m) = \sum_{m=-\infty}^{\infty} h(m)\,u_s(k-m) = \sum_{m=-\infty}^{k} h(m)$$

a) If the impulse response is the unit step sequence, then

$$y(k) = \sum_{m=-\infty}^{k} u_s(m) = \sum_{m=0}^{k} 1 = (k+1)\,u_s(k)$$

b) If the impulse response is finite, then

$$\begin{aligned} y(k) &= h(0)x(k) + h(1)x(k-1) + h(2)x(k-2) \\ &= u_s(k) + 2\,u_s(k-1) + 3\,u_s(k-2) \end{aligned}$$

c) If the impulse response is a decaying exponential, then

$$y(k) = \sum_{m=-\infty}^{k} (0.9)^m\,u_s(m) = \sum_{m=0}^{k} (0.9)^m = \frac{1-(0.9)^{k+1}}{1-0.9}\,u_s(k) = \left(10 - 9\,(0.9)^k\right)u_s(k)$$

Problem 28: FIR approximation of an IIR system

a) Plot the impulse response and step response of the discrete-time system whose operational transfer function is

$$H_1(E) = \frac{E}{E - 0.5}$$

b) Plot the impulse response and step response of the discrete-time system whose operational transfer function is

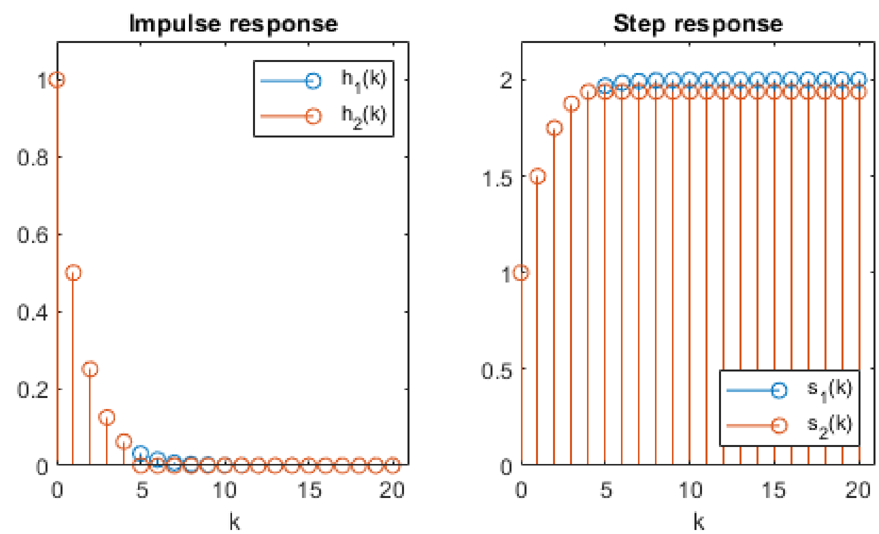
$$H_2(E) = \frac{E^4 + 0.5\,E^3 + 0.25\,E^2 + 0.125\,E + 0.0625}{E^4}$$

c) Compare the results of parts **(a)** and **(b)**, and comment on the suitability of the FIR system $H_2(E)$ as an approximation of the IIR system $H_1(E)$.

SOLUTION: The following MATLAB code plots the impulse and step response of both systems.

```
% input signals
delta = @(k) (k-1 == 0);  % unit impulse
ustep = @(k) (k-1 >= 0);  % unit step

% number of iterations
N = 21;

% pre-allocate arrays
h1 = zeros(1,N);  % impulse response in part (a)
s1 = zeros(1,N);  % step response in part (a)

% system in part (a)
h1(1) = 1;
s1(1) = 1;
for k = 1:N-1
    h1(k+1) = 0.5*h1(k) + delta(k+1);
    s1(k+1) = 0.5*s1(k) + ustep(k+1);
end

% system in part (b)
k = 1:N;
h2 = delta(k) + 0.5*delta(k-1) + 0.25*delta(k-2) ...
    + 0.125*delta(k-3) + 0.0625*delta(k-4);
s2 = ustep(k) + 0.5*ustep(k-1) + 0.25*ustep(k-2) ...
    + 0.125*ustep(k-3) + 0.0625*ustep(k-4);

figure;

subplot(1,2,1);
stem(0:N-1,h1); hold on;
stem(0:N-1,h2);
axis([0,N,0,1.1]);
xlabel('k');
legend('h_1(k)','h_2(k)');
title('Impulse response');
```

```
subplot(1,2,2);
stem(0:N-1,s1); hold on;
stem(0:N-1,s2);
axis([0,N,0,2.2]);
xlabel('k');
legend('s_1(k)','s_2(k)','Location','southeast');
title('Step response');
```



The impulse and step responses of both system match exactly for the first four values and then differ slightly. Therefore, the FIR system $H_2(E)$ provides a reasonable approximation to the IIR system $H_1(E)$. Note, however, that the final values of the step responses are different, so the systems have different dc gains.

Problem 29: Impulse response

a) Design a discrete-time LTI system that has the impulse response

$$h_1(k) = b^k\, u_s(k-1)$$

For your solution, you may give either a simulation diagram or the operational transfer function.

b) Design another discrete-time LTI system that has the impulse response

$$h_2(k) = \begin{cases} b & k = 1 \\ b^2 & k = 2 \\ 0 & \text{otherwise} \end{cases}$$

For your solution, you may give either a simulation diagram or the operational transfer function.

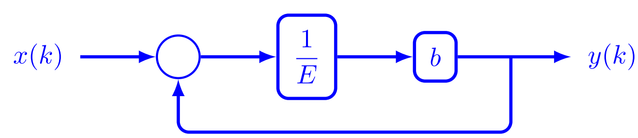
c) Note that the impulse responses for the two systems in parts **(a)** and **(b)** match for $k \leq 2$. For what range of values of the parameter b would you consider the second system to be a good approximation of the first one? Explain your reasoning.

SOLUTION:

a) A system with impulse response $h_1(k)$ is the system with operational transfer function

$$H_1(E) = \frac{b}{E-b}$$

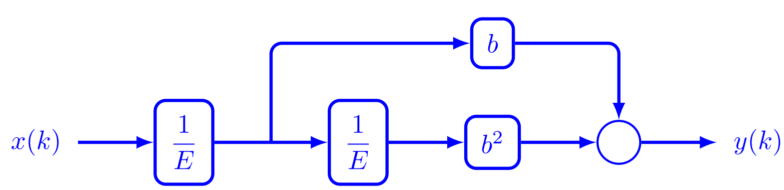
which has a simulation diagram



b) A system with impulse response $h_2(k)$ is the system with operational transfer function

$$H_1(E) = \frac{bE + b^2}{E^2}$$

which has a simulation diagram



c) The FIR system in part **(b)** is a good approximation of the IIR system in part **(a)** when $|b|$ is small, since then the impulse response of the IIR system decays to zero quickly which closely matches that of the FIR system which is zero for $k \geq 3$.

Problem 30: Radar

Consider a radar system, where an electromagnetic pulse is used to determine the distance to some target. The target reflects the pulse back to the system with a delay propertional to the distance to the target. Ideally, the received signal will simply be a shifted and possibly scaled (due to attenuation) version of the original transmitted signal.

A *matched filter* is commonly used to detect the presence of the target. If the transmitted signal is denoted by $p[n]$, then the matched filter is an LTI system with impulse response $h(k) = p(-k)$.

Suppose the radar system sends out the pulse

$$p(k) = \begin{cases} 1 & \text{if } 0 \leq k \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

and that the signal received after being reflected off the target is a delayed version of the transmitted signal,

$$x(k) = p(k-5)$$

Determine and plot the output $y(k) = (x * h)(k)$ of the matched filter. For what value of k does the output reach its maximum value?

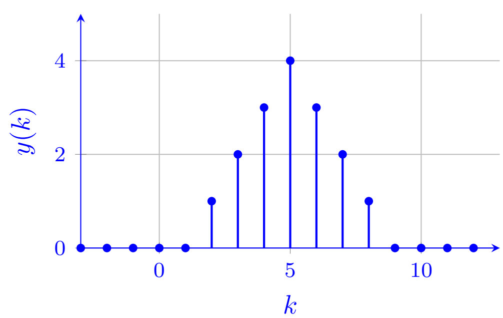
SOLUTION: The output of the matched filter is

$$y(k) = p(k-5) * p(-k)$$

Substituting the expression for the transmitted signal,

$$\begin{aligned} y(k) = \delta(k-2) + 2\,\delta(k-3) + 3\,\delta(k-4) \\ + 4\,\delta(k-5) + 3\,\delta(k-6) + 2\,\delta(k-7) + \delta(k-8) \end{aligned}$$

The maximum value is attained at $k = 5$, which is the amount of the delay between the transmitted pulse $p(k)$ and the received signal $x(k)$.



Problem 31: Stability

Determine whether or not each of the following discrete-time systems is BIBO stable.

a) $y(k) = x(k-2) \cdot x(k+2)$

b) $y(k+1) = 0.5\, y(k) + 3\, x(k+1) - 2\, x(k)$

c) $H(E) = \dfrac{E-5}{(E^2+E+1)}$

d) $h(k) = 2^k\, u_s(k)$

SOLUTION:

a) Yes, the system is BIBO stable. This is a nonlinear system, so to prove stability we must use the definition. If the input is bounded, then there exists a constant M such that $|x(k)| < M$ for all k, which implies that

$$|y(k)| = |x(k-2) \cdot x(k+2)| = |x(k-2)| \cdot |x(k+2)| < M^2$$

so the output is also bounded. Note that the bounds on the input and output are not the same, although this is not required for BIBO stability.

b) Yes, the system is BIBO stable since the characteristic polynomial is $E - 0.5$, which has a single real at 0.5 inside the unit circle.

c) No, the system is *not* BIBO stable since the characteristic polynomial has two complex roots at $-0.5 \pm j\sqrt{2} = \exp(\pm j1.9106)$ which are both on the unit circle. In particular, the bounded input

$$x(k) = \cos(1.9106\, k)$$

resonates with the system and produces an unbounded output. This system is *marginally stable*.

d) No, the system is *not* BIBO stable since the impulse response is not absolutely summable:

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} 2^k = \infty$$

Problem 32: Computing the z-Transform

Find the z-transform of each of the following discrete-time signals. Express each answer as a single rational function.

a) $x(k) = 2\,\delta(k) - \delta(k-2) = \begin{cases} 2, & k = 0 \\ -1, & k = 2 \\ 0, & \text{otherwise} \end{cases}$

b) $x(k) = \begin{cases} 0, & k < 0 \\ 1, & 0 \leq k \leq 5 \\ 2^{k-5}, & k \geq 5 \end{cases}$

c) $x(k) = \{2, 2, 0, -1, -1, -1, 2, 2, 0, -1, -1, -1, 2, 2, 0, -1, -1, -1, \ldots\}$
Hint: express $x(k)$ as the sum of shifted, identical, finite sequences

d) $x(k) = \begin{cases} 2, & k \text{ odd} \\ 0, & k \text{ even} \\ 0, & k \leq 0 \end{cases}$

e) $x(k) = a^{k-1}\,u_s(k-1) + \dfrac{1}{a}\delta(k)$

f) $x(k) = (0.5)^k \cos\!\left(k\frac{\pi}{4}\right) u_s(k)$

g) $x(k) = (0.5)^{k-2} \cos\!\left((k-2)\frac{\pi}{4}\right) u_s(k-2)$

h) $x(k) = (0.5)^k \cos\!\left(k\frac{\pi}{4}\right) u_s(k-2)$

i) $x(k) = (0.2)^k\,u_s(k) + (-2)^{k-1}\,u_s(k)$

j) $x(k) = (0.2)^k\,u_s(k) + (-2)^{k-1}\,u_s(k-1)$

SOLUTION:

a) From the table, the z-transform of an impulse is

$$\delta(k) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad 1$$

Using the delay property of the z-transform, we have

$$\delta(k-2) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{1}{z^2}$$

Since $x(k)$ is a linear combination of these two signals, we can use the linearity property of the z-transform to write $X(z)$ as the same linear combination of the corresponding

z-transforms,

$$X(z) = 2 - \frac{1}{z^2}$$

Getting a common denominator, we can express this by the single rational function

$$X(z) = \frac{2z^2 - 1}{z^2}$$

b) We first note that the signal can be expressed in terms of unit step signals as

$$x(k) = u_s(k) - u_s(k-5) + 2^{k-5}\, u_s(k-5)$$

From the table, we have the z-transform pair

$$a^k\, u_s(k) \quad \stackrel{Z}{\longleftrightarrow} \quad \frac{z}{z-a}$$

Using the delay property of the z-transform, we have

$$a^{k-5}\, u_s(k-5) \quad \stackrel{Z}{\longleftrightarrow} \quad \frac{z}{z^5\,(z-a)}$$

Since $x(k)$ is a linear combination of these signals (with appropriate values for a), we can use the linearity property of the z-transform to write $X(z)$ as the same linear combination of the corresponding z-transforms,

$$X(z) = \frac{z}{z-1} - \frac{1}{z^5}\frac{z}{z-1} + \frac{1}{z^5}\frac{z}{z-2}$$

Getting a common denominator, we can express this by the single rational function

$$X(z) = \frac{z^5 - z^4 - z^3 - z^2 - z - 1}{z^4\,(z-2)}$$

c) We first note that $x(k)$ repeats every six iterations. Therefore, define the sequence

$$f(k) = \{2, 2, 0, -1, -1, -1, 0, 0, 0, 0, 0, \ldots\}$$

that is equal to $x(k)$ for the first six iterations and then zero for $k \geq 6$. This is a finite duration signal with z-transform

$$F(z) = 2 + \frac{2}{z} - \frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} = \frac{2z^5 + 2z^4 - z^2 - z - 1}{z^5}$$

We can now write $x(k)$ in terms of this auxilliary sequence as

$$x(k) = f(k) + E^{-6} f(k) + E^{-12} f(k) + \ldots = \left(\sum_{m=0}^{\infty} E^{-6m} \right) f(k)$$

since the sequence repeats every six iterations. Then using the linearity and delay properties of the z-transform,

$$X(z) = \left(\sum_{m=0}^{\infty} z^{-6m} \right) F(z)$$

To evaluate the infinite sum, we rewrite this as

$$X(z) = \left(\sum_{m=0}^{\infty} \left(\frac{1}{z^6} \right)^m \right) F(z)$$

We can now use the formula for the geometric sum and substitute the value for $F(z)$ to obtain

$$X(z) = \frac{z^6}{z^6 - 1} \cdot \frac{2z^5 + 2z^4 - z^2 - z - 1}{z^5}$$

After simplification, the final solution is

$$X(z) = \frac{z\,(2z^5 + 2z^4 - z^2 - z - 1)}{z^6 - 1}$$

d) Similar to part **(c)**, we first note that $x(k)$ repeats every two iterations. Therefore, define the sequence

$$f(k) = \{0, 2, 0, 0, \ldots\}$$

that is equal to $x(k)$ for the first two iterations and then zero for $k \geq 2$. This is just a scaled and delayed impulse, which has z-transform $F(z) = 2/z$. We can now write $x(k)$ in terms of this auxilliary sequence as

$$x(k) = f(k) + E^{-2} f(k) + E^{-4} f(k) + \ldots = \left(\sum_{m=0}^{\infty} E^{-2m} \right) f(k)$$

since the sequence repeats every two iterations. Then using the linearity and delay properties of the z-transform,

$$X(z) = \left(\sum_{m=0}^{\infty} z^{-2m} \right) F(z)$$

To evaluate the infinite sum, we rewrite this as

$$X(z) = \left(\sum_{m=0}^{\infty} \left(\frac{1}{z^2} \right)^m \right) F(z)$$

We can now use the formula for the geometric sum and substitute the value for $F(z)$ to obtain

$$X(z) = \frac{z^2}{z^2 - 1} \cdot \frac{2}{z} = \frac{2z}{z^2 - 1}$$

e) Using the delay property of the z-transform, we have the pair

$$a^{k-1}\, u_s(k-1) \quad \overset{Z}{\longleftrightarrow} \quad \frac{1}{z - a}$$

Using this along with the table entry for the z-transform of an impulse, we have that

$$X(z) = \frac{1}{z - a} + \frac{1}{a}(1) = \frac{z}{a\,(z - a)}$$

f) Using the table entry

$$a^k \cos(k\theta)\, u_s(k) \quad \overset{z}{\longleftrightarrow} \quad \frac{z\,(z - a\,\cos\theta)}{z^2 - 2a\cos\theta\, z + a^2}$$

with $a = 0.5$ and $\theta = \pi/4$, we have that

$$X(z) = \frac{z\left(z - 0.5\cos(\frac{\pi}{4})\right)}{z^2 - 2\,(0.5)\cos(\frac{\pi}{4})z + (0.5)^2}$$

Since $\cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$, this simplifies to

$$X(z) = \frac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}}$$

g) This signal $x(k)$ is the same as that in part **(f)** except shifted to the right by two, so from the delay property of the z-transform, $X(z)$ is the same except divided by z^2,

$$X(z) = \frac{z - \frac{\sqrt{2}}{4}}{z\left(z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}\right)}$$

h) This signal is similar to that in part **(f)** except that the first two terms are zero, so it can be written as

$$x(k) = \underbrace{(0.5)^k \cos(k\tfrac{\pi}{4})\, u_s(k)}_{\text{signal in part (f)}} - \underbrace{(0.5)^0\,\cos(0)}_{\text{term at time } k=0}\,\delta(k) - \underbrace{(0.5)^1\,\cos(\tfrac{\pi}{4})}_{\text{term at time } k=1}\,\delta(k-1)$$

Therefore, we can use the linearity property of the z-transform to obtain

$$X(z) = \frac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}} - 1 - \frac{\sqrt{2}}{4}\cdot\frac{1}{z}$$

Getting a common denominator and simplifying, we obtain

$$X(z) = \frac{-\sqrt{2}}{16z\left(z^2 - \frac{\sqrt{2}}{2}z + \frac{1}{4}\right)}$$

i) To use the entries in the table, we first write the signal as

$$x(k) = (0.2)^k\, u_s(k) + (-2)^{-1}(-2)^k\, u_s(k)$$

Then using the table entry

$$a^k\, u_s(k) \quad \overset{z}{\longleftrightarrow} \quad \frac{z}{z-a}$$

twice with $a = 0.2$ and $a = -2$ along with the linearity property of the z-transform, we have that

$$X(z) = \frac{z}{z - 0.2} - \frac{1}{2}\cdot\frac{z}{z-(-2)}$$

Getting a common denominator and simplifying gives

$$X(z) = \frac{z\left(0.5z + 2.1\right)}{z^2 + 1.8z - 0.4}$$

j) In this case, we can directly use the delay and linearity properties of the z-transform to obtain

$$X(z) = \frac{z}{z-0.2} + \frac{1}{z} \cdot \frac{z}{z+2} = \frac{z^2+3z-0.2}{z^2+1.8z-0.4}$$

Problem 33: Inverse z-Transform

Find the inverse z-transform of each of the following signals.

a) $X(z) = 2 + \dfrac{3}{z} + \dfrac{6}{z^3} + \dfrac{4}{z^7}$

b) $X(z) = \dfrac{z+1}{z^6\,(z-1)}$

c) $X(z) = \dfrac{z+1}{(z-2)(z-1)}$

d) $X(z) = \dfrac{z^2+2z+1}{(z+0.5)^3\,(z-1)}$

e) $X(z) = \dfrac{z+1}{z^2-2z+2}$

SOLUTION:

a) Using the z-transform pair

$$\delta(k-m) \quad \overset{Z}{\longleftrightarrow} \quad z^{-m}$$

in the table along with the linearity property, we have that

$$x(k) = 2\,\delta(k) + 3\,\delta(k-1) + 6\,\delta(k-3) + 4\,\delta(k-7)$$

b) We could perform a partial fraction expansion on $X(z)$ directly, although it would contain a lot of terms due to the factor of z^6 in the denominator. Instead, we can write $X(z)$ as

$$X(z) = \frac{z+1}{z^6\,(z-1)} = \frac{1}{z^6}\cdot\frac{z}{z-1} + \frac{1}{z^7}\cdot\frac{z}{z-1}$$

Then using the table entry

$$u_s(k) \quad \overset{Z}{\longleftrightarrow} \quad \frac{z}{z-1}$$

along with the delay property, the signal is

$$x(k) = u_s(k-6) + u_s(k-7)$$

c) The partial fraction expansion is

$$X(z) = \frac{z+1}{(z-2)(z-1)} = \frac{Az}{z-2} + \frac{Bz}{z-1} + C$$

- To solve for A, multiply both sides by $(z-2)$ and set $z=2$.

$$(z-2)\,X(z)\big|_{z=2} = 2A = 3$$

- To solve for B, multiply both sides by $(z-1)$ and set $z=1$.

$$(z-1)\,X(z)\big|_{z=1} = B = -2$$

- To solve for C, set $z=0$.

$$X(0) = C = \frac{1}{2}$$

Therefore, the signal is

$$x(k) = \frac{3}{2}\cdot 2^k\,u_s(k) - 2\,u_s(k) + \frac{1}{2}\,\delta(k)$$

d) Since the factor of $(z+0.5)$ is repeated three times, the partial fraction expansion is

$$X(z) = \frac{z^2+2z+1}{(z+0.5)^3\,(z-1)} = \frac{Az}{z+0.5} + \frac{Bz}{(z+0.5)^2} + \frac{Cz}{(z+0.5)^3} + \frac{Dz}{z-1} + E$$

- To solve for C, multiply both sides by $(z+0.5)^3$ and set $z=-0.5$.

$$(z+0.5^3)\,X(z)\big|_{z=-0.5} = -\frac{1}{2}\,C = -\frac{1}{6} \qquad \Longrightarrow \qquad C = \frac{1}{3}$$

- To solve for D, multiply both sides by $(z-1)$ and set $z=1$.

$$(z-1)\,X(z)\big|_{z=1} = D = \frac{32}{27}$$

- To solve for E, set $z=0$.

$$X(0) = E = -8$$

- To find the remaining coefficients, we can substitute in several values of z to get other equations that must hold:

$$0 = 2A - 4B + 8C + \frac{1}{2}D + E \qquad (z=-1)$$
$$-\frac{9}{2} = \frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C - D + E \qquad (z=0.5)$$

  Substituting in the values of the known parameters, we obtain the linear system of equations

$$\begin{bmatrix} \frac{128}{27} \\ \frac{122}{27} \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{184}{27} \\ \frac{20}{9} \end{bmatrix}$$

Therefore, the signal is

$$x(k) = \left[\frac{184}{27}\,(-0.5)^k + \frac{20}{9}\,k\,(-0.5)^{k-1} + \frac{1}{3}\cdot\frac{k\,(k-1)}{2}\,(-0.5)^{k-2} + \frac{32}{27}\right] u_s(k) - 8\,\delta(k)$$

e) The denominator of $X(z)$ is z^2-2z+2, which has complex roots at

$$z = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = 1 \pm j$$

The roots have magnitude $a = \sqrt{2}$ and angle $\theta = \pi/4$, so the relevant entries from the table are

$$\left(\sqrt{2}\right)^k \sin\left(k\tfrac{\pi}{4}\right) u_s(k) \quad \xleftrightarrow{\mathcal{Z}} \quad \frac{z}{z^2 - 2z + 2}$$
$$\left(\sqrt{2}\right)^k \cos\left(k\tfrac{\pi}{4}\right) u_s(k) \quad \xleftrightarrow{\mathcal{Z}} \quad \frac{z\,(z-1)}{z^2 - 2z + 2}$$

where we used that $\cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$. Then writing $X(z)$ as

$$X(z) = \frac{z+1}{z^2-2z+2} = \frac{z}{z^2-2z+2} + \frac{1}{z} \cdot \frac{z}{z^2-2z+2}$$

and using the delay property, the inverse z-transform is

$$x(k) = \left(\sqrt{2}\right)^k \sin\left(k\tfrac{\pi}{4}\right) u_s(k) + \left(\sqrt{2}\right)^{k-1} \sin\left((k-1)\tfrac{\pi}{4}\right) u_s(k-1)$$

Problem 34: Final value theorem

For each of the following z-transforms $X(z)$, use the final value theorem (if applicable) to compute the limit of the signal $x(k)$ as $k \to \infty$.

a) $X(z) = 2 + \frac{3}{z} + \frac{6}{z^3} + \frac{4}{z^7}$

b) $X(z) = \frac{z+1}{z^6\,(z-1)}$

c) $X(z) = \frac{z+1}{(z-2)(z-1)}$

d) $X(z) = \frac{z^2+2z+1}{(z+0.5)^3\,(z-1)}$

e) $X(z) = \frac{z+1}{z^2-2z+2}$

SOLUTION:

a) $X(z) = \frac{2z^7+3z^6+6z^4+4}{z^7}$ has seven poles at $z=0$ which are all inside the unit circle in the complex plane, so the final value theorem trivially gives that the limit is zero.

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} (z-1)\,X(z) = 0$$

b) $X(z)$ has six poles at $z=0$ and a pole at $z=1$, so the final value theorem applies and gives

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} (z-1)\,X(z) = \lim_{z\to 1} \frac{z+1}{z^6} = 2$$

c) $X(z)$ has poles at $z=1$ and $z=2$. Since one of the poles is outside the unit circle, the final value theorem does not apply and the limit does not exist.

d) $X(z)$ has three poles at $z=-0.5$ and a pole at $z=1$, so the final value theorem applies and gives

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1} (z-1)\,X(z) = \lim_{z\to 1} \frac{z^2+2z+1}{(z+0.5)^3} = \frac{32}{27} \approx 1.1852$$

e) $X(z)$ has poles at $z = 1 \pm j$ which are both outside the unit circle, so the final value theorem does not apply and the limit does not exist (the signal oscillates while growing in magnitude).

Problem 35: Step response

Use the z-transform to compute the step response of the following systems.

a) $h(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$

b) $h(k) = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \\ 3, & k = 2 \\ 0, & \text{otherwise} \end{cases}$

c) $h(k) = \begin{cases} (0.9)^k, & k \geq 0 \\ 0, & k < 0 \end{cases}$

SOLUTION: In the z-domain, the zero-state response of the system is $Y(z) = H(z)\,X(z)$, where $X(z) = \frac{z}{z-1}$ is the z-transform of a unit step signal. To find the step response, we first compute the z-transform of the impulse response, multiply by $X(z)$, and then take the inverse z-transform to obtain $y(k)$.

a) The impulse response is a unit step which has z-transform $H(z) = \frac{z}{z-1}$. The step response is then

$$Y(z) = H(z)\,X(z) = \left(\frac{z}{z-1}\right)\left(\frac{z}{z-1}\right) = z \cdot \frac{z}{(z-1)^2}$$

which has inverse z-transform

$$y(k) = (k+1)\,u_s(k+1)$$

Note that the signal is zero when $k = -1$, so this can also be written as

$$y(k) = (k+1)\,u_s(k)$$

b) The z-transform of the impulse response is

$$H(z) = 1 + \frac{2}{z} + \frac{3}{z^2} = \frac{z^2 + 2z + 3}{z^2}$$

The output is then

$$Y(z) = H(z)\,X(z) = \left(\frac{z^2+2z+3}{z^2}\right)\left(\frac{z}{z-1}\right) = \frac{z^2+2z+3}{z\,(z-1)}$$

Rewriting the expression as

$$Y(z) = \frac{z}{z-1} + \frac{2}{z} \cdot \frac{z}{z-1} + \frac{3}{z^2} \cdot \frac{z}{z-1}$$

and using the delay and linearity properties, the step response is

$$y(k) = u_s(k) + 2\,u_s(k-1) + 3\,u_s(k-2)$$

c) The z-transform of the impulse response is $H(z) = \frac{z}{z-0.9}$, so the step response is

$$Y(z) = H(z)\,X(z) = \left(\frac{z}{z-0.9}\right)\left(\frac{z}{z-1}\right)$$

Performing a partial fraction expansion,

$$Y(z) = \frac{z^2}{(z-0.9)(z-1)} = \frac{Az}{z-0.9} + \frac{Bz}{z-1}$$

- To solve for A, multiply by $z - 0.9$ and set $z = 0.9$.

$$(z-0.9)\,Y(z)\big|_{z=0.9} = 0.9A = \frac{0.9^2}{0.9-1} \qquad \Longrightarrow \qquad A = -9$$

- To solve for B, multiply by $z - 1$ and set $z = 1$.

$$(z-1)\,Y(z)\big|_{z=1} = B = \frac{1^2}{0.1} \qquad \Longrightarrow \qquad B = 10$$

Therefore, the step response is

$$y(k) = \left[10 - 9\,(0.9)^k\right] u_s(k)$$

Problem 36: Zero-state response

Find the zero-state response $y(k)$ of a discrete-time LTI system with impulse response $h(k)$ to the input $x(k)$.

a) $h(k) = (0.5)^k\, u_s(k) + \delta(k)$ and $x(k) = u_s(k)$

b) $H(z) = \dfrac{z^2 + 2z + 3}{z^2}$ and $X(z) = \dfrac{z}{z-1}$

c) $h(k) = u_s(k)$ and $x(k) = (0.5)^k\, u_s(k) + \delta(k)$

d) $h(k) = (0.5)^k \cos\!\left(k\frac{\pi}{4}\right) u_s(k)$ and $x(k) = u_s(k)$

e) $h(k) = 2^k\, u_s(k-1)$ and $x(k) = k\, u_s(k)$

f) $H(z) = \dfrac{z}{(z-0.25)^2}$ and $X(z) = \dfrac{z}{z-0.25}$

SOLUTION: The zero-state response is given by $Y(z) = H(z)\, X(z)$. Therefore, we need to compute the transfer function $H(z)$, the z-transform of the input signal $X(z)$, multiply them together to get $Y(z)$, and then take the inverse z-transform to get $y(k)$.

a) The z-transform of the impulse response is

$$H(z) = \frac{z}{z-0.5} + 1 = \frac{2z - 0.5}{z - 0.5}$$

and the z-transform of the input is $X(z) = \frac{z}{z-1}$. Then

$$Y(z) = H(z)\, X(z) = \left(\frac{2z-0.5}{z-0.5}\right)\left(\frac{z}{z-1}\right)$$

Performing a partial fraction expansion,

$$Y(z) = \frac{z\,(2z-0.5)}{(z-0.5)(z-1)} = \frac{Az}{z-0.5} + \frac{Bz}{z-1}$$

where the coefficients are $A = -1$ and $B = 3$, so the zero-state response is

$$y(k) = \left[3 - (0.5)^k\right] u_s(k)$$

b) The z-transform of the zero-state response is

$$Y(z) = H(z)\, X(z) = \left(\frac{z^2+2z+3}{z^2}\right)\left(\frac{z}{z-1}\right) = \frac{z}{z-1} + \frac{2}{z}\cdot\frac{z}{z-1} + \frac{3}{z^2}\cdot\frac{z}{z-1}$$

so the zero-state response is

$$y(k) = u_s(k) + 2\, u_s(k-1) + 3\, u_s(k-2)$$

c) This is the same as in part **(a)** except that the input signal and impulse response are switched. This does not change the zero-state response, so the response is the same as in part **(a)**,

$$y(k) = \left[3 - (0.5)^k\right] u_s(k)$$

d) The z-transform of the impulse response is

$$H(z) = \frac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2} z + \frac{1}{4}}$$

and the z-transform of the input is $X(z) = \frac{z}{z-1}$. For the complex roots, the relevant z-transform pairs (with $a = 1/2$ and $\theta = \pi/4$) are

$$\left(\tfrac{1}{2}\right)^k \sin\left(k\tfrac{\pi}{4}\right) u_s(k) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{\frac{\sqrt{2}}{4} z}{z^2 - \frac{\sqrt{2}}{2} z + \frac{1}{4}}$$

$$\left(\tfrac{1}{2}\right)^k \cos\left(k\tfrac{\pi}{4}\right) u_s(k) \quad \overset{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2} z + \frac{1}{4}}$$

A partial fraction expansion using these transforms is

$$Y(z) = \frac{z^2\left(z - \frac{\sqrt{2}}{4}\right)}{(z-1)\left(z^2 - \frac{\sqrt{2}}{2} z + \frac{1}{4}\right)} = A\frac{z}{z-1} + B\frac{\frac{\sqrt{2}}{4} z}{z^2 - \frac{\sqrt{2}}{2} z + \frac{1}{4}} + C\frac{z\left(z - \frac{\sqrt{2}}{4}\right)}{z^2 - \frac{\sqrt{2}}{2} z + \frac{1}{4}}$$

where the coefficients are

$$A = \frac{3\sqrt{2} + 16}{17} \approx 1.1907 \qquad B = \frac{5\sqrt{2} + 4}{17} \approx 0.6512 \qquad C = \frac{1 - 3\sqrt{2}}{17} \approx -0.1907$$

so the zero-state response is

$$y(k) = \left[1.1907 - 0.1907\,(0.5)^k \cos\left(k\tfrac{\pi}{4}\right) + 0.6512\,(0.5)^k \sin\left(k\tfrac{\pi}{4}\right)\right] u_s(k)$$

e) The z-transform of the impulse response is $H(z) = \frac{2}{z-2}$ and the z-transform of the input signal is $X(z) = \frac{z}{(z-1)^2}$, so the output is

$$Y(z) = H(z)\,X(z) = \frac{2z}{(z-2)(z-1)^2} = \frac{Az}{z-2} + \frac{Bz}{z-1} + \frac{Cz}{(z-1)^2}$$

- To find A, multiply by $z - 2$ and set $z = 2$.

$$(z-2)\,Y(z)\big|_{z=2} = 2A = \frac{2(2)}{(2-1)^2} \quad \Longrightarrow \quad A = 2$$

- To find C, multiply by $(z-1)^2$ and set $z = 1$.

$$(z-1)^2\,Y(z)\big|_{z=1} = C = \frac{2(1)}{(1-2)} \quad \Longrightarrow \quad C = -2$$

- To find B, set $z = 3$ (or some other number).

$$Y(3) = 3A + \frac{3}{2}B + \frac{3}{4}C = \frac{2(3)}{(3-2)(3-1)^2} \quad \Longrightarrow \quad B = -2$$

Therefore, the zero-state response is

$$y(k) = \left[2\,(2)^k - 2 - 2k\right] u_s(k)$$

f) The z-transform of the output is

$$Y(z) = H(z)\,X(z) = \frac{z^2}{(z-0.25)^3} = z \cdot \frac{z}{(z-0.25)^3}$$

Then using the shift property and the appropriate entry in the table,

$$y(k) = E\left\{\frac{k\,(k-1)}{2}\,(0.25)^{k-2}\,u_s(k)\right\}$$

which simplifies to

$$y(k) = \frac{(k+1)\,k}{2}\,(0.25)^{k-1}\,u_s(k+1)$$

Problem 37: Solving difference equations

Find the zero-input response $y_{\text{ZI}}(k)$, the zero-state response $y_{\text{ZS}}(k)$, and the complete response $y(k)$ for each of the following systems.

a) $(E + 0.5)\{y(k)\} = 0$, $y(0) = 0$

b) $(E^2 - 3E + 2)\{y(k)\} = u_s(k)$, $y(0) = 2$, $y(1) = 0$

c) $(E^2 - 2E + 2)\{y(k)\} = 2^{-k}\, u_s(k)$, $y(0) = y(1) = 0$

SOLUTION:

a) The system has zero input and zero initial conditions, so

$$y_{\text{ZI}}(k) = y_{\text{ZS}}(k) = y(k) = 0$$

b) Taking the z-transform of the difference equation

$$y(k+2) - 3y(k+1) + 2y(k) = u_s(k) \qquad y(0) = 2,\ y(1) = 0$$

gives

$$\big[z^2 Y(z) - z^2 y(0) - z y(1)\big] - 3\big[zY(z) - zy(0)\big] + 2Y(z) = X(z)$$

Solving for the output,

$$Y(z) = \underbrace{\frac{1}{(z-1)(z-2)} X(z)}_{\text{ZSR}} + \underbrace{\frac{z^2\, y(0) + z\,\big(y(1) - 3\,y(0)\big)}{(z-1)(z-2)}}_{\text{ZIR}}$$

Substituting the initial conditions and input $X(z) = \frac{z}{z-1}$ and performing partial fraction expansions,

$$\begin{aligned} Y_{\text{ZS}}(z) &= \frac{z}{(z-1)^2(z-2)} = \frac{z}{z-2} - \frac{z}{z-1} - \frac{z}{(z-1)^2} \\ Y_{\text{ZI}}(z) &= \frac{2z\,(z-3)}{(z-1)(z-2)} = \frac{4z}{z-1} - \frac{2z}{z-2} \end{aligned}$$

Then taking the inverse z-transforms gives

$$\begin{aligned} y_{\text{ZS}}(k) &= \big(2^k - 1 - k\big)\, u_s(k) \\ y_{\text{ZI}}(k) &= \big(4 - 2^{k+1}\big)\, u_s(k) \end{aligned}$$

so the complete output is

$$y(k) = \big(3 - k - 2^k\big)\, u_s(k)$$

c) Since the initial conditions are zero, $y_{\text{ZI}}(k) = 0$. The zero-state response is

$$Y_{\text{ZS}}(z) = \frac{1}{z^2 - 2z + 2} \cdot \frac{z}{z - 0.5}$$

which has a real pole at $z = 0.5$ and two complex poles at $z = 1 \pm j$. The corresponding table entries are

$$\left(\sqrt{2}\right)^k \sin\left(k\frac{\pi}{4}\right) u_s(k) \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z}{z^2 - 2z + 2}$$

$$\left(\sqrt{2}\right)^k \cos\left(k\frac{\pi}{4}\right) u_s(k) \quad \stackrel{\mathcal{Z}}{\longleftrightarrow} \quad \frac{z\,(z-1)}{z^2 - 2z + 2}$$

Then the partial fraction expansion has the form

$$Y_{\mathsf{ZS}}(z) = \frac{z}{(z-0.5)(z^2-2z+2)} = \frac{Az}{z-0.5} + \frac{Bz}{z^2-2z+2} + \frac{Cz\,(z-1)}{z^2-2z+2}$$

with $A = 0.8$, $B = 0.4$, and $C = -0.8$. Therefore, the zero-state and complete responses are

$$y(k) = y_{\mathsf{ZS}}(k) = \left[0.8\,(0.5)^k + 0.4\left(\sqrt{2}\right)^k \sin\left(k\frac{\pi}{4}\right) - 0.8\left(\sqrt{2}\right)^k \cos\left(k\frac{\pi}{4}\right)\right] u_s(k)$$

Problem 38: Response of LTI systems

Consider a causal LTI system described by the difference equation

$$y(k+1) = 0.8\,y(k) + u(k) - u(k-1)$$

where $u(k)$ is the input signal and $y(k)$ is the output signal.

a) Find the step response of the system.

b) Find the zero-input response of the system due to the initial condition $y(0) = 2$.

SOLUTION:

a) Taking the z-transform (with zero initial conditions) gives

$$Y(z) = \frac{0.8}{z}\,Y(z) + \frac{1}{z}\,U(z) - \frac{1}{z^2}\,U(z)$$

Rearranging, the transfer function is

$$H(z) = \frac{Y(z)}{U(z)} = \frac{z-1}{z\,(z-0.8)}$$

b) The step response is

$$S(z) = H(z)\,\frac{z}{z-1} = \frac{1}{z-0.8}$$

which, in the time domain, is

$$s(k) = (0.8)^{k-1}\,u_s(k-1)$$

c) Taking the z-transform including initial conditions with the input set to zero gives

$$z\,Y(z) - z\,y(0) = 0.8\,Y(z)$$

so the zero-input response is

$$Y(z) = \frac{z}{z-0.8}\,y(0)$$

Substituting the initial condition and taking the inverse transform,

$$y(k) = 2\,(0.8)^k\,u_s(k)$$

Problem 39: Transfer function

A certain discrete-time LTI system has the transfer function

$$H(z) = \frac{3\,(z+0.5)}{z\,(z-0.5)}$$

a) Write the difference equation that relates the input $x(k)$ to the output $y(k)$.

b) Find the impulse response of the system.

c) Find the step response of the system.

d) Find the dc gain of the system.

SOLUTION:

a) The operational transfer function is

$$H(E) = \frac{3\,(E+0.5)}{E\,(E-0.5)}$$

which satisfies $y(k) = H(E)\{x(k)\}$. Therefore, the difference equation is

$$y(k+2) - 0.5\,y(k+1) = 3\,x(k+1) + 1.5\,x(k)$$

b) The impulse response is the inverse z-transform of the transfer function. Performing a partial fraction expansion, the transfer function can be written as

$$H(z) = \frac{12z}{z-0.5} - \frac{3}{z} - 12$$

Using the table of z-transform pairs, the impulse response is then

$$h(k) = 12\,(0.5)^k\,u_s(k) - 3\,\delta(k-1) - 12\,\delta(k)$$

c) In the z-domain, the step response is

$$Y(z) = H(z)\,\frac{z}{z-1} = \frac{3\,(z+0.5)}{(z-1)(z-0.5)} = \frac{9z}{z-1} - \frac{12z}{z-0.5} + 3$$

Taking the inverse z-transform, the step response is

$$y(k) = \left(9 - 12\,(0.5)^k\right) u_s(k) + 3\,\delta(k)$$

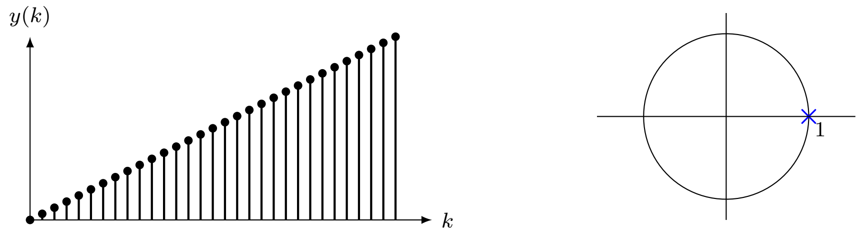
d) The dc gain of the system is the transfer function evaluated at $z = 1$, which is

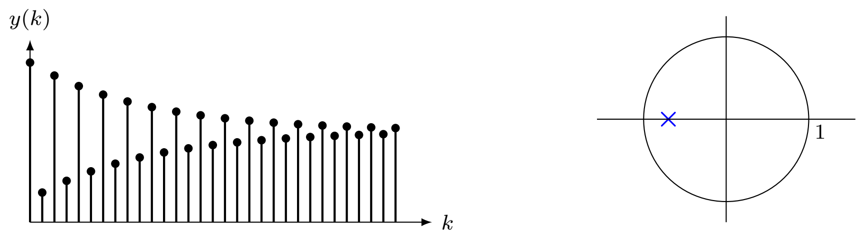
$$H(1) = \frac{3\,(1.5)}{0.5} = 9$$

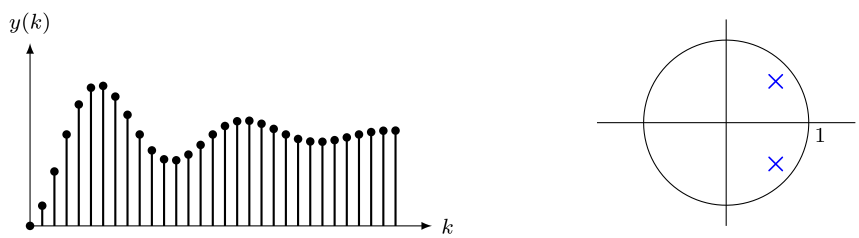
Note that this is also the limit of the step response as $k \to \infty$.

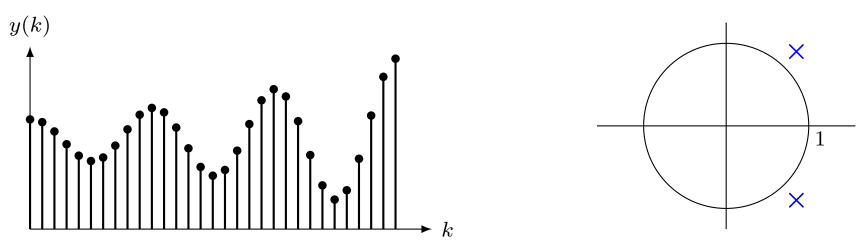
Problem 40: Step response

Four different discrete-time LTI systems have the step responses shown on the left. Mark the approximate z-plane pole location(s) of the corresponding system transfer functions $H(z)$.









Problem 41: Frequency response

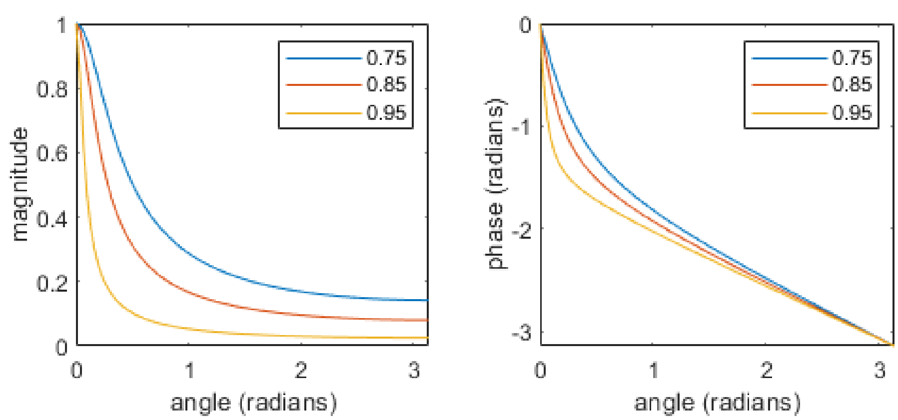
Use MATLAB to plot the frequency response of the system

$$H(z) = \frac{1-a}{z-a}$$

for the cases $a = 0.95$, $a = 0.85$, and $a = 0.75$. How does the pole location affect the magnitude and phase of the frequency response?

SOLUTION: The code and plot are shown below. As the pole moves closer to $z = 1$, both the magnitude and phase have sharper transitions near an angle of $\theta = 0$.

```
theta = linspace(0,pi,100);  % theta = omega*T
z = exp(1i*theta);           % z variable (on unit circle)

for a = [0.75 0.85 0.95]     % for each pole location

    H = (1-a) ./ (z-a);      % frequency response

    % plot the magnitude
    subplot(121); plot(theta,abs(H)); hold on;
    xlim([0,pi]);
    xlabel('angle (radians)');
    ylabel('magnitude');
    legend('0.75','0.85','0.95');

    % plot the phase
    subplot(122); plot(theta,angle(H)); hold on;
    xlim([0,pi]);
    xlabel('angle (radians)');
    ylabel('phase (radians)');
    legend('0.75','0.85','0.95');
end
```



Problem 42: FIR approximation to IIR system

The transfer function

$$G(z) = \frac{z^3 + 0.5z^2 + 0.25z + 0.125}{z^4}$$

is an FIR approximation to the transfer function

$$H(z) = \frac{1}{z - 0.5}$$

a) Find the impulse response of each system.

b) Find the step response of each system and determine the dc gain of each system from its step response.

c) Plot the frequency response of each system and determine the dc gain of each system from its frequency response.

SOLUTION:

a) The impulse response of each system is

$$g(k) = \delta(k-1) + 0.5\,\delta(k-2) + 0.25\,\delta(k-3) + 0.125\,\delta(k-4)$$
$$h(k) = (0.5)^{k-1}\,u_s(k-1)$$

b) The step response of each system is

$$y_G(k) = u_s(k-1) + 0.5\,u_s(k-2) + 0.25\,u_s(k-3) + 0.125\,u_s(k-4)$$
$$y_H(k) = 2\left[1 - (0.5)^k\right] u_s(k)$$

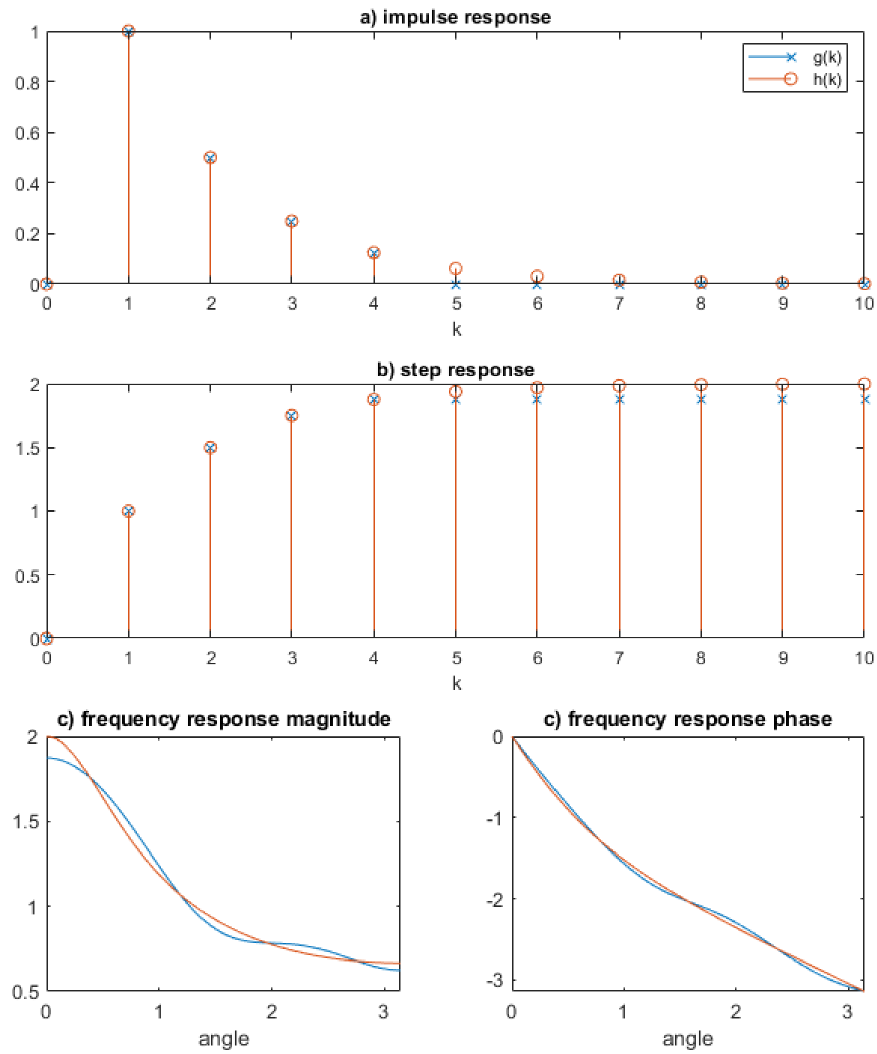
The dc gain is the final value of the step response, so

$$\text{dc gain of } G(z) = 1.875$$
$$\text{dc gain of } H(z) = 2$$

c) The dc gain is the value of the frequency response magnitude at an angle of $\theta = 0$, which gives the same values as in part **(b)**.

```
% impulse response
k = 0:10;
g = [0 1 0.5 0.25 0.125 0 0 0 0 0 0];
h = (0.5).^(k-1) .* (k >= 1);

% step response
yg = cumsum(g);
yh = cumsum(h);

% frequency response
theta = linspace(0,pi,100);
z = exp(1i*theta);
G = (z.^3 + 0.5*z.^2 + 0.25*z + 0.125) ./ z.^4;
```

```
H = 1 ./ (z-0.5);

subplot(3,2,[1,2]);
stem(k,g,'x'); hold on;
stem(k,h);
xlabel('k');
title('a) impulse response');
legend('g(k)','h(k)');

subplot(3,2,[3,4]);
stem(k,yg,'x'); hold on;
stem(k,yh);
xlabel('k');
title('b) step response');

subplot(3,2,5);
plot(theta,abs(G)); hold on;
plot(theta,abs(H));
xlabel('angle');
title('c) frequency response magnitude');

subplot(3,2,6);
plot(theta,angle(G)); hold on;
plot(theta,angle(H));
xlabel('angle');
title('c) frequency response phase');
```


a) impulse response
g(k)
h(k)
1
0.8
0.6
0.4
0.2
0
0 1 2 3 4 5 6 7 8 9 10
k
b) step response
2
1.5
1
0.5
0
0 1 2 3 4 5 6 7 8 9 10
k
c) frequency response magnitude
2
1.5
1
0.5
0 1 2 3
angle
c) frequency response phase
0
-1
-2
-3
0 1 2 3
angle

Problem 43: Frequency response

Consider the discrete-time system having the transfer function

$$H(z) = \frac{z+1}{z-0.9}$$

a) Plot the magnitude and phase of the frequency response of the system.

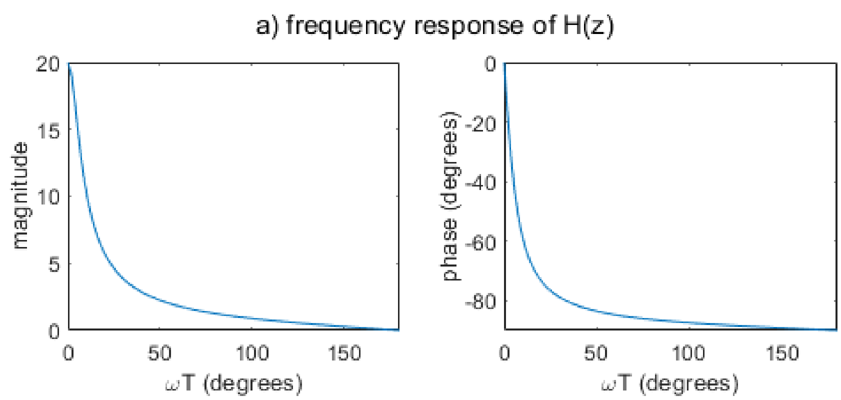
b) Program the corresponding difference equation, and iterate to determine the responses with initial condition $y(0) = 1$ to the following inputs:

$$x_1(k) = u_s(k)$$
$$x_2(k) = \cos\big(k\tfrac{\pi}{4}\big)\, u_s(k)$$
$$x_3(k) = (-1)^k\, u_s(k)$$

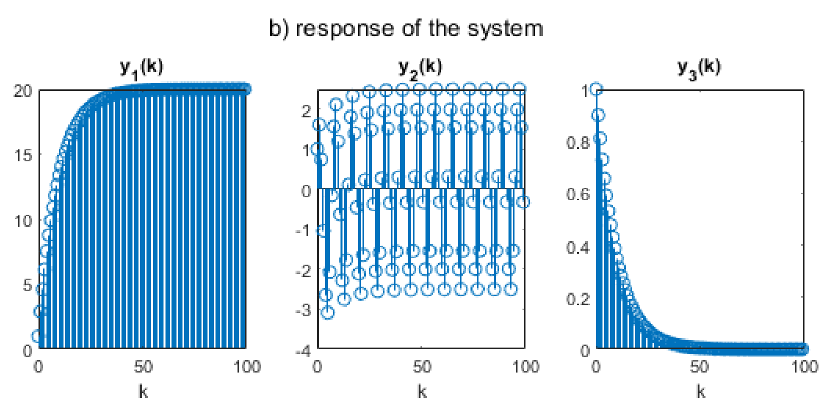
c) Explain the amplitudes of the steady-state responses from part **(b)** in terms of the frequency response of the system.

SOLUTION:

a) The following plot shows the frequency response of the system.



b) The response due to each input signal is as follows.



c) The steady-state values observed in part **(b)** can be explained in terms of the frequency response from part **(a)** as follows:

- $x_1(k)$ oscillates with frequency $\theta = 0$ at which the magnitude of the frequency response is 20, and the output $y_1(k)$ has a steady-state amplitude of 20.
- $x_2(k)$ oscillates with frequency $\theta = \frac{\pi}{4} = 90^\circ$ at which the magnitude of the frequency response is about 2.5, and the output $y_2(k)$ has a steady-state amplitude of about 2.5.
- $x_3(k)$ oscillates with frequency $\theta = \pi = 180^\circ$ at which the magnitude of the frequency response is zero, and the output $y_3(k)$ has a steady-state value of zero.

```
% a) plot the frequency response
theta_rad = linspace(0,pi,100);   % angle (radians)
theta_deg = rad2deg(theta_rad);   % angle (degrees)
z = exp(1i*theta);                % z-domain variable
H = (z+1) ./ (z-0.9);             % frequency response

sgtitle('a) frequency response of H(z)');

% plot the magnitude
subplot(121); plot(theta_deg,abs(H));
axis([0,180,0,20]);
xlabel('\omegaT (degrees)');
ylabel('magnitude');

% plot the phase
subplot(122); plot(theta_deg,angle(H)*180/pi);
axis([0,180,-90,0]);
xlabel('\omegaT (degrees)');
ylabel('phase (degrees)');

% b) iterate the difference equation
ustep = @(k) (k >= 0);  % unit step signal
x1 = @(k) ustep(k);
x2 = @(k) cos(k*pi/4)*ustep(k);
x3 = @(k) (-1)^k*ustep(k);

y0 = 1;     % initial condition
N  = 100;   % number of iterations

y1 = iterate(x1,y0,N);
y2 = iterate(x2,y0,N);
y3 = iterate(x3,y0,N);

sgtitle('b) response of the system');

subplot(131); stem(0:N-1,y1); xlabel('k'); title('y_1(k)');
subplot(132); stem(0:N-1,y2); xlabel('k'); title('y_2(k)');
subplot(133); stem(0:N-1,y3); xlabel('k'); title('y_3(k)');
```

```
function y = iterate(x,y0,N)
    y = zeros(1,N);
```

```
    y(1) = y0;
    for k = 1:N-1
        y(k+1) = 0.9*y(k) + x(k+1) + x(k);
    end
end
```

Problem 44: Filtering a noisy signal

A filter has the transfer function

$$H(z) = K\,\frac{(z^2+1)(z+1)}{(z^2+a^2)(z+a)}$$

where the gain K is chosen so that $H(1) = 1$ and a is a parameter. The sampling period of the filter is $T_s = 25 \times 10^{-6}$ seconds.

a) Plot the frequency response magnitude of the filter for $a = 0$, $a = 0.5$, and $a = 0.9$.

b) Suppose the input to the filter is given by

$$x(t) = \cos(2\pi \cdot 3000t) + \cos(2\pi \cdot 10000t)$$

with $t = kT_s$, where the first term (at 3000 Hz) represents the desired signal and the second term (at 10000 Hz) represents some unwanted noise.
Use MATLAB to plot the input and output of the filter for $a = 0.5$.

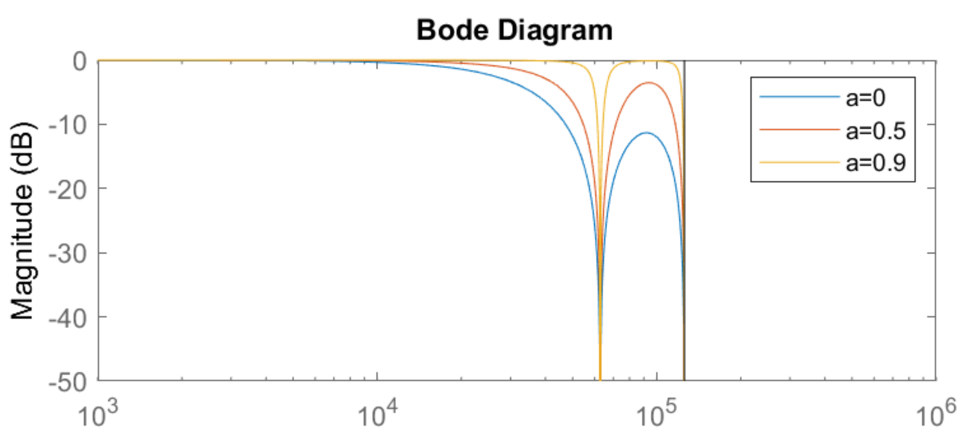
c) Suppose the noise component is changed to 9000 Hz, making the input to the filter

$$x(t) = \cos(2\pi \cdot 3000t) + \cos(2\pi \cdot 9000t)$$

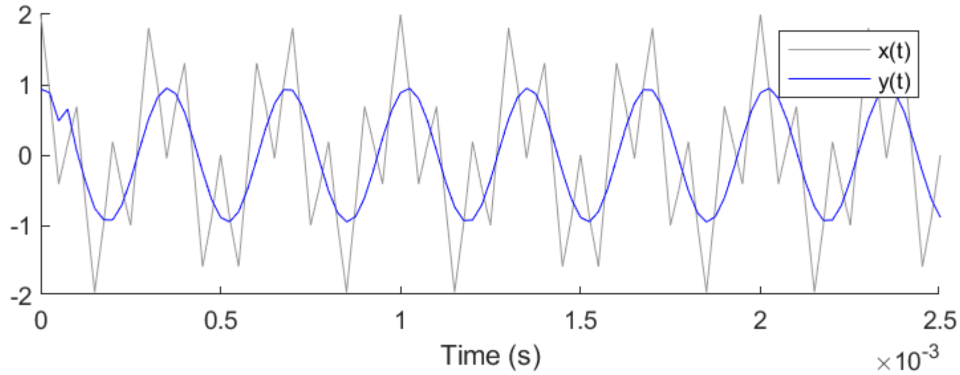
Use MATLAB to plot the input and output of the filter for $a = 0.5$.
The filter is tuned to reject any noise at 10000 Hz. Does it still work when the noise is at 9000 Hz?

SOLUTION:

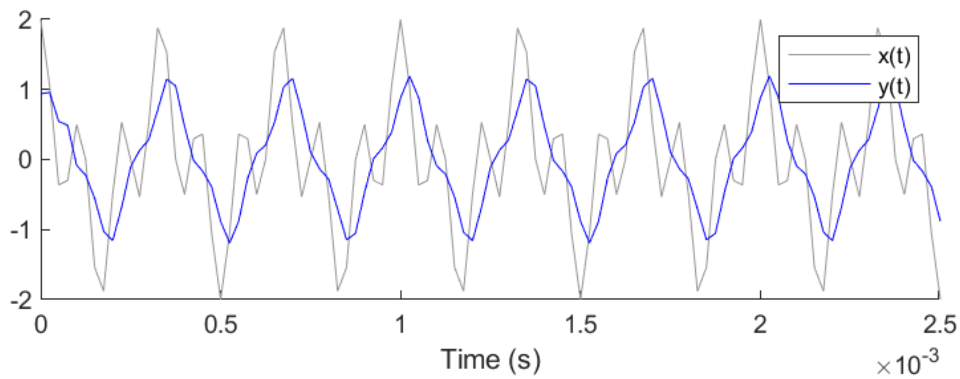
a) The plot of the frequency response magnitude of the system is as follows.



b) The response of the system is as follows. Note that the output only contains the sinusoid at 3000 Hz, while the noise has been completely removed.



c) The response of the system is as follows. The noise now contaminates the response, indicating that the filter is much less effective at eliminating noise at 9000 Hz. This is due to the sharp notch in the frequency response magnitude at 10000 Hz.



The MATLAB code used to produce the plots is as follows.

```
clc; clear; close all;

% sampling period (s)
T = 0.000025;

% z-domain variable (using the sampling period)
z = tf('z',T);

% transfer function with parameter a
K = @(a) (1+a^2)*(1+a)/4;
H = @(a) K(a)*(z^2+1)*(z+1)/((z^2+a^2)*(z+a));

% frequency response magnitude
figure; hold on;
bodemag(H(0)); hold on;
bodemag(H(0.5));
bodemag(H(0.9));
legend('a=0','a=0.5','a=0.9');
ylim([-50 0]);

k = 0:100;  % discrete-time index
t = k*T;    % continuous-time vector

% input signals
x1 = cos(2*pi*3000*t) + cos(2*pi*10000*t);
x2 = cos(2*pi*3000*t) + cos(2*pi*9000*t);

% output signals
y1 = lsim(H(0.5),x1,t);
y2 = lsim(H(0.5),x2,t);

% plot for part (b)
figure; hold on;
plot(t,x1,'color',150/255*[1 1 1]);
plot(t,y1,'b');
legend('x(t)','y(t)');
xlabel('Time (s)');

% plot for part (c)
figure; hold on;
plot(t,x2,'color',150/255*[1 1 1]);
plot(t,y2,'b');
legend('x(t)','y(t)');
xlabel('Time (s)');
```

Problem 45: Frequency response

Consider the discrete-time LTI system with z-domain transfer function

$$H(z) = \frac{z}{z^2 - 2 \cdot (0.95)\, \cos(\pi/6)\, z + (0.95)^2}$$

a) Plot the magnitude and phase of the frequency response of the system.

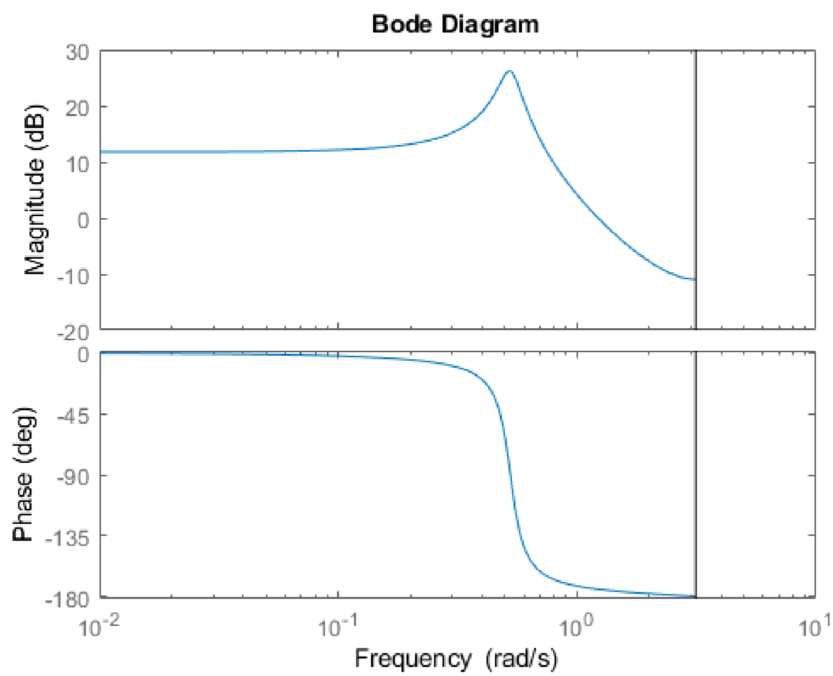
b) Program the corresponding difference equation, and iterate to determine the responses to the inputs

$$\begin{aligned} x_1(k) &= \cos(k\pi/12)\, u_s(k) \\ x_2(k) &= \cos(k\pi/6)\, u_s(k) \\ x_3(k) &= \cos(k\pi/3)\, u_s(k) \end{aligned}$$

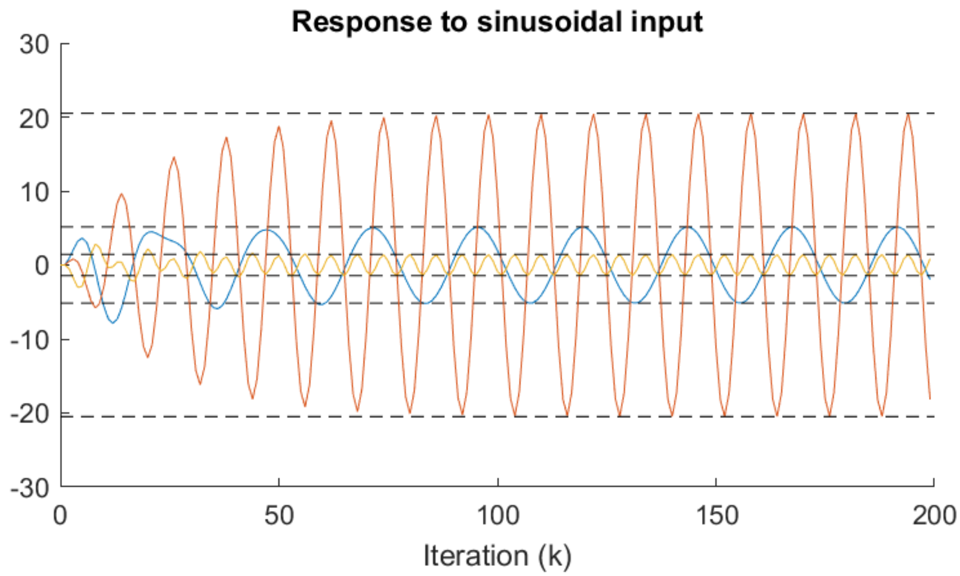
c) Explain the amplitudes of the steady-state responses from part **(b)** in terms of the frequency response from part **(a)**.

SOLUTION:

a) The plot of the frequency response of the system is as follows.



b) The response of the system to each input signal is as follows.



c) The amplitude of the steady-state response to a sinusoidal input is equal to the magnitude of the frequency response at the corresponding frequency.

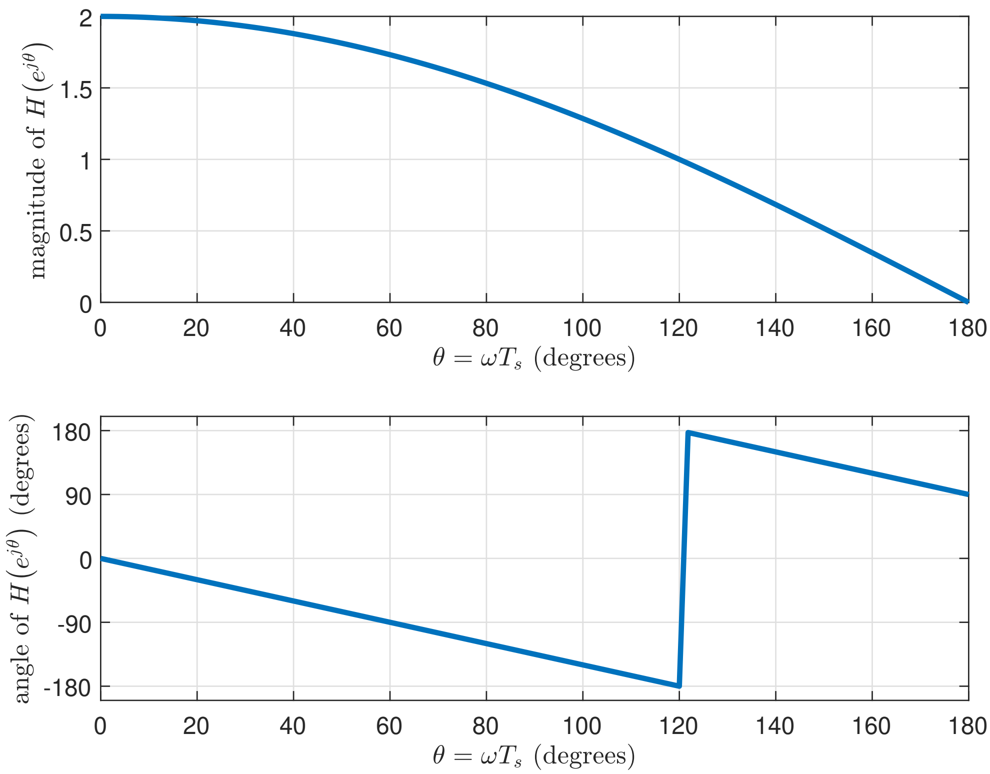
The MATLAB code used to produce the plots is as follows.

```
clc; clear; close all;

% transfer function
z = tf('z');
H = z/(z^2 - 2*0.95*cos(pi/6)*z + (0.95)^2);

% bode plot
figure; bode(H);

% number of iterations
N = 200;

% pre-allocate output array
y = zeros(1,N);

% figure for system response
figure; hold on;
title('Response to sinusoidal input');
xlabel('Iteration (k)');

% iterate over input signal frequencies
for theta = [pi/12, pi/6, pi/3]
```

```

    % initial conditions
    y(1) = 0;
    y(2) = 0;

    % iterate difference equation
    for k = 3:N
        y(k) = 2*(0.95)*cos(pi/6)*y(k-1) - (0.95)^2*y(k-2) + cos((k-1)*theta);
    end

    % frequency response magnitude
    M = abs(evalfr(H,exp(1i*theta)));

    % plot response and dashed lines showing frequency response magnitude
    plot(0:N-1,y);
    plot([0 N],[M M],'--k');
    plot([0 N],-[M M],'--k');
end
```

Problem 46: Frequency response

A certain second-order discrete-time system has the following frequency response with sampling period $T_s = 0.1$ seconds.



a) Determine the dc gain of the system.

b) Find the forced response of the system if the input is $\cos(k\pi/3)$.

c) The transfer function of the system has a single zero. What is the location of the zero?

d) For the given system, find the approximate range of frequencies (in Hz) for which the output signal amplitude is greater than that of the input signal.

SOLUTION:

a) The dc gain is the magnitude of the frequency response when $\theta = 0$, which is two.

b) The sinusoid has frequency $\theta = \pi/3 = 60^\circ$, at which the frequency response has magnitude $M = 1.75$ and phase $\phi = -90^\circ = -\pi/2$. Therefore, the forced response is

$$y(k) = 1.75 \cos\left(k\frac{\pi}{3} - \frac{\pi}{2}\right)$$

c) The magnitude of the transfer function is zero at $\theta = 180^\circ = \pi$, so the transfer function must have a zero at $z = e^{j\pi} = -1$.

d) The amplitude of the output signal is greater than that of the input signal when the magnitude of the frequency response is greater than one, which occurs for $\theta \leq 120^\circ = 2\pi/3$. Since $\theta = \omega T$ where $T = 0.1$ is the sampling frequency, we need

$$\omega \leq \frac{1}{T}\frac{2\pi}{3}$$

in units of radians per second. The angular frequency is $\omega = 2\pi f$, so we need to divide by 2π to convert to Hertz. Therefore, the magnitude of the output is greater than the magnitude of the input for frequencies less than 3.33 Hz.

Problem 47: Transfer function

A certain discrete-time LTI system has the transfer function

$$H(z) = \frac{3\,(z+0.5)}{z\,(z-0.5)}$$

a) Write the difference equation that relates the input $x(k)$ to the output $y(k)$.

b) Find the impulse response of the system.

c) Find the step response of the system.

d) Find the dc gain of the system.

e) Draw a pole-zero plot of the system.

f) Sketch the frequency response of the system. Label the values of the graphs at $\theta = 0$ and $\theta = \pi$, and label the axes.

SOLUTION:

a) The operational transfer function is

$$H(E) = \frac{3\,(E+0.5)}{E\,(E-0.5)}$$

which satisfies $y(k) = H(E)\{x(k)\}$. Therefore, the difference equation is

$$y(k+2) - 0.5\,y(k+1) = 3\,x(k+1) + 1.5\,x(k)$$

b) The impulse response is the inverse z-transform of the transfer function. Performing a partial fraction expansion, the transfer function can be written as

$$H(z) = \frac{12z}{z-0.5} - \frac{3}{z} - 12$$

Using the table of z-transform pairs, the impulse response is then

$$h(k) = 12\,(0.5)^k\,u_s(k) - 3\,\delta(k-1) - 12\,\delta(k)$$

c) In the z-domain, the step response is

$$Y(z) = H(z)\,\frac{z}{z-1} = \frac{3\,(z+0.5)}{(z-1)(z-0.5)} = \frac{9z}{z-1} - \frac{12z}{z-0.5} + 3$$

Taking the inverse z-transform, the step response is

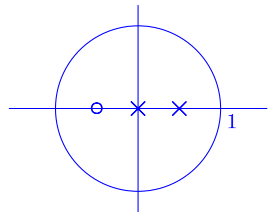
$$y(k) = \left(9 - 12\,(0.5)^k\right)u_s(k) + 3\,\delta(k)$$

d) The dc gain of the system is the transfer function evaluated at $z = 1$, which is

$$H(1) = \frac{3\,(1.5)}{0.5} = 9$$

Note that this is also the limit of the step response as $k \to \infty$.

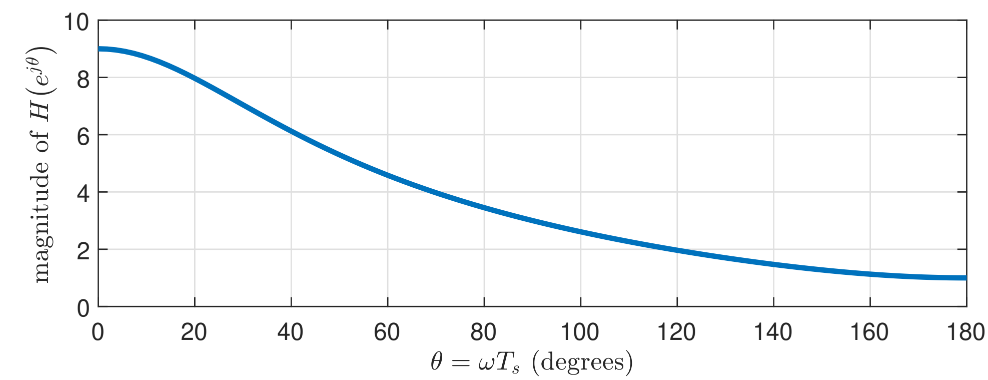
e) The system has a zero at $z = -0.5$ and poles at $z = 0$ and $z = 0.5$.

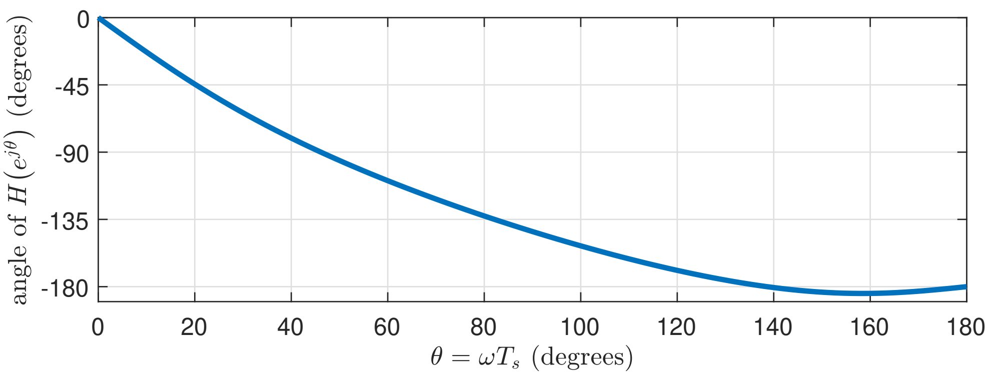


f) The frequency response is the transfer function evaluated on the unit circle, $z = e^{j\theta}$.

- At an angle $\theta = 0$, the frequency response is $H(e^{j0}) = H(1) = 9$, which has magnitude $M = 9$ and angle $\phi = 0$.
- At an angle $\theta = \pi$, the frequency response is $H(e^{j\pi}) = H(-1) = -1$, which has magnitude $M = 1$ and angle $\phi = \pm 180°$.

Using these two points, the frequency response should look something like this:





Problem 48: Inverse Laplace transform

Find the inverse Laplace transform of each of the following.

a) $X(s) = \dfrac{s+2}{s^2+7s+12}$

b) $X(s) = \dfrac{3s+2}{s^2+6s+25}$

SOLUTION:

a) The poles of the Laplace transform are $s = -4$ and $s = -3$. Then using partial fraction expansion, the Laplace transform can be written as

$$X(s) = \frac{s+2}{(s+3)(s+4)} = \frac{A}{s+3} + \frac{B}{s+4}$$

Multiplying by the denominator gives the equation

$$\begin{aligned} s+2 &= A\,(s+4) + B\,(s+3) \\ &= (A+B)\,s + (4A+3B) \end{aligned}$$

Since this must hold for all s, each of the corresponding coefficients must be equal. This gives the two equations

$$\begin{aligned} 1 &= A + B && s \text{ equation} \\ 2 &= 4A + 3B && \text{constant equation} \end{aligned}$$

which have the solution $A = -1$ and $B = 2$. The partial fraction expansion is then

$$X(s) = \frac{-1}{s+3} + \frac{2}{s+4}$$

so the inverse Laplace transform is

$$x(t) = \left(-e^{-3t} + 2\,e^{-4t}\right) u_s(t)$$

b) The poles of the Laplace transform are $s = -3 \pm j4$ which are complex, so the inverse Laplace transform will contain sinusoids. Splitting the numerator into two terms (one corresponding to the sine term and the other to the cosine term), we have

$$X(s) = 3 \cdot \frac{s+3}{(s+3)^2 + 4^2} - \frac{7}{4} \cdot \frac{4}{(s+3)^2 + 4^2}$$

Therefore, the inverse Laplace transform is

$$x(t) = e^{-3t} \left(3\cos(4t) - \tfrac{7}{4}\sin(4t)\right) u_s(t)$$

Problem 49: Fourier transform and LTI systems

Consider an LTI system with input signal $x(t)$ and impulse response $h(t)$, where

$$x(t) = e^{-(t-2)}\, u(t-2) \qquad \text{and} \qquad h(t) = \delta(t-1)$$

Determine the output signal $y(t)$ in two ways:

- by convolving $x(t)$ with $h(t)$
- by taking the inverse transform of $Y(j\omega) = H(j\omega)\, X(j\omega)$

Show your work. You should get the same result using both methods, which verifies the convolution property of the Fourier transform.

SOLUTION: Note that the LTI system with impulse response $h(t)$ is a one-second delay, so we expect the output to be the input signal shifted to the right by one.

- Using convolution, the output of the system is

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, \mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} e^{-(\tau-2)}\, u(\tau-2)\, \delta(t-\tau-1)\, \mathrm{d}\tau \end{aligned}$$

  To evaluate the integral, we can use the sampling property of the unit impulse which says that the integral of a signal $g(t)$ multiplied by an impulse is the value of the function at the location of the impulse, that is,

$$\int_{-\infty}^{\infty} g(t)\, \delta(t-t_0)\, \mathrm{d}t = g(t_0)$$

  In our case, the impulse is located at $\tau = t-1$. Therefore, the output is

$$y(t) = e^{-(t-3)}\, u(t-3)$$

- To compute the Fourier transform of the input signal, we can use the transform pair

$$e^{-t}\, u(t) \quad \xleftrightarrow{\mathcal{F}} \quad \frac{1}{j\omega+1}$$

  along with the time-shifting property

$$x(t-t_0) \quad \xleftrightarrow{\mathcal{F}} \quad e^{-j\omega t_0}\, X(j\omega)$$

  Applying the time-shifting property to the decaying exponential,

$$x(t) = e^{-(t-2)}\, u(t-2) \quad \xleftrightarrow{\mathcal{F}} \quad X(j\omega) = e^{-j\omega 2}\, \frac{1}{j\omega+1}$$

To find the Fourier transform of the impulse response, we can apply the time-shifting property of the Fourier transform to the pair

$$\delta(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad 1$$

which gives that

$$h(t) = \delta(t-1) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad H(j\omega) = e^{-j\omega}$$

The Fourier transform of the output is then the product of the Fourier transform of the impulse response with the Fourier transform of the input,

$$\begin{aligned} Y(j\omega) &= H(j\omega)\, X(j\omega) \\ &= e^{-j\omega}\, e^{-j\omega 2}\, \frac{1}{j\omega + 1} \\ &= e^{-j\omega 3}\, \frac{1}{j\omega + 1} \end{aligned}$$

Using the time-shifting property to take the inverse Fourier transform gives

$$y(t) = e^{-(t-3)}\, u(t-3)$$

which is the same as before.

Problem 50: System identification

Consider a continuous-time LTI system with step response

$$s(t) = (-5 + 9\,e^{2t} - 4\,e^{-3t})\,u_s(t)$$

a) Find the transfer function $H(s)$ of the system.

b) Draw a pole-zero plot of the transfer function.

c) Find the impulse response $h(t)$ of the system.

d) Determine whether or not the system is stable.

e) For a certain input signal $x(t)$, the output is observed to be

$$y(t) = 10\,(e^{2t} - e^{-t})\,u_s(t)$$

Determine the input signal.

SOLUTION:

a) Using the pair

$$e^{at}\,u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a}$$

the Laplace transform of the step response is

$$S(s) = -\frac{5}{s} + \frac{9}{s-2} - \frac{4}{s+3} = \frac{30\,(s+1)}{s\,(s-2)(s+3)}$$

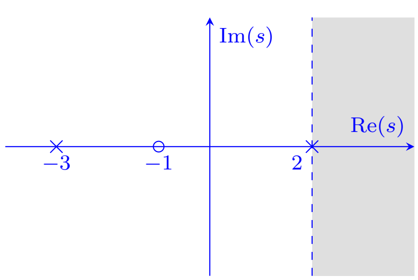
The step response is the output due to a unit step signal $u_s(t)$, which has Laplace transform $U(s) = 1/s$. Since the system is LTI, the Laplace transform of the output is given by

$$S(s) = H(s)\,U(s)$$

where $H(s)$ is the transfer function. Solving for $H(s)$ gives

$$H(s) = \frac{S(s)}{U(s)} = \frac{30\,(s+1)}{(s-2)(s+3)}$$

b) $H(s)$ has poles at $s=-3$ and $s=2$, and it has a zero at $s=-1$, so its pole-zero plot is as follows.



c) The impulse response is the inverse Laplace transform of $H(s)$. To compute the inverse transform, we can use partial fraction expansion,

$$H(s) = \frac{30\,(s+1)}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

Multiplying by the denominator gives the equation

$$30\,(s+1) = A\,(s+3) + B\,(s-2)$$

Equating the coefficients in s gives the two equations

$$\begin{aligned} 30 &= A + B && s \text{ equation} \\ 30 &= 3A - 2B && \text{constant equation} \end{aligned}$$

which have the solution $A = 18$ and $B = 12$, so the partial fraction expansion is

$$H(s) = \frac{18}{s-2} + \frac{12}{s+3}$$

Now we can take the inverse Laplace transform using a table to obtain the impulse response

$$h(t) = (18\,e^{2t} + 12\,e^{-3t})\,u(t)$$

d) The system is **not** stable since the transfer function has a pole in the right-half plane.

e) The Laplace transform of the output signal is

$$Y(s) = \frac{10}{s-2} - \frac{10}{s+1} = \frac{30}{(s-2)(s+1)}$$

The output of the system is related to the input signal by

$$Y(s) = H(s)\,X(s)$$

So the Laplace transform of the input signal is

$$X(s) = \frac{Y(s)}{H(s)} = \frac{s+3}{(s+1)^2}$$

which has partial fraction expansion

$$X(s) = \frac{s+3}{(s+1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2}$$

Multiplying both sides by the denominator gives

$$s + 3 = A\,(s+1) + B$$

Equating coefficients in s gives that $A = 1$ and $B = 2$, so the partial fraction expansion is

$$X(s) = \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

Taking the inverse Laplace transform, the input signal is

$$x(t) = (1 + 2t)\,e^{-t}\,u(t)$$

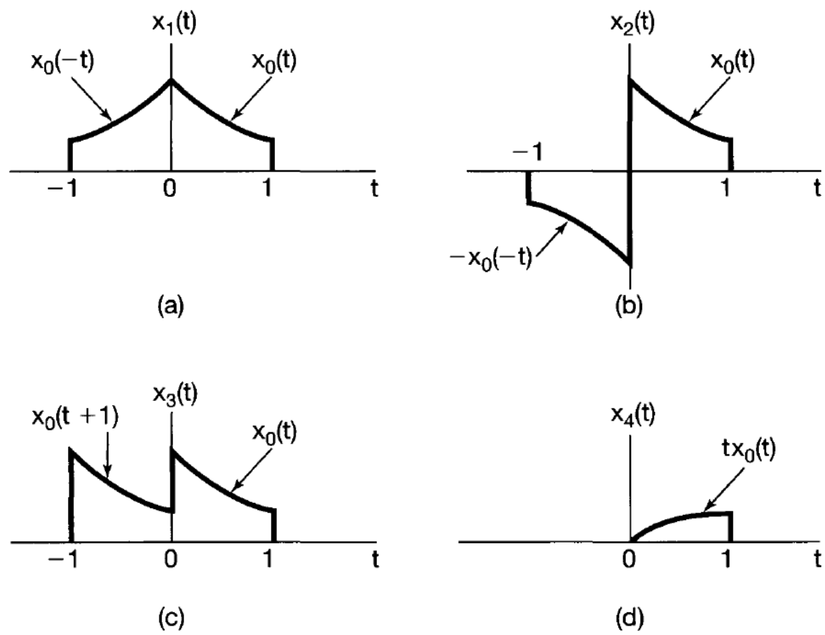
Note that the input signal $x(t)$ is bounded, but the output signal $y(t)$ is unbounded. Since a bounded input produces an unbounded output, the system is not stable (as already seen from the transfer function).

Problem 51: Properties of the Fourier transform

Consider the signal

$$x_0(t) = \begin{cases} e^{-t} & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Determine the Fourier transform of each signal shown below.



Hint: One way to solve this problem is to directly apply the definition of the Fourier transform to each signal. A simpler approach, however, is to evaluate *only* the Fourier transform of $x_0(t)$ and then use properties of the Fourier transform to obtain the transform of each signal.

SOLUTION: Using the definition of the Fourier transform, the spectrum of $x_0(t)$ is

$$\begin{aligned} X_0(j\omega) &= \int_{-\infty}^{\infty} x_0(t)\, e^{-j\omega t}\, \mathrm{d}t \\ &= \int_0^1 e^{-t}\, e^{-j\omega t}\, \mathrm{d}t \\ &= -\frac{1}{1+j\omega}\, e^{-(1+j\omega)t}\Big|_{t=0}^{1} \\ &= \frac{1}{1+j\omega}\left(1 - e^{-(1+j\omega)}\right) \end{aligned}$$

a) The signal $x_1(t)$ can be written in terms of $x_0(t)$ as

$$x_1(t) = x_0(t) + x_0(-t)$$

Then using the linearity and time reversal properties of the Fourier transform,

$$X_1(j\omega) = X_0(j\omega) + X_0(-j\omega)$$

Substituting the expression for $X_0(j\omega)$ and simplifying gives

$$\begin{aligned} X_1(j\omega) &= \frac{1}{1+j\omega}\left(1 - e^{-(1+j\omega)}\right) + \frac{1}{1-j\omega}\left(1 - e^{-(1-j\omega)}\right) \\ &= \frac{1}{1+\omega^2}\left(2 - \frac{1}{e}(e^{j\omega} + e^{-j\omega}) - \frac{j\omega}{e}(e^{j\omega} - e^{-j\omega})\right) \end{aligned}$$

From Euler's relation, we have the trigonometric identities

$$\cos(\omega) = \frac{e^{j\omega} + e^{-j\omega}}{2} \qquad \text{and} \qquad \sin(\omega) = \frac{e^{j\omega} - e^{-j\omega}}{j2}$$

Using these to simplify $X_1(j\omega)$ gives

$$X_1(j\omega) = \frac{2}{1+\omega^2}\left(1 - \frac{\cos\omega}{e} + \frac{\omega\,\sin\omega}{e}\right)$$

Note that the spectrum is purely real since the signal $x_1(t)$ is even.

b) The signal $x_2(t)$ can be written in terms of $x_0(t)$ as

$$x_2(t) = x_0(t) - x_0(-t)$$

Then using the linearity and time reversal properties of the Fourier transform,

$$X_1(j\omega) = X_0(j\omega) - X_0(-j\omega)$$

Substituting the expression for $X_0(j\omega)$ and simplifying gives

$$\begin{aligned} X_2(j\omega) &= \frac{1}{1+j\omega}\left(1 - e^{-(1+j\omega)}\right) - \frac{1}{1-j\omega}\left(1 - e^{-(1-j\omega)}\right) \\ &= \frac{1}{1+\omega^2}\left(-j2\omega + \frac{1}{e}(e^{j\omega} - e^{-j\omega}) + \frac{j\omega}{e}(e^{j\omega} + e^{-j\omega})\right) \end{aligned}$$

Using the trigonometric identities from part **(a)** and simplifying gives

$$X_2(j\omega) = \frac{j2}{1+\omega^2}\left(-\omega + \frac{\sin\omega}{e} + \frac{\omega\,\cos\omega}{e}\right)$$

Note that the spectrum is purely imaginary since the signal $x_2(t)$ is odd.

c) The signal $x_3(t)$ can be written in terms of $x_0(t)$ as

$$x_3(t) = x_0(t) + x_0(t+1)$$

Then using the linearity and time shifting properties of the Fourier transform,

$$X_3(j\omega) = X_0(j\omega) + e^{j\omega}\, X_0(j\omega)$$

Substituting the expression for $X_0(j\omega)$ gives

$$X_3(j\omega) = \frac{1}{1+j\omega}\left(1 + e^{j\omega}\right)\left(1 - e^{-(1+j\omega)}\right)$$

d) The signal $x_4(t)$ can be written in terms of $x_0(t)$ as

$$x_4(t) = t\, x_0(t)$$

Then using the differentiation in frequency property of the Fourier transform,

$$X_4(j\omega) = j\,\frac{\mathrm{d}}{\mathrm{d}\omega} X_0(j\omega) = j\,\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{1 - e^{-(1+j\omega)}}{1+j\omega}\right)$$

Using the quotient rule to compute the derivative and simplifying gives

$$X_4(j\omega) = j\left(\frac{j\, e^{-(1+j\omega)}(1+j\omega) - j\,(1 - e^{-(1+j\omega)})}{(1+j\omega)^2}\right)$$

Then using that $j^2 = -1$ to simpify the expression gives

$$X_4(j\omega) = \frac{1 - e^{-(1+j\omega)}}{(1+j\omega)^2} - \frac{e^{-(1+j\omega)}}{1+j\omega}$$

Problem 52: System identification

Consider an LTI system whose zero-state response to the input signal

$$u(t) = e^{-t}\, u_s(t)$$

is given by

$$y(t) = (1 - e^{-2t})\, u_s(t)$$

a) Find the transfer function of the system.

b) Draw a pole-zero plot of the transfer function.

c) Find the impulse response of the system.

d) Is the system stable? Justify your answer.

SOLUTION:

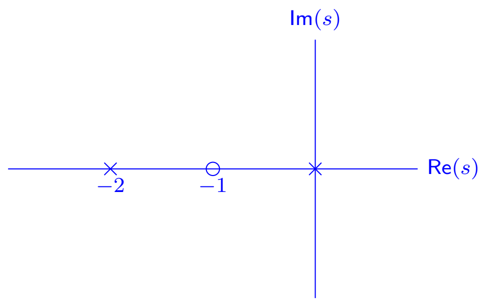
a) The Laplace transform of the input signal is $U(s) = 1/(s+1)$ and the Laplace transform of the output signal is

$$Y(s) = \frac{1}{s} - \frac{1}{s+2} = \frac{2}{s\,(s+2)}$$

The transfer function is then

$$H(s) = \frac{Y(s)}{U(s)} = \frac{2\,(s+1)}{s\,(s+2)}$$

b) The pole-zero plot of the transfer function is as follows:



c) Using the partial fraction expansion

$$H(s) = \frac{2\,(s+1)}{s\,(s+2)} = \frac{1}{s} + \frac{1}{s+2}$$

the impulse response is

$$h(t) = (1 + e^{-2t})\, u_s(t)$$

d) The system is *not* stable since there is a pole at $s = 0$.