CS 228 : Logic in Computer Science

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CNF Explosion

Consider the formula $\varphi = (p_1 \wedge p_2 \cdots \wedge p_n) \vee (q_1 \wedge q_2 \ldots q_m)$

- What is the equivalent CNF formula?
- $\bigwedge_{i \in \{1,\ldots,n\}, j \in \{1,\ldots,m\}} (p_i \vee q_j)$ has mn clauses
- Distributivity explodes the formula

Tseitin Encoding : The Idea

- Introducing fresh variables, Tseitin encoding can give an equisatisfiable formula without exponential explosion.
- $\varphi = p \vee (q \wedge r)$
- Replace $q \wedge r$ with a fresh variable x and add a clause which asserts that x simulates $q \wedge r$
- $(p \vee x) \wedge (x \rightarrow (q \wedge r))$ which is $(p \vee x) \wedge (\neg x \vee q) \wedge (\neg x \vee r)$

Tseitin Encoding

- Assume the input formula is in NNF (all negations attached only to literals) and has only $\wedge, \vee$
- Replace each $G_1 \wedge \cdots \wedge G_n$ just below a $\vee$ with a fresh variable p and conjunct $(\neg p \vee G_1) \wedge \cdots \wedge (\neg p \vee G_n)$ (same as $p \rightarrow G_1 \wedge \cdots \wedge G_n$).

Tseitin Encoding

- $\varphi = (p_1 \wedge p_2 \cdots \wedge p_n) \vee (q_1 \wedge q_2 \cdots \wedge q_m)$
- Choose fresh variables x, y
- $\psi = (x \vee y) \wedge \bigwedge_{i \in \{1,\ldots,n\}} (\neg x \vee p_i) \wedge \bigwedge_{j \in \{1,\ldots,m\}} (\neg y \vee q_j)$ has $m + n + 1$ clauses
- φ and ψ are equisatisfiable. Prove.

(DPLL)Davis-Putnam-Loveland-Logemann Method

DPLL

- DPLL combines search and deduction to decide CNF satisfiability
- Underlies most modern SAT solvers

Partial Assignments

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- Under a partial assignment α, the state of a variable v is true if $\alpha(v) = 1$, false if $\alpha(v) = 0$, and unassigned otherwise.
- Let $V = \{x, y, z\}$ and let $\alpha(x) = 1, \alpha(y) = 0$. Then the state of x under α is true, state of y is false, and the state of z is unassigned.

State of a Clause

Assume we have a formula in CNF. Under a partial assignment α,

- a clause C is true if there exists a literal ℓ in C whose state is true
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State of a Formula

Under a partial assignment α,

- A CNF formula F is true if for each $C \in F$, C is true
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Unit Clause and Unit Literal

Let C be a clause and α a partial assignment. Then

- C is a unit clause under α if there is a literal $\ell \in C$ which is unassigned, and the rest are false.
- Then ℓ is a unit literal under α.

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- $C = x \vee \neg y \vee \neg z \vee w$ is not a unit clause
- $C = x \vee \neg y$ is not a unit clause

DPLL

DPLL maintains a partial assignment. To begin, it has an empty assignment.

- Assigns unassigned variables 0 or 1 randomly
- Forced to assign 0 or 1 to unit literals appropriately

DPLL Actions

- DPLL has 3 actions : decisions, unit propagation and backtracking
- An assignment is either a decision assignment or an implied assignment
  - Decisions : Decide an assignment for a variable (random choice)
  - Implied assignments or unit propagation : to deal with unit literals
- Backtrack when in a conflict

DPLL Algorithm States

- At any time, the state of the algorithm is a pair (F, α) where F is the given CNF and α is a partial assignment
- A state (F, α) is successful if α sets some literal in each clause of F to be true
- A conflict state is one where α sets all literals in some clause of F to be false

DPLL Algorithm States

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  1. deleting from F, any clause containing a true literal under α, and
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  2. deleting from each remaining clause, all literals false under α.

  Let $\alpha(x) = 0, \alpha(y) = 1$.
- For $F = (x \vee y \vee z) \wedge (x \vee \neg y \vee \neg z)$, $F|\alpha = \{\neg z\}$
- For $F = (x \vee y) \wedge (\neg x \vee \neg y)$, $F|\alpha = \{\}$.
- For $F = (x \vee \neg y)$, $F|\alpha$ is the empty clause, denoted $\perp$
- If (F, α) is successful, then $F|\alpha$ is the empty set.
- If (F, α) is a conflict state, then the empty clause is in $F|\alpha$.

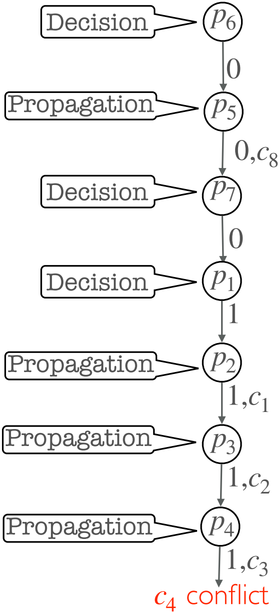
The DPLL Algorithm

Input : CNF formula F.

1. Initialise α as the empty assignment
2. While there is a unit clause L in $F|\alpha$, add $L = 1$ to α (unit propagation)
3. If $F|\alpha$ contains no clauses, then stop and output α
4. If $F|\alpha$ contains the empty clause, then apply the learning procedure to add a new clause C to F. If it is the empty clause, output UNSAT. Otherwise, backtrack to the highest level at which C is a unit clause, go to line 2.
5. Decide on a new assignment $p = b$ to be added to α, goto line 2. p is then called a decision variable.

DPLL Example

$$c_1 = \neg p_1 \vee p_2$$
$$c_2 = \neg p_1 \vee p_3 \vee p_5$$
$$c_3 = \neg p_2 \vee p_4$$
$$c_4 = \neg p_3 \vee \neg p_4$$
$$c_5 = p_1 \vee p_5 \vee \neg p_2$$
$$c_6 = p_2 \vee p_3$$
$$c_7 = p_2 \vee \neg p_3 \vee p_7$$
$$c_8 = p_6 \vee \neg p_5$$



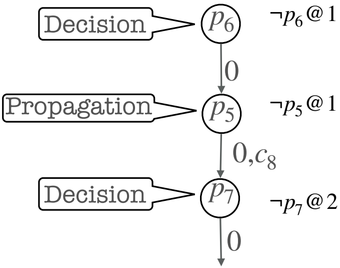
Clause Learning

Run of DPLL

The partial assignment in construction is called a a run of DPLL. In the previous slide, the run ended in a conflict.

Decision Level

During a run, the decision level of an assignment $\alpha(p) = b$ is the number of decisions that precede $\alpha(p) = b$. We begin with decision level 1 to avoid terminology of a 0th level decision.



Implication Graphs

During a DPLL run, we maintain a data structure called an implication graph.
Under a partial assignment α, the implication graph $G = (V, E)$,

- V is the set consisting of the true literals under α, as well as the conflict node
- E contains all edges $\{(\ell_1, \ell_2) \mid \neg\ell_1, \ell_2$ belong to the same clause and $\alpha(\ell_1)$ is involved in forcing unit propagation to make ℓ_2 true$\}$
- E also contains edges to a conflict node : in this case, the source node ℓ is such that $\alpha(\ell)$ is responsible for the conflict.

Each node is annotated with the decision level.

$c_1 = \neg p_1 \vee p_2$

$c_2 = \neg p_1 \vee p_3 \vee p_5$

$c_3 = \neg p_2 \vee p_4$

$c_4 = \neg p_3 \vee \neg p_4$

$c_5 = p_1 \vee p_5 \vee \neg p_2$

$c_6 = p_2 \vee p_3$

$c_7 = p_2 \vee \neg p_3 \vee p_7$

$c_8 = p_6 \vee \neg p_5$