CS 228 : Logic in Computer Science

Krishna. S

Recap of Basics

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- A conjunction of literals $L_1 \wedge L_2 \wedge \ldots L_n$ is satisfiable iff ...

Normal Forms : CNF Validity

Let $\varphi = C_1 \wedge C_2 \wedge \cdots \wedge C_n$ be in CNF.

- Checking if φ is satisfiable is NP-complete.
- Checking if φ is valid is polynomial time. Why?

Normal Forms : DNF Satisfiability

Let $\varphi = D_1 \vee D_2 \vee \cdots \vee D_n$ be in DNF.

- Checking if φ is valid is NP-complete. Why?
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- Consider for example $\varphi = p \leftrightarrow q$.
- Truth table of φ : φ is false when $p = T, q = F$ and $p = F, q = T$.
- DNF equivalent is $(p \wedge q) \vee (\neg p \wedge \neg q)$
- CNF equivalent is $(\neg p \vee q) \wedge (p \vee \neg q)$.

CNF to DNF Sizes

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- $$\varphi' = \bigvee_{S \subseteq \{1,\ldots,n\}} (\bigwedge_{i \in S} p_i \wedge \bigwedge_{i \notin S} q_i)$$
- Prove that any equivalent DNF formula has 2^n clauses

On an Example

- $\varphi = (p_1 \vee q_1) \wedge (p_2 \vee q_2)$
- DNF(φ) : $(p_1 \wedge p_2) \vee (p_1 \wedge q_2) \vee (q_1 \wedge p_2) \vee (q_1 \wedge q_2)$

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- $p_1 = T, q_1 = F, p_2 = T, q_2 = F$
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- $p_1 = F, q_1 = T, p_2 = F, q_2 = T$
- 4 minimal assignments; all satisfy φ

On an Example

- Choose 2 minimal assignments α, β such that $\alpha(p_i) \neq \beta(p_i)$ for some i.
- Define a new assignment $\mathsf{min}(\alpha, \beta)$ as a pointwise min of α, β

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- Choose 2 minimal assignments α, β such that $\alpha(p_i) \neq \beta(p_i)$ for some i.
- Define a new assignment $\min(\alpha, \beta)$ as a pointwise min of α, β
  - $\alpha = (p_1 = T, q_1 = F, p_2 = T, q_2 = F)$,
  - $\beta = (p_1 = F, q_1 = T, p_2 = T, q_2 = F)$,
  - $\min(\alpha, \beta) = (p_1 = F, q_1 = F, p_2 = T, q_2 = F)$.

On an Example

- $\min(\alpha, \beta) \not\models p_1 \vee q_1$
- $\min(\alpha, \beta) \not\models (p_1 \vee q_1) \wedge (p_2 \vee q_2) = \varphi$

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- Assume both α, β satisfy (some) same clause in DNF(φ).
- Then $\mathtt{min}(\alpha, \beta)$ will also satisfy the same clause (why?)

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- Assume both α, β satisfy (some) same clause in DNF(φ).
- Then $\mathsf{min}(\alpha, \beta)$ will also satisfy the same clause (why?)
- Then $\mathsf{min}(\alpha, \beta) \models$ DNF(φ), a contradiction
- Therefore, no two minimal assignments can satisfy the same clause in DNF(φ)
- Hence, we need as many clauses in DNF(φ) as the number of minimal assignments

Explosion CNF to DNF

Generalise the reasoning for n . Think of an example where DNF to CNF explodes.

Given a formula φ in CNF, compute in polytime a formula $\beta \wedge \gamma$ which is equisatisfiable to φ such that β is Horn and γ is 2-CNF.