CS 228 : Logic in Computer Science

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Completeness

$\varphi_1, \ldots, \varphi_n \models \psi \Rightarrow \varphi_1, \ldots, \varphi_n \vdash \psi$

Whenever $\varphi_1, \ldots, \varphi_n$ semantically entail ψ, then ψ can be proved from $\varphi_1, \ldots, \varphi_n$. The proof rules are complete

Completeness : 3 steps

- Given $\varphi_1, \ldots, \varphi_n \models \psi$
- Step 1: Show that $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$
- Step 2: Show that $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$
- Step 3: Show that $\varphi_1, \ldots, \varphi_n \vdash \psi$

Completeness : Step 1

- Assume $\varphi_1, \ldots, \varphi_n \models \psi$. Whenever all of $\varphi_1, \ldots, \varphi_n$ evaluate to true, so does ψ.
- If $\not\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$, then ψ evaluates to false when all of $\varphi_1, \ldots, \varphi_n$ evaluate to true, a contradiction.
- Hence, $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

Completeness : Step 2

- Given $\models \psi$, show that $\vdash \psi$
- Assume $p_1, \ldots, p_n$ are the propositional variables in ψ. We know that all the 2^n assignments of values to $p_1, \ldots, p_n$ make ψ true.
- Using this insight, we have to give a proof of ψ.

Completeness : Step 2

Truth Table to Proof

Let φ be a formula with variables $p_1, \ldots, p_n$. Let $\mathcal{T}$ be the truth table of φ, and let l be a line number in $\mathcal{T}$. Let $\hat{p}_i$ represent p_i if p_i is assigned true in line l, and let it denote $\neg p_i$ if p_i is assigned false in line l. Then

1. $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
2. $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi$ if φ evaluates to false in line l

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  - Assume φ evaluates to true in line l of $\mathcal{T}$. Then φ_1 evaluates to false in line l. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\varphi_1$.
  - Assume φ evaluates to false in line l of $\mathcal{T}$. Then φ_1 evaluates to true in line l. By inductive hypothesis, $\hat{p}_1, \ldots, \hat{p}_n \vdash \varphi_1$.
  - Use the $\neg\neg i$ rule to obtain a proof of $\hat{p}_1, \ldots, \hat{p}_n \vdash \neg\neg\varphi_1$.

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  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \models \varphi_1$ and $\hat{r}_1, \ldots, \hat{r}_j \models \neg\varphi_2$. Then, $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \neg\varphi_2$.

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  - By inductive hypothesis, $\hat{q}_1, \ldots, \hat{q}_k \models \varphi_1$ and $\hat{r}_1, \ldots, \hat{r}_j \models \neg\varphi_2$. Then, $\hat{p}_1, \ldots, \hat{p}_n \models \varphi_1 \wedge \neg\varphi_2$.
  - Prove that $\varphi_1 \wedge \neg\varphi_2 \vdash \neg(\varphi_1 \rightarrow \varphi_2)$.

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  - If φ evaluates to true in line l, then there are 3 possibilities. If both φ_1, φ_2 evaluate to true, then we have $\hat{p_1}, \ldots, \hat{p_n} \models \varphi_1 \wedge \varphi_2$. Proving $\varphi_1 \wedge \varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.

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  - If both φ_1, φ_2 evaluate to false, then we have $\hat{p_1}, \ldots, \hat{p_n} \models \neg\varphi_1 \wedge \neg\varphi_2$. Proving $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.

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  - If both φ_1, φ_2 evaluate to false, then we have $\hat{p_1}, \ldots, \hat{p_n} \models \neg\varphi_1 \wedge \neg\varphi_2$. Proving $\neg\varphi_1 \wedge \neg\varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.
  - Last, if φ_1 evaluates to false and φ_2 evaluates to true, then we have $\hat{p_1}, \ldots, \hat{p_n} \models \neg\varphi_1 \wedge \varphi_2$. Proving $\neg\varphi_1 \wedge \varphi_2 \vdash \varphi_1 \rightarrow \varphi_2$, we are done.

Truth Table to Proof

- Prove the cases $\wedge, \vee$.

On An Example

We know $\models (p \wedge q) \rightarrow p$. Using this fact, show that $\vdash (p \wedge q) \rightarrow p$.

- $p, q \vdash (p \wedge q) \rightarrow p$
- $\neg p, q \vdash (p \wedge q) \rightarrow p$
- $p, \neg q \vdash (p \wedge q) \rightarrow p$
- $\neg p, \neg q \vdash (p \wedge q) \rightarrow p$

Now, combine the 4 proofs above to give a single proof for $\vdash (p \wedge q) \rightarrow p$.

Completeness : Steps 2, 3

- Step 2: From $\models \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$, use LEM on all the propositional variables of $\varphi_1, \ldots, \varphi_n, \psi$ to obtain $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$.

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- Step 3: Take the proof $\vdash \varphi_1 \to (\varphi_2 \to (\ldots(\varphi_n \to \psi)\ldots))$. This proof has *n* nested boxes, the *i*th box opening with the assumption φ_i. The last box closes with the last line ψ. Hence, the line immediately after the last box is $\varphi_n \to \psi$.

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- In a similar way, the $(n-1)$th box has as its last line $\varphi_n \to \psi$, and hence, the line immediately after this box is $\varphi_{n-1} \to (\varphi_n \to \psi)$ and so on.

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- Add premises $\varphi_1, \ldots, \varphi_n$ on the top. Use MP on the premises, and the lines after boxes 1 to n in order to obtain ψ.

Summary

Propositional Logic is sound and complete.

Satisfiability and Validity

A formula φ over variables $\{p_1, \ldots, p_n\}$ is satisfiable if there exists a valuation for $p_1, \ldots, p_n$ which makes φ true.

A formula φ over variables $\{p_1, \ldots, p_n\}$ is unsatisfiable if there does not exist any valuation for $p_1, \ldots, p_n$ which makes φ true.

A formula φ over variables $\{p_1, \ldots, p_n\}$ is valid if φ true for all valuations to $p_1, \ldots, p_n$.

Checking Satisfiability

- The SAT problem : Given a formula φ, is it satisfiable?
- Given a formula φ, is it valid?
- SAT is NP-complete. NP represents non-deterministic polynomial time.
- Given a witness, it is easy to check if the witness is a valid witness in polynomial time, Finding a witness is not as easy.
  - Given a valuation α for the variables $p_1, \ldots, p_n$ of a formula φ, we can check if φ evaluates to true under α in time polynomial in n.
  - Finding whether such a valuation α exists is not as easy.
- SAT solvers are tools which implement heuristics to check satisfiability.
- The input to a SAT solver is a formula in some specified form.

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- A formula F is in DNF if it is a disjunction of a conjunction of literals.

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Every formula F is equivalent to some formula F_1 in CNF and some formula F_2 in DNF.

CNF Algorithm

Given a formula F, $(x \to [\neg(y \vee z) \wedge \neg(y \to x)])$

- Replace all subformulae of the form $F \to G$ with $\neg F \vee G$, and all subformulae of the form $F \leftrightarrow G$ with $(\neg F \vee G) \wedge (\neg G \vee F)$. When there are no more occurrences of $\to, \leftrightarrow$, proceed to the next step.

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- Get rid of all double negations : Replace all subformulae
  - $\neg\neg G$ with G,
  - $\neg(G \wedge H)$ with $\neg G \vee \neg H$
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  When there are no more such subformulae, proceed to the next step.
- Distribute $\vee$ wherever possible.

The resultant formula F_1 is in CNF and is provably equivalent to F.
$[(\neg x \vee \neg y) \wedge (\neg x \vee \neg z)] \wedge [(\neg x \vee y) \wedge (\neg x \vee \neg x)]$

Polynomial Time Formula Classes

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- $p \wedge (\neg p \vee \neg q \vee r) \wedge (\neg a \vee \neg b)$ is Horn, but $a \vee b$ is not Horn.
- A basic Horn formula is one which has no $\wedge$. Every Horn formula is a conjunction of basic Horn formulae.

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- Basic Horn with no positive literals are written as $p \wedge q \wedge \cdots \wedge r \rightarrow \bot$.

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- Thus, a Horn formula is written as a conjunction of implications.

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- If there is a subformula of the form $(p_1 \wedge \cdots \wedge p_m) \to q$, where each p_i is marked, and q is not marked, mark q. Repeat this until there are no subformulae of this form and proceed to the next step.
- Consider subformulae of the form $(p_1 \wedge \cdots \wedge p_m) \to \bot$. If there is one such subformula with all p_i marked, then say Unsat, otherwise say Sat.

An Example

$$(\top \to A) \wedge (C \to D) \wedge ((A \wedge B) \to C) \wedge ((C \wedge D) \to \bot) \wedge (\top \to B).$$

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The Horn Algorithm

The Horn algorithm concludes Sat iff H is satisfiable.

Complexity of Horn

- Given a Horn formula ψ with *n* propositions, how many times do you have to read ψ?
- Step 1: Read once
- Step 2: Read atmost *n* times
- Step 3: Read once

2-CNF

- 2-CNF : CNF where each clause has at most 2 literals.

$$(\neg p \vee q) \wedge p \wedge (r \vee \neg q) \wedge (\neg r \vee p)$$