CS 228 : Logic in Computer Science

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Natural Deduction: Proof Engines

- Proof engines provide a systematic and reliable way to derive new true statements from existing true statements
- Must be sound and complete
  - Completeness: Any formula that can be inferred can be derived by the proof engine. For instance, starting with p, q, we can infer $p \vee q, p \wedge q, \neg p \vee q, p \vee \neg q, \neg\neg p \wedge q$ and so on. The proof engine must generate all these possibilities.

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  - Completeness: Any formula that can be inferred can be derived by the proof engine. For instance, starting with p, q, we can infer $p \lor q, p \land q, \neg p \lor q, p \lor \neg q, \neg\neg p \land q$ and so on. The proof engine must generate all these possibilities.
  - Soundness: Any formula that is derived by the proof engine is indeed valid. For instance, starting with premises p, q, the proof engine should not churn out $\neg p$!

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- $\varphi_1, \ldots, \varphi_n \vdash \psi$: This is called a sequent. $\varphi_1, \ldots, \varphi_n$ are premises, and ψ, the conclusion.
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- For example, $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?

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- For example, $\neg p \to q, q \to r, \neg r \vdash p$ is a sequent. How do you prove this?
- Proof rules to be carefully chosen, for instance you should not end up proving something like $p \wedge q \vdash \neg q$

The Rules of the Proof Engine

Rules for Natural Deduction

The and introduction rule denoted $\wedge i$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

Rules for Natural Deduction

The and elimination rule denoted $\wedge e_1$

$$\frac{\varphi \wedge \psi}{\varphi}$$

The and elimination rule denoted $\wedge e_2$

$$\frac{\varphi \wedge \psi}{\psi}$$

A first proof using $\wedge i, \wedge e_1, \wedge e_2$

- Show that $p \wedge q, r \vdash q \wedge r$

1.	$p \wedge q$	premise
2.		

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1.	$p \wedge q$	premise
2.	r	premise
3.	q	$\wedge e_2$ 1
4.		

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3.	q	$\wedge e_2$ 1
4.	$q \wedge r$	$\wedge i$ 3,2

Rules for Natural Deduction

The rule of double negation elimination $\neg\neg e$

$$\frac{\neg\neg\varphi}{\varphi}$$

The rule of double negation introduction $\neg\neg i$

$$\frac{\varphi}{\neg\neg\varphi}$$

Rules for Natural Deduction

The implies elimination rule or Modus Ponens MP

$$\frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Another Proof

- Show that $p, p \to q, p \to (q \to \neg\neg r) \vdash r$

1.	$p \to (q \to \neg\neg r)$	premise
2.		

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3.	p	premise
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1.	$p \to (q \to \neg\neg r)$	premise
2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.		

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5.	q	MP 2,3
6.		

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2.	$p \to q$	premise
3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.		

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3.	p	premise
4.	$q \to \neg\neg r$	MP 1,3
5.	q	MP 2,3
6.	$\neg\neg r$	MP 4,5
7.	r	$\neg\neg e$ 6

Rules for Natural Deduction

Another implies elimination rule or Modus Tollens MT

$$\frac{\varphi \to \psi \qquad \neg\psi}{\neg\varphi}$$

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

$$\begin{array}{lll} 1. & p \to \neg q & \text{premise} \\ 2. & & \end{array}$$

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.		

A Proof

- Show that $p \rightarrow \neg q, q \vdash \neg p$

1.	$p \rightarrow \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.		

A Proof

- Show that $p \to \neg q, q \vdash \neg p$

1.	$p \to \neg q$	premise
2.	q	premise
3.	$\neg\neg q$	$\neg\neg i$ 2
4.	$\neg p$	MT 1,3

More Rules

- Thanks to MT, we have $p \to q, \neg q \vdash \neg p$.

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- So far, no proof rule that can do this.
- Given $p \to q$, let us assume $\neg q$. Can we then prove $\neg p$?

More Rules

- Thanks to MT, we have $p \rightarrow q, \neg q \vdash \neg p$.
- Can we prove $p \rightarrow q \vdash \neg q \rightarrow \neg p$?
- So far, no proof rule that can do this.
- Given $p \rightarrow q$, let us assume $\neg q$. Can we then prove $\neg p$?
- Yes, using MT.

The implies introduction rule $\rightarrow i$

- $p \rightarrow q \vdash \neg q \rightarrow \neg p$

1.	$p \rightarrow q$	premise
2.	$\neg q$	assumption
3.	$\neg p$	MT 1,2
4.	$\neg q \rightarrow \neg p$	$\rightarrow i$ 2-3

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	*true*	premise
2.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
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3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.		

More on $\rightarrow i$

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1.	*true*	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.		

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2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.		

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7

More on $\rightarrow i$

- $\vdash (q \rightarrow r) \rightarrow [(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)]$

1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8

More on $\rightarrow i$

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1.	$true$	premise
2.	$q \rightarrow r$	assumption
3.	$\neg q \rightarrow \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \rightarrow r$	$\rightarrow i$ 4-8
10.	$(\neg q \rightarrow \neg p) \rightarrow (p \rightarrow r)$	$\rightarrow i$ 3-9
11.		

More on $\to i$

- $\vdash (q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$

1.	$true$	premise
2.	$q \to r$	assumption
3.	$\neg q \to \neg p$	assumption
4.	p	assumption
5.	$\neg\neg p$	$\neg\neg i$ 4
6.	$\neg\neg q$	MT 3,5
7.	q	$\neg\neg e$ 6
8.	r	MP 2,7
9.	$p \to r$	$\to i$ 4-8
10.	$(\neg q \to \neg p) \to (p \to r)$	$\to i$ 3-9
11.	$(q \to r) \to [(\neg q \to \neg p) \to (p \to r)]$	$\to i$ 2-10

Transforming Proofs

- $(q \to r), (\neg q \to \neg p), p \vdash r$
- Transform any proof $\varphi_1, \ldots, \varphi_n \vdash \psi$ to $\vdash \varphi_1 \to (\varphi_2 \to \ldots (\varphi_n \to \psi) \ldots))$ by adding n lines of the rule $\to i$