CS 228 : Logic in Computer Science

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Propositional Logic

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- $\neg$ binds tighter than $\vee, \wedge$, which bind tighter than $\rightarrow$.
- $\neg p \vee q$ is read as $(\neg p) \vee (q)$; $p \vee q \rightarrow r$ is read as $(p \vee q) \rightarrow r$
- $\rightarrow$ is right associative : in the absence of parantheses, $p \rightarrow q \rightarrow r$ is read as $p \rightarrow (q \rightarrow r)$

Encoding and Natural Deduction

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- It is raining, and Alice is outside, and is not wet. $\psi = (R \wedge \mathit{AliceOut} \wedge \neg \mathit{AliceWet})$

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- If it rains, Alice is outside and does not have any raingear with her, she will get wet. $\varphi = (R \land AliceOut \land \neg RG) \to AliceWet$
- It is raining, and Alice is outside, and is not wet. $\psi = (R \land AliceOut \land \neg AliceWet)$
- So, Alice has her rain gear with her. RG
- Thus, $\chi = \varphi \land \psi \to RG$. You can deduce RG from $\varphi \land \psi$.
- Is χ valid? Is χ satisfiable?

Two Examples

Solve Sudoku

Consider the following kid's version of Sudoku.

	2	4	
1			3
4			2
	1	3	

Rules:

- Each row must contain all numbers 1-4
- Each column must contain all numbers 1-4
- Each 2×2 block must contain all numbers 1-4
- No cell contains 2 or more numbers

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  - Row 1: $[P(1,1,1) \vee P(1,2,1) \vee P(1,3,1) \vee P(1,4,1)] \wedge$
    $[P(1,1,2) \vee P(1,2,2) \vee P(1,3,2) \vee P(1,4,2)] \wedge$
    $[P(1,1,3) \vee P(1,2,3) \vee P(1,3,3) \vee P(1,4,3)] \wedge$
    $[P(1,1,4) \vee P(1,2,4) \vee P(1,3,4) \vee P(1,4,4)]$

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    $[P(1,1,4) \vee P(1,2,4) \vee P(1,3,4) \vee P(1,4,4)]$
  - Row 2: $[P(2,1,1) \vee \ldots$
  - Row 3: $[P(3,1,1) \vee \ldots$
  - Row 4: $[P(4,1,1) \vee \ldots$

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  $[P(1,1,4) \vee P(2,1,4) \vee P(3,1,4) \vee P(4,1,4)]$
- Column 2: $[P(1,2,1) \vee \ldots$
- Column 3: $[P(1,3,1) \vee \ldots$
- Column 4: $[P(1,4,1) \vee \ldots$

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- Upper left block contains all numbers 1-4:

$$[P(1,1,1) \vee P(1,2,1) \vee P(2,1,1) \vee P(2,2,1)] \wedge$$
$$[P(1,1,2) \vee P(1,2,2) \vee P(2,1,2) \vee P(2,2,2)] \wedge$$
$$[P(1,1,3) \vee P(1,2,3) \vee P(2,1,3) \vee P(2,2,3)] \wedge$$
$$[P(1,1,4) \vee P(1,2,4) \vee P(2,1,4) \vee P(2,2,4)]$$

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$$[P(1,1,4) \vee P(1,2,4) \vee P(2,1,4) \vee P(2,2,4)]$$

- Upper right block contains all numbers 1-4:

$$[P(1,3,1) \vee P(1,4,1) \vee P(2,3,1) \vee P(2,4,1)] \wedge \dots$$

- Lower left block contains all numbers 1-4:

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No cell contains 2 or more numbers

- For cell(1,1):

$$P(1,1,1) \rightarrow [\neg P(1,1,2) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,2) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,3) \wedge \neg P(1,1,4)] \wedge$$

$$P(1,1,3) \rightarrow [\neg P(1,1,1) \wedge \neg P(1,1,2) \wedge \neg P(1,1,4)] \wedge$$

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- Similar for other cells

Encoding as Propositional Satisfiability

Encoding Initial Configuration:

$$P(1,2,2) \wedge P(1,3,4) \wedge P(2,1,1) \wedge P(2,4,3) \wedge$$

$$P(3,1,4) \wedge P(3,4,2) \wedge P(4,2,1) \wedge P(4,3,3)$$

Solving Sodoku

To solve the puzzle, just conjunct all the above formulae and find a satisfiable truth assignment!

Gold Rush

(**Box1**) *The gold is not here*

(**Box2**) *The gold is not here*

(**Box3**) *The gold is in Box 2*

Only one message is true; the other two are false. Which box has the gold?

Solve Gold Rush

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  - $M1 \leftrightarrow \neg G1$, $M2 \leftrightarrow \neg G2$, $M3 \leftrightarrow G2$
  - $\neg(M1 \land M2 \land M3)$, $M1 \lor M2 \lor M3$,
  - $(\neg M1 \land \neg M2) \lor (\neg M1 \land \neg M3) \lor (\neg M2 \land \neg M3)$

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  - $M1 \leftrightarrow \neg G1$, $M2 \leftrightarrow \neg G2$, $M3 \leftrightarrow G2$
  - $\neg(M1 \wedge M2 \wedge M3)$, $M1 \vee M2 \vee M3$,
  - $(\neg M1 \wedge \neg M2) \vee (\neg M1 \wedge \neg M3) \vee (\neg M2 \wedge \neg M3)$
  - Conjunct all these, and call the formula φ.

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  - Conjunct all these, and call the formula φ.
  - Is there a unique satisfiable assignment for φ?
  - For example, is $M1 = \textit{true}$ a part of the satisfiable assignment?

A Proof Engine for Natural Deduction

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- So, Alice has her rain gear with her : RG
- Thus, given premises φ, ψ, we can conclude RG. Is it ok to conclude $\neg RG$?
- Given premises φ, ψ, can we infer RG via a proof?
- In general, given premises $\varphi_1, \ldots, \varphi_n$, we can infer many more formulae from them.