

# **CS 228 : Logic in Computer Science**

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  - ▶  $p \wedge q \models p \vee q$  (why?)
  - ▶  $p \vee q \not\models q \vee r$  (why?)
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- ▶ Formulae  $\varphi$  and  $\psi$  are **provably equivalent** iff  $\varphi \vdash \psi$  and  $\psi \vdash \varphi$
- ▶ Formulae  $\varphi$  and  $\psi$  are **semantically equivalent** iff  $\varphi \models \psi$  and  $\psi \models \varphi$

# Soundness of Propositional Logic

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$$\varphi_1, \dots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \dots, \varphi_n \vDash \psi$$

Whenever  $\psi$  can be proved from  $\varphi_1, \dots, \varphi_n$ , then  $\psi$  evaluates to true whenever  $\varphi_1, \dots, \varphi_n$  evaluate to true

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- ▶ When  $k = 1$ , there is only one line in the proof, say  $\varphi$ , which is the premise. Then we have  $\varphi \vdash \varphi$ , since  $\varphi$  is given. But then we also have  $\varphi \vDash \varphi$ .

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- ▶ Assume that whenever  $\varphi_1, \dots, \varphi_n \vdash \psi$  using a proof of length  $\leq k - 1$ , we have  $\varphi_1, \dots, \varphi_n \vDash \psi$ .

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- ▶ Consider now a proof with  $k$  lines.

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- ▶ We have the shorter proofs  $\varphi_1, \dots, \varphi_n \vdash \psi_1$  and  $\varphi_1, \dots, \varphi_n \vdash \psi_2$
- ▶ By inductive hypothesis, we have  $\varphi_1, \dots, \varphi_n \models \psi_1$  and  $\varphi_1, \dots, \varphi_n \models \psi_2$ . By semantics, we have  $\varphi_1, \dots, \varphi_n \models \psi_1 \wedge \psi_2$ .

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- ▶ The line just after the box was  $\psi$ .
- ▶ Consider adding  $\psi_1$  in the premises along with  $\varphi_1, \dots, \varphi_n$ . Then we will get a proof  $\varphi_1, \dots, \varphi_n, \psi_1 \vdash \psi_2$ , of length  $k - 1$ . By inductive hypothesis,  $\varphi_1, \dots, \varphi_n, \psi_1 \models \psi_2$ . By semantics, this gives  $\varphi_1, \dots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- ▶ The equivalence of  $\varphi_1, \dots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$  and  $\varphi_1, \dots, \varphi_n, \psi_1 \vdash \psi_2$  gives the proof.

# Soundness : Other cases

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# Completeness

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$$\varphi_1, \dots, \varphi_n \models \psi \Rightarrow \varphi_1, \dots, \varphi_n \vdash \psi$$

Whenever  $\varphi_1, \dots, \varphi_n$  semantically entail  $\psi$ , then  $\psi$  can be proved from  $\varphi_1, \dots, \varphi_n$ . The proof rules are **complete**

# Completeness : 3 steps

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- ▶ Step 1: Show that  $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$
- ▶ Step 2: Show that  $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$

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- ▶ Step 2: Show that  $\vdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$
- ▶ Step 3: Show that  $\varphi_1, \dots, \varphi_n \vdash \psi$

# Completeness : Step 1

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- ▶ Assume  $\varphi_1, \dots, \varphi_n \models \psi$ . Whenever all of  $\varphi_1, \dots, \varphi_n$  evaluate to true, so does  $\psi$ .

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- ▶ Assume  $\varphi_1, \dots, \varphi_n \models \psi$ . Whenever all of  $\varphi_1, \dots, \varphi_n$  evaluate to true, so does  $\psi$ .
- ▶ If  $\nvdash \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$ , then  $\psi$  evaluates to false when all of  $\varphi_1, \dots, \varphi_n$  evaluate to true, a contradiction.

# Completeness : Step 1

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- ▶ Assume  $\varphi_1, \dots, \varphi_n \models \psi$ . Whenever all of  $\varphi_1, \dots, \varphi_n$  evaluate to true, so does  $\psi$ .
- ▶ If  $\not\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$ , then  $\psi$  evaluates to false when all of  $\varphi_1, \dots, \varphi_n$  evaluate to true, a contradiction.
- ▶ Hence,  $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$ .

# Completeness : Step 2

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- ▶ Given  $\models \psi$ , show that  $\vdash \psi$
- ▶ Assume  $p_1, \dots, p_n$  are the propositional variables in  $\psi$ . We know that all the  $2^n$  assignments of values to  $p_1, \dots, p_n$  make  $\psi$  true.

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- ▶ Using this insight, we have to give a proof of  $\psi$ .

# Completeness : Step 2

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## Truth Table to Proof

Let  $\varphi$  be a formula with variables  $p_1, \dots, p_n$ . Let  $\mathcal{T}$  be the truth table of  $\varphi$ , and let  $l$  be a line number in  $\mathcal{T}$ . Let  $\hat{p}_i$  represent  $p_i$  if  $p_i$  is assigned true in line  $l$ , and let it denote  $\neg p_i$  if  $p_i$  is assigned false in line  $l$ . Then

1.  $\hat{p}_1, \dots, \hat{p}_n \vdash \varphi$  if  $\varphi$  evaluates to true in line  $l$
2.  $\hat{p}_1, \dots, \hat{p}_n \vdash \neg\varphi$  if  $\varphi$  evaluates to false in line  $l$