CS 228 : Logic in Computer Science

Krishna. S

Resolution

- Given a propositional logic formula φ, is it unsatisfiable?

Resolution

- Given a propositional logic formula φ, is it unsatisfiable?
- How does a solver do it?
- Assume φ is in CNF.

Resolution

- Given a propositional logic formula φ, is it unsatisfiable?
- How does a solver do it?
- Assume φ is in CNF.
- Let $C_1, \ldots, C_n$ be the clauses in φ. We denote each C_i as a set of literals.
- If $C_1 = \neg p \vee q \vee r$, we denote C_1 as the set $\{\neg p, q, r\}$

Resolution

- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p.

Resolution

- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p. Then the clause $R = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ is a resolvent of C_1 and C_2.
- Let $C_1 = \{p_1, \neg p_2, p_3\}$ and $C_2 = \{p_2, \neg p_3, p_4\}$.

Resolution

- Let C_1, C_2 be two clauses. Assume $p \in C_1$ and $\neg p \in C_2$ for some literal p. Then the clause $R = (C_1 - \{p\}) \cup (C_2 - \{\neg p\})$ is a resolvent of C_1 and C_2.
- Let $C_1 = \{p_1, \neg p_2, p_3\}$ and $C_2 = \{p_2, \neg p_3, p_4\}$. As $p_3 \in C_1$ and $\neg p_3 \in C_2$, we can resolve C_1, C_2 with the resolvent is $\{p_1, p_2, \neg p_2, p_4\}$.
- Resolvents are not unique : $\{p_1, p_3, \neg p_3, p_4\}$ is also a resolvent.

3 rules in Resolution

- Let G be any formula. Let F be the CNF formula resulting from the CNF algorithm applied to G. Then $G \vdash F$ (Prove!)

3 rules in Resolution

- Let G be any formula. Let F be the CNF formula resulting from the CNF algorithm applied to G. Then $G \vdash F$ (Prove!)
- Let F be a formula in CNF, and let C be a clause in F. Then $F \vdash C$ (Prove!)

3 rules in Resolution

- Let G be any formula. Let F be the CNF formula resulting from the CNF algorithm applied to G. Then $G \vdash F$ (Prove!)
- Let F be a formula in CNF, and let C be a clause in F. Then $F \vdash C$ (Prove!)
- Let F be a formula in CNF. Let R be a resolvent of two clauses of F. Then $F \vdash R$ (Prove!)

Completeness of Resolution

Show that resolution can be used to determine whether any given formula is unsatisfiable.

- Given F in CNF, let $Res^0(F) = \{C \mid C \text{ is a clause in } F\}$.

Completeness of Resolution

Show that resolution can be used to determine whether any given formula is unsatisfiable.

- Given F in CNF, let $Res^0(F) = \{C \mid C \text{ is a clause in } F\}$.
- $Res^n(F) = Res^{n-1}(F) \cup \{R \mid R \text{ is a resolvent of two clauses in } Res^{n-1}(F)\}$

Completeness of Resolution

Show that resolution can be used to determine whether any given formula is unsatisfiable.

- Given F in CNF, let $Res^0(F) = \{C \mid C \text{ is a clause in } F\}$.
- $Res^n(F) = Res^{n-1}(F) \cup \{R \mid R \text{ is a resolvent of two clauses in } Res^{n-1}(F)\}$
- $Res^0(F) = F$, there are finitely many clauses in F,
- $Res^1(F)$ is finite, and there are finitely many fresh clauses that can be derived from F,
- $Res^2(F)$ is finite, there are finitely many fresh clauses that can be derived from F and $Res^1(F)$,

Completeness of Resolution

Show that resolution can be used to determine whether any given formula is unsatisfiable.

- Given F in CNF, let $Res^0(F) = \{C \mid C \text{ is a clause in } F\}$.
- $Res^n(F) = Res^{n-1}(F) \cup \{R \mid R \text{ is a resolvent of two clauses in } Res^{n-1}(F)\}$
- $Res^0(F) = F$, there are finitely many clauses in F,
- $Res^1(F)$ is finite, and there are finitely many fresh clauses that can be derived from F,
- $Res^2(F)$ is finite, there are finitely many fresh clauses that can be derived from F and $Res^1(F)$,
- There is some $m \geqslant 0$ such that $Res^m(F) = Res^{m+1}(F)$. Denote it by $Res^*(F)$.

Example

Let $F = \{\{p\}, \{\neg p\}\}$

- $Res^0(F) = F$

Example

Let $F = \{\{p\}, \{\neg p\}\}$

- $Res^0(F) = F$
- $Res^1(F) = F \cup \emptyset = Res^*(F)$

Example

Let $F = \{\{p\}, \{\neg p\}\}$

- $Res^0(F) = F$
- $Res^1(F) = F \cup \emptyset = Res^*(F)$

Let $F = \{\{p_1, p_2, \neg p_3\}, \{\neg p_2, p_3\}\}$.

- $Res^0(F) = F$

Example

Let $F = \{\{p\}, \{\neg p\}\}$

- $Res^0(F) = F$
- $Res^1(F) = F \cup \emptyset = Res^*(F)$

Let $F = \{\{p_1, p_2, \neg p_3\}, \{\neg p_2, p_3\}\}$.

- $Res^0(F) = F$
- $Res^1(F) = F \cup \{p_1, p_2, \neg p_2\} \cup \{p_1, \neg p_3, p_3\}$.

Example

Let $F = \{\{p\}, \{\neg p\}\}$

- $Res^0(F) = F$
- $Res^1(F) = F \cup \emptyset = Res^*(F)$

Let $F = \{\{p_1, p_2, \neg p_3\}, \{\neg p_2, p_3\}\}$.

- $Res^0(F) = F$
- $Res^1(F) = F \cup \{p_1, p_2, \neg p_2\} \cup \{p_1, \neg p_3, p_3\}$.
- $Res^2(F) = Res^1(F) \cup \{p_1, p_2, \neg p_3\} \cup \{p_1, p_3, \neg p_2\}$

Soundness and Completeness of Resolution

Soundness of Resolution

Let F be a formula in CNF. If $\emptyset \in Res^*(F)$, then F is unsatisfiable.

- If $\emptyset \in Res^*(F)$. Then $\emptyset \in Res^n(F)$ for some n.

Soundness and Completeness of Resolution

Soundness of Resolution

Let F be a formula in CNF. If $\emptyset \in Res^*(F)$, then F is unsatisfiable.

- If $\emptyset \in Res^*(F)$. Then $\emptyset \in Res^n(F)$ for some n.
- Since $\emptyset \notin Res^0(F)$ ($\emptyset$ is not a clause), there is an $m > 0$ such that $\emptyset \notin Res^m(F)$ and $\emptyset \in Res^{m+1}(F)$.

Soundness and Completeness of Resolution

Soundness of Resolution

Let F be a formula in CNF. If $\emptyset \in \mathit{Res}^*(F)$, then F is unsatisfiable.

- If $\emptyset \in \mathit{Res}^*(F)$. Then $\emptyset \in \mathit{Res}^n(F)$ for some n.
- Since $\emptyset \notin \mathit{Res}^0(F)$ ($\emptyset$ is not a clause), there is an $m > 0$ such that $\emptyset \notin \mathit{Res}^m(F)$ and $\emptyset \in \mathit{Res}^{m+1}(F)$.
- Then $\{p\}, \{\neg p\} \in \mathit{Res}^m(F)$. By the rules of resolution, we have $F \vdash p, \neg p$, and thus $F \vdash \bot$. Hence, F is unsatisfiable.

Soundness and Completeness of Resolution

Completeness of Resolution

If F is unsatisfiable, then $\emptyset \in Res^*(F)$.

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in Res^*(F)$.

- Let F have k clauses $C_1, \ldots, C_k$.

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in Res^*(F)$.

- Let F have k clauses $C_1, \ldots, C_k$.
- wlg, assume that no C_i has both p and $\neg p$

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in Res^*(F)$.

- Let F have k clauses $C_1, \ldots, C_k$.
- wlg, assume that no C_i has both p and $\neg p$
- Induct on the number n of propositional variables that occur in F.

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in Res^*(F)$.

- Let F have k clauses $C_1, \ldots, C_k$.
- wlg, assume that no C_i has both p and $\neg p$
- Induct on the number n of propositional variables that occur in F.
- If $n = 1$, then the possible clauses are p, $\neg p$ and $p \vee \neg p$. The third one is ruled out, by assumption.

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in Res^*(F)$.

- Let F have k clauses $C_1, \ldots, C_k$.
- wlg, assume that no C_i has both p and $\neg p$
- Induct on the number n of propositional variables that occur in F.
- If $n = 1$, then the possible clauses are p, $\neg p$ and $p \vee \neg p$. The third one is ruled out, by assumption.
- If $F = \{\{p\}\}$ or $F = \{\{\neg p\}\}$, F is satisfiable.

Resolution

If F in CNF is unsatisfiable, then $\emptyset \in Res^*(F)$.

- Let F have k clauses $C_1, \ldots, C_k$.
- wlg, assume that no C_i has both p and $\neg p$
- Induct on the number n of propositional variables that occur in F.
- If $n = 1$, then the possible clauses are p, $\neg p$ and $p \vee \neg p$. The third one is ruled out, by assumption.
- If $F = \{\{p\}\}$ or $F = \{\{\neg p\}\}$, F is satisfiable.
- Hence, F must be $\{\{p\}, \{\neg p\}\}$. Clearly, $\emptyset \in Res^1(F)$.

Resolution

- Inductive hypothesis : If F has $\leqslant n$ variables and is unsat, then $\emptyset \in Res^*(F)$.

Resolution

- Inductive hypothesis : If F has $\leqslant n$ variables and is unsat, then $\emptyset \in Res^*(F)$.
- Let F have $n+1$ variables $p_1, \ldots, p_{n+1}$.

Resolution

- Inductive hypothesis : If F has $\leqslant n$ variables and is unsat, then $\emptyset \in Res^*(F)$.
- Let F have $n+1$ variables $p_1, \ldots, p_{n+1}$.

- Let G_0 be the conjunction of all C_i in F such that $\neg p_{n+1} \notin C_i$.
- Let G_1 be the conjunction of all C_i in F such that $p_{n+1} \notin C_i$.

Resolution

- Inductive hypothesis : If F has $\leqslant n$ variables and is unsat, then $\emptyset \in Res^*(F)$.
- Let F have $n+1$ variables $p_1, \ldots, p_{n+1}$.

- Let G_0 be the conjunction of all C_i in F such that $\neg p_{n+1} \notin C_i$.
- Let G_1 be the conjunction of all C_i in F such that $p_{n+1} \notin C_i$.

- Clauses in F= Clauses in G_0 $\cup$ Clauses in G_1

Resolution

- Inductive hypothesis : If F has $\leqslant n$ variables and is unsat, then $\emptyset \in Res^*(F)$.
- Let F have $n+1$ variables $p_1, \ldots, p_{n+1}$.

  - Let G_0 be the conjunction of all C_i in F such that $\neg p_{n+1} \notin C_i$.
  - Let G_1 be the conjunction of all C_i in F such that $p_{n+1} \notin C_i$.

- Clauses in F= Clauses in $G_0 \cup$ Clauses in G_1

- Let $F_0 = \{C_i - \{p_{n+1}\} \mid C_i \in G_0\}$
- Let $F_1 = \{C_i - \{\neg p_{n+1}\} \mid C_i \in G_1\}$

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$

- If p_{n+1} (here p_3) is assigned false in F, then F is satisfiable whenever F_0 is satisfiable

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$

- If p_{n+1} (here p_3) is assigned false in F, then F is satisfiable whenever F_0 is satisfiable
- If p_{n+1} (here p_3) is assigned true in F, then F is satisfiable whenever F_1 is satisfiable

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$

- If p_{n+1} (here p_3) is assigned false in F, then F is satisfiable whenever F_0 is satisfiable
- If p_{n+1} (here p_3) is assigned true in F, then F is satisfiable whenever F_1 is satisfiable
- Hence F is satisfiable whenever $F_0 \vee F_1$ is satisfiable.

Resolution

Let $F = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}, \{\neg p_2, \neg p_3\}\}$ and $n = 2$.

- $G_0 = \{\{p_1, p_3\}, \{p_2\}, \{\neg p_1, \neg p_2, p_3\}\}$, $G_1 = \{\{p_2\}, \{\neg p_2, \neg p_3\}\}$.
- $F_0 = \{\{p_1\}, \{p_2\}, \{\neg p_1, \neg p_2\}\}$ and $F_1 = \{\{p_2\}, \{\neg p_2\}\}$

- If p_{n+1} (here p_3) is assigned false in F, then F is satisfiable whenever F_0 is satisfiable
- If p_{n+1} (here p_3) is assigned true in F, then F is satisfiable whenever F_1 is satisfiable
- Hence F is satisfiable whenever $F_0 \vee F_1$ is satisfiable.
- As F is unsatisfiable, F_0 and F_1 are both unsatisfiable.

Resolution

- By induction hypothesis, $\emptyset \in \mathit{Res}^*(F_0)$ and $\emptyset \in \mathit{Res}^*(F_1)$.

Resolution

- By induction hypothesis, $\emptyset \in Res^*(F_0)$ and $\emptyset \in Res^*(F_1)$.
- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.

Resolution

- By induction hypothesis, $\emptyset \in Res^*(F_0)$ and $\emptyset \in Res^*(F_1)$.
- Hence, $\emptyset \in Res^*(G_0)$ or $\{p_{n+1}\} \in Res^*(G_0)$, and $\emptyset \in Res^*(G_1)$ or $\{\neg p_{n+1}\} \in Res^*(G_1)$.
- If $\emptyset \in Res^*(G_0)$ or $\emptyset \in Res^*(G_1)$, then $\emptyset \in Res^*(F)$.

Resolution

- By induction hypothesis, $\emptyset \in \mathit{Res}^*(F_0)$ and $\emptyset \in \mathit{Res}^*(F_1)$.
- Hence, $\emptyset \in \mathit{Res}^*(G_0)$ or $\{p_{n+1}\} \in \mathit{Res}^*(G_0)$, and $\emptyset \in \mathit{Res}^*(G_1)$ or $\{\neg p_{n+1}\} \in \mathit{Res}^*(G_1)$.
- If $\emptyset \in \mathit{Res}^*(G_0)$ or $\emptyset \in \mathit{Res}^*(G_1)$, then $\emptyset \in \mathit{Res}^*(F)$.
- Else, $\{p_{n+1}\} \in \mathit{Res}^*(G_0)$ and $\{\neg p_{n+1}\} \in \mathit{Res}^*(G_1)$.

Resolution

- By induction hypothesis, $\emptyset \in \mathit{Res}^*(F_0)$ and $\emptyset \in \mathit{Res}^*(F_1)$.
- Hence, $\emptyset \in \mathit{Res}^*(G_0)$ or $\{p_{n+1}\} \in \mathit{Res}^*(G_0)$, and $\emptyset \in \mathit{Res}^*(G_1)$ or $\{\neg p_{n+1}\} \in \mathit{Res}^*(G_1)$.
- If $\emptyset \in \mathit{Res}^*(G_0)$ or $\emptyset \in \mathit{Res}^*(G_1)$, then $\emptyset \in \mathit{Res}^*(F)$.
- Else, $\{p_{n+1}\} \in \mathit{Res}^*(G_0)$ and $\{\neg p_{n+1}\} \in \mathit{Res}^*(G_1)$.
- Hence $\emptyset \in \mathit{Res}^*(F)$.

Resolution Summary

Given a formula ψ, convert it into CNF, say ζ. ψ is satisfiable iff $\emptyset \notin Res^*(\zeta)$.

- If ψ is unsat, we might get $\emptyset$ before reaching $Res^*(\zeta)$.
- If ψ is sat, then truth tables are faster : stop when some row evaluates to 1.

Pop Quiz

1. Prove using natural deduction

$$p \to (q \vee r \vee s),\ q \to (\neg p \vee \neg s),\ (r \vee s) \to q \ \vdash \ \neg p \vee \neg s$$

   You cannot use LEM directly or indirectly (that is, proving LEM)

2. Is this formula satisfiable? Check using HornSAT.

$$\neg q \wedge (\neg p \vee \neg r \vee s) \wedge (\neg p \vee \neg s \vee t) \wedge (\neg r \vee \neg t \vee q) \wedge p \wedge$$

$$(\neg s \vee \neg t \vee r) \wedge (\neg r \vee \neg s \vee \neg t) \wedge (\neg p \vee r)$$