



CS 228 : Logic in Computer Science

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 - ▶ $p \vee q \not\models q \vee r$ (why?)
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- ▶ Formulae φ and ψ are **semantically equivalent** iff $\varphi \models \psi$ and $\psi \models \varphi$

Soundness of Propositional Logic

$$\varphi_1, \dots, \varphi_n \vdash \psi \Rightarrow \varphi_1, \dots, \varphi_n \models \psi$$

Whenever ψ can be proved from $\varphi_1, \dots, \varphi_n$, then ψ evaluates to true whenever $\varphi_1, \dots, \varphi_n$ evaluate to true

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- ▶ Consider now a proof with k lines.

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- ▶ By inductive hypothesis, we have $\varphi_1, \dots, \varphi_n \models \psi_1$ and $\varphi_1, \dots, \varphi_n \models \psi_2$. By semantics, we have $\varphi_1, \dots, \varphi_n \models \psi_1 \wedge \psi_2$.

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- ▶ Assume ψ was obtained using $\rightarrow i$. Then ψ is of the form $\psi_1 \rightarrow \psi_2$.
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- ▶ The line just after the box was ψ .
- ▶ Consider adding ψ_1 in the premises along with $\varphi_1, \dots, \varphi_n$. Then we will get a proof $\varphi_1, \dots, \varphi_n, \psi_1 \vdash \psi_2$, of length $k - 1$. By inductive hypothesis, $\varphi_1, \dots, \varphi_n, \psi_1 \models \psi_2$. By semantics, this gives $\varphi_1, \dots, \varphi_n \models \psi_1 \rightarrow \psi_2$

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- ▶ The equivalence of $\varphi_1, \dots, \varphi_n \vdash \psi_1 \rightarrow \psi_2$ and $\varphi_1, \dots, \varphi_n, \psi_1 \vdash \psi_2$ gives the proof.

Soundness : Other cases

Completeness

$$\varphi_1, \dots, \varphi_n \models \psi \Rightarrow \varphi_1, \dots, \varphi_n \vdash \psi$$

Whenever $\varphi_1, \dots, \varphi_n$ semantically entail ψ , then ψ can be proved from $\varphi_1, \dots, \varphi_n$. The proof rules are **complete**

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- ▶ Step 3: Show that $\varphi_1, \dots, \varphi_n \vdash \psi$

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- ▶ If $\not\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$, then ψ evaluates to false when all of $\varphi_1, \dots, \varphi_n$ evaluate to true, a contradiction.

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- ▶ If $\not\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$, then ψ evaluates to false when all of $\varphi_1, \dots, \varphi_n$ evaluate to true, a contradiction.
- ▶ Hence, $\models \varphi_1 \rightarrow (\varphi_2 \rightarrow (\dots (\varphi_n \rightarrow \psi) \dots))$.

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- ▶ Using this insight, we have to give a proof of ψ .

Completeness : Step 2

Truth Table to Proof

Let φ be a formula with variables p_1, \dots, p_n . Let \mathcal{T} be the truth table of φ , and let l be a line number in \mathcal{T} . Let \hat{p}_i represent p_i if p_i is assigned true in line l , and let it denote $\neg p_i$ if p_i is assigned false in line l . Then

1. $\hat{p}_1, \dots, \hat{p}_n \vdash \varphi$ if φ evaluates to true in line l
2. $\hat{p}_1, \dots, \hat{p}_n \vdash \neg \varphi$ if φ evaluates to false in line l