~ Chapter - 6 ~

System of Particles & Rotational motion

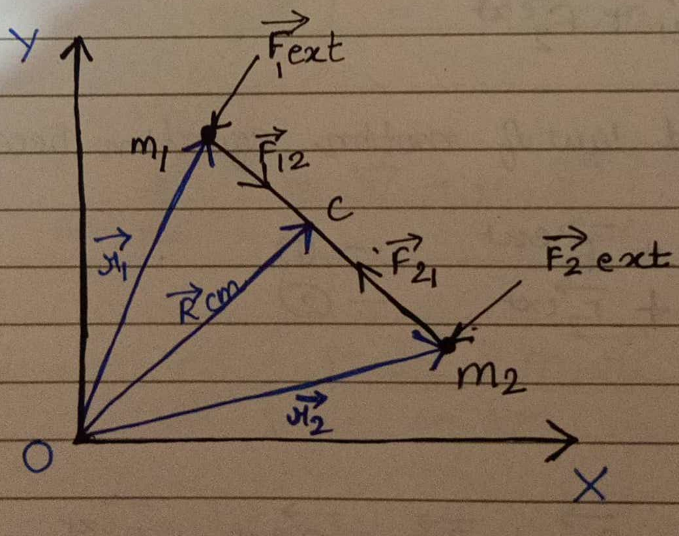
* CENTRE OF MASS *

The centre of mass of a system of particals is a fixed point at which whole mass of the system may be supposed to be concentrated for describing its transratory motion.

from AB-INITIO

Derivation of expression for the centre of mass of a two particle system.

Consider a system of two particles P_1 and P_2 of masses m_1 and m_2. Let $\vec{r_1}$ and $\vec{r_2}$ be their position vectors at any instant t with respect to the origin O.



$$\vec{V_1} = \frac{d\vec{r_1}}{dt} \qquad \vec{V_2} = \frac{d\vec{r_2}}{dt}$$

$$a_1 = \frac{dv_1}{dt} = \frac{d}{dt}\left(\frac{d\vec{r_1}}{dt}\right)$$

$$= \boxed{\frac{d^2 r_1}{dt^2}}$$

$$a_2 = \frac{dv_2}{dt} = \frac{d}{dt}\left(\frac{dr_2}{dt}\right)$$

$$= \boxed{\frac{d^2 r_2}{dt^2}}$$

* Total force $\vec{F_1}$ acting on particle P_1 is the sum of the internal force $\vec{F}_{12}$ due to P_2 and external force $\vec{F_1}^{ext}$ on it.

Thus,

$$\vec{F_1} = \vec{F}_{12} + \vec{F_1}^{ext}$$

Similarly, total force acting on Particle P_2 is given by

$$\vec{F_2} = \vec{F_{21}} + \vec{F_2}^{ext}$$

According to Newton 2nd law of motion equation becomes

$$m_1\vec{a_1} = \vec{F_{12}} + \vec{F_1}^{ext} \quad -①$$

$$m_2\vec{a_2} = \vec{F_{21}} + \vec{F_2}^{ext} \quad -②$$

Adding ① and ②

$$m_1\vec{a_1} + m_2\vec{a_2} = \vec{F_{12}} + \vec{F_{21}} + \vec{F_1}^{ext} + \vec{F_2}^{ext} \quad -③$$

According to Newton 3rd law.

$$\vec{F_{12}} = -\vec{F_{21}}$$

$$\vec{F_{12}} + \vec{F_{21}} = 0$$

Also $\vec{F_1}^{ext} + \vec{F_2}^{ext} = F$

Now equation (3) becomes

$$m_1\vec{a_1} + m_2\vec{a_2} = 0 + F = F$$
$$m_1\vec{a_1} + m_2\vec{a_2} = \vec{F}$$

Let total mass of two particle system is 'M' then
$$M = m_1 + m_2$$

Now,
$$\vec{F} = M\vec{a}_{cm}$$

$$m_1\vec{a_1} + m_2\vec{a_2} = M\vec{a}_{cm}$$

$$m_1 \frac{d^2\vec{r_1}}{dt^2} + m_2 \frac{d^2\vec{r_2}}{dt^2} = M\vec{a}_{cm}$$

$$\frac{d^2}{dt^2}\left[m_1\vec{r_1} + m_2\vec{r_2}\right] = M\vec{a}_{cm}$$

$$\vec{a}_{cm} = \frac{1}{M}\frac{d^2}{dt^2}\left[m_1\vec{r_1} + m_2\vec{r_2}\right]$$

$$\frac{\cancel{d^2} R_{cm}}{\cancel{dt^2}} = \frac{\cancel{d^2}}{\cancel{dt^2}}\left[\frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}\right] \quad (\because M = m_1 + m_2)$$

$$R_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

For n particle system

$$\boxed{R_{cm} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \ldots\ldots\ldots + m_n\vec{r_n}}{m_1 + m_2 + \ldots\ldots\ldots m_n}}$$

* Equation of motion for centre of mass *

Let $\vec{F_1}, \vec{F_2}, \vec{F_3}, \ldots\ldots, \vec{F_n}$ be the external forces acting on the particles of masses $m_1, m_2, m_3, \ldots\ldots, m_n$ respectively. If $\vec{a}_{cm}$ is the acceleration of the centre of of mass and $\vec{F}_{tot}$ ($\vec{F}_{total}$) be the vector sum of all external forces acting on a system then,

$$M\vec{a}_{CM} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots\ldots + \vec{F_n}$$

$$M\vec{a}_{CM} = \vec{F}_{tot}$$

Where

$$\vec{a}_{cm} = \frac{d^2 R_{cm}}{dt^2}$$

That is center of the system moves as if the entire mass of the system is concentrated at this point and total external force acts on this point.

Velocity of centre of mass is constant in the absence of external force.

$$\vec{F}_{tot} = M\vec{a}_{CM}$$

In the absence of any external force

$$\vec{F}_{tot} = 0$$

$$M\vec{a}_{CM} = 0$$

$$M \neq 0$$

$$\vec{a}_{CM} = 0$$

$$\frac{d\vec{V}_{CM}}{dt} = 0 \qquad \vec{V}_{CM} = \text{constant}$$

As the derivative of constant is zero therefore velocity of centere of mass is constant.

right answer

Momentum Conservation and Centre of mass motion

Consider a system of 'n' particles of masses $m_1, m_2, \ldots, m_n$
Suppose the force $F_1, F_2, \ldots, F_n$ ~~ext~~ exerted on them
produce acceleration $a_1, a_2 \ldots a_n$ respectively. In the
absence of any external force.

$$\vec{F}_{tot} = 0$$

$$\vec{F_1} + \vec{F_2} + \ldots + \vec{F_n} = 0$$

$$m_1\vec{a_1} + m_2\vec{a_2} + \ldots + \frac{m d\vec{n}}{dt} = 0$$

$$\frac{d}{dt}\left[m_1 v_1 + m_2 v_2 + \ldots + m_n v_n\right] = 0$$

$$\frac{d}{dt}\left[P_1 + P_2 + \ldots + P_n\right] = 0$$

$$\vec{P_1} + \vec{P_2} + \ldots \vec{P_n} = \text{constant}$$

Hence if no net external force acts on a system the total
linear momentum of system is conserved.

Now,

Position vector of centre of mass of n ~~particle~~ particle
system is given by :-

$$\vec{R}_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots\dots + m_n\vec{r_n}}{m_1 + m_2 + \dots\dots + m_n}$$

$$\vec{R}_{CM} = \frac{m_1\vec{r_1} + m_2\vec{r_2} + \dots\dots + m_n\vec{r_n}}{M}$$

* Differentiating b/w with respect to 't' *

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M}\frac{d}{dt}\left(m_1\frac{d\vec{r_1}}{dt} + m_2\frac{d\vec{r_2}}{dt} + m_n\frac{d\vec{r_n}}{dt}\right)$$

$$\vec{V}_{CM} = \frac{1}{M}\left(m_1\vec{V_1} + m_2\vec{V_2} + \dots\dots + m_n\vec{V_n}\right)$$

$$\vec{V}_{CM} = \frac{1}{M}\left(\vec{P_1} + \vec{P_2} + \dots\dots + \vec{P_n}\right)$$

$$\vec{V}_{CM} = \frac{1}{M}\vec{P}$$

$$\vec{P} = M\vec{V}_{CM}$$

Hence, total linear momentum of a system of particles is equal to the product of the total mass of the system and the velocity of its centre of mass.

Rigid body ?

A body is said to be rigid if it does not undergo any change in its size and shape, however large the external force may be acting on it.

Centre of mass of Rigid body

The centre of mass of a rigid body is a point at a fixed position with respect to ~~time~~ body as a whole.

The position of centre of a mass of a rigid body depends on 2 factors:

1) Geometrical Shape of the body.
2) The distribution of mass in the body.

S.No	Shape of Body	Position of Centre of mass
1	Long thin rod	Middle point of the rod
2	Thin circular ring	Geometrical centre of the ring
3	Circular disc	" " " disc
4	Rectangular lamina	Point of intersection of diagonals
5	" " cubical block	" " " "
6	Cylinder	Middle point of the axis
7	Solid or hollow sphere	Geometrical centre of the sphere.
8	Triangular lamina	Point of intersection of the medians
9	Right circular cone	A point on its axis at a distance of $\frac{h}{4}$ from its base, h = height of cone

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Equations of Rotational Motion

Linear

1. $v = u + at$
2. $s = ut + \frac{1}{2}at^2$
3. $v^2 - u^2 = 2as$

Rotation

$v = \omega \quad u = \omega_0 \quad a = \alpha \quad t = t$

1. $\omega = \omega_0 + \alpha t$
2. $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
3. $\omega^2 - \omega_0^2 = 2\alpha\theta$

Derivation of Eq of rotational motion

Consider a rigid body rotating about a fixed axis with constant angular acceleration α

(1) $\omega = \omega_0 + \alpha t$

Proof

$$\alpha = \frac{d\omega}{dt}$$

$$d\omega = \alpha dt \quad \text{— (1)}$$

At time

$t = 0 \quad vel = \omega_0$

$t = t \quad vel = \omega$

Integrating eq (1) b/s within proper limits

$$\int_{\omega_0}^{\omega} d\omega = \int_0^t \alpha dt$$

$$[\omega]_{\omega_0}^{\omega} = \alpha \int_0^t dt$$

$$[\omega]_{\omega_0}^{\omega} = \alpha \, [t]_0^t$$

$$\omega - \omega_0 = \alpha(t-0)$$

$$\boxed{\omega = \omega_0 + \alpha t}$$

② $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

$$\omega = \frac{d\theta}{dt}$$

$$d\theta = \omega \, dt$$

$$d\theta = (\omega_0 + \alpha t)\, dt \quad -①$$

At time

$t = 0 \qquad \theta = 0$

$t = t \qquad \theta = \theta$

Integrating eq ① b/s within proper limits

$$\int_0^\theta d\theta = \int_0^t \omega_0 \, dt + \int_0^t \alpha t \, dt$$

$$[\theta]_0^\theta = \omega_0 (t)_0^t + \alpha\left(\frac{t^2}{2}\right)$$

$$\boxed{\theta = \omega_0 t + \frac{1}{2}\alpha t^2}$$

③ $\omega^2 - \omega_0^2 = 2\alpha\theta$

$$\alpha = \frac{d\omega}{dt} \times \frac{d\theta}{d\theta} \qquad (d\theta/dt = \omega)$$

$$\alpha = \omega \frac{d\omega}{d\theta}$$

$$\omega\, d\omega = \alpha\, d\theta \quad -①$$

At time

$t = 0 \qquad \theta = 0 \qquad \omega = \omega_0$

$t = t \qquad \theta = \theta \qquad \omega = \omega$

Integrating eq ① b/s within proper limits

$$\int_{\omega_0}^{\omega} \omega\, d\omega = \int_0^{\theta} \alpha\, d\theta$$

$$\left[\frac{\omega^2}{2}\right]_{\omega_0}^{\omega} = \alpha\theta$$

$$\frac{\omega^2 - \omega_0^2}{2} = \alpha\theta$$

$$\boxed{\omega^2 - \omega_0^2 = 2\alpha\theta}$$

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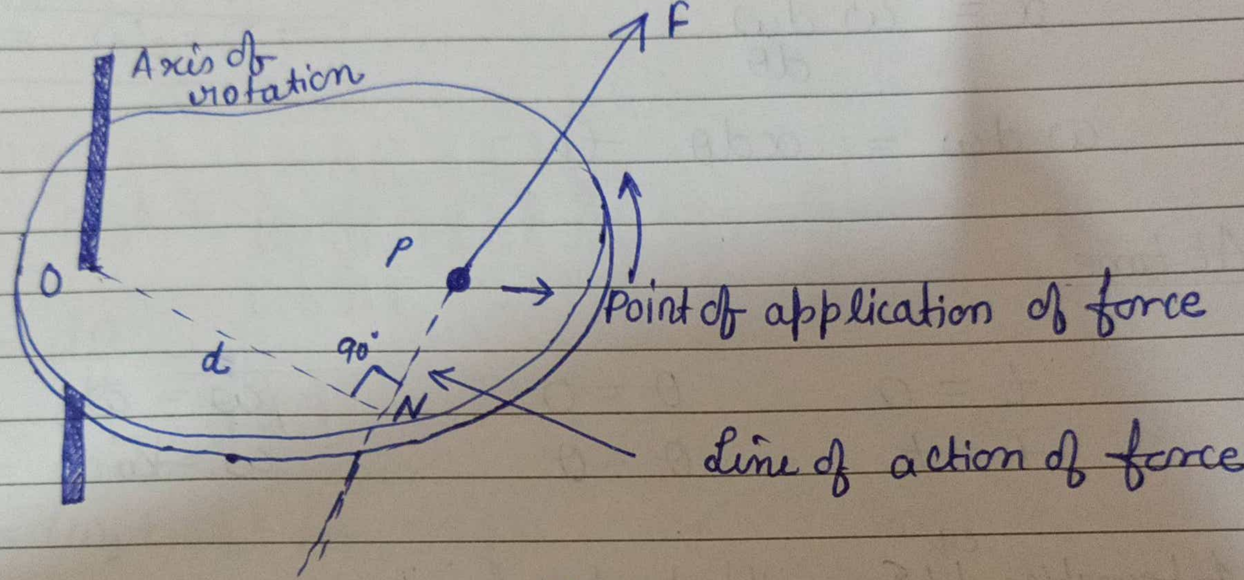
TORQUE OR MOMENT OF FORCE

Torque (τ) → tau (टॉअ)



** It is the turning effect of the force about the axis of rotation.

** It is measured as the product of the magnitude of force and the ⊥ distance b/w the line of action of force and the axis of rotation

$$\tau = F \times ON$$

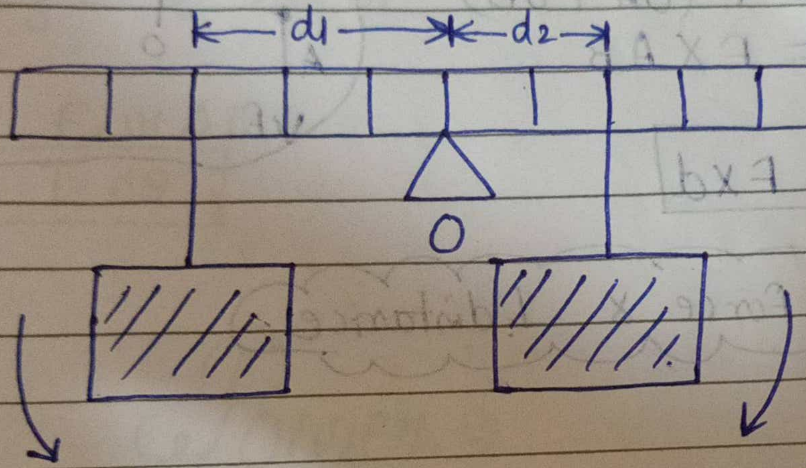
$$\tau = F \times d$$

Torque = Force X Lever arm.

Unit = Newton metre (Nm)

* Principle of Moments *

When a body is in rotational equilibrium the sum of the clockwise moments about any point is equal to the sum of the anticlockwise moments about that point (OR) the algebric sum of moments about any point is zero



$f_1 = m_1 g = W_1$ $\quad$ $F_2 = m_2 g = W_2$

Anticlockise moment about O $= f_1 \times d_1 = W_1 \times d_1$

Clockwise moment about O $= F_2 \times d_2 = W_2 \times d_2$

According to principle of moment, anticlockwise moment is equal to clockwise moment.

$$W_1 \times d_1 = W_2 \times d_2$$

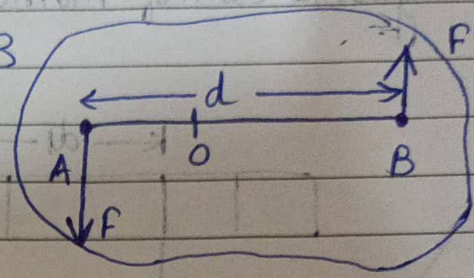
Load X Load arm = Effort X Effort arm

Sometime called **Lever Principle**

* COUPLE * (τ)

A pair of equal and opposite forces acting on a body along two different lines of action constitutes a couple.

Moment of Couple (τ) $= F \times OA + F \times OB$
$= F(OA + OB)$
$= F \times AB$

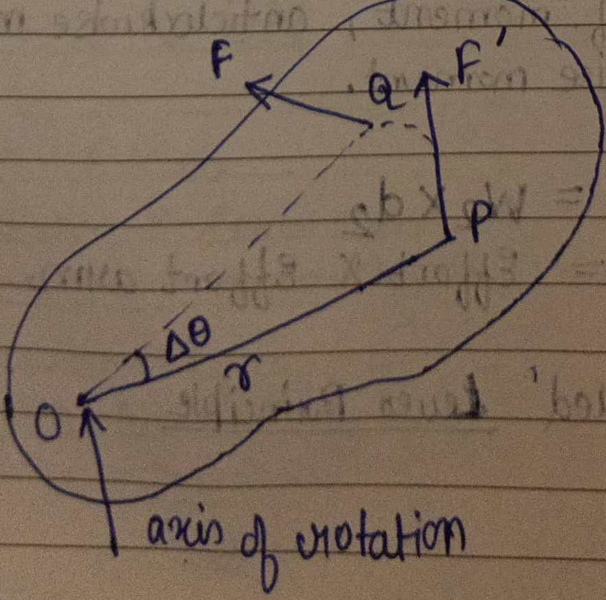


$$\boxed{\tau = F \times d}$$

Torque = force × ⊥ distance

* WORD DONE BY A TORQUE *

Suppose a body undergoes an angular displacement $\Delta\theta$ under the action of tangential force F.



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Work done by Torque is

$$W = F \times \text{distance Arc } PQ$$

Now

$$\text{angle} = \frac{\text{arc } PQ}{\text{radius}}$$

$$\Delta\theta = \frac{\text{Arc } PQ}{r}$$

$$\text{Arc } PQ = \Delta\theta\, r$$

$$W = F \cdot r\,\Delta\theta$$

$$\boxed{W = \tau\,\Delta\theta}$$

Power

Work done by variable force

$$W = \int_{\theta_1}^{\theta_2} \tau\, d\theta$$

Power delivered by torque

we know

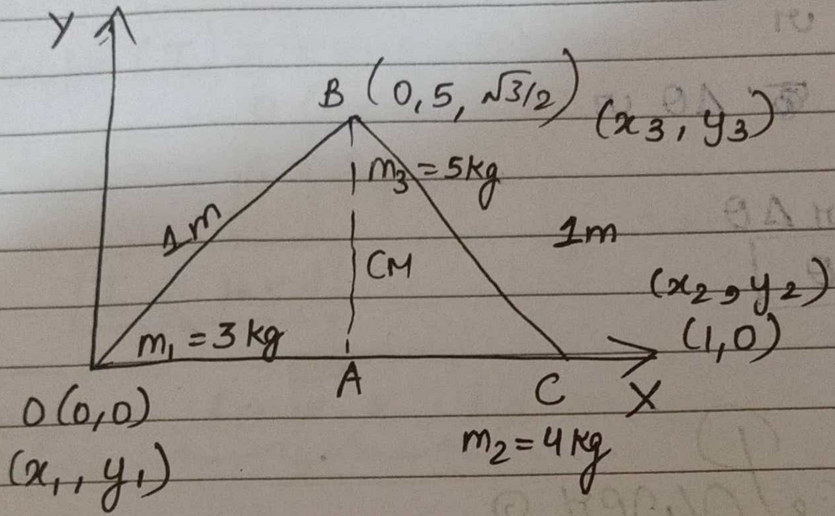
$$\Delta W = \tau\,\Delta\theta$$

Divide by Δt both sides,

$$\frac{\Delta W}{\Delta t} = \tau \frac{\Delta\theta}{\Delta t}$$

$$P = \tau\omega$$

(Q) 3 masses 3, 4, 5 kg are located at the corners of an equilateral triangle of side 1 ~~cm~~ m. Locate the centre of mass.



$$x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$x = \frac{(3 \times 0) + (4 \times 1) + (5 \times 0.5)}{12}$$

$$x = \frac{13}{24} = 0.24$$

$$y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$= \frac{5\sqrt{3}}{24} = 0.36$$

* Angular momentum *

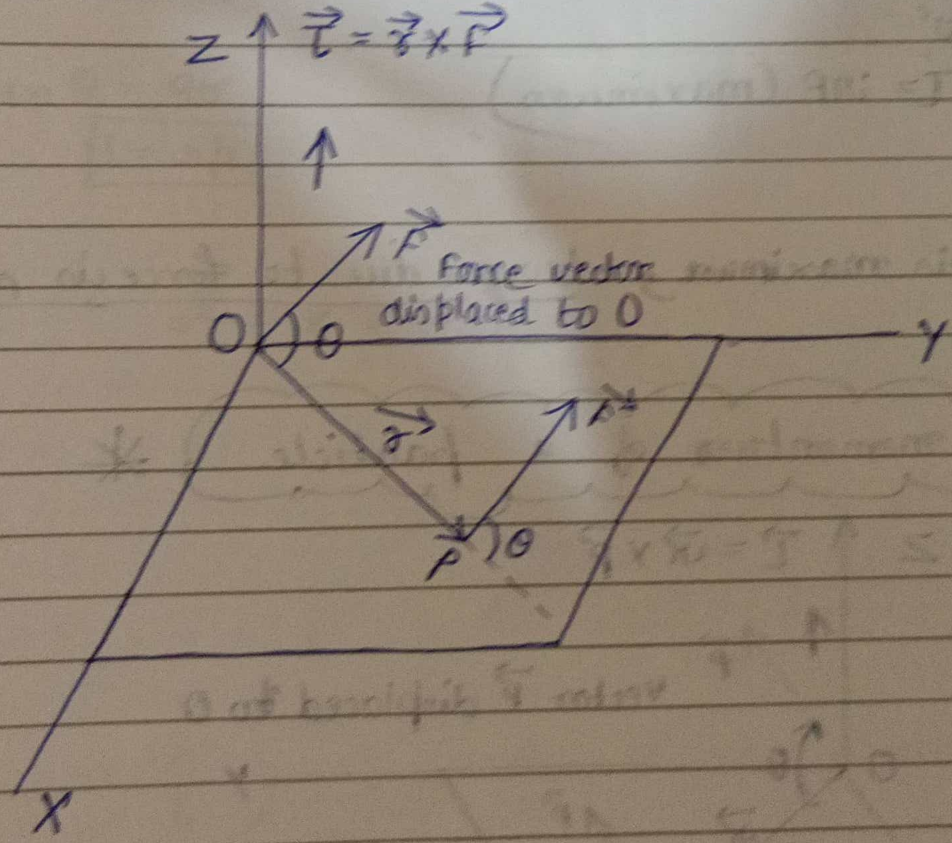
It is defined as the moment of linear momentum of the particle about the axis. It is measured as product of linear momentum and ⊥ distance of its line of action.

$L = P \times \perp \text{distance}$

$L = Pd$

$\tau = Fd$

$L = Pd$



* Torque acting on a particle *

Consider a particle P in X-Y plane. The torque acting on the particle is defined as the vector product of position & force vectors

$$\vec{\tau} = \vec{r} \times \vec{F}$$
$$\vec{\tau} = rF\sin\theta$$

The direction of Torque is given by right hand rule

Special cases,

(i) when $\theta = 0°$ or $180°$,

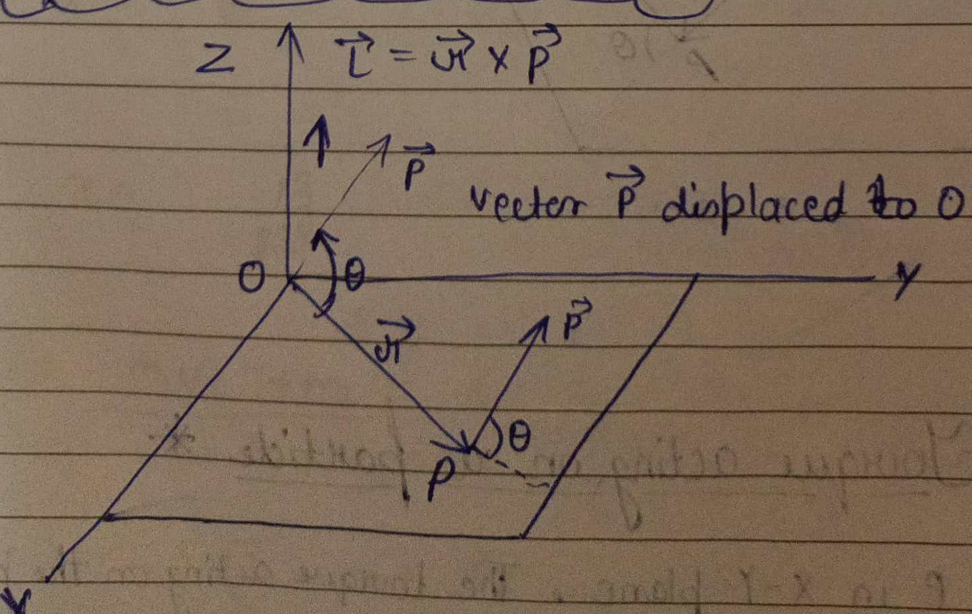
$$\tau = 0$$

(ii) When $\theta = 90°$

$$\tau = rF \text{ (maximum)}$$

(iii) when r is maximum, torque due to force is maximum

* Angular momentum of a particle *



Consider a particle P in x-y plane. The ~~t~~ angular momentum of the particle is defined as the vector product of position vector and linear momentum

$$\vec{\tau} = \vec{r} \times \vec{p}$$

$$\tau = rp\sin\theta$$

The direction is given by right hand rule

<u>Special cases</u>

1. When $\theta = 0^\circ$ or 180°

$$\boxed{L=0}$$

2. when $\theta = 90^\circ$

$$\boxed{L = rp}$$

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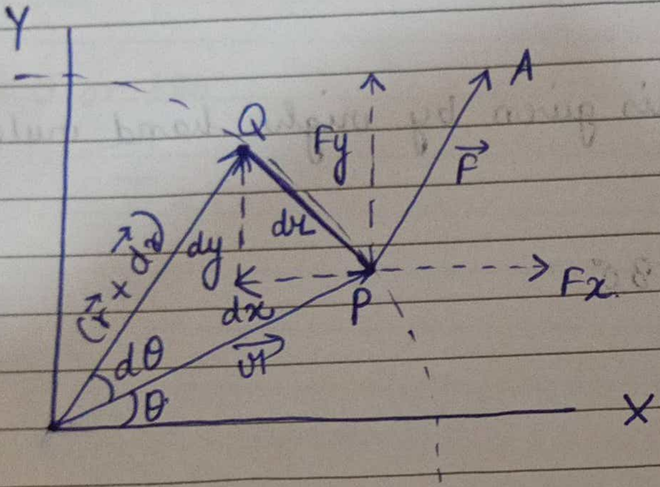
Expression for Torque in Cartesian Coordinates
(Physical meaning of Torque)

①



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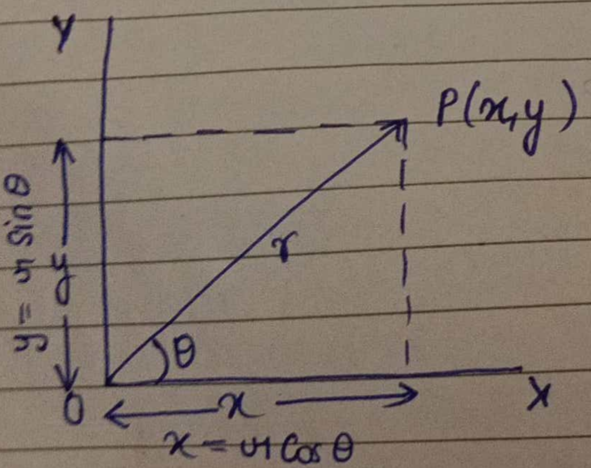


Consider a particle of mass m rotating in x-y plane.

$$\text{Let } OP = \vec{r}, \quad OQ = \vec{r} + \vec{dr}$$

From Δ law of vector addition

$$\vec{OP} + \vec{PQ} = \vec{OQ}$$

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$\vec{PQ} = \cancel{\vec{r}} + \vec{dr} - \cancel{\vec{r}}$$

$$\boxed{\vec{PQ} = \vec{dr}}$$

Small amount of work done

$$dW = F \cdot dr \quad — (1)$$

$$\left.\begin{aligned} F &= \hat{i} F_x + \hat{j} F_y \\ dr &= \hat{i} dx + \hat{j} dy \end{aligned}\right] \text{Put in (1)}$$

$$dW = (\hat{i} F_x + \hat{j} F_y) \cdot (\hat{i} dx + \hat{j} dy)$$

$$\boxed{dW = F_x dx + F_y dy} \quad — (2)$$

$$x = r\cos\theta \qquad y = r\sin\theta$$

differentiating above eqⁿ with respect to θ

$$\frac{dx}{d\theta} = r[-\sin\theta] \qquad \frac{dy}{d\theta} = r\cos\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta \qquad \frac{dy}{d\theta} = x$$

$$\boxed{dx = -y\,d\theta} \qquad \boxed{dy = x\,d\theta}$$

* Put in (2) *

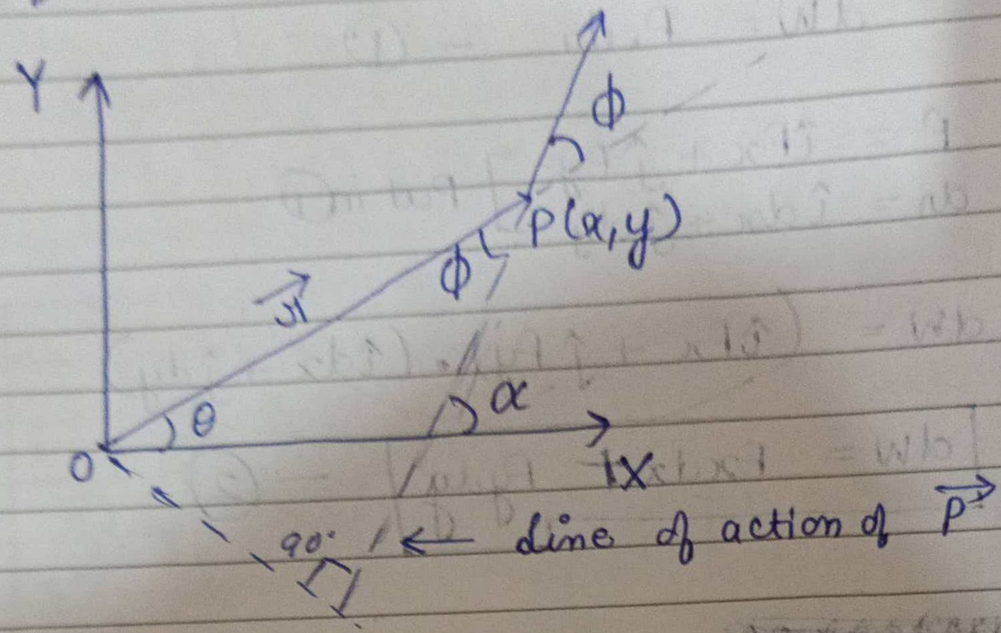
$$dW = F_x(-y\,d\theta) + F_y(x\,d\theta)$$

$$dW = (xF_y - yF_x)\,d\theta$$

$$dW = \tau_z\,d\theta$$

$$\tau_z = xF_y - yF_x$$

Expression for Torque in Polar ~~Quad~~ Coordinates



Suppose a line of action of force F makes an angle α with x-axis. Two rectangular components of F

$$F_x = F\cos\alpha$$
$$F_y = F\sin\alpha$$

Also

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Now

$$\tau = xF_y - yF_x$$
$$\tau = r\cos\theta\, F\sin\alpha - r\sin\theta\, F\cos\alpha$$
$$\tau = rF(\sin\alpha\cos\theta - \cos\alpha\sin\theta) \quad \left[\because \sin a\cos b - \cos a\sin b = \sin(a-b)\right]$$
$$\tau = rF\sin(\alpha - \theta) \quad -(1)$$

From diagram

$$\theta + \phi = \alpha \quad \text{(Exterior angle property (E.A.P))}$$
$$\phi = \alpha - \theta$$

Eq ① becomes

$$\boxed{\tau = rF \sin\phi}$$

Expression for angular momentum in Cartesian Coordinates

We know

$$\tau = xF_y - yF_x \quad \text{—①}$$

According to Newton's 2nd law of motion

Put in ①

$$F_x = \frac{dp_x}{dt} = \frac{d(mv_x)}{dt} = m\frac{dv_x}{dt}$$

$$F_y = \frac{dp_y}{dt} = \frac{d(mv_y)}{dt} = m\frac{dv_y}{dt}$$

$$\tau = x\, m\frac{dv_y}{dt} - y\, m\frac{dv_x}{dt}$$

$$\tau = m\left(x\frac{dv_y}{dt} - y\frac{dv_x}{dt}\right) \quad \text{—②}$$

Using rule for differentiation of Products

$$\frac{d}{dt}(xV_y - yV_x) = x\frac{dV_y}{dt} + V_y\frac{dx}{dt} - y\frac{dV_x}{dt} - V_x\frac{dy}{dt}$$

$$= x\frac{dV_y}{dt} + \cancel{V_yV_x} - y\frac{dV_x}{dt} - \cancel{V_xV_y}$$

$$= x\frac{dV_y}{dt} - y\frac{dV_x}{dt}$$

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Now eq. ②

$$\tau = m\frac{d}{dt}(xV_y - yV_x)$$

$$\tau = \frac{d}{dt}(xmV_y - ymV_x)$$

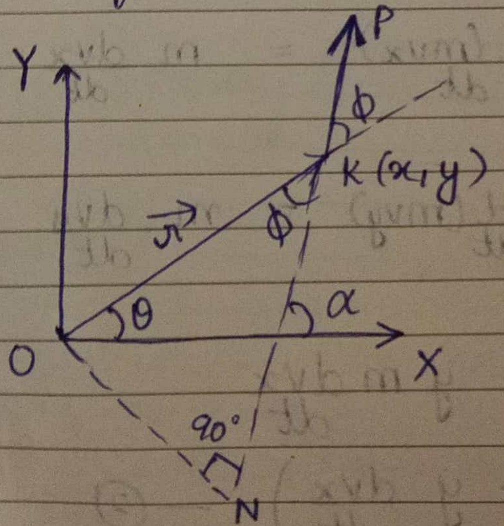
$$\tau = \frac{d}{dt}(xP_y - yP_x)$$

$$\tau = \frac{dL}{dt}$$

$$\boxed{L = xP_y - yP_x}$$

$$\boxed{\begin{array}{c} F = \frac{dp}{dt} \\ \downarrow \\ \tau = \frac{dL}{dt} \end{array}}$$

Expression for angular momentum in polar coordinates



Suppose K(x, y) is a position of particle of mass m, linear momentum P rotating in x-y plane

$$P_x = P\cos\alpha$$
$$P_y = P\sin\alpha$$

Now,

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$L = xp_y - yp_x$$

$$L = r\cos\theta\, P\sin\alpha - r\sin\theta\, P\cos\alpha$$
$$L = rP(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$$
$$L = rP\sin(\alpha-\theta) \quad -①$$

From diagram

$$\theta + \phi = \alpha$$
$$\phi = \alpha - \theta$$

Eq ① becomes

$$\boxed{L = rp\sin\phi} \quad -②$$

ΔOKN

$$\sin\phi = \frac{ON}{OK}$$

$$\sin\phi = \frac{ON}{r}$$

$$ON = r\sin\phi$$

Eq - ②

$$L = rp\sin\theta$$
$$L = p(ON)$$
$$L = p(r\perp)$$

Also

$$\boxed{L = \vec{r} \times \vec{p}}$$

_/_/_

* Relation b/w Torque & Angular momentum *

We know

$$\vec{\tau} = \vec{r} \times F$$

Also

$$\vec{L} = \vec{r} \times \vec{P} \quad — (1)$$

differentiating (1) w.r.t time

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{P})$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{P}}{dt} + \vec{P} \times \frac{d\vec{r}}{dt}$$

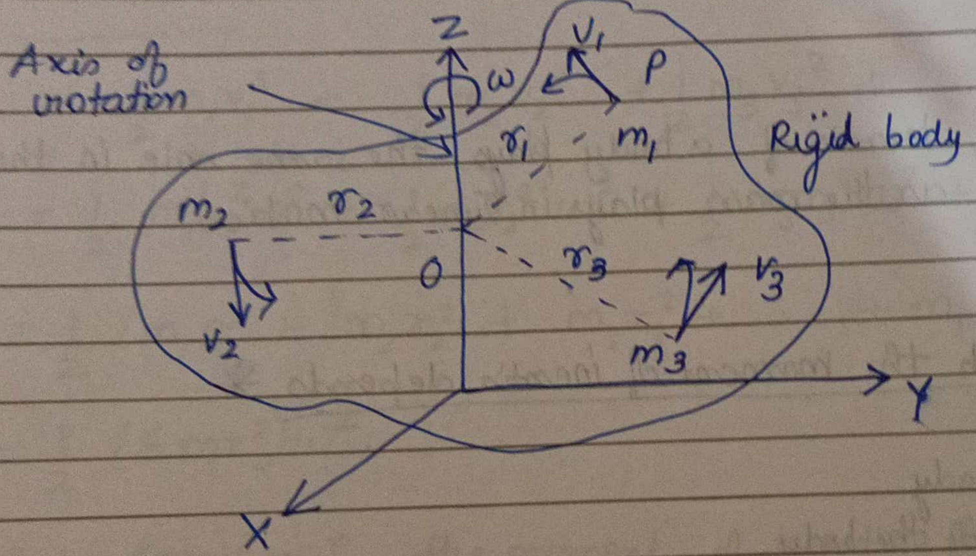
$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} + m\vec{V} \times \vec{V}$$

$$\frac{d\vec{L}}{dt} = \tau + 0$$

$$\tau = \frac{d\vec{L}}{dt}$$

* Moment of Inertia *

It is defined as the sum of the products of the masses of the particles constituting the body and the squares of their respective distances from the axis of rotation.



$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots \quad m_n r_n^2$$

$$I = \Sigma m r_i^2$$

* Physical significance of moment of inertia *

The moment of inertia of a body about an axis of rotation resist a change in its rotational motion. The greater the moment of Intertia of a body, the greater is the τ required to change its state of rotation. Thus, moment of inertia can be regarded as the measure of rotational ~~motion~~ inertia of the body.

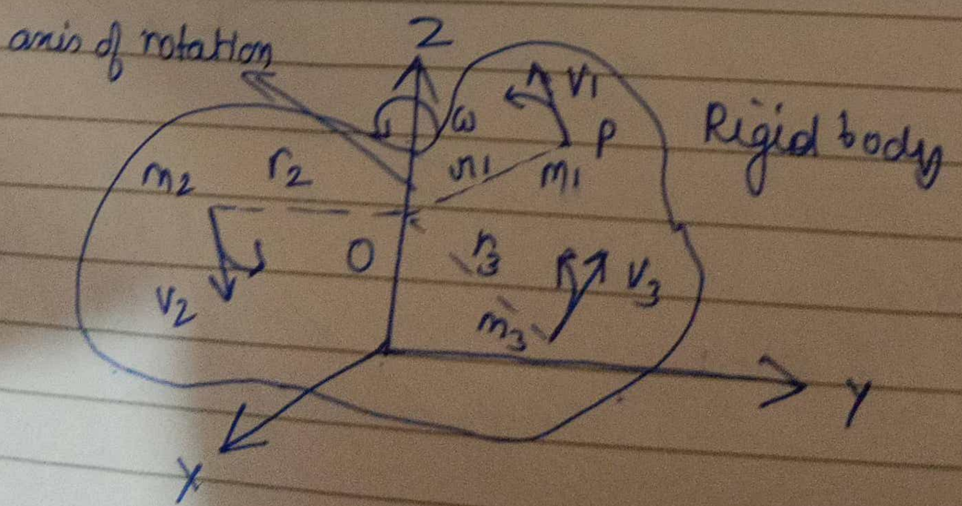
Note

The moment of Inertia of a body plays the same role in the rotational motion as the mass plays in linear motion

* Factors on which the moment of inertia depends *

1. Mass of the body
2. Size and shape of the body
3. Distribution of mass about the axis of rotation.
4. Position & orientation of the axis of rotation w.r.t the body.

Relation b/w Rotational K.E and moment of intertia



Consider a rigid body rotating about an axis OZ with uniform angular velocity ω. It consists of n particles of masses m_1, m_2 and m_3 situated at distances $r_1, r_2, r_3 \ldots, r_n$ from the axis of rotation. Since ω is same then their linear velocities are

$$v_1 = r_1\omega \quad v_2 = r_2\omega \quad v_3 = r_3\omega \ldots \quad v_n = r_n\omega$$

Rotational K.E

$$= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots + \frac{1}{2}m_nv_n^2$$

$$= \frac{1}{2}m_1r_1^2\omega^2 + \frac{1}{2}m_2r_2^2\omega^2 + \frac{1}{2}m_3r_3^2\omega^2 + \ldots + \frac{1}{2}m_nr_n^2\omega^2$$

$$= \frac{1}{2}(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots + m_nr_n^2)\omega^2$$

$$= \frac{1}{2}(\Sigma mr^2)\omega^2$$

But $\Sigma mr^2 = I$, the moment of inertia of the body about the axis of rotation.

$\therefore$ Rotational K.E $= \frac{1}{2}I\omega^2$

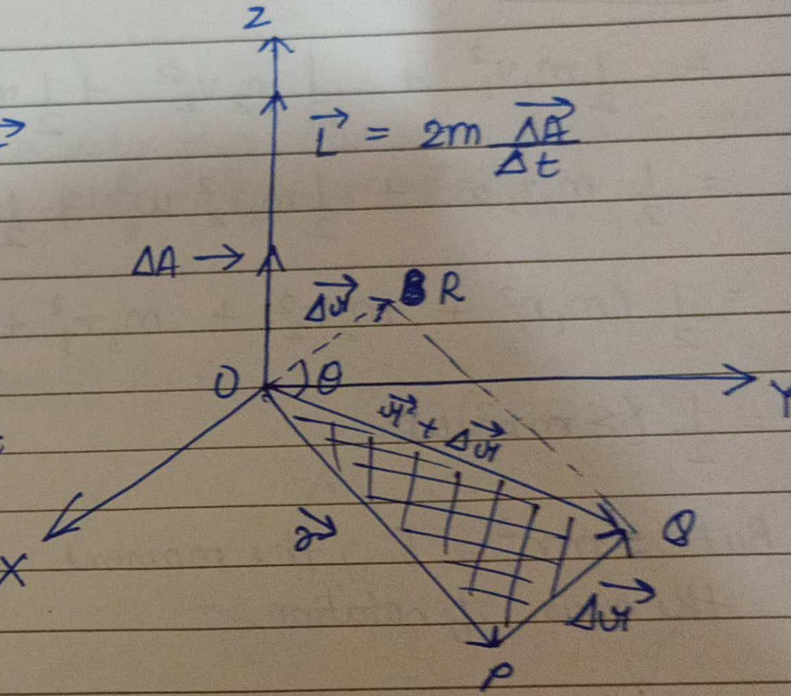
When $\omega = 1$, rotational K.E $= \frac{1}{2}I$

$I = 2 \times$ Rotational K.E

Hence the moment of inertia of a rigid body about an axis of rotation is numerically equal to twice the rotational K.E of the body when it is rotating with unit angular velocity about that axis.

Geometorical Meaning of angular momentum

Consider a particle of masses m rotating in the X-Y plane about the origin O. Let $\vec{r}$ and $(\vec{r} + \Delta\vec{r})$ be the position vectors of the particle at instants t and $(t + \Delta t)$ respectively, ~~as shown~~. The displacement of the particle in small time Δt is



$$\overrightarrow{PQ} = (\vec{r} + \overrightarrow{\Delta r}) - \vec{r} = \overrightarrow{\Delta r}$$

If $\vec{v}$ is the velocity of the particle at point P, then the small displacement covered in time Δt may be expressed as

$$\overrightarrow{\Delta r} = \vec{v}\,\Delta t$$

Complete the ||gram OPQR $= \vec{r} \times \overrightarrow{\Delta r}$

$$\therefore \text{Area of } \Delta OPQ = \frac{1}{2}(\vec{r} \times \overrightarrow{\Delta r})$$

The shaded area ΔOPQ represents the area swept by the position vector in time Δt. By right hand rule, its direction is along Z-axis. If this area is represented by $\overrightarrow{\Delta A}$ then

$$\overrightarrow{\Delta A} = \frac{1}{2}(\vec{r} \times \overrightarrow{\Delta r}) = \frac{1}{2}(\vec{r} \times \vec{v}\,\Delta t)$$

If $\vec{P}$ is the linear momentum of the particle then

$$\vec{p} = m\vec{v} \quad \text{or} \quad \vec{v} = \frac{\vec{p}}{m}$$

$$\therefore \quad \Delta\vec{A} = \frac{1}{2}\vec{r} \times \frac{\vec{p}}{m}\Delta t$$

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{1}{2m}(\vec{r} \times \vec{p})$$

But $\vec{r} \times \vec{p} = \vec{L}$, the angular momentum of the particle about Z-axis, so we have

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{L}}{2m} = \frac{1}{2} \times \text{angular momentum per unit mass}$$

$$\vec{L} = 2m\frac{\Delta\vec{A}}{\Delta t}$$

The quantity $\Delta\vec{A}/\Delta t$ is the area swept out by the position vector per unit time is called the areal velocity of the particle. Thus.

Angular momentum = 2 × mass × areal velocity

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* Radius of Gyration *

The radius of gyration of a body about its axis of rotation may be defined as the distance from the axis of rotation at which if the whole mass of the body were concentrated, its moment of inertia about the given axis would be the same as with the actual distribution of mass.

Expressions for K

Suppose a ~~rigid~~ rigid body consists of n particles of mass m each, situated at distances $r_1, r_2, r_3 \ldots r_n$ from the axis of rotation AB

The moment of inertia of the body about the axis AB is

$$I = mr_1^2 + mr_2^2 + mr_3^2 + \ldots mr_n^2$$

$$= m(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)$$

$$= m \times n \frac{(r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2)}{n}$$

$$I = M \frac{(r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2)}{n}$$

where $M = m \times n$ = total mass of the body

If K is the radius of gyration about the axis AB then

$$I = MK^2$$

$$\cancel{M}K^2 = \cancel{M}\left(\frac{r_1^2 + r_2^2 + r_3^2 + \ldots + r_n^2}{n}\right)$$

$$K = \sqrt{\frac{r_1^2 + r_2^2 + r_3^2 + \ldots r_n^2}{n}}$$

Hence the radius of gyration of a body about an axis of rotation may also be defined as the root mean square distance of its particles from the axis of rotation

Factors on which radius of gyration of a body ~~depends~~ depends :-
① Position and direction of the axis of rotation
② Distribution of mass about the axis of rotation

* Relation b/w Torque & Moment of Inertia *

Suppose a rigid body consists of n particles of masses $m_1, m_2, m_3, \ldots\ldots, m_n$; situated at distances $r_1, r_2, r_3, \ldots\ldots r_n$ from the axis of rotation AB. When a torque is applied on a body its angular velocity changes. This produces angular acceleration α which is same for each particle but their linear " are different.

Linear acceleration of the 1st particle, $a_1 = r_1 \alpha$

Force acting on the 1st particle, $F_1 = m_1 r_1 \alpha$ $\left(F = ma_1 = m_1 r_1 \alpha \right.$

Moment of force F_1 about the axis rotation is

$$\tau_1 = F_1 r_1 = m_1 r_1^2 \alpha$$

Total Torque acting on the rigid body is

$$\tau = \tau_1 + \tau_2 + \tau_3 + \ldots\ldots + \tau_n$$

$$= m_1 r_1^2 \alpha + m_2 r_2^2 \alpha + m_3 r_3^2 \alpha + \ldots\ldots + m_n r_n^2 \alpha$$

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2)\alpha = (\Sigma mr^2)\alpha$$

But $\Sigma mr^2 = I$, moment of Inertia about the given axis

$$\boxed{\tau = I\alpha}$$

Torque = moment of Inertia x angular acceleration

When $\alpha = 1$, $\tau = I$

Thus, the moment of inertia of a rigid body about an axis of rotation is numerically equal to the external torque required to produce unit angular acceleration in the body about that axis.

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Relation b/w angular momentum and Moment of Inertia

Consider a rigid body rotating about a fixed axis with uniform angular velocity ω. The body consists of n particles of masses $m_1, m_2, m_3, \ldots, m_n$; situated at distance $r_1, r_2, r_3, \ldots r_n$ from the axis of rotation. The angular velocity ω of all the n particles will be same but their linear velocities will be different and are given by

$$v_1 = r_1\omega, \quad v_2 = r_2\omega, \quad v_3 = r_3\omega, \ldots \quad v_n = r_n\omega$$

Linear momentum of 1st particle

$$P_1 = m_1 v_1 = m_1 r_1 \omega$$

Moment of linear momentum of the 1st particle about the axis YY',

$$L_1 = P_1 r_1 = m_1 r_1^2 \omega$$

The angular momentum of a rigid body about an axis is the sum of moments of linear momenta of all its particles about the axis. Thus

$$L = L_1 + L_2 + L_3 + \ldots + L_n = m_1 r_1^2 \omega + m_2 r_2^2 \omega + m_3 r_3^2 \omega + \ldots + m_n r_n^2 \omega$$

$$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots + m_n r_n^2)\omega = (\Sigma m r^2)\omega$$

But $\Sigma m r^2 = I$, moment of inertia of the body about the given axis

$$L = I\omega$$

Angular momentum = M.I x Angular velocity

When $\omega = 1$, $L = I$

Thus the moment of inertia of a body about an axis is numerically equal to the angular momentum of the rigid body when rotating with unit angular velocity about the axis.

* Law of conservation of angular momentum *

Suppose the external torque acting on a rigid body due to external forces is zero. Then

$$\tau = \frac{dL}{dt} = 0$$

Hence, $L = \text{constant}$

So, when the total external torque acting on a rigid body is 0, the total angular momentum of the body is conserved. This is law of conservation of angular momentum.

Clearly,

when $\tau = 0$, $L = I\omega = \text{constant}$ or $I_1\omega_1 = I_2\omega_2$

This means that when no external torque is acting, the angular velocity ω of the body can be increased or decreased by decreasing or increasing the moment of inertia of the body.

$$I \propto \frac{1}{\omega}$$

Illustrations of the law of conservation of angular momentum

① Planetary motion

The angular velocity of a planet revolving in an elliptical orbit around the sun increases, when it comes closer to the sun because its moment of inertia about the axis through the sun decreases. When it goes far away from the sun, its moment of inertia increases and hence angular velocity decreases so as to conserve angular momentum.

② A man carrying heavy weights in his hands and standing on a rotating turn-table can change the angular speed of the turn-table

If a person stands on a turn-table with some heavy weights in his hands stretched out and the table is rotated slowly, his angular speed at once increases, as he draws his hands to his chest. The moment of inertia of man & weights taken together decreases, as he draws his arms inward. As moment of inertia decreases, the angular speed increases so as to conserve total angular momentum.

③ A diver jumping from a spring board exhibits somersaults in air before touching the water surface

After leaving the spring board, a diver curls his body by pulling his arms and legs towards the centre of his body. This decreases his moment of inertia and he spins fast in midair. Just before hitting the water surface, he stretches out his arms. This decreases his moment of inertia and the diver enters water at a gentle speed.

④ An ice-skater or ballet dancer can increase her angular velocity by folding her arms and bringing the stretched leg close to the other leg

When she stretches her hands and a leg outward, her moment of inertia increases and hence angular speed decreases to conserve angular momentum. When she folds her arms and brings the stretched leg close to the other leg, her moment of inertia decreases & hence angular speed increases.

⑤ The speed of the inner layers of the whirlwind in a tornado is ~~atm~~ alarmingly high.

The angular velocity of air in a tornado increases as it goes towards the centre. This is because as the air moves towards the centre, its moment of inertia (I) decreases and to conserve angular momentum ($L = I\omega$), the angular velocity (ω) increases.

* Equilibrium of Rigid bodies *

A rigid body is said to be in equilibrium if both linear momentum & angular momentum of rigid body remains constant with time.

Translational equilibrium

The resultant of all the external forces acting on a body must be zero ~~or~~ otherwise they would produce linear acceleration.

$$\sum \vec{F_i}^{ext} = 0$$

Rotational equilibrium

The resultant of Torque due to all the forces acting on the body about any point must be zero otherwise they would produce angular acceleration.

$$\sum \tau_i^{ext} = \sum r_i \times \sum f_i^{ext} = 0$$

Linear motion		Rotational motion	
Quantities			
Displacement	S	angular displacement	θ
Velocity	v	angular velocity	ω
Acceleration	a	angular acceleration	α or a_θ
force	F	torque	τ
mass	m	moment of inertia.	I
Expressions			
Velocity	$v = \frac{ds}{dt}$	angular velocity	$\omega = \frac{d\theta}{dt}$
Acceleration	$a = \frac{dv}{dt}$	angular acceleration	$\alpha = \frac{d\omega}{dt}$
force	$F = ma = \frac{d}{dt}(mv)$	Torque	$\tau = I\alpha = \frac{d}{dt}(I\omega)$
Work done	$W = Fs$	Work done	$W = \tau\theta$
linear K.E	$E = \frac{1}{2}mv^2$	Rotational K.E	$E = \frac{1}{2}I\omega^2$
Power	$P = Fv$	Power	$P = \tau\omega$
linear momentum	$p = mv$	angular momentum	$L = I\omega$
Impulse	$F\Delta t = mv - mu$	angular impulse	$\tau\Delta t = I\omega_f - I\omega_i$

Equations of motion

Linear	Rotational
(i) $v = u + at$	(i) $\omega = \omega_0 + \alpha t$
(ii) $s = ut + \frac{1}{2}at^2$	(ii) $\theta = \theta_0 t + \frac{1}{2}\alpha t^2$
(iii) $v^2 - u^2 = 2as$	(iii) $\omega^2 = \omega_0^2 = 2\alpha\theta$