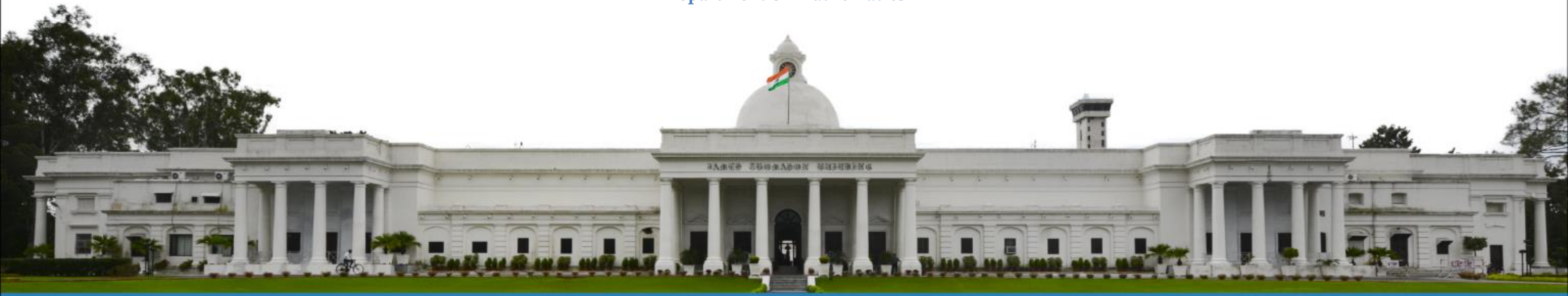
IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELS : ANALYSIS AND APPLICATIONS

Lecture 6 : Discrete Time Non – Linear Models in Population Dynamics - I

Dr. Ameeya Kumar Nayak
Department of Mathematics



Contents:

$$\boxed{C_{n+1} \propto C_n}$$

- Non-Linear Cell Division Model.
- Cob-web Graphs for Non-Linear Cell Models.
- Non-Linear Stability Analysis.
- Logistic Growth Function.

Non-Linear Cell Division Model:

- In the previous lectures, we have studied about the analytical and graphical solution of linear models of cell division. Mathematically (considering no migration) they were represented as:

$$C_{n+1} = \alpha C_n.$$

$C_{n+1} \propto C_n$

- The meaning associated with this basic model is that each cell will survive till end. But does it happen in actual ? No ! To survive each cell, they need to fight for resources. So, there is a need to include the survival rate.

- Again, on which factor(s) should this survival function depend on? And what should be the nature of this function?

Non-Linear Cell Division Model:

- To answer previous questions, let's model out the cell division process with the survival function S(C),

$$C_n = \alpha C_{n-1} S(C_{n-1}) \text{ where } n \in \mathbb{N}$$

$C_{n+1} \propto C_n$ $S(C_n)$ critical

- Now, let's analyze this survival function S(C)

1. No Competition: If S(C) = 1 for all C.

2. Contest Competition: Here there are finite number of resources. The cells which are able to take one resource will survive and take part in next generation. So

$$S(C) = 1 \quad \text{for } C \le C_{critical}$$
$$S(C) = C_{critical} / C \quad \text{for } C > C_{critical}$$

3. Scramble Competition: Here each individual is assumed to share equal amount of limited resource. If this quantity is sufficient, they will survive otherwise not. Hence,

$$S(C) = 1 \quad \text{for } C \le C_{critical}$$
$$S(C) = 0 \quad \text{for } C > C_{critical}$$

Non-Linear Cell Division Model:

- Now, practically it's not possible for critical value to be fixed. Also, 0 survival seems unrealistic at-least for large populations ! Then how should this survival function to act?

- Let us assume that $f(C) = CS(C)$.

- Now, the contest competition describes the exact compensation as $\lim_{C \to \infty} f(C) = l$ where l is a constant. This represents the situation of compensating any numbers of new cells with already extinct cells. Hence $s(c) \sim \frac{l}{c}$ for large C.

- Now, let's consider the in-general case for $S(C) \sim \frac{l}{C^b}$

❖ when 0<b<1 it's called under compensation – resources are less utilized

❖ when b>1 it's called over compensation – resources are over utilized.

Non-Linear Cell Division Model:

- Now, if $b \approx 1$ then there is contest competition which means f(C) eventually levels out a non-zero level for large population hence population will be stabilized by avoiding too many newborns.

- Again, come back to the model equation

$$C_n = \alpha C_{n-1} S(C_{n-1}) = \alpha\, f(C_{n-1}) \text{ where } n \in N.$$

- For exhibition of compensatory behavior, f(C) ~ constant and S(C) ~ 1 for small C assuming very small competition among cells. Hence growth must be exponential with growth rate α.

- One of the simple functions of S(C) can be

$$S(C) = \frac{1}{1 + aC},$$ where a is any constant.

Non-Linear Cell Division Model:

- This leads to the model equation as,

$$C_n = \alpha \frac{C_{n-1}}{1 + aC_{n-1}} \quad \text{where } n \in N$$

- If we assume the carrying capacity of environment is E, and the population reached E will stay there i.e. if $C_k = E$ for some E then $C_{k+m} = E$ for all $m \geq 0$ and for some k.

- By substituting in model equation will lead to $a = \frac{\alpha - 1}{E}$, and the resulting model is given as follows. It's also known as Beverton-Holt Model.

$$C_k = \alpha \frac{C_{k-1}}{1 + \frac{\alpha - 1}{E} C_{k-1}}$$

Non-Linear Cell Division Model:

- This model can further be generalized as:

$$C_k = \alpha \frac{C_{k-1}}{(1 + a C_{k-1})^b}$$

- This is also known as Hassel Model. Depending on b, it compensates both contest competition (for b=1) and scramble competition (for b>1).

Stability of Non-Linear System:

- There are relatively very few cases where analytical solution of non-linear difference equations can be computed directly.
- A general non-linear difference equation is in form of:

$$x_{n+1} = f(x_n, x_{n-1}, \ldots) \text{ where } n \in \mathbb{N}.$$

- Thus, we must have to be satisfied with the nature of system (stability analysis) or with the computer generated solution (numerical analysis).
- In order to study the nature of system, let's consider a general first order equation (because higher order equations can be converted into system of first order equations. How?).

Stability of First Order Non-Linear Equation:

- The general first order difference equation is:

$$x_{n+1} = f(x_n) \text{ where } n \in N.$$

- There are a lot of importance of steady state solution especially in problems of dynamics where growth, propagation or reproduction takes place.

- An equilibrium or a steady state relates to the absence of change in system.

- In context of difference equation, a steady state solution λ is defined as: $\boxed{x_{n+1} = x_n = \lambda}$ which describes no change from generation n to n+1.

$x = f(x)$

- This implies $\boxed{\lambda = f(\lambda)}$. Mathematically, this point λ is referred as fixed point of function f(.) (the value that function f leaves unchanged).

- Invariant

- Quite often, solving for fixed point is simpler/easier than solving non-linear difference equation for general solution.

Stability of First Order Non-Linear Equation:

- A steady state is termed to be stable if all the neighboring states are attracting to it. If it is not so, steady state is unstable.

$|f(x)-4|<\epsilon$

$|x_0-4|<\delta$

- If the system is unstable, population may crash or the number of competing group may shift in favor of few one.

- Now, let us take any arbitrary point x_n and we want to study that whether this point move away or move towards the steady state. (remember cob-web? The same thing was done graphically.)

- Let's assume $\boxed{x_n = \lambda + \varepsilon_n}$, where ε_n is a very small quantity termed as perturbation of steady state.

$\Rightarrow \boxed{x_{n+1} = \lambda + \varepsilon_{n+1}}$, $\quad x_{n+1}=f(x_n) \to \alpha=f(\alpha)$

$\Rightarrow \varepsilon_{n+1} = x_{n+1} - \lambda$; $\quad \alpha = x_0+h = f(x_0+h)$

$\Rightarrow \boxed{\varepsilon_{n+1} = f(x_n) - \lambda}$, (from fixed point equation) $\quad = f(x_0)+hf'(x_0)$

Stability of First Order Non-Linear Equation:

➔ $\varepsilon_{n+1} = f(\lambda + \varepsilon_n) - \lambda = f(\lambda) + \lambda \frac{dy}{dx}\Big|_{x=\lambda} + O(\lambda^2).$

Since ε_n is very small, so we can expand function f by Taylor's series. After neglecting 2nd and higher order terms,

➔ $\varepsilon_{n+1} = f(\lambda) - \lambda + \lambda \frac{dy}{dx}\Big|_{x=\lambda} = \lambda \frac{dy}{dx}\Big|_{x=\lambda}$

0

- This equation is in form of:

$$\varepsilon_{n+1} = a\varepsilon_n \text{ where } n \in N \text{ and } a = \frac{dy}{dx}\Big|_{x=\lambda}$$

- Recall from previous lectures, the system will be stable if $|a| < 1$ and unstable if $|a| > 1$.

Stability of Beverton Holt Model of Cell Division:

- Consider the Beverton Holt model of cell division process and assume parameter a = 1/p and $\alpha = k/p$ where both k and p are positive. Now the model will be in form of:

$$\boxed{C_{n+1} = k\frac{C_n}{p+C_n}}$$

$$\lambda = K\frac{\lambda}{p+\lambda} \Rightarrow \lambda\left(1-\frac{K}{p+\lambda}\right)=0$$

$\lambda \neq 0$ or

- To calculate the steady state, substitute $C_{n+1} = C_n = \lambda$. There are two steady states:

$$\lambda = 0 \text{ and } \lambda = k - p.$$

$$\frac{K}{p+\lambda} = 1 \Rightarrow K = p+\lambda$$

$$\lambda = K - p$$

- Obviously, k > p because there is no meaning of negative population. To check for stability find the differentiation of function on right hand side of equation at both steady state points.

$$f(C_n) = k\frac{\lambda}{p+\lambda} \quad \text{and } f'(C_n) = k\frac{p}{(p+\lambda)^2}$$

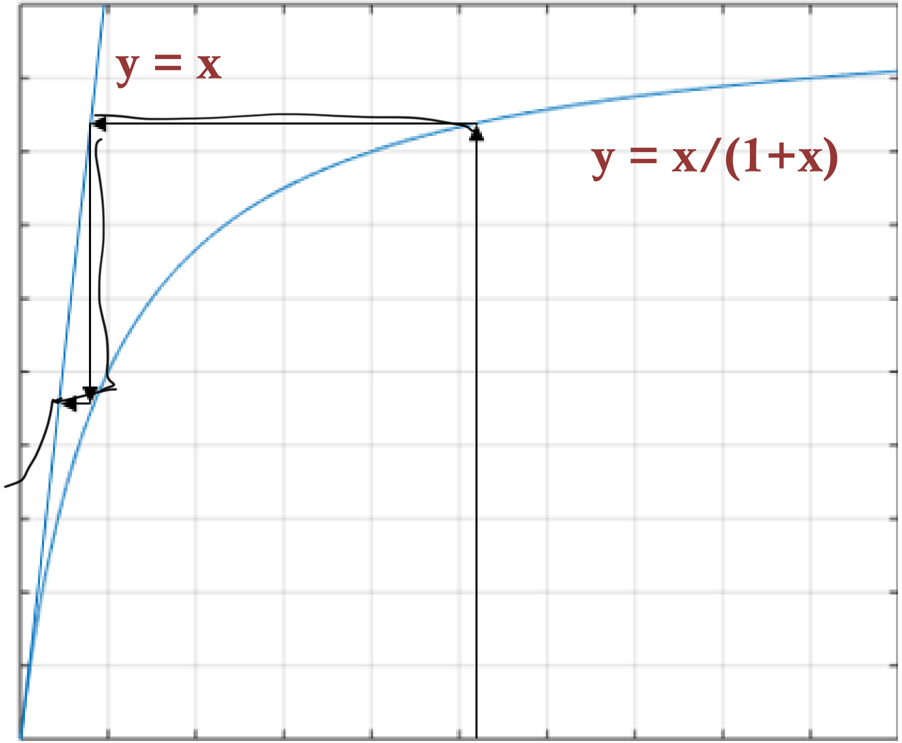
- Clearly, $\lambda = 0$ is stable when k < p and $\lambda = k - p$ is stable when k > p.

Stability of Beverton Holt Model of Cell Division:

- Now's let's verify the same result by cob-web graphs.

- As one can verify that, all the vertical and horizontal motions are monotonically decreasing, hence the model is stable.

- (Exercise) Observe the nature of cob-web for Hassel model for b=0.5 and 2.



Logistic Difference Equation:

- We have already discussed several discrete time models in population dynamics. All these model were specific to problem. Can't we think of a model which can govern many classes of systems? E.g. different populations, distinct species, one species at different stage of evolution or development.

- The model can have variety of meaning depending on the parameters involved in it.

- Now, let's consider a particular type of first order difference equation.

$$y_{n+1} = y_n (r - dy_n) .$$

- This equation has 2 parameters r and d. Can we reduce the parameters? YES!. The method used to reduce the parameters is called as non-dimensionalization.

- Why do we need to get rid of parameters? Because, the number of computations are directly proportional to the number to parameters. So, if we are reducing the parameters means we are reducing the commutations almost proportionally.

Logistic Difference Equation:

- Now put $x_n = (d/r)\, x_n$. This leads to the following form which is also know as Pearl-Verhulst equation.

$$x_{n+1} = r\, x_n (1 - x_n)\ . = 0$$

- We are left with only one parameter r. To find the steady state solution, let's substitute $x_{n+1} = x_n = \lambda$.

- There are two steady states, $\lambda = 0$ and $\lambda = 1 - 1/r$.

- For the stability, $F(x_n) = r\, x_n (1 - x_n)$ so $F'(x_n) = r(1 - 2x_n)$. At $\lambda = 0$, $F'(\lambda) = r$ and at $\lambda = 1 - 1/r$, $F'(\lambda) = 2 - r$.

- The steady state $\lambda = 0$ is stable for $|r| < 1$ and $\lambda = 1 - 1/r$ is stable for $|2 - r| < 1$ or $1 < r < 3$.

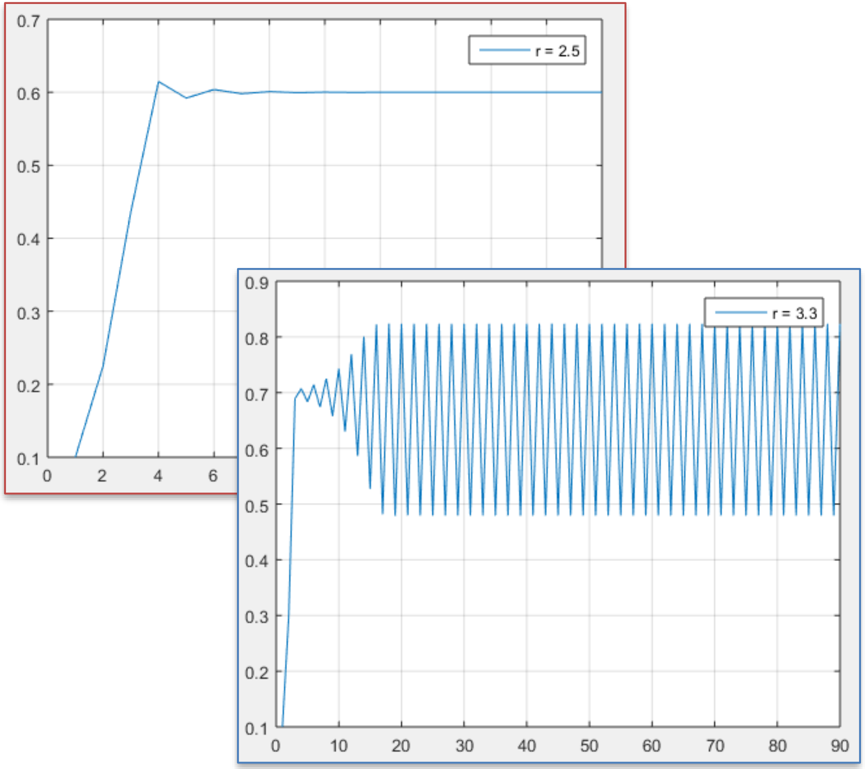
Observation on Logistic Difference Equation:

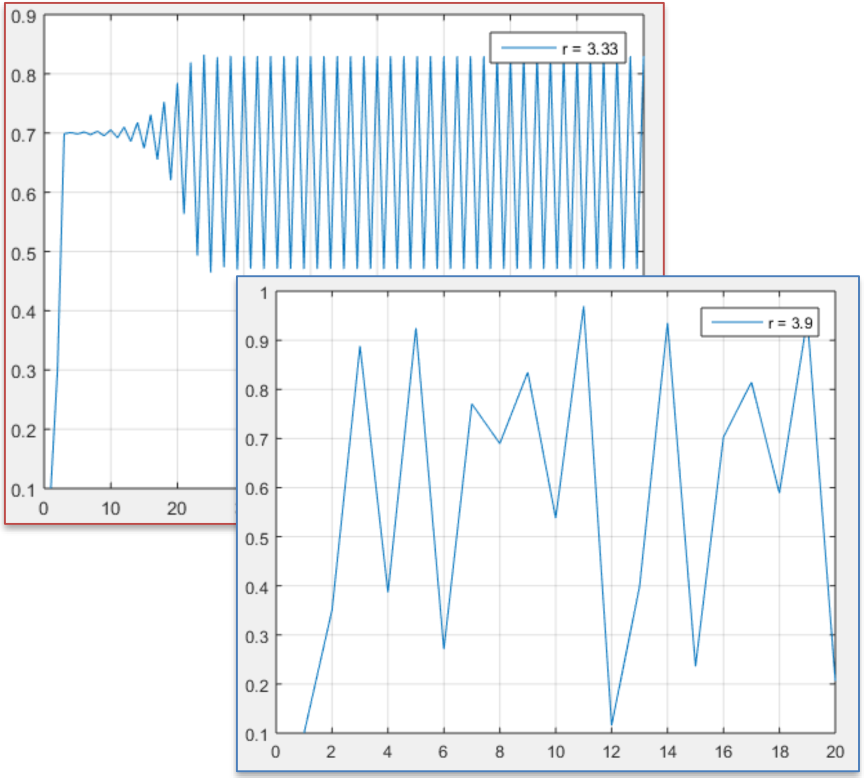
- (Exercise) For $x_0 = 0.1$, perform 20 iterations for r=2.5, r=3.3, r=3.33 and r=3.9. Observe the pattern of numbers. Are you able to find anything interesting?

- r = 2.5 Steady State.
- r = 3.3 Periodic Oscillations with period = 2.
- r = 3.33 Periodic Oscillations with period = 2.
- r = 3.9 Chaos.

S. No.	r = 2.5	r = 3.3	r = 3.33	r = 3.9
1	0.1000	0.1000	0.1000	0.1000
2	0.2250	0.2970	0.2997	0.3510
3	0.4359	0.6890	0.6989	0.8884
4	0.6147	0.7071	0.7008	0.3866
5	0.5921	0.6835	0.6983	0.9249
6	0.6038	0.7139	0.7016	0.2710
7	0.5981	0.6740	0.6972	0.7705
8	0.6010	0.7251	0.7030	0.6896
9	0.5995	0.6577	0.6953	0.8348
10	0.6002	0.7429	0.7055	0.5379
11	0.5999	0.6303	0.6918	0.9694
12	0.6001	0.7690	0.7100	0.1157
13	0.6000	0.5863	0.6857	0.3992
14	0.6000	0.8004	0.7177	0.9353
15	0.6000	0.5271	0.6747	0.2358
16	0.6000	0.8226	0.7309	0.7029
17	0.6000	0.4816	0.6550	0.8145
18	0.6000	0.8239	0.7525	0.5892
19	0.6000	0.4788	0.6202	0.9439
20	0.6000	0.8235	0.7844	0.2064

Observation on Logistic Difference Equation:





Summary:

- Different non-linear cell division models.
- Formulation based on survival rate function.
- Non-linear stability analysis.
- Logistic growth difference equation.
- Observation of logistic growth function with a set of parameter values.
- Steady and transient behaviors.


Thank you!
STABILO