IIT ROORKEE

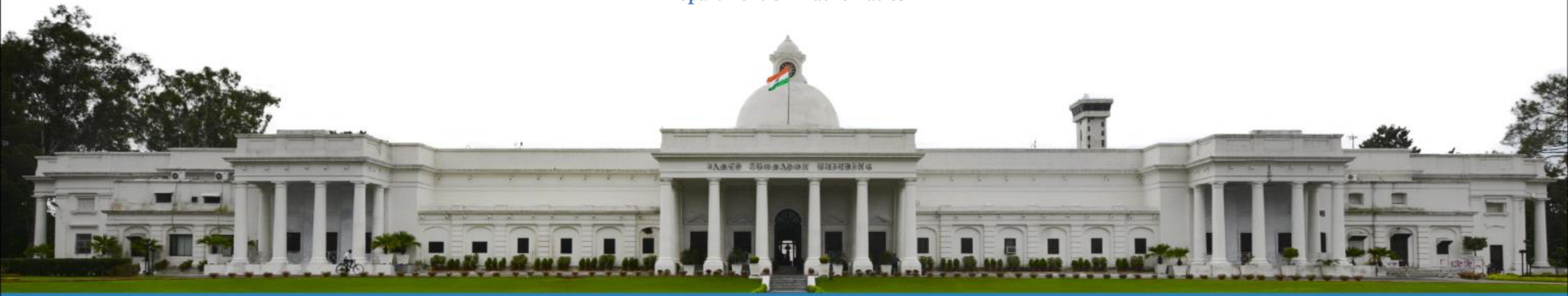
NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 2.2 : Analysis on Logistic Difference Equation

Dr. Ameeya Kumar Nayak

Department of Mathematics



Contents:

- Analysis on Logistic Difference Equation.
- Periodically Stable Solutions.
- Introduction of Bifurcation.

Analysis on Logistic Difference Equation:

- In the previous lecture, we have observed the nature of logistic difference equation $\boxed{x_{n+1} = r\, x_n (1 - x_n)}$ for different values of r.

- Using fixed point criterion we found the stability region of steady state.

  For steady state $\lambda = 0$, stability exist when $|r| < 1$,

  For steady state $\lambda = 1 - 1/r$, stability exist when $1 < r < 3$.

  $|2-r| < 1$

- We also verified the same result by finding the each iteration value of logistic equation for r = 2.5, 3.3, 3.33 and 3.9.

- We concluded with graphs of iterative solution that except for r = 2.5, no other case showed stability.

Analysis on Logistic Difference Equation:

- Beyond $r>3$, we observed some periodic oscillations in juxtapose of $r \sim 3.3$ with period 2.

- Period 2 implies that successive generations will alternate between two different values of x. Let's call these values as ζ_1 and ζ_2.

- Mathematical representation of period 2 (also called two-point cycles) oscillation can be represented by 2 simultaneous difference equations as

$$\boxed{x_{n+1} = f(x_n)} \text{ and } x_{n+2} = x_n. \qquad \text{.... 5.1}$$

- After combining these 2 equations, $x_{n+2} = f(x_{n+1}) = f[f(x_n)]$. Let's call $\boxed{g(x) = f(f(x))}$. Hence the equation is $\boxed{x_{n+2} = g(x_n)}$. 5.2

Analysis on Logistic Difference Equation:

- Let k be a new index which skips every alternate value. Define k = n/2 for every even n numbers. Hence the equation will be in form of

$$x_{k+1} = g(x_k). \quad \text{.... 5.3}$$

$x_{n+2} = g(x_n)$ $n=2$ $x_4 = g(x_2)$ $x_2 =$ $x_{k+1} = g(x_k)$

- The steady state solution of above equation ζ will be 2-periodic. Since ζ is oscillating between two different values so it will have two solutions and these will be ζ_1 and ζ_2.

- With this, one can easily analyze the stability of above equation which says $\left|\frac{dg}{dx}\right|$ at $x = \zeta$ should be less than 1.

$|a| < 1$

- This condition is equivalent to $$\left|\left(\frac{df}{dx}\right)_{x=\zeta_1}\left(\frac{df}{dx}\right)_{x=\zeta_2}\right| < 1$$ (how?) 5.4

Analysis on Logistic Difference Equation:

- For the logistic difference equation, $\boxed{f(x) = r\,x(1\text{-}x)}$. So $g(x) = f(f(x))$

$$g(x) = r\,[r\,x(1\text{-}x)][1\text{-}\,r\,x(1\text{-}x)] = r^2\,x(1\text{-}x)(1 - rx + rx^2). \quad \ldots.\ 5.5$$

- Hence the difference equation of the interest can be obtained by using equations 5.3 and 5.5:

$$x_k = r^2 x_k (1 - r x_k - r x_k^2)(1-x_k) \qquad x_{k+1} = r^2\,x_k(1\text{-}x_k)(1 - rx_k + rx_k^2). \qquad \frac{dx}{dt} = 0 \quad \ldots.\ 5.6$$

- To find the steady state put $\boxed{x_{k+1} = x_k = \zeta}$ in eq. 5.6. The algebraic equation will be

$$\zeta = r^2\,\zeta(1\text{-}\,\zeta)(1 - r\zeta + r\zeta^2). \qquad \zeta[1 - r^2(1-\zeta)(1 - r\zeta + r\zeta^2)] = 0 \quad \ldots.\ 5.7$$

- One steady state is $\zeta = 0$ and the other will satisfy a cubic equation in ζ.

$$1 = r^2\,(1\text{-}\,\zeta)(1 - r\zeta + r\zeta^2). \quad \ldots.\ 5.8$$

- This cubic equation is very difficult to solve. Notice that the steady state of equation $x_{n+1} = f(x_n)$ will also be the steady state of $\boxed{x_{n+2} = f[f(x_n)]}$. This implies

$$\boxed{x_n = x_{n+1} = x_{n+2} = \zeta.} \quad \ldots.\ 5.9$$

Analysis on Logistic Difference Equation:

$x_{n+1}^{\zeta} - x_n^{\zeta} = 0$

$x_{n+2}^{\zeta} - x_{n+1}^{\zeta} = 0$

- The non-zero steady state value of the equation $x_{n+1} = r\, x_n (1 - x_n)$ is $\zeta = 1 - 1/r$. So this ζ value will satisfy the cubic equation given in eq. 5.8.

- One of the factors of cubic equation 5.8 is $\zeta - 1 + 1/r$ and hence the reduced form of equation is (Exercise! divide eq. 5.8 by $\zeta - 1 + 1/r$ to get eq. 5.10)

$$(\zeta - 1 + 1/r)\,[\zeta^2 - \zeta(1 + 1/r) + (1/r + 1/r^2)] = 0. \qquad \ldots.\ 5.10$$

- The roots of quadratic factor of above reduced form of cubic equation 5.10 are

$$\zeta = \frac{r + 1 \pm \sqrt{(r+1)(r-3)}}{2r} \quad \text{for } r < -1 \text{ and } r > 3. \qquad \ldots.\ 5.11$$

Analysis on Logistic Difference Equation:

- The calculated two values for steady states are defined only for r<-1 and r>3.

- To check the stability, we will use the eq. 5.4

$$\left(\frac{df}{dx}\right)_{x=\zeta_1} = \left(-1-\sqrt{(r+1)(r-3)}\right) \qquad \left(\frac{df}{dx}\right)_{x=\zeta_2} = \left(1-\sqrt{(r+1)(r-3)}\right) \quad \ldots.5.12$$

- After substituting equation 5.12 in eq. 5.4, we will get the condition for stability as:

$$|-r^2+2r+4| < 1 \quad \ldots.\ 5.13$$

- For both the cases r>3 and r<-1, condition for stability eq. 5.13 is quietly satisfied.

- Thus, we can conclude that for positive values of r, steady state of two generation map $f(f(x_n))$ exist only for r > 3. This occurs only when steady state 1 – 1/r brings stability.

Analysis on Logistic Difference Equation:

- Now, let's find the upper limiting case i.e. $3 < r < r_1$ for r up to which the concept of two generations will work.
- By substituting this in stability condition, the value of r_1 will turn out to be $r_1 = 1 + \sqrt{6} \sim 3.4494$. Refer the following steps for this calculation:

$$r < r_1$$
$$r - 1 < r_1 - 1$$
$$(r-1)^2 - 5 < (r_1 - 1)^2 - 5$$
$$r^2 - 2r - 4 < (r_1 - 1)^2 - 5$$
$$-1 < (r_1 - 1)^2 - 5 < 1 \qquad \text{(satisfy stability condition, eq. 5.13)}$$
$$4 < (r_1 - 1)^2 < 6$$
$$\mathbf{3 < r_1 < 1 + \sqrt{6}}.$$

$|r_1| \le 1$

Analysis on Logistic Difference Equation:

- Again one can question, what will happen beyond $r \sim 3.4494$? The same has already been analyzed with graph of iteration values for $r=3.9$ and found that it was CHAOS!. After 3.4494, for some values nature will be 4-periodic, then 8-periodic and so on. So in general, we call it chaos.
- Here, we used the methodology to solve for 2-periodic solution. However, the same can be used for any value of n. But it's very cumbersome to solve for higher values of n.
- We have analyzed different scenarios of stable periodic oscillations with variations in parameter r.
- What will happen when parameter values will be perturbed?
- The phenomenon of analyzing significant changes in output with small perturbations in parameter(s) is termed as bifurcation.

Summary:

Linearization:

Ill Conditioned:

- Analysis of logistic difference equation beyond r > 3.
- Periodic stable solutions.
- Introduction to bifurcations.

$a_{11}x_1 + a_{12}x_2 + a_{11}x_3 = b_1$

$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$

$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$

$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 = b_4$

$\boxed{x_1 = x_1 + \epsilon_1}$

Transcendental equations:

$f(x) = xe^x - \sin x = 0$ {Iterative method}

(1) Newton's method

$u^{n+1} = u^n + \Delta u$ $\boxed{u_{0j}^{n+1} = u_{0j}^{n} + \Delta u_{0j}}$