IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 2 : Discrete Time Linear Models in Population Dynamics - I

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Contents:

- Population Dynamics.
- Fibonacci Rabbit Model.
- Linear Cell Division Model.
- Linear Difference Equation with Constant Coefficients.

Population Dynamics:

- With the current world human population of more than 7.2 billion, you might be questioned about the rate of increasing and their survival conditions.
- J.E. Cohen wrote the book on *"How Many People Can the Earth Support?"* . In this he relates the historical and scientific factors of the same.
- But the question here is that can we study these factors mathematically? Yes ! The stream of bio-science which deals with such questions is popularly known as Population Dynamics.
- Modeling the growth of population as a dynamical system driven by biological and environmental processes is called population dynamics.
- This stream includes the study of survival capability of populations with limited resources and by fighting with diseases.

Fibonacci Rabbit Model:

- Consider the problem of rabbit generation which was originally postulated by Fibonacci.

- A young pair of rabbits, one of each sex is placed on an Iceland. A pair of rabbit does not breed until they are 2 months old. When, they are 2 months old, each pair will produce another pair in each month.

- Let's assume that f_n is the number of pairs of rabbits present after n months, for n = 1, 2, 3, …. so on.

  $f_1 = f(1) =$ Number of pair of rabbits after 1 month = 1.

  $f_2 = f(2) =$ Number of pair of rabbits after 2 months = 1.

  $f_3 = f(3) =$ Number of pair of rabbits after 3 months = 1+1 = 2, and so on.

- Hence, the mathematical model (linear difference equation) will of the form:

$$f(n) = f(n-1) + f(n-2), \text{ for } n \geq 3. \qquad \ldots.\ 2.1$$

Linear Homogeneous Difference Equation:

- The general form of linear homogenous difference equation of order 'k' with constant coefficients is:

$$A_n = c_1 A_{n-1} + c_2 A_{n-2} + c_3 A_{n-3} + \dots\dots\dots + c_k A_{n-k} \qquad \dots.\ 2.2$$

$$m^n = c_1 m^{n-1} + c_2 m^{n-2} + c_3 m^{n-3} + \text{----} + c_k m^{n-k}$$

- Since it's of order k, the same number of initial conditions are required. Let's say $A_0 = a_0$ $A_1 = a_1$ ..... $A_{k-1} = a_{k-1}$.

$(D-1)^2 \quad D = 1, 1 \quad \frac{d^2y}{dx^2} - 4y = 50ae^{2x} \quad D^2 = 4 \quad c_1e^{2x} + c_2 \quad ae^{2x}$

- **<u>Procedure to solve above type of equation:</u>**

1. Assume that $A_n = m^n$ is the solution of the equation and substitute this in equation 2.2.

2. This will get simplified in a polynomial of order k. This is also termed as auxiliary or characteristic equation. The roots of equation are called auxiliary or characteristic roots and these roots help in getting explicit form of solution.

Linear Homogeneous Difference Equation:

3. If all the roots of characteristic polynomial are real and distinct then, solution will be in the form of

$$A_n = b_1\alpha_1^n + b_2\alpha_2^n + \ldots\ldots + b_k\alpha_k^n \qquad \ldots.\ 2.3$$

where α_i 's are roots of poly. and b_i 's are obtained after substituting those k initial conditions for i = 1, 2,…,k.

4. If any one of the roots, say α_1 has multiplicity m then solution will be in the form of:

$$A_n = b_1\alpha_1^n + nb_2\alpha_1^n + \ldots + n^{m-1}\, b_m\alpha_1^n + b_{m+1}\alpha_2^n + \ldots + b_{m+k-1}\alpha_{k-m-1}^n \qquad \ldots.\ 2.4$$

5. If the roots are in complex conjugate pairs say $\alpha = u \pm iv$ then this will lead to the solution in the form of (for order 2)

$$b_1\, r^n \sin(\theta n) + b_2\, r^n \cos(\theta n), \qquad \ldots.\ 2.5$$

where $r^2 = u^2 + v^2$ and $\tan(\theta) = v/u$ for $0 \le \theta < \pi$.

Stability of Linear Homogeneous Difference Equation:

- If $|\alpha_i| < 1$ for all $i = 1, 2, \ldots, k$ then all the roots lie inside the unit circle and hence tends to 0 as n approaches towards ∞. This leads to the stable condition asymptotically.

- If $|\alpha_i| \leq 1$ for all $i = 1, 2, \ldots, k$ (at least one of the α_i has magnitude 1) then it's called stable condition.

- If any of the roots say $|\alpha_i| > 1$ then it diverges and hence make the system unstable.

Analysis of Fibonacci Rabbit Model:

- The mathematical representation of Fibonacci Rabbit Model is $f(n) = f(n-1) + f(n-2)$ for all $n \geq 3$.

$m^n = m^{n-1} + m^{n-2}$ $m^2 - m - 1 = 0$

- To solve this second order linear homogeneous difference equation, we'll substitute $f(n) = m^n$, hence the characteristic equation will be $m^2 - m - 1 = 0$. The characteristic roots are $m = \dfrac{1 \pm \sqrt{5}}{2}$

- The required solution will be in the form of $f_n = c_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_2 \left(\frac{1-\sqrt{5}}{2}\right)^n$ where c_1, c_2 are constants. Using initial conditions, the solution will be

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$ for all $n \in N$. …. 2.6

- See the beauty of mathematics !. Irrational numbers are giving a natural number in every iteration i.e., for every value of n. (Interesting !!)

- Again, since one of the auxiliary roots is > 1, so this equation will approach to infinity as n grows with an order of $O(\varepsilon^{1.618})$, where $\varepsilon = f_{n+1} / f_n$.

Linear Cell Division Model:

- Assume that a cell division takes place concurrently and at each time, cell is producing α numbers of daughter cells.

- Number or density of cells at the beginning of cell division process is C_0.

- After n generations, number of cells shall be C_n , given as

$$C_n = \alpha C_{n-1} \text{ where } n \in N \qquad \ldots\ 2.7$$

n = generation number.

- This model is in the form of simple linear difference equation and also known as Malthus model named after famous English Economist Thomas Robert Malthus.

Linear Cell Division Model:

- Now assume that, cells are producing themselves at a rate of b and simultaneously cells are extincting at a rate of d.

- If cell population is changing only due to births and deaths of cells i.e. without any migration effect, then $C_n - C_{n-1}$ gives the difference of number of births and deaths over the interval t_{n-1} to t_n . If b and d are constants then, $C_n - C_{n-1} = (b - d)\, C_{n-1}$, or

$$C_n = (1 + b - d)\, C_{n-1} = \alpha^n C_0 \text{ where } n \in \mathbb{N}, (\text{for } \alpha = 1 + b - d). \qquad \ldots.\ 2.8$$

- If further migration of cells is allowed with a constant rate of migration M (M>0 for immigration and M<0 for emigration), the evolved model will be:

$$C_n = \alpha C_{n-1} + M, \text{ where } n \in \mathbb{N}. \qquad \ldots.\ 2.9$$

- This equation is in the form of linear non-homogeneous difference equation. **Stability** of the same can be analyzed from the previous analysis.

Linear Non-Homogeneous Difference Equation:

- The general form of linear non-homogeneous difference equation of order 'k' is:

$$A_n + d_1 A_{n-1} + d_2 A_{n-2} + d_3 A_{n-3} + \ldots\ldots\ldots + d_k A_{n-k} = f(n). \qquad \ldots.\ 2.10$$

- If f(n) is identically zero then above equation turns into linear homogeneous equation.

- If f(n) is not identically zero then the solution $\boxed{A_n = C.F. + P.F.}$,where C.F. is called complimentary function and P.F. is called particular function. C.F. can be calculated from the method described for linear homogeneous equation. The value of P.F. depends on the forcing function f(n).

- For calculating P.F., there is a big role of auxiliary equation (recall!). Let g(m) is the corresponding auxiliary equation of homogeneous part of non-homogeneous equation after substituting $A_n = m^n$.

Linear Non-Homogeneous Difference Equation:

- Now let's discuss different possible cases of f(n).

1. For $\boxed{f(n) = a^n}$, then $\boxed{\text{P.F.} = \frac{1}{g(a)}a^n}$, provided $g(a) \neq 0$. $\frac{1}{(D-a)^m} f(a)$ degree n

   If $g(a) = 0$, then P.F. $= \frac{1}{(m-a)^k}a^n = (nCk)\,a^{n-k}$ where k is the multiplicity factor by which a is occurring as root in $g(m)$.

2. For $f(n) = n^p$, then P.F. $= \frac{1}{g(m)}n^p = \frac{1}{g(1+\Delta)}n^p$,

   where, Δ operator is defined as: $\Delta([n]^p) = p.[n]^{p-1}$. and [.] is factorial notation defined as $\boxed{[n]^r = n(n-1)\ldots(n-r+1)}$.

3. For f(n) = $cos(kn)$ or $sin(kn)$, then P.F. $= \boxed{\frac{1}{2i}\left[\frac{1}{g(m)}e^{ikn} \pm \frac{1}{g(m)}e^{-ikn}\right]}$

   where $m = \exp(ik)$. Again, if $g(m)=0$, then the same procedure will be followed as mentioned in step 1.

Example on Linear Non-Homogeneous Difference Equation:

Question : Solve $y_{n+2} - 4y_n = 2^n + n^2 - 1$ with $y(0)=y(1)=0$.

$\underline{\ \ }$ C.F. $f(n) \rightarrow$ P.F.

Solution: For C.F. Put $y_n = m^n$, this leads to the auxiliary equation as : $g(m) = m^2 - 4 = 0$. The auxiliary roots are $m = \pm 2$. So C.F. $y_n = c_1(2)^n + c_2(-2)^n$.

$m_1 = 2$, $m_2 = -2$ $= c_1(m_1)^n + c_2(m_2)^n$

For P.F.

P.F. 1 (for 2^n)$= 0.25(n\,2^{n-1} - 2^n/4)$;

P.F. 3 (for 1)$= 1/3$

P.F. 2 (for n^2) $= [(1+\Delta)^2 - 4]^{-1}(n^2)$

$= [(1+\Delta)^2 - 4]^{-1}(n(n-1) + n)$

$= [(1+\Delta)^2 - 4]^{-1}([n]^2 + [n])$

$= -1/3\,[[n]^2 + 7[n]/3 + 20/9]$

$= -1/3\,[n^2 + 4n/3 + 20/9]$

So the general solution will be

$$y_n = c_1(2)^n + c_2(-2)^n + 0.25(n\,2^{n-1} - 2^n/4) - 1/3\,[n^2 + 4n/3 + 20/9] + 1/3.$$

After substituting these two initial conditions, the particular solution will be

$$y_n = 713/1296(2)^n - 26/324(-2)^n + 0.25(n\,2^{n-1} - 2^n/4) - 1/3\,[n^2 + 4n/3 + 20/9] + 1/3.$$

Analysis of Linear Cell-Division Model:

- The mathematical representation of linear cell-division model is given by eq. 2.9 : $C_n - \alpha C_{n-1} = M$, where $n \in N$.

- If M is treated as constant, the solution of above equation after substituting initial condition can be written as:

$$\boxed{C_n = C_0\alpha^n + (M/M\text{-}\alpha)(1 - \alpha^n)} \quad \ldots.\ 2.11$$

$C_n = C_0.$

- For stability analysis, α may have following cases:
  1. $\alpha = 1$ implies constant population of cells. (stable)
  2. $\alpha > 1$ implies no. of cells goes to ∞ as generation progress. (unstable)
  3. $\alpha < 1$ implies population will get leveled as generation progress. (asymptotically stable)

Summary:

- Introductory population dynamics.
- Fibonacci rabbit model formulation and solution.
- Solution method for linear difference equation with constant coefficients.
- Different linear cell models and their formulation.


Thank you!
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