MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 2.5 : Discrete Time Prey-Predator Model

↓ Linear growth
(i) model

$$\frac{dx}{dt} = (b-d)x$$

$$= \alpha x \text{ (Linear growth model)}$$

$$\frac{dx}{dt} = x(a-bx)$$

$$= \boxed{ax - bx^2}$$

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Contents:

- Prey – Predator Model.
- A Resource Limiting Prey – Predator Model.

Prey – Predator Model:

- In the last lecture, we studied about prey – predator model where each individual species were having geometrical growth rate i.e.

$$\boxed{x_{n+1} = (1 - a)\, x_n .}$$

$y_{n+1} - y_n = c\, y_n + d\, x_n y_n$

$\frac{dy}{dt} = (c\, y_n + x_n)\, d$

$\frac{dx}{dt} = -(b-d)\, x = \alpha x_n$

$\boxed{x_{n+1} = x_n + \alpha x_n}$

$= x_n(\alpha + 1)$

- We found that both the fixed points were un-stable.

logistic (or) population growth

$\frac{dx}{dt} = x(a - bx) = ax - bx^2$

- While introducing logistic growth rate function, we learnt that this function comes in picture when there are limited resource or competition for food. The same concept will be applied here for prey species (because they are the only resources for predators) in order to re-formulate the model. The new model is:

$F(x_n, y_n)$

$G(x_n, y_n)$

$$x(n+1) = (1+a)\, x(n) - \beta x^2(n) - b\, x(n)\, y(n),$$

$$y(n+1) = (1- c)\, y(n) + d\, x(n)\, y(n). \qquad \ldots\ 10.1$$

self limiting

growth of population

due to limited resource of food.

Stability of Prey – Predator Model with Limited Resources:

Competition model

- The fixed points are given as:

$x(n+1) = x(n) = \lambda_1$

$y(n+1) = y(n) = \lambda_2$

$$\lambda_1 = (1+a)\lambda_1 - \beta\,\lambda_1^2 - b\lambda_1\,\lambda_2\,,$$
$$\lambda_2 = (1-c)\lambda_2 + d\lambda_1\,\lambda_2\,.$$

$\lambda_1(a - \beta\lambda_1 - b\lambda_2) = 0$

$\lambda_2(-c + d\lambda_1) = 0$

$(\lambda_1, \lambda_2) = (0, 0)$

- After solving for those equations, we will get

$$\lambda_1 = 0 \text{ or } a - \beta\,\lambda_1 - b\lambda_2 = 0,$$
$$\lambda_2 = 0 \text{ or } d\lambda_1 - c = 0.$$

$\lambda_1 = c/d$

$\lambda_2 = (a - \beta\, c/d)\,1/b$

- Hence the steady states are given as: $(\lambda_1, \lambda_2) = (0, 0)$; $(a/\beta, 0)$; $(c/d, (a/b - \beta c/bd))$.

Stability of Prey – Predator Model with Limited Resources:

- For fixed point (0, 0), the Jacobian is $J = \begin{bmatrix} 1+a & 0 \\ 0 & 1-c \end{bmatrix}$

$F(x_n, y_n) = (1+a)x(n) - \beta x^2(n) - b x(n) y(n)$

$\frac{\partial F}{\partial x_n} = (1+a) - 2\beta x(n) - b y(n) = (1+a) - 2\beta x - b\lambda_2$

$-2\beta \frac{a}{\beta}$

$\frac{\partial F}{\partial y_n} = -b x(n) = -b\lambda_1$

Clearly, the eigen values are (1+ a) and (1 – c). For fixed point (0, 0) to be stable,

$|1 + a| < 1$ implies $-2 < a < 0$.

$G(x_n, y_n) = (1-c) y(n) + d x(n) y(n)$

$G_y = (1-c) + dx$

$G_x = d\lambda_2$

$\lambda = 0$

This is not possible and hence the fixed point (0, 0) is un-stable.

- For fixed point (a/β, 0), the Jacobian is $J = \begin{bmatrix} 1-a & -b\frac{a}{\beta} \\ 0 & (1-c)+d\frac{a}{\beta} \end{bmatrix}$.

Since J is upper triangular matrix, the eigen values are given by principle diagonal entries only. The eigen values are (1 – a) and (1 – c) + dK where K = a/β called as carrying capacity.

Stability of Prey – Predator Model with Limited Resources:

- The stability of fixed point $(a/\beta, 0)$ will depend on the magnitude of both the eigen values.

  We have already assumed $a>0$ while formulating the model. If a is bounded at 2 i.e. $0 < a < 2$ then the $|1 - a| < 1$.

- Now, consider the second eigen value $(1 – c) + dK$. Since we know that $0 < c < 1$, so $0 < 1 – c < 1$. Define a gain-loss ratio term $\mu = K/(c/d)$. So the eigen value can be written as $1 – c + \mu c$. For stability,

$$|1 – c + \mu c| < 1 \text{, or}$$

$$-1 < 1 – c + \mu c < 1 \text{, or}$$

$$0 < c(1 – \mu) < 2.$$

Stability of Prey – Predator Model with Limited Resources:

$1-c>0$

- Now observe the last inequality carefully. Since $0 < c < 1$, so if $\mu > 1$, it will not satisfy the stability condition and hence will make the fixed point un-stable.

  Also if $0 < \mu < 1$, it will make fixed point asymptotically stable.

$\frac{dy}{dt} = -cy + dxy = (-c + dx)y$

- Now, what is the biological meaning of the gain-loss ratio μ ? We concluded mathematically that if $\mu < 1$ i.e. $K < c/d$ then it will make fixed point stable. K is carrying capacity and the ratio c/d signifies ratio of decline rate of predator in absence of prey to effect of interaction with prey on predator's growth.

➢ If c is high then it will make predator to extinct at faster rate in absence of prey's which also implies that interaction effect will be less on predator (and vice-versa). So overall c/d will show the loss of predator population.

➢ If d is high, it means predator has high effect due to interaction, which will make prey population to decline at much faster rate and the same will affect the predator's population at later stage.

Stability of Prey – Predator Model with Limited Resources:

- So in both the cases, c/d signifies the loss of predator population. Hence the ratio μ termed as gain to loss ratio.

- Biologically, if gain > loss then system should be stable about fixed point (a/β, 0) for μ < 1 and the system should be un-stable about (a/β, 0) for μ > 1.

- Now, for steady state (c/d, (a/b – βc/bd)), the Jacobian is $J = \begin{bmatrix} 1 - \frac{\beta c}{d} & -\frac{bc}{d} \\ d\left(\frac{a}{b} - \frac{\beta c}{bd}\right) & 1 \end{bmatrix}$.

- The characteristic equation is:

$$\lambda^2 - \lambda\left(2 - \frac{\beta c}{d}\right) + \left(1 - \frac{\beta c}{d}\right) + ac\left(1 - \frac{1}{\mu}\right) = 0$$

Stability of Prey – Predator Model with Limited Resources:

- The eigen values are given as:

$$\boxed{\lambda = 0.5\left(2-\frac{\beta c}{d}\right) \pm 0.5\sqrt{\left(\frac{\beta c}{d}\right)^2 - 4ac\left(1-\frac{1}{\mu}\right)}} < 0$$

- Observe that if $\boxed{\mu < 1}$ then $(1 - 1/\mu) < 0$ implies one of the eigen values

$$\boxed{\lambda > 0.5\left(2-\frac{\beta c}{d}\right) + 0.5\sqrt{\left(\frac{\beta c}{d}\right)^2}} \text{ or } \lambda > 1.$$

- So fixed point will not be stable for $\mu < 1$ and will be asymptotically stable for $\mu > 1$.

Summary:

- Prey – predator models.
- Effect of limited resources on prey-predator model.

Mathematical Models → Deterministic, Stochastic

Discrete Time System (involve difference equations)

- Linear Models
- **Non-Linear Models**

Continuous time system (involve ODE's, PDE's)

1. **Non- linear Cell Division Model**

✓ Survival function (No competition, contest competition, scramble competition)

✓ Beverton Holt Model

✓ Hassel Model

2. **Non- linear Prey-Predator Model**

✓ Without limited resources.

✓ With limited resources.

Logistic Difference Equation:

1. Applications.
2. Formulation.
3. **Graphical Analysis**

✓ Stable solution.

✓ Periodic Solution.

✓ Chaos.

4. **Introduction to bifurcation**

a. Saddle-node bifurcation.

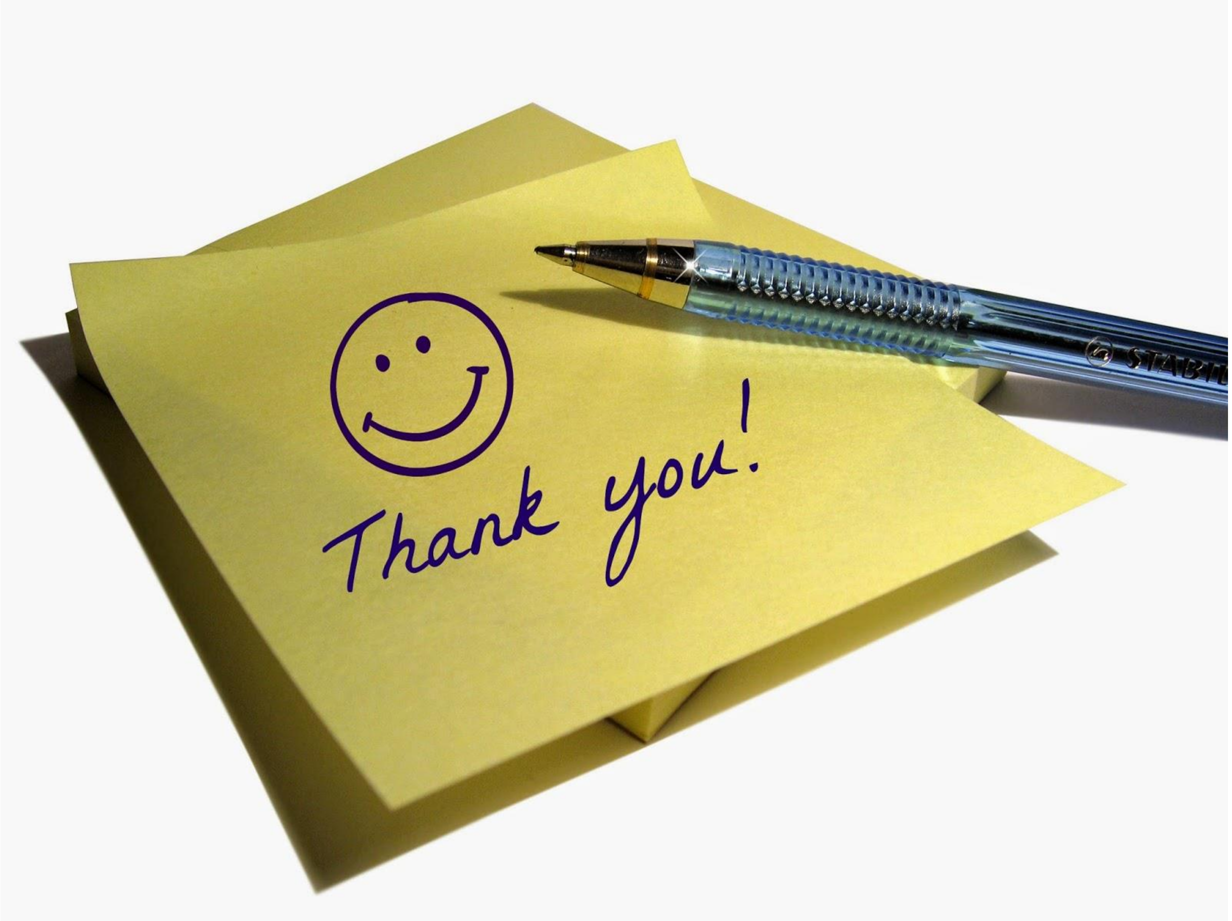
b. Trans-critical bifurcation.

c. Pitch-fork bifurcation.

d. Periodic doubling (flip) bifurcation.

Stability Analysis:

1. Fixed point.
2. Linearization – Taylor's series method.
3. Jacobian.


Thank you!
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