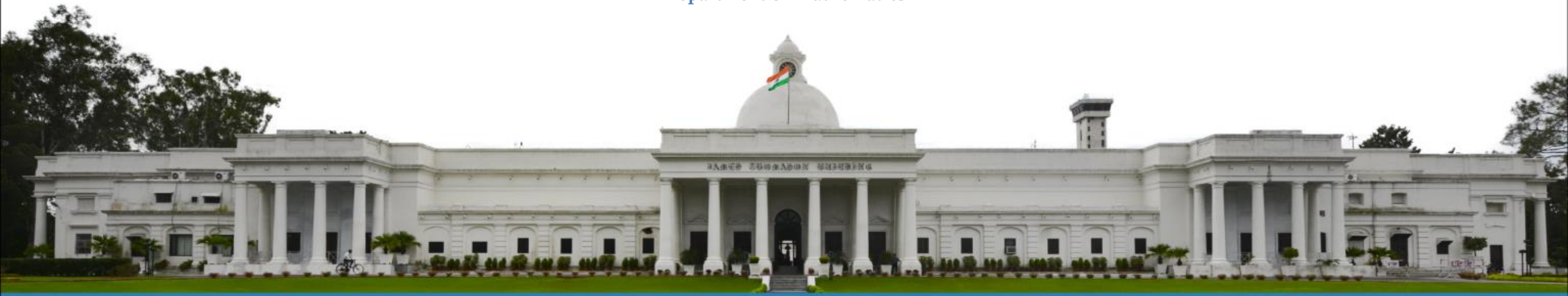
IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 3.3 : Continuous Time Models in Population Dynamics - I

Dr. Ameeya Kumar Nayak
Department of Mathematics



Contents:

- Growth of Micro-Organisms.
- Dependency of Rate of Reproduction on Time and Resources.
- Steady States.
- Concept of Limited Resources.

Growth of Micro-organisms:

- One of the simplest model using differential equations is the growth of micro-organisms.

- The growth of unicellular micro-organisms like bacteria and the changes occurring in their population can be studied by simple differential equation. We have already studied the discrete model for same. (Recall cell-division model)

- Suppose a droplet of bacterial suspension is put in to a flask or a test tube having some nutrient medium. The culture is also maintained at compatible conditions to help bacteria to reproduce by cell division process.

- If after time t, K number of new cells are formed by cell division process i.e.

$$K = \text{rate of reproduction per unit time}$$

Growth of Micro-organisms:

- Hence the model for growth of micro-organisms will be:

$$\boxed{\frac{dN}{dt} = KN(t)} \quad \text{.... 13.1}$$

- For constant reproduction rate, the solution of above equation is given as:

$$N(t) = N_0 e^{Kt}, \quad \text{where } N_0 = N(0) \quad \text{.... 13.2}$$

- This equation is also known as Malthus Law (Recall the same in discrete time).

- If instead of cell reproduction, they are extincting at the rate of K, then the equation will be (also referred as decaying population model):

$$\boxed{N(t) = N_0 e^{-Kt}}, \quad \text{where } N_0 = N(0) \quad \text{.... 13.3}$$

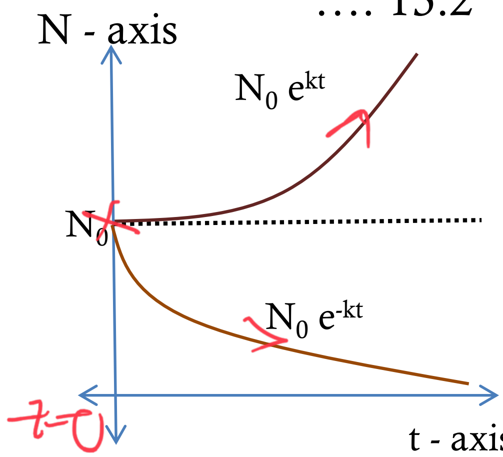


Fig. 13.1: Exponential Nature

Growth of Micro-organisms:

- Doubling time τ is defined as the time at which the number of cells will get doubled as that of the initial population. This is defined as:

$N(t)$

$$2N_0 = N_0 e^{K\tau} \quad \text{or} \quad \boxed{\tau = \frac{ln2}{K}} \qquad \text{…. 13.4}$$

- Similarly, for decaying population model, half life period can be defined as the time at which the number of cells will get halved as that of the initial population.

$$\frac{N_0}{2} = N_0 e^{-K\tau} \quad \text{or} \quad \boxed{\tau = \frac{ln2}{K}} \qquad \text{…. 13.5}$$

- Now, think for parameter K. Should it be a function of time or something else? If K depends on time, it's generally known as Gompertz law. Generally there is no exact information of time dependence behavior of K, but practically we can analyze that K should have direct or in-direct dependency on available resources.

Growth of Micro-organisms:

- Let C represent the concentration of resources available. Then for K = K(C), the Malthus model will be reframed as:

$$\boxed{\frac{dN}{dt} = K(C)N(t)} \quad \text{.... 13.6}$$

- Now the question is, how resource-concentration is varying with time i.e. C = C(t)? Is it a constant function or a varying one?

- Experimentally, the rate of resource concentration should decrease as rate of population increases i.e. $C' = -\alpha N'$ where α is a constant describing consumption of food by each new bacteria. Hence the mathematical model for growth of population with resource dependent production rate will be:

$$\begin{aligned} N'(t) &= K(C)N(t) \\ C'(t) &= -\alpha N'(t) = -\alpha K(C) N(t) \end{aligned} \quad \text{.... 13.7}$$

Growth of Micro-organisms:

- Let's consider a simple case where rate of reproduction per unit time, K depends linearly on available concentration i.e. $K(C) = \kappa C$. So the model will be:

$$N'(t) = \kappa C N(t)$$
$$C'(t) = -\alpha N'(t) = -\alpha\kappa C N(t) \qquad \text{.... 13.8}$$

- To convert this system of two differential equations into a single differential equation, let's first solve the second differential equation of system 13.8. This simplifies as:

$C(t) = -\alpha N(t) + C_0$ where C_0 is a constant.

- Substituting this expression of C(t) back in first equation, we will get logistic growth equation as:

$= \kappa C_0 N(t) - \alpha\kappa N^2(t)$

$$N'(t) = \kappa N(t)(C_0 - \alpha N(t)) \qquad \text{.... 13.9}$$

$\to 0$

Growth of Micro-organisms:

$N(t)=0$ or $C_0 - \alpha N(t) = 0$

$N(t) = C_0/\alpha$

- Recall, we have solved the same equation in previous lecture as an example in method of separation of variables. The solution so obtained was:

$N(t) \to A$ as $t \to \infty$

$$\boxed{N(t) = \frac{A\cancel{N_0}}{\cancel{N_0} + (A - N_0)e^{-\gamma t}}} \quad \text{.... 13.10}$$

$C(t) = -\alpha N(t) + C_0$

$t \to \infty$

$N(t) \to A$

where $N_0 = N(0)$, $A = C_0/\alpha$ and $\gamma = \kappa C_0$.

$t=0$

- The steady state is defined as a state where the rate of population growth is zero (refer eq. 13.9) i.e. where the population is approaching a constant value. There are two steady states, $N = 0$ and $N = C_0/\alpha = A$.

Growth of Micro-organisms:

- Now if the population size is small, then equation 13.9 will turn into

$$\boxed{N'(t) \sim \kappa C_0 N(t),} \qquad \text{.... 13.11}$$

  i.e. it will have exponential growth instead of logistic growth rate.

- Now analyze the graph (fig. 13.2), for both high and small initial population, the population is approaching to steady state N = A. Hence the steady state N = A is stable. The steady state N = 0 is unstable because for population starting near 0, it's approaching to A only.

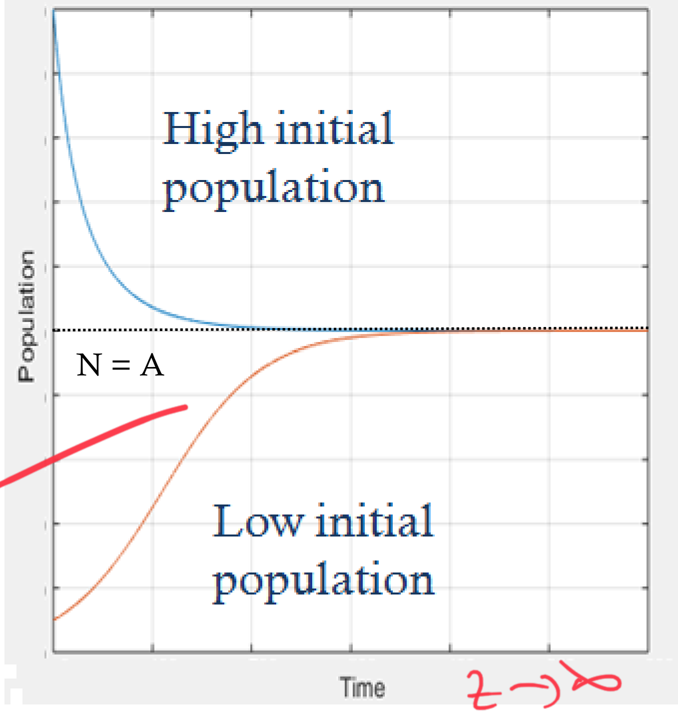


Fig. 13.2 Graph of Cell Division Model

Growth of Micro-organisms:

- So we have analyzed the behavior of population growth for both high and small initial population when population rate function has linear variation with respect to available resources.
- This linear variation of resources implies that resources are unlimited. But practically we have limited resources.
- Then how to deal with limited resources in our model? To model this kind of culture, we use a devise called **Chemostat**.
- We will study this in next lecture.

Summary:

- Growth of Micro-organisms.
- Doubling time and half life time.
- Variation of rate of reproduction with respect to time and available resources.
- Linear variation of rate of reproduction.
- Stable and unstable steady states.
- Limited resources.


Thank you!