IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 2.4 : Discrete Time Non-Linear Models in Population Dynamics - II

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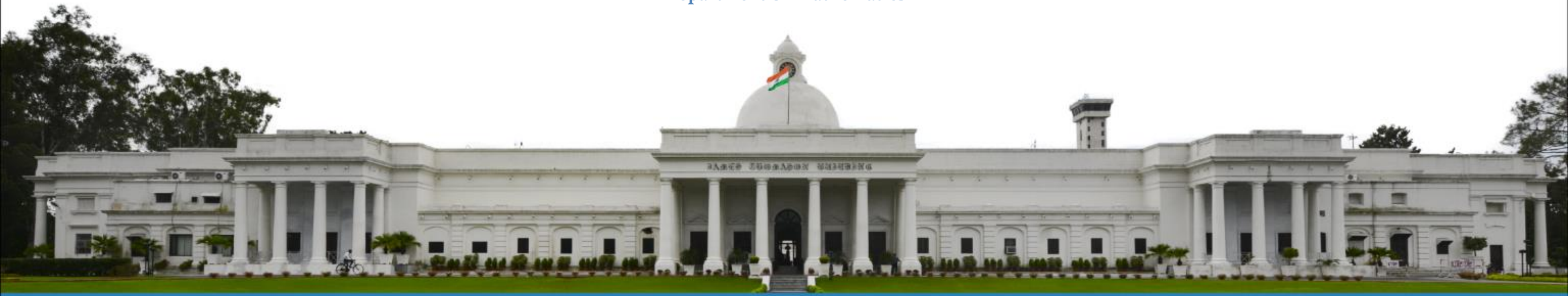

Contents:

- Logistic Difference Equation.
- Prey-Predator Model.

Logistic Difference Equation:

- Consider the general logistic difference equation,

$$\boxed{x_{n+1} = r\, x_n (1 - x_n).} = f(x_n)$$

$x_1 = r x_0$
$x_2 = r x_1 = r^2 x_0$
$x_3 = r x_2 = r^3 x_0$
$\boxed{x_{n+1} = r^{n+1} x_0}$

- If x_n is very small then $1 - x_n \sim 1$ and hence the equation will reduce in form of

$$\boxed{x_{n+1} \approx r\, x_n.}$$

So logistic equation turns out be an exponential growth function for very small values of x_n. ($x_{n+1} \approx r^{n+1} x_0$) → very small

- Hence the linear models which we considered so far are only valid for small populations.

- Also we have already analyzed logistic equation. Now in this lecture we will study about some population models which are based on logistic growth.

Non-Linear Prey-Predator Model: Discrete model

- Assume $x(n)$ be the population of prey

  and $y(n)$ be the population of predator.

- Let us analyze following points for simplified model formulation:

1. (Recall the linear prey-predator model) x is the only prey and y is the only predator.
2. If there are no predators, the prey population will grow at a rate proportional to the population of the prey.
3. If there are no prey, the predator population will decline at a rate proportional to the population of the predator.
4. When both are present, it is beneficial for growth of predator while harmful for prey. Hence predator population will increase and prey will decrease at the rate proportional to product of two population.

Non-Linear Prey-Predator Model:

$x_{n+1} = ((1+a) - b y_n) x_n$

- Based on these points, the mathematical model will be:

$F(x(n), y(n))$

$$x(n+1) = (1+a)\, x(n) - b\, x(n)\, y(n),$$

$\Delta x_n = a x_n - b x_n y_n$

$G(x_n, y_n)$

$\Delta y_n = y_{n+1} - y_n = -c y_n + d x_n y_n$

$$y(n+1) = (1- c)\, y(n) + d\, x(n)\, y(n).$$

$x_{n+1} - x_n = a x_n - b x_n y_n$

$\Rightarrow \boxed{x_{n+1} = (1+a) x_n - b x_n y_n}$

where all the constants a, b, c and d > 0 (why?) and

a = growth rate of prey population, c = death rate of predator population,

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b, d = effect of the interaction between prey and predator. (Recall the linear prey-predator model)Observe that if c > 1 it will make predator to grow geometrically in absence of prey species, which is un-realistic. Hence 0 < c < 1.

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- In previous lecture we learnt about stability analysis of first order non-linear difference equation. (Recall fixed point and cob-web graphs !). But here we have a system of two non-linear coupled equations. Can we apply the same fixed point method to deal this situation ? YES, it can be done with a matrix approach.

Jacobian Matrix:

$Z_1(n+1), Z_2(n+1)$ - - - $Z_m(n+1)$

$Z_1(n+1)$ $Z_2(n+1)$

- Consider a general system of m non-linear difference equations

$$Z_i\,(n+1) = F_i(Z_1\,(n), Z_2\,(n), \ldots, Z_m\,(n)) \text{ for } i = 1, 2, 3, \ldots, m.$$

For m = 2, let $\boxed{A = [\lambda_1 \quad \lambda_2]^T}$, where A represent a vector of corresponding fixed points.

- To find these fixed points, we need to solve two simultaneous non-linear algebraic equations:

$Z_1(n+1) = f_1(Z_1(n), Z_2(n))$ $Z_i(n) = \lambda_i$

$$\boxed{\lambda_1 = F_1(\lambda_1, \lambda_2)} \text{ and } \boxed{\lambda_2 = F_2(\lambda_1, \lambda_2)}.$$

$Z_i(n+1) = \lambda_i$

- Now to check the stability of system around these fixed points, we need a Jacobian matrix. The Jacobian matrix for system of 2 non-linear equations is given as:

$$J(A) = \begin{bmatrix} \frac{\partial F_1}{\partial Z_1}(\lambda_1,\lambda_2) & \frac{\partial F_1}{\partial Z_2}(\lambda_1,\lambda_2) \\ \frac{\partial F_2}{\partial Z_1}(\lambda_1,\lambda_2) & \frac{\partial F_2}{\partial Z_2}(\lambda_1,\lambda_2) \end{bmatrix}$$

$$\left|\frac{\partial f}{\partial n}\right| < 1$$

Analysis of Prey-Predator Model:

- If the magnitude of all the eigen values of Jacobian matrix J(A) are strictly less than 1, then system is stable. $|J(A)| < 1$
- If the magnitude of any one of the eigen values is greater than or equal to 1, system will be unstable.

- Go back to the non-linear prey-predator model. The fixed points are given as:

$$x_{n+1} \quad \lambda_1 = (1+a)\lambda_1 - b\lambda_1 \lambda_2 \qquad \Rightarrow \lambda_1[1 - 1 - a + b\lambda_2] = 0$$

$$y_{n+1} \quad \lambda_2 = (1\text{-}c)\lambda_2 + d\lambda_1 \lambda_2 \qquad \Rightarrow \lambda_2[1 - 1 + c - d\lambda_1] = 0$$

- By solving those coupled simultaneous equations, we will get (0, 0) and (c/d, a /b) as fixed points.
- The Jacobian matrix is given as: $J = \begin{bmatrix} (1+a) - by & -bx \\ dy & (1-c) + dx \end{bmatrix}$ at $(0, 0)$

Analysis of Prey-Predator Model:

- For fixed point (0, 0), the Jacobian matrix is: $J = \begin{bmatrix} (1+a) - \lambda & 0 \\ 0 & (1-c) - \lambda \end{bmatrix}$

  The eigen values are (1+a) and (1– c).

  For fixed point (0, 0) to be stable |1 + a| should be less than 1. or -1 < 1+ a < 1 implies -2 < a < 0. But we considered a > 0 so the fixed point (0, 0) is un-stable.

- Similarly for fixed point (c/d, a/b), the Jacobian matrix is: $J = \begin{bmatrix} 1 - \lambda & -\frac{bc}{d} \\ \frac{da}{b} & 1 - \lambda \end{bmatrix}$

  $(1-\lambda)^2 + ac = 0$

  The eigen values are given as $\lambda = 1 \pm i\, ac$

  $\lambda = 1 \pm i\, ac$.

  The magnitude of eigen values are > 1, hence this steady state is also un-stable.

Example on Prey-Predator Model:

Question: Consider a forest where only fox and rabbits are living. Fox are entirely depending on the rabbit population. If the rabbit's growth rate is 1.2 and that of fox is 0.4. The effect of interaction of both the species on rabbits is 0.5 while on fox is 2. Formulate a discrete time model and calculate the number of each species after 2 years.

Solution: The model corresponding to this data is:

$$R(n+1) = (1+1.2)\, R(n) - 0.5\, R(n)\, F(n),$$

$$F(n+1) = (1-0.4)\, F(n) + 2\, R(n)\, F(n).$$

Now proceed for calculation of Jacobian after finding fixed points. (Exercise)

We will get two fixed points (0, 0) and (0.2, 2.4). The Jacobian of system is

$$J = \begin{bmatrix} 2.2 - 0.5F_n & -0.5R_n \\ 2F_n & 0.6 + 2R_n \end{bmatrix}$$

Example on Prey-Predator Model:

- For fixed point (0, 0) the eigen values are 2.2 and 0.6 (one of them is > 1 so unstable) (Exercise).
- For fixed point (0.2, 2.4) the Jacobian is

$$J = \begin{bmatrix} 1 & -0.1 \\ 4.8 & 1 \end{bmatrix}$$

- The magnitude of eigen values is greater than 1 , so unstable.

Summary:

- Applications of logistic difference equation.
- Prey-predator model.
- Jacobian matrix.
- Stability of non-linear system.
- Analysis of prey-predator model.


Thank you!
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