IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 3 : Discrete Time Linear Models in Population Dynamics - II

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Contents:

- Linear Prey-Predator Model.
- Stability Analysis of Linear Systems – Matrix Approach.
- Graphical Solution of First Order Linear Difference Equations.

Linear Prey-Predator Model:

- Consider a forest containing only Tigers (predator) and $\boxed{\text{Deer}}$ (prey). The tigers kill the prey for food.

- Let T_n = population of tigers at the end of year 'n' and
  D_n = population of deer at the end of year 'n'.

- In order to formulate model, following assumptions are considered:

1. Deer are the only prey for tigers and Tigers are the only predators for deer.
2. Tigers will die out in absence of deer but deer population will grow in absence of tigers, also.
3. In presence of deer population the rate of tiger population growth increases.
4. In presence of tiger population the growth rate of deer population declines.

Linear Prey-Predator Model:

- Further, let's assume that
  a : rate at which tiger dies in absence of deer.
  b : rate at which tiger grows in presence of deer.
  c : rate at which deer grows in absence of tigers.
  d : rate at which deer dies in presence of tigers.

- Hence the model equations will be:

$$\Delta T_n = T_{n+1} - T_n = -a T_n + b D_n,$$
$$\Delta D_n = D_{n+1} - D_n = c D_n - d T_n \text{ where, } n \in N. \qquad \ldots 3.1$$

- In matrix form the model can be written as:

$$\begin{bmatrix} T_{n+1} \\ D_{n+1} \end{bmatrix} = \begin{bmatrix} 1-a & b \\ -d & 1+c \end{bmatrix} \begin{bmatrix} T_n \\ D_n \end{bmatrix}$$

Stability of Linear System – Matrix Approach:

- Suppose, a linear system is represented as:

  $[W_{n+1}] = [A][W_n]$; where [A] is called coefficient matrix. …. 3.2

- Any point in a system is called an equilibrium point (critical point or steady state) when $W_{n+1} = 0$.

- Characteristic equation of the system 3.2 is given by

$$\det(A - \lambda I) = 0.$$

- Let λ_i 's be the characteristic roots or eigen values of matrix [A].

  1. If any one of $|\lambda_i| > 1$, then the system is unstable about equilibrium point.
  2. If all $|\lambda_i| < 1$ then the system is said to be stable about equilibrium point.
  3. If all $|\lambda_i| \leq 1$ (at-least one $\lambda_i = 1$) then the system is called asymptotically stable about equilibrium point.

Example on Linear Prey-Predator Model:

Question : Consider a prey-predator model for two species X and Y where X is predator and Y is prey. The prey's growth rate is 1.2 while that of the predator is 1.3. Prey's population reduces by factor of 0.3 times that of predator while predator's population increases by 0.4 times that of prey's. Construct the discrete time model and comment on stability.

Solution : From the given data, a=0.4 ; b=1.3 ; c=1.2 ; d=0.3. Hence the formulated model is:

$$X_{n+1} = 0.6\, X_n + 1.3\, Y_n ,$$
$$Y_{n+1} = 2.2\, Y_n - 0.3\, X_n \text{ where, } n \in N. \qquad \ldots.\ 3.3$$

- The coefficient matrix [A] = $\begin{bmatrix} 0.6 & 1.3 \\ -0.3 & 2.2 \end{bmatrix}$ $|A - \lambda I| = 0$

Example on Linear Prey-Predator Model:

- The eigen values of the coefficient matrix are: $\boxed{\lambda_1 = 1.4}$ and $\boxed{\lambda_2 = 1.9.}$ Hence the solution will be $X_n = c_1(1.4)^n + c_2(1.9)^n$, $Y_n = c_3(1.4)^n + c_4(1.9)^n$. By substituting these equations back in the system, one can get rid of 2 constants out of 4 constants (c_1 ,c_2, c_3, c_4). (Exercise. Verify that $\boxed{8c_1 = 13c_3}$; $\boxed{c_2 = c_4}$) and with given initial condition one can find the particular solution.

α^n α_1 α_2

$$\boxed{X_{n+1} - 0.6X_n = 1.3Y_n}$$
$$c_1(1.4)^{n+1} + c_2(1.9)^{n+1} - 0.6(c_1(1.4)^n - c_2(1.9)^n = 1.3Y_n) = 1.3Y_n$$
$$0.8c_1(1.4)^n + 1.3c_2(1.9)^n = 1.3Y_n$$
$$\frac{8}{13}c_1(1.4)^n + c_2(1.9)^n = Y_n$$

- The absolute values of both the eigen values are greater than 1, and hence the system is **unstable** about it's equilibrium point (0, 0).

- Now, what do you mean by unstable system here?- This means the process will go on and will never settle. When, predators will increase, prey population will grow on increasing and vice – versa.

- Now, we can analyze the situation better, when one of the λ is less than one and when both λ's are less than one. (Analyze!!)

Graphical Solution of First Order Difference Equation:

- So far we have discussed only the analytical solution of linear difference equations with constant coefficients. But if, we have been given any general difference equation (of first order) to solve!! Will these methods help us? NO !.

- With analytical methods, it's not always possible to obtain the solution due to certain limitations. Hence we need some other general method to deal with any kind of difference equation.

- In this case, graphical method may help us in analyzing the situation. Although, it will not give us the equation form of model, but it will help us to understand the behavior of the system.

- The way of solving difference equations graphically is called cob-webbing method. **This method is only applicable for first order equations.**

Graphical Solution of Difference Equation:

- Let a general first order difference equation is given as:

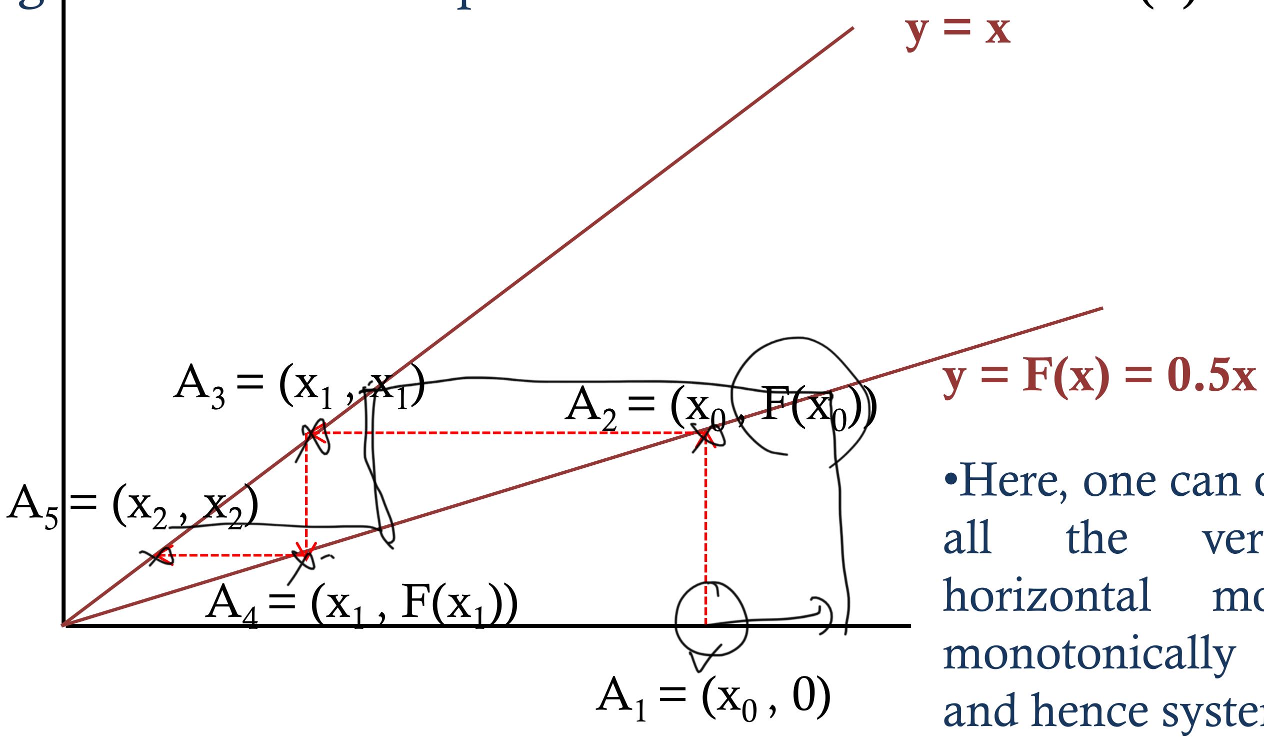
$$\boxed{X_{n+1} = F(X_n).} \qquad \text{.... 3.4}$$

- The methodology to draw cob-webs is as following:

1. Draw the graph of function y=F(x).
2. Draw the line y = x. Mark all the intersection points with curve of y = F(x).
3. Take any arbitrary point $A_1 = (x_0 , 0)$ on positive x-axis and put a vertical line up-to graph of $y= F(x)$. Here you'll get the point $A_2 = (x_0 , F(x_0))$.
4. From point A_2 ,move horizontally towards line y=x and stop at $A_3 = (F(x_0), F(x_0))$. Now say $x_1=F(x_0)$.
5. Again, move vertically from point A_3 to $A_4 = (x_1 , F(x_1))$ and call $x_2=F(x_1)$. Repeat this process moving alternately in horizontal and vertical direction till you find the pattern.
6. The resulting polygonal path is termed as the cob-web graph or simply a cob-web.

Graphical Solution of Difference Equation:

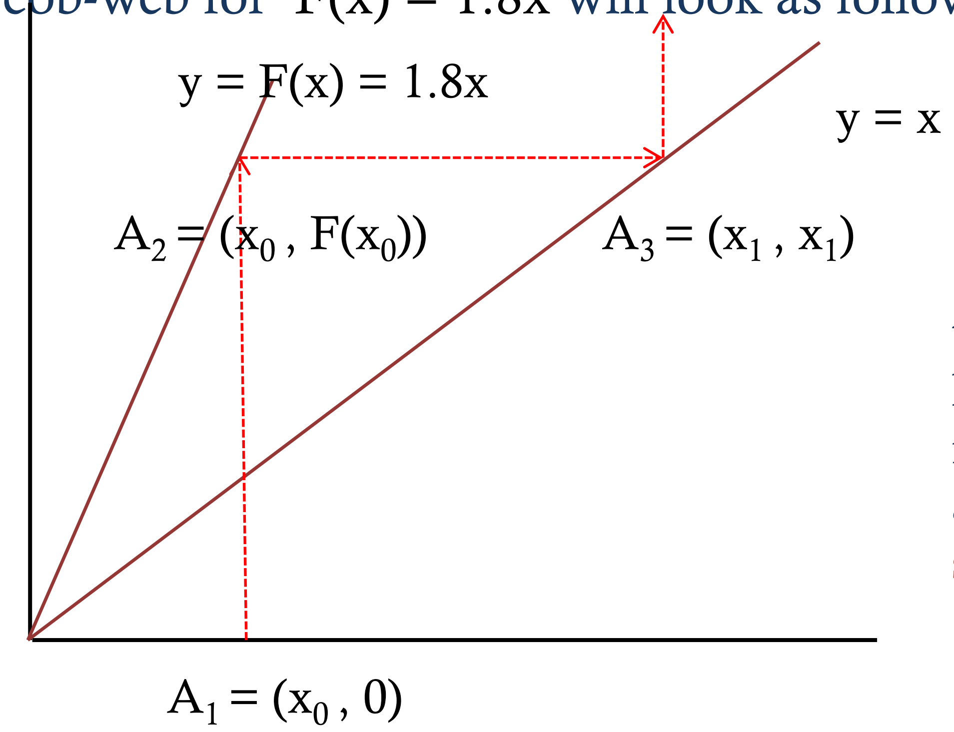
We are giving here a basic example of cob-web method with $F(x) = 0.5x$.



- Here, one can observe that all the vertical and horizontal motions are monotonically decreasing and hence system is **stable**.

Graphical Solution of Difference Equation:

Similarly the cob-web for $F(x) = 1.8x$ will look as follows:



- Here, one can observe that all the vertical and horizontal motions are monotonically increasing and hence system is **un-stable**.

Analysis of Linear Cell Model by Cobweb:

- In previous lecture we formulated the cell model as $\boxed{C_{n+1} = \alpha C_n}$ One can easily verify analytical solution with cob-web graphs for different values of α.

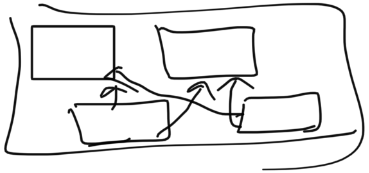
- The equations which have analyzed by cob webbing so far has equilibrium point at origin (i.e. homogenous linear difference equations only).

- Now think for nature of cob-web graphs for non-homogenous equations. Will they differ by homogenous one graphically? (YES!) And what about stability? (NO change!) (match your results with analytical solution.)

$[\quad] \neq 0$

- Since, we have already seen that non-homogeneous equation differ only at equilibrium point and will not affect the stability. Graphically, it means the graph of F(x) will get shifted by the parameter situated at right hand side (here it is M).

Summary:



- Linear Prey-Predator Models – Formulation and Solution.
- Solution of Linear Systems and Stability Analysis.
- Graphical Method for First Order Difference Equations.

$$\frac{dY}{dX} = 0 = \boxed{\text{Stability}}$$

$$F'(x) = 0$$


Thank you!
STABILO