IIT ROORKEE

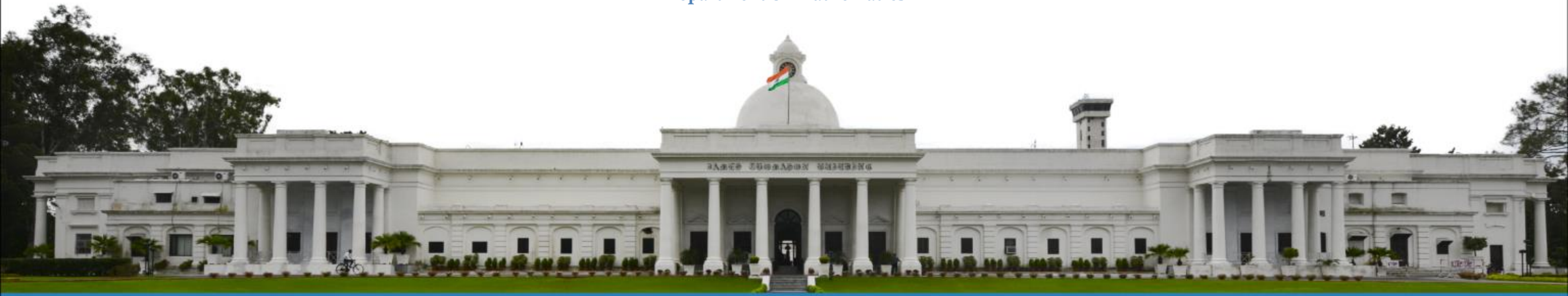
NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 2.3 : Classifications of Bifurcation

Dr. Ameeya Kumar Nayak

Department of Mathematics



Contents:

- Definition of Bifurcation.
- Saddle-Node Bifurcation.
- Trans-Critical Bifurcation.
- Pitchfork Bifurcation.

Definition of Bifurcation:

- In the previous lecture, we have analyzed the behavior of logistic difference equation $x_{n+1} = r\, x_n (1 - x_n)$ for different values of parameter r.

- For r = 2.5 the behavior is stable.
  For r = 3.3, 3.33 the behavior is periodic with period 2.
  For r = 3.9 the behavior is chaotic.

- Other than stable nature, a small change in model parameters may lead to a significant changes in solution behavior.

- The phenomenon of very significant changes in solution behavior with small changes in model parameter is termed as bifurcation. And the critical values of model parameter(s) where this phenomena occurs is (are) bifurcation points.

Classification of Bifurcation:

- There are 3 basic canonical types of bifurcation occurs in first order difference equation.

1. Saddle-node bifurcation.
2. Trans-critical bifurcation.
3. Pitchfork bifurcation.

- Why they are canonical?

1. They are experienced by higher order difference equations also.
2. Other bifurcations are either the combination of these three or with minor variation in these.

Saddle-Node Bifurcation:

$\boxed{y_{n+1} = x = y_n}$

- A prototype equation of saddle-node bifurcation is:

$$x = f(y_n) = \boxed{y_{n+1} = y_n - y_n^2 + \mu = F(y_n, \mu).} \quad \ldots .8.1$$

- In general a saddle-node bifurcation occurs (near the bifurcation point) if the model possesses an unique curve of fixed points in the (y, μ) plane.

- This curve lies on one side of the line $\mu = \mu_c$ and passes through bifurcation point.

$F' = 1 - 2y_n + 0$ $\quad y_n = \frac{1}{2}$

- The steady states of equation 8.1 are given by:

$$\lambda = \lambda - \lambda^2 + \mu \text{ or } \boxed{\lambda = \pm\sqrt{\mu}} \quad \ldots .\ 8.2$$

- Clearly, equation 8.1 has no steady states for $\mu < 0$ and $\pm\sqrt{\mu}$ for $\mu > 0$ (steady state). Also, the positive steady state is stable and negative is unstable (How? Find $F'(y_n, \mu)$ and proceed).

Saddle-Node Bifurcation:

- The bifurcation point is at $(y_c, \mu_c) = (0, 0)$. The graph of $\lambda = \pm\sqrt{\mu}$ is parabola opening right-ward. The eigen value corresponding to this bifurcation point is 1.

- The conditions for such bifurcation to occur are :

  1. $F(y_c, \mu_c) = 0,$
  2. $F_y(y_c, \mu_c) = 1,$
  3. $F_\mu(y_c, \mu_c) \neq 0,$
  4. $F_{yy}(y_c, \mu_c) \neq 0.$

J. D. Murray

Math Biology

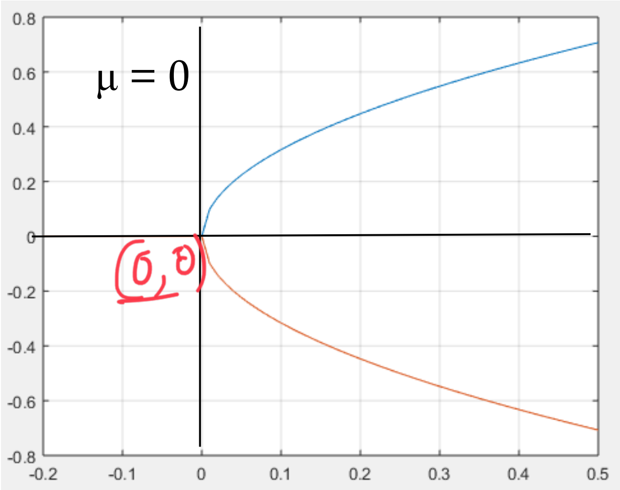


Fig. 8.1 : Graph of $\lambda = \pm\sqrt{\mu}$.

- Example: Insect Pest Models.

Trans-critical Bifurcation:

- The prototype equation for trans-critical bifurcation is

$$y_{n+1} = y_n - y_n^2 + \mu\, y_n = F(y_n, \mu). \qquad \dots 8.3$$

$y_{n+1} = \lambda$

$y = \lambda$

- In general a trans-critical bifurcation occurs (near the bifurcation point) if the model possesses two curves of fixed points in the (y, μ) plane.

- Both these curves pass through bifurcation point (y_c, μ_c) and both lie on both the sides of line $\mu = \mu_c$.

$\lambda = \lambda - \lambda^2 + \mu\lambda$ $\boxed{\lambda = 0,\ \lambda = \mu}$

$\lambda(\mu - \lambda) = 0$

- Clearly, it has two steady states $\lambda = 0$ (trivial) and $\lambda = \mu$ (non-trivial).

$\boxed{\mu\lambda - \lambda\lambda = 0}$ $\boxed{\mu \geq 0}$

- For stability of steady states, $F'(\lambda, \mu) = 1 - 2\lambda + \mu$. The trivial steady state is stable if $\mu < 0$ and non-trivial is stable if $\mu > 0$.

Trans-critical Bifurcation: $Ax = b \;\; |A| \neq 0$

- Again, the bifurcation point is at $(y_c, \mu_c) = (0, 0)$. Both the graphs $y = 0$ and $y = \mu$ are straight lines. And exchange of stability occurs at bifurcation point.

- Conditions for such bifurcation to occur are:
    1. $\boxed{F(y_c, \mu_c) = 0,}$
    2. $F_y(y_c, \mu_c) = 1,$
    3. $F_\mu(y_c, \mu_c) = 0,$ ✓
    4. $F_{yy}(y_c, \mu_c) \neq 0.$

$y = \alpha$

$\lambda = \mu$

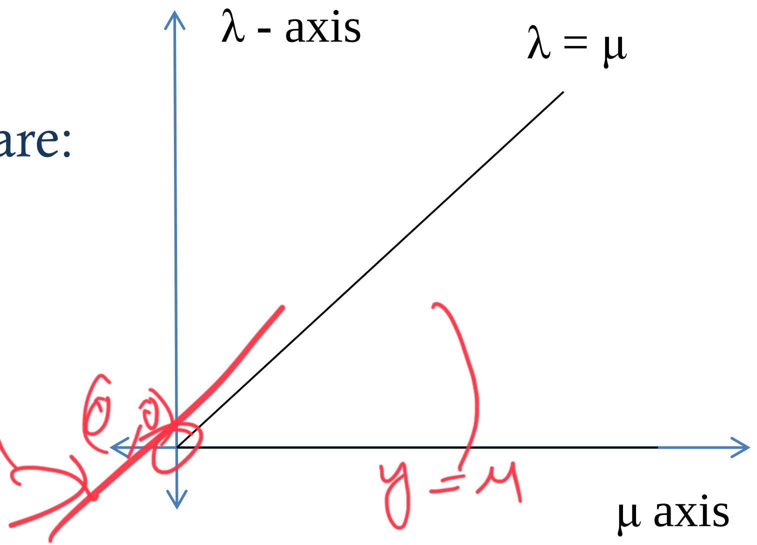


Fig. 8.2 : Graph of $\lambda = 0$ and $\lambda = \mu$.

Pitchfork Bifurcation:

- The prototype equation for pitchfork bifurcation is

$$y_{n+1} = y_n - y_n^3 + \mu y_n = F(y_n, \mu). \qquad \ldots.8.4$$

$x = x - x^3 + \mu x$

$x(\mu - x^2) = 0$

- In general a pitch-fork bifurcation occurs (near the bifurcation point) if the model possesses two curves of fixed points in the (y, μ) plane.

- Both these curves pass through bifurcation point (y_c, μ_c) and one of which lies on both the sides of line $\mu = \mu_c$.

- Clearly, it has three steady states $\lambda = 0$ (trivial) and $\lambda = \pm\sqrt{\mu}$ (non-trivial).

- $F'(\lambda, \mu) = 1 - 3\lambda^2 + \mu$. The trivial steady state is stable if $\mu < 0$ and both non-trivial steady states are stable if $\mu > 0$.

Pitchfork Bifurcation:

- Again, the bifurcation point is at $(y_c, \mu_c) = (0, 0)$. The graphs $y = 0$ and $y = \pm\sqrt{\mu}$ are respectively straight line and parabola opening right-ward. And exchange of stability occurs at bifurcation point.

- Conditions for such bifurcation to occur are:
    1. $F(y_c, \mu_c) = 0$,
    2. $F_y(y_c, \mu_c) = 1$,
    3. $F_\mu(y_c, \mu_c) = 0$,
    4. $F_{yy}(y_c, \mu_c) = 0$,
    5. $F_{y\mu}(y_c, \mu_c) \neq 0$,
    6. $F_{yyy}(y_c, \mu_c) \neq 0$.

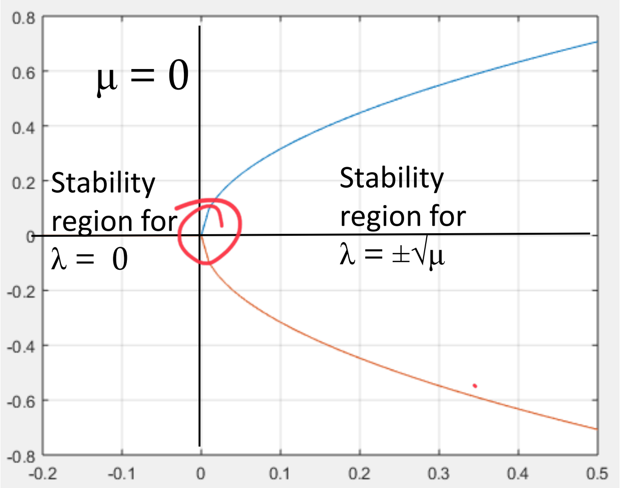


Fig. 8.3 : Graph of $\lambda = \pm\sqrt{\mu}$ for pitch-fork bifurcation.

Summary:

- Definition of bifurcation.
- Saddle-node bifurcation.
- Trans-critical bifurcation.
- Pitchfork bifurcation.
- The logistic difference equation $x_{n+1} = r\, x_n (1 - x_n)$ which we analyzed in last lecture pertains to periodic doubling or flip bifurcation. The typical from of the same is given as:

$$y_{n+1} = - y_n + y_n^3 - \mu\, y_n = F(y_n, \mu). \qquad \text{....8.5}$$

You can observe that right hand side function of equation 8.5 is negative of the same of pitchfork bifurcation.


Thank you!