9.1 Solutions to Chapter 1

Exercises 1.1

(**i**) The expectation of Y is:

$$E[Y] = E[\alpha + \beta X] = \alpha + \beta E[X] = \alpha + \beta\mu.$$

Setting $E[Y] = 0$ gives $\alpha + \beta\mu = 0$, and consequently we have $\alpha = -\beta\mu$. The variance of Y is:

$$var(Y) = var(\alpha + \beta X) = \beta^2 var(X) = \beta^2\sigma^2.$$

Setting $var(Y) = 1$ gives $\beta^2\sigma^2 = 1$ or equivalently $\beta = \frac{1}{\sigma}$. Thus, the required values are $\alpha = -\frac{\mu}{\sigma}$ and $\beta = \frac{1}{\sigma}$ and consequently, we have $Y = \frac{X-\mu}{\sigma}$

(**ii**) The correlation coefficient between X and Y is:

$$corr(X, Y) = \frac{cov(X, Y)}{\sqrt{var(X)var(Y)}}.$$

Since $Y = \alpha + \beta X$, we have:

$$cov(X, Y) = cov(X, \alpha + \beta X) = \beta cov(X, X) = \beta var(X) = \frac{\sigma^2}{\sigma} = \sigma.$$

Thus,

$$corr(X, Y) = \frac{\sigma}{\sigma \cdot 1} = 1.$$

So, X and Y are perfectly correlated.

(**iii**) The cumulative distribution function (CDF) of Y is:

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\alpha + \beta X \le y) \\ &= P\left(X \le \frac{y-\alpha}{\beta}\right) \\ &= F_X\left(\frac{y-\alpha}{\beta}\right). \end{aligned}$$

By substituting $\alpha = -\frac{\mu}{\sigma}$ and $\beta = \frac{1}{\sigma}$ we whave:

$$F_Y(y) == F_X(\sigma y + \mu).$$

(**iv**) We assume that X is symmetrically distributed about μ, meaning $X - \mu \stackrel{d}{=} -(X - \mu)$. This implies $F_X(\mu + x) = 1 - F_X(\mu - x)$, for all x. By considering $Y = \frac{X-\mu}{\sigma}$, we can rewrite X as $X = \sigma Y + \mu$. If $X - \mu$ and $-(X - \mu)$ have the same distribution, then $\frac{X-\mu}{\sigma}$ and $-\frac{X-\mu}{\sigma}$ have the same distribution as well. Since $Y = \frac{X-\mu}{\sigma}$ and $-Y = -\frac{X-\mu}{\sigma}$, we can conclude that Y is also symmetric about its mean (which is 0).

Exercises 1.2

We need to add the assumption that $p = 2$ to ensure that the matrices are compatible. If you want to keep p as an arbitrary value, you need to add $p - 2$ columns of zeros to $\Sigma^{1/2}$ in the following solution to ensure that $\Sigma^{1/2}Y$ is well-defined.

Let us consider the transformation $X = \mu + \Sigma^{1/2}Y$. We know that a linearly transformed normally distributed random vector is again normally distributed. From the rules for the mean and variance matrix of the linearly transformed random variable we know that $E(X) = \mu + \Sigma^{1/2}E(Y) = \mu$ and $var(X) = \Sigma^{1/2}var(Y)\Sigma^{1/2} = \Sigma$. On a computer, the square root matrix $\Sigma^{1/2}$ can be easily calculated from Σ using spectral decomposition:

$$\Sigma^{1/2} = \begin{pmatrix} -0.38 & -0.92 \\ 0.92 & -0.38 \end{pmatrix} \begin{pmatrix} 4.62 & 0 \\ 0 & 0.38 \end{pmatrix}^{1/2} \begin{pmatrix} -0.38 & -0.92 \\ 0.92 & -0.38 \end{pmatrix} = \begin{pmatrix} 0.84 & -0.54 \\ -0.54 & 1.95 \end{pmatrix}.$$

One then applies the above formula that linearly transforms Y into X.

Note 9.1 *In* R *you can use the following to calculate the spectral decomposition of Σ, which leads to $\Sigma^{1/2}$.*

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A = matrix(c(1,-1.5,-1.5,4), nrow=2)
r = eigen(A)
V = r$vectors
lam = r$values
sqrt_Lambda = diag(sqrt(lam))

sqrt_A = V %*% sqrt_Lambda %*% solve(V)

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Exercises 1.3

Clearly,

$$Y = X_1 + X_2 = AX = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

and $var(AX) = A\ var(X)\ A^T$, where ^{T} indicates the transpose of a matrix. Hence,

$$\begin{aligned} var(Y) &= A\ \Sigma\ A^T \\ &= \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ &= 3 \end{aligned}$$

Another possibility is to write

$$var(Y) = var(X_1 + X_2) = var(X_1) + 2cov(X_1, X_2) + var(X_2) = 3.$$

Exercises 1.4

(i-a) Set $X^{(1)} = (X_1, X_2)^T$ and $X^{(2)} = X_3$. Since $(X_1, X_2, X_3)^T$ is normally distributed, the conditional distribution of $X_3|X_1, X_2$ is also normal:

$$X_3|X_1, X_2 \sim N(\mu_{3|12}, \sigma^2_{3|12})$$

where the conditional mean is given by

$$\begin{aligned}
\mu_{3|12} &= \mu_3 + \begin{pmatrix} \phi^2 & \phi \end{pmatrix} \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}^{-1} \begin{pmatrix} X_1 - \mu_1 \\ X_2 - \mu_2 \end{pmatrix} \\
&= \begin{pmatrix} \phi^2 & \phi \end{pmatrix} \times \frac{1}{1-\phi^2} \begin{pmatrix} 1 & -\phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\
&= \phi X_2
\end{aligned}$$

and the conditional variance is

$$\begin{aligned}
\sigma^2_{3|12} &= 1 - \begin{pmatrix} \phi^2 & \phi \end{pmatrix} \begin{pmatrix} 1 & \phi \\ \phi & 1 \end{pmatrix}^{-1} \begin{pmatrix} \phi^2 \\ \phi \end{pmatrix} \\
&= 1 - \begin{pmatrix} \phi^2 & \phi \end{pmatrix} \times \frac{1}{1-\phi^2} \begin{pmatrix} 1 & -\phi \\ -\phi & 1 \end{pmatrix} \begin{pmatrix} \phi^2 \\ \phi \end{pmatrix} \\
&= 1 - \phi^2.
\end{aligned}$$

(**i-b**) Set $X^{(1)} = X_2$ and $X^{(2)} = X_3$. We know that $(X_2, X_3)^T$ is normally distributed, Therefore the conditional distribution of $X_3|X_2$ is also normal:

$$X_3|X_2 \sim N(\mu_{3|2}, \sigma^2_{3|2})$$

where the conditional mean is given by

$$\begin{aligned}
\mu_{3|2} &= \mathbb{E}[X_3|X_2] \\
&= \mu_3 + \phi(X_2 - \mu_2) \\
&= \phi X_2
\end{aligned}$$

and the conditional variance is

$$\begin{aligned}
\sigma^2_{3|2} &= \sigma^2_{3|2} \\
&= 1 - \phi \times \phi \\
&= 1 - \phi^2.
\end{aligned}$$

(**ii**) As can be seen in Part (i), $X_3|X_1, X_2$ and $X_3|X_2$ are identically distributed (have the same distribution). Therefore, we can conclude that $X_3|X_2$ and X_1 are independent.

Extra Point: It can be shown that $cov(X_1, X_3|X_2) = 0$. For this purpose you need to show that

$$\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} |X_2 \sim N\left(\begin{pmatrix} \phi X_2 \\ \phi X_2 \end{pmatrix}, \begin{pmatrix} 1-\phi^2 & 0 \\ 0 & 1-\phi^2 \end{pmatrix} \right)$$