SCHOOL OF MATHEMATICS AND STATISTICS

UNSW Sydney

MATH5425 Graph Theory Term 1, 2025

Assignment 2

1. Let $k \geq 2$ be an integer. Suppose that G is a graph with $\chi(G) \geq k+1$ such that any subgraph H of G with $|H| < |G|$ is k-colourable.

   (a) Briefly explain why G is not bipartite.
   Solution :
   A bipartite graph has two disjoint sets of vertices where none of the vertices in a set are adjacent to each other allowing it to be coloured with at most 2 colours or $\chi = 2$.

   The given graph G with $\chi(G) \geq k+1$ where $k \geq 2$ or $\chi(G) \geq 3$. Since G needs more than 2 colors, it cannot be bipartite.

   (b) Prove that G is 2-connected.

   Solution :
   Assuming G is disconnected then every other component of G is a proper subgraph and k-colurable. Every component when coloured independently with k colours yields a k colouring of G contradicting $\chi(G) \geq k+1$. Hence, G must be connected.

   If G is connected but not 2-connected, $v \in V(G)$ is a cut-vertex which disconnects disconnects G into at least two components $C_1, C_2, \ldots, C_m$. For each disconnected graph component C_i, define the induced subgraph $H_i = G[C_i \cup \{v\}]$. We can say that H_i is k-colorable since it is given $|H_i| < |G|$.

   For every subgraph H_i, we can permute the order of colors so that vertex v always receives the same color. This is possible because permuting colours preserves the validity of the coloring.

   Combining these colorings gives a proper k-coloring of G, v is the only shared and has the same color in all subgraphs and there are no edges between different components.

   This produces a k-coloring of G where its chromatic number would be at most k, However, this contradicts the given condition $\chi(G) \geq k+1$.

   Therefore, the initial assumption is false, and G must be 2-connected.

   (c) Let x, y be distinct vertices of G. Prove that there exists a path from x to y of odd length and a path from x to y of even length.

   Hint: By (a) we know that G contains an odd cycle C. Menger's Theorem (Theorem 3.3.1) may help.

Solution :
From part (a), G is not bipartite which means there exists an odd cycle C. From part (b), since G is 2-connected, Menger's Theorem(Theorem 3.3.1) guarantees there exist two internally disjoint paths P_1 and P_2 connecting x and y.

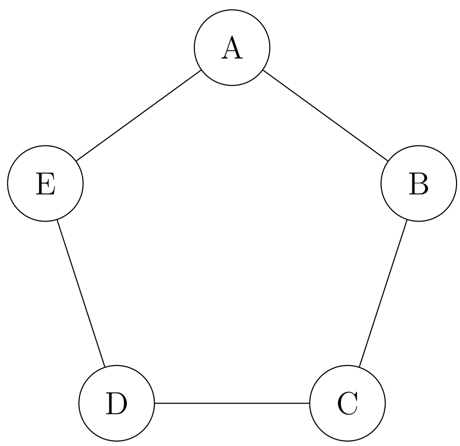
Case 1: P_1 has even length and P_2 odd length or vice versa If one path is even-length and the other is odd-length, this satisfies the requirement.(Menger's Theorem - Theorem 3.3.6)

Case 2: P_1 and P_2 are both even or odd length
Consider the cycle $C' = P_1 \cup P_2$, which will have even length. Using part (a), we know G contains an odd cycle C. Since G is 2-connected and by applying Menger's Theorem (Theorem 3.3.1) there exist two vertex-disjoint path Q_1 (from $u \in C'$ to $w \in C$) and Q_2 (from $v \in C'$ to $z \in C$) that have no common vertices except their endpoints. The odd cycle C contains two paths between w and z, one of odd length and one of even length. These paths allow to reroute the even cycle C' through the odd cycle C.

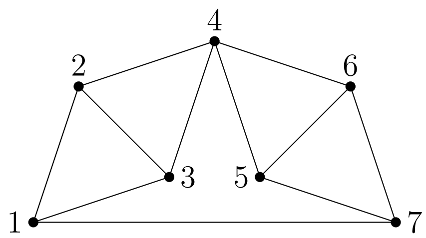
Follow P_1 from x to u, traverse the odd-length subpath of C from w to z, take Q_2 to v, and follow P_2 to y. This creates a new path with parity:

$$(\text{Length of } P_1) + (\text{Length of odd detour or C subpath}) + (\text{Length of } P_2)$$

The detour adds an odd number of edges and $P_1 \cup P_2$ was of even length, this produces a path with odd length. Similarly, using the even-length subpath of C results in satisfying the requirement. Thus we obtain both even and odd x-y-paths.



(d) Let G_0 be the following graph, which is called *Moser's spindle*:



Choose **one** of the following options:

Option 1: Prove that G_0 is not 3-connected, $\chi(G_0) = 4$, and every subgraph H of G_0 with $|H| < |G_0|$ is 3-colourable.

Option 2: If you prefer, instead of using Moser's spindle, define your own graph G_0 and prove that your graph G_0 has the above properties.

(*Remark: This example illustrates that part* (b) *cannot be strengthened.*)

Solution :
Using Option 1 (Moser's spindle).

Not 3-connected: Removing vertices 1 and 4 disconnects the edge with vertices 2 and 3 from the rest of the graph. Thus the Moser's spindle is 2-connected but not 3-connected.

Chromatic number $\chi(G_0) = 4$: The triangles (1-2-3), (2-3-4), (4-5-6), and (5-6-7) in the graph share vertices. A 3 coloring system creates conflicts at the shared vertices. For example, vertex 1 and 4 can share the same colour and vertex 2 and 3 can have the same coluring as 5 and 6. However since vertex 7 is adjacent to 1, it creates a need for a new colour making it $\chi(G_0) = 4$.

Proper subgraphs are 3-colorable: Any subgraph formed by removing a single vertex from Moser's spindle becomes 3-colorable. For example,

- **Remove Vertex 1**: Subgraph has vertices {2,3,4,5,6,7}. Coloring: 2-Red, 3-Blue, 4-Green, 5-Red, 6-Blue, 7-Green.
- **Remove Vertex 4**: Subgraph splits into {1,2,3,5,6,7}. Coloring: 1-Red, 2-Blue, 3-Green; 5-Red, 6-Blue, 7-Green.
- **Remove Vertex 7**: Subgraph has vertices {1,2,3,4,5,6}. Coloring: 1-Red, 2-Blue, 3-Green, 4-Red, 5-Blue, 6-Green.
- **Remove Vertex 2**: Subgraph has vertices {1,3,4,5,6,7}. Coloring: 1-Red, 3-Blue, 4-Green, 5-Red, 6-Blue, 7-Green.
- **Remove Vertex 3**: Subgraph has vertices {1,2,4,5,6,7}. Coloring: 1-Red, 2-Blue, 4-Green, 5-Red, 6-Blue, 7-Green.
- **Remove Vertex 5**: Subgraph has vertices {1,2,3,4,6,7}. Coloring: 1-Red, 2-Blue, 3-Green, 4-Red, 6-Blue, 7-Green.
- **Remove Vertex 6**: Subgraph has vertices {1,2,3,4,5,7}. Coloring: 1-Red, 2-Blue, 3-Green, 4-Red, 5-Blue, 7-Green.

In all cases, adjacent vertices receive distinct colors. Since every proper subgraph H formed by removing any single vertex admits a valid 3-coloring.

2. Let $G = (V, E)$ be a graph with $\chi(G) = k \geq 2$, and let $c : V \to \{1, 2, \ldots, k\}$ be a k-colouring of G. Define sets $S_1, S_2, \ldots, S_k \subseteq V$ as follows:

$$S_1 = \{u \in V : c(u) = 1\},$$
$$S_j = \{u \in V : c(u) = j \;\text{ and }\; uw \in E \text{ for some } w \in S_{j-1}\}$$

for $j = 2, \ldots, k$. That is, S_1 is the set of all vertices coloured 1 under c, and S_j is the set of all vertices coloured j with a neighbour in S_{j-1}, for $j = 2, \ldots, k$.

(a) Prove that the sets S_1, $S_2, \ldots, S_k$ are all non-empty.

Let $G = (V, E)$ be a graph with $\chi(G) = k \geq 2$, and let $c : V \to \{1, 2, \ldots, k\}$ be a proper k-coloring. We prove that the sets $S_1, S_2, \ldots, S_k$ defined recursively as:

$$S_1 = \{u \in V : c(u) = 1\},$$
$$S_j = \{u \in V : c(u) = j \text{ and } uw \in E \text{ for some } w \in S_{j-1}\} \quad (j \geq 2),$$

are all non-empty.

Solution :

If S_1 is empty because $\chi(G) = k = |G|$, it means G can be colored with $k - 1$ colors, contradicting $\chi(G) = k$. Thus, S_1 is non empty.

Using induction, assuming the sets $S_1, S_2, \ldots, S_{j-1}$ are non empty for $j \geq 2$ and for contradiction, suppose that S_j is empty. It means that no vertex colored j is adjacent to any vertex in S_{j-1}.

Because S_{j-1} is empty, S_{j-1} forms an independent set and has no neighbors in S_{j-2}. Recolor these vertices with $j - 2$. This is valid because they are not adjacent to S_{j-2} and remain non-adjacent to other recolored vertices in S_{j-1}

All vertices originally colored j, are now recolored to with color $j-1$. Because S_{j-1} is empty, no vertex in the original coloring j is adjacent to S_{j-1} meaning no conflicts. The neighbors of coloring j are either in S_{j-1} now coloured $j-2$ or other colour classes not equal to $j - 1$.

The new coloring uses only $k - 1$ colors, contradicting $\chi(G) = k$. Hence, S_j is not empty.

By induction, all sets $S_1, S_2, \ldots, S_k$ are non-empty.

(b) Prove that G contains a path P of length $k - 1$ such that the k vertices of P are all coloured with distinct colours under c.

Solution :

Since S_k is not empty from part (a), choose any vertex $v_k \in S_k$ coloured k. By definition of S_k, v_k must have a neighbor $v_{k-1} \in S_{k-1}$. Similarly, v_{k-1} must have a neighbor $v_{k-2} \in S_{k-2}$, continue this till $v_1 \in S_1$

$$v_1 \leftrightarrow v_2 \leftrightarrow \cdots \leftrightarrow v_{k-1} \leftrightarrow v_k$$

The path contains exactly k vertices and $k - 1$ edges. Adjacent vertices have different colours meaning each v_k belongs to $S_k \implies c(v_k) = k$ Each vertex in P belongs to a distinct set S_k, ensuring all k colors 1, 2, $\ldots$, k appear uniquely. $\chi(G) = k$ prevents color reduction, maintaining all color classes

Thus a path P with length k-1 and distinct colors must exist in G.

(c) Suppose that H is a graph such that the longest path in H has length $r \geq 1$. Prove that $\chi(H) \leq r + 1$.

Solution :
Assume for contradiction that $\chi(H) \geq r+2$. Then H admits a proper $(r+2)$-coloring. By part (b), any proper $(r+2)$-coloring of H must contain a path P of length $r+1$ (since $(r+2)-1 = r+1$), where all $r+2$ vertices on P are distinctly colored.

However, this contradicts the assumption that the longest path in H has length r. Therefore, our assumption $\chi(H) \geq r+2$ must be false, and we conclude that $\chi(H) \leq r+1$.

$$\boxed{\chi(H) \leq r+1}$$

3. Fix integers $r \geq t \geq 4$ and let $\mathcal{U}$ be a set of subsets of $[n] = \{1, 2, \ldots, n\}$, such that each subset $S \in \mathcal{U}$ has size r.

Suppose that
$$|\mathcal{U}| \leq \frac{t^{r-1}}{(t-1)^r}.$$
Use the probabilistic method to prove that there exists a map $c : [n] \to \{1, 2, \ldots, t\}$ such that for every $S \in \mathcal{U}$,
$$\{c(u) \mid u \in S\} = \{1, 2, \ldots, t\}.$$

Solution :

Consider a random coloring where each element of $[n]$ is independently assigned a color from $\{1, 2, \ldots, t\}$ uniformly at random. Each color is chosen with probability $1/t$. A subset $S \in \mathcal{U}$ is called bad if it does not contain all t colors under the random coloring c. Let X denote the random variable counting the number of bad subsets in $\mathcal{U}$. The probability that S it fails to contain all t colors :
$$\mathbb{P}(S \text{ does not contain all colors } t) \leq t \cdot \left(\frac{t-1}{t}\right)^r,$$
Expected number of bad subsets is:
$$\mathbb{E}[\text{Bad events}] = \sum_{S \in \mathcal{U}} \mathbb{P}(S \text{ is bad }) \leq |\mathcal{U}| \cdot t \cdot \left(\frac{t-1}{t}\right)^r.$$
Substituting the given bound :
$$\mathbb{E}[X] \leq \frac{t^{r-1}}{(t-1)^r} \cdot t \cdot \left(\frac{t-1}{t}\right)^r$$
$$\mathbb{E}[X] \leq 1$$
Since X is a non-negative (since its a count) integer-valued random variable, there must exist at least one colouring where $X = 0$. If every colouring had $X \geq 0$ then $\mathbb{E}[X] \geq 1$ contradicting $\mathbb{E}[X] \leq 1$. Thus, there exists a coloring where no subset $S \in \mathcal{U}$ is bad i.e S contains all t colours