SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney

MATH5425 Graph Theory   Term 1, 2025

Problem Sheet 5, Connectivity

1. Show that the block graph of any connected graph is a tree.

2. Without using Menger's Theorem (Theorem 3.3.1/Theorem 3.3.5), show that any two vertices of a 2-connected graph lie on a common cycle.

   Hint: use Proposition 3.1.3 and induction.

3. Let G be a graph with connectivity $\kappa(G) = k \geq 1$ and let S be a minimal separating set for G. That is, S separates G and $|S| = k$. Prove that every vertex in S has a neighbour in every component of $G - S$.

4. Let G be a 3-edge-connected cubic graph. (Recall that "cubic" means 3-regular.) Prove that G is 3-connected.

5. Let G be a k-connected graph, where $k \geq 2$ is an integer. Given a vertex $u \in V(G)$ and $W \subseteq V(G)$, a (u, W)-*fan* is a set of paths from u to W such that any two of the paths have only u in common. (Note: for each path P in a (u, W)-fan, the only vertex in $P \cap W$ is the endvertex of P.)

   (a) Explain why $d(u) \geq k$.

   (b) Suppose that $u \notin W$ and $|W| \geq k$. Prove that G has k paths which form a (u, W)-fan.   (*Hint:* Menger's Theorem.)

   (c) Hence show that if G has at least $2k$ vertices then G has a cycle of length at least $2k$.