SCHOOL OF MATHEMATICS AND STATISTICS
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MATH5425 Graph Theory Term 1, 2025

Problem Sheet 2, Matchings and Hamilton cycles

1. Let G be a bipartite graph and let M be a matching in G which is not maximum (so there is some larger matching in G). Prove that G contains an augmenting path with respect to M. Is the same true of nonbipartite graphs?

2. Let $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be a set of finite sets and let $U = \bigcup_{i=1}^n A_i$.

   (a) A *system of distinct representatives* (SDR) for $\mathcal{A}$ is a set $\{x_1, \ldots, x_n\}$ of n distinct elements of U such that $x_k \in A_k$ for $k = 1, \ldots, n$. Prove that $\mathcal{A}$ has a system of distinct representatives if and only if

   $$\left| \bigcup_{i \in I} A_i \right| \geq |I|$$

   for all subsets $I \subseteq \{1, \ldots, n\}$.

   (*Hint: Define a suitable bipartite graph and apply Hall's theorem.*)

   (b) Now let $d_1, \ldots, d_n$ be positive integers. By adapting your proof of (a), show that there exist disjoint subsets $D_k \subseteq A_k$ with $|D_k| = d_k$ for $k = 1, \ldots, n$ if and only if

   $$\left| \bigcup_{i \in I} A_i \right| \geq \sum_{i \in I} d_i$$

   for all $I \subseteq \{1, \ldots, n\}$. (The case where $d_k = 1$ for all k corresponds to an SDR.)

3. Suppose that a graph G has a Hamilton cycle. Under what simple condition on $n = |G|$ can we immediately deduce that G has a perfect matching?

4. Let G be a graph with $n = |G| \geq 3$ and with minimum degree $\delta(G) \geq \lceil n/2 \rceil + 1$. Prove that G contains at least two distinct Hamilton cycles. (Two Hamilton cycles are distinct if their edge sets are distinct.)

5. (a) Prove that any Hamiltonian graph G is 2-connected.

   (b) Let $d \geq 2$ be a fixed constant. Construct a connected graph G with minimum degree $\delta(G) = d$ such that G has no Hamilton cycle.

   (c) Suppose that $n = |G|$ is odd. Construct a connected graph with minimum degree $(n-1)/2$ which is not Hamiltonian. (This shows that the lower bound in Dirac's theorem (Theorem 10.1.1) is best possible.)

(... Please turn over for Question 6)

6. In this question, we will prove that a graph $G = (V, E)$ contains a matching of size k if and only if

$$q(G - S) \leq |S| + |G| - 2k \quad \text{for all} \quad S \subseteq V \tag{**}$$

(Here $q(G - S)$ denotes the number of components of $G - S$ of odd order, as in lectures.)

Let $n = |G|$ and $W = \{w_1, \ldots, w_{n-2k}\}$. Define the graph G' with vertex set $V \cup W$ and edge set

$$E \cup \{w_i w_j \mid 1 \leq i < j \leq n - 2k\} \cup \{w_i v \mid 1 \leq i \leq n - 2k,\, v \in V\}.$$

That is, $G'[W]$ is a complete graph of order $n - 2k$ and every vertex in W is also joined to every vertex in G.

(a) Prove that G has a matching of size k if and only if G' has a perfect matching.

(b) Suppose that Tutte's condition

$$q(G' - S') \leq |S'| \quad \text{for all} \quad S' \subseteq V(G') \tag{*}$$

holds for G'. Prove that condition (**) holds for G. (Hint: given $S \subseteq V$, consider $S' = S \cup W$.)

(c) Now suppose that condition (**) holds for all $S \subseteq V$. Prove that condition (*) holds for all $S' \subseteq V(G')$. (Hint: consider the two cases $W - S' \neq \emptyset$ and $W - S' = \emptyset$.)

(d) Explain why this completes the proof.