$\{u, v\}$ or uv

Chapter 1. Introduction

The main reference for this section is Diestel Graph Theory, Chapter 1.

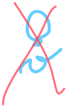
A graph $G = (V, E)$ is a set V of vertices and a set E of unordered pairs of distinct vertices, called edges.

Sometimes we write $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G, respectively.

Write vw or $\{v, w\}$ for the edge joining v and w, and say that v and w are neighbours or that they are adjacent.

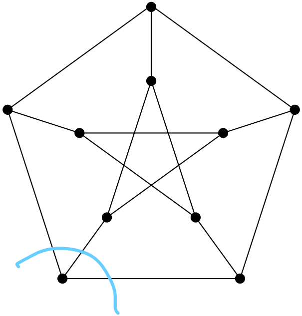
In this course, unless otherwise stated, graphs are

- finite: so $|V| \in \mathbb{N}$.
- labelled: we can distinguish vertices from each other. Usually $V = [n] := \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
- undirected, since edges are unordered pairs of vertices.
- simple: no loops $\{v, v\}$ or multiple edges (since E is a set, not a multiset).



Usually we draw a graph with vertices as black dots and edges as a line between them. (We often don't display the vertex labels.)

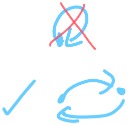
This is the Petersen graph:

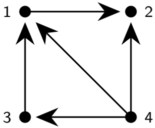


Related objects

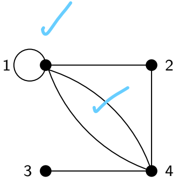
Directed graphs have ordered edges (v, w), drawn as an arrow from v to w.

one way streets
FSA

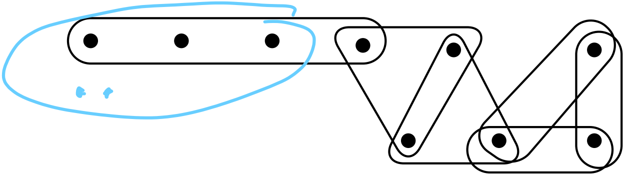




Multigraphs are undirected but have loops and multiple edges allowed.



A hypergraph $H = (V, E)$ consists of a set of vertices V and a set E of hyperedges, where each hyperedge is a nonempty subset of V.



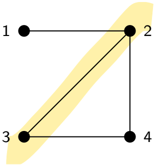
A graph G with vertex set $\{v_1, \ldots, v_n\}$ has adjacency matrix $A(G) = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E, \\ 0 & \text{otherwise.} \end{cases}$$

So $A(G)$ is a symmetric $n \times n$ 0-1 matrix with zero diagonal.

For example:

G



$$A(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

Similarly, a multigraph corresponds to a symmetric matrix over $\mathbb{N}$ and a directed graph corresponds to a square 0-1 matrix with a zero diagonal. no longer symmetric

Hypergraphs are described by a different matrix, called an incidence matrix (defined in the lecture notes).

In this course we do not work much with adjacency or incidence matrices.

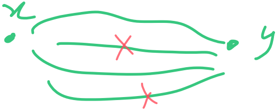
Usefulness of graphs

We claim that graphs are useful in the real world.

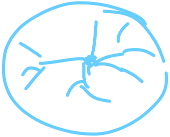
Computer or communications networks.

Redundancy:

⇒ connectivity



Efficiency.



logarithmic diameter in n

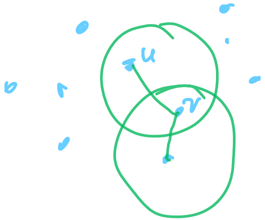
Gene sequencing.

"Smash" many copies of a gene into bits

V = set of these segments

edge if the two segments have a "big" overlap

A mobile phone company wants to build transmitters to cover the sites in some set V. Each transmitter can transmit for r km. What is the least number of transmitters required, and where should they be built?



Smallest dominating set

Random graphs

Often in applications we want to know how an algorithm or hypothesis will work on a "typical" instance of the problem. If the problem can be modelled using graphs, then we want a random graph with certain properties. We study these later.

"null model"

We will also use probabilistic methods to prove some deterministic results.

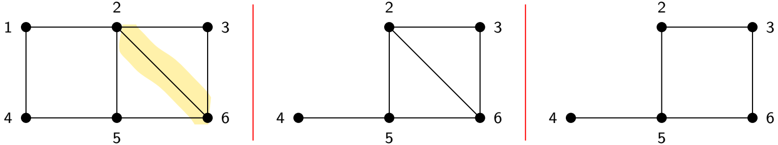
We will only use the basics of discrete probability theory, which we will revise when needed.

More definitions

$\cancel{(\emptyset, \emptyset)}$

The trivial graph has at most one vertex. Hence it has no edges. (We often ignore the graph with zero vertices.)

A subgraph of a graph $G = (V, E)$ is a graph $H = (W, F)$ such that $W \subseteq V$ and $F \subseteq E$.

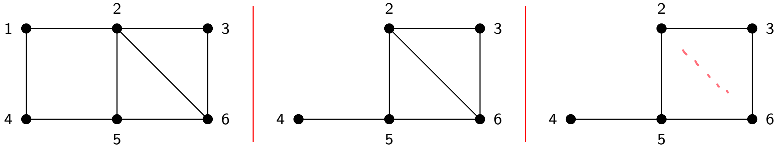


G H_1 H_2

deleted vertex 1 from G

We say that H is an induced subgraph if for all $v, w \in W$, if $vw \in E(G)$ then $vw \in E(H)$.

Write $H = G[W]$, and say that H is the subgraph of G induced by the vertex set W.



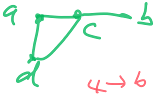
G $\qquad$ H_1 induced ✓ $\qquad$ H_2 ✗

The number of vertices of G, written $|G| = |V(G)|$, is called the order of G.

The number of edges of G, sometimes written $\|G\| = |E(G)|$, is called the size of G.

Note: I usually just write $|E(G)|$.

one-one + onto



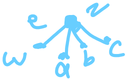
$4 \to b$
$3 \to c$
$\{1,2\} \to \{a,d\}$

Two graphs $G = (V, E)$ and $H = (W, F)$ are isomorphic if there exists a bijection $\varphi : V \to W$ such that $\varphi(v)\varphi(w) \in F$ if and only if $vw \in E$.

The map φ is called a graph isomorphism, or isomorphism.

We often say that two graphs are equal if they are equal up to isomorphism, which means that we can relabel the vertices of one graph to obtain the other.

Any graph property which is invariant under isomorphism is a graph invariant. Examples?



$d_G(v) = 4$

1.2 The degree of a vertex

If $v \in e$ where v is a vertex and e is an edge, then we say that e is incident with v.

The degree $d_G(v)$ of vertex v in a graph G is the number of edges of G which are incident with v.

We write $d(v)$ if the graph G is clear from the context.

Let $N_G(v)$ (or just $N(v)$) be the set of all neighbours of v in G. Then $d(v) = |N(v)|$.

$N_G(v) = \{a, b, c, w\}$

"theory builders"
"problem solvers". methods

Tim Gowers: the two cultures of mathematics

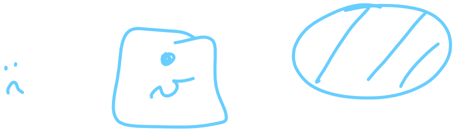
The Handshaking Lemma
In any graph $G = (V, E)$,

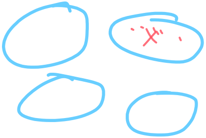
$$\sum_{v \in V} d(v) = 2|E|.$$

Proof uses "double counting" of $\{(v,e) : v \in e, e \in E\}$

Let $\delta(G) = \min_{v \in V} d(v)$ be the minimum degree in G, and $\Delta(G) = \max_{v \in V} d(v)$ be the maximum degree in G.

A vertex of degree 0 is an isolated vertex.





Some special graphs

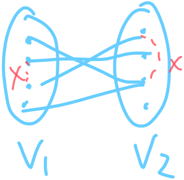
A graph is *k-partite* if there exists a partition of its vertex set

$$V = V_1 \cup V_2 \cup \cdots \cup V_k$$

into k nonempty disjoint subsets (parts) such that there are no edges between vertices in the same part.

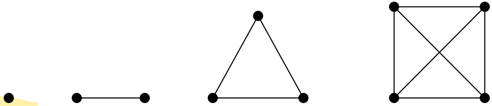
We say bipartite and tripartite instead of 2-partite and 3-partite, respectively.

bipartite



The complete graph on r vertices, denoted K_r, has all $\binom{r}{2}$ edges present.

(We say "the" complete graph K_r because it is unique up to isomorphism.)

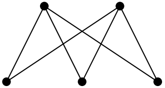


K_1 K_2 K_3 K_4

$\binom{4}{2} = 6$ edges.

The complete bipartite graph $K_{r,s}$ has r vertices in one part of the vertex bipartition, s vertices in the other, and all rs edges present.

$K_{2,3}$



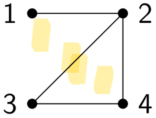
6 edges $(= 2 \times 3)$

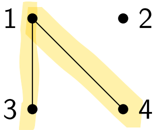
A graph is regular if every vertex has the same degree. If every vertex of a graph has degree d then we say that the graph is d-regular.

Observe that K_r is regular of degree $\underline{\quad r-1 \quad}$.
When is $K_{r,s}$ regular? When $r = s$.

The complement of a graph G is the graph $\overline{G} = (V, \overline{E})$ where $vw \in \overline{E}$ if and only if $vw \notin E$.

G

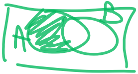




$\overline{G}$

Note that $\overline{K_n}$ is the graph with n vertices and no edges.

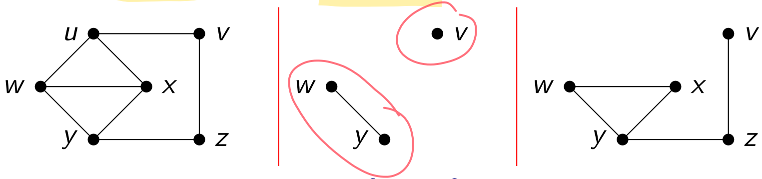
$A - B$ set difference $= \{a \in A : a \notin B\}$



If $G = (V, E)$ and $X \subset V$ then $G - X$ denotes the graph obtained from G by deleting all vertices in X and all edges which are incident with vertices in X.

Write $G - v$ instead of $G - \{v\}$, for $v \in V$.

Also write $G - H$ instead of $G - V(H)$, if H is a subgraph of G.



G $\quad$ $G - \{u, x, z\}$ $\quad$ $G - u$

often written $A \setminus B$.

$G = (V, E)$

$e = \{x, y\}$ ambiguous

$G - \{x, y\}$ ✗

$\left.\begin{array}{l} G - e \\ G - xy \end{array}\right\}$ delete the edge

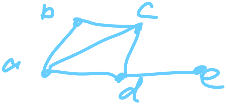
If $F \subseteq E$ then $G - F$ denotes the graph $(V, E - F)$ obtained from G by deleting the edges in F.

Write $G - e$ or $G - xy$ when $F = \{e\}$ and $e = xy$.

Also write $G + xy$ or $G + e$ to denote the graph obtained from G by adding the edge e. That is, $G + e = (V, E \cup \{e\})$.

Note: if e is already present as an edge of G then $G + e = G$.

$G - \{\{a, b\}, \{c, d\}\}$



cdadedab

1.3 Paths and cycles

A walk in the graph G is a sequence of vertices

$$v_0\, v_1\, v_2 \cdots v_k$$

such that $v_i v_{i+1} \in E$ for $i = 0, 1, \ldots, k-1$.

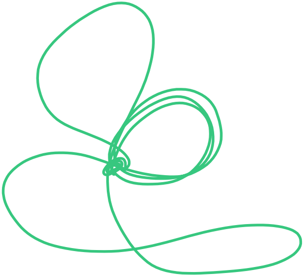
Sometimes we write this walk as

$$v_0\, e_0\, v_1\, e_1 \cdots v_{k-1}\, e_{k-1}\, v_k$$

where $e_i = v_i v_{i+1}$.

The length of this walk is k.

The walk is closed if $v_0 = v_k$.



An Euler tour is a closed walk in a graph which uses every edge precisely once. A graph is Eulerian if it has an Euler tour.

Theorem 1.8.1 (Euler, 1736)
A connected graph is Eulerian if and only if every vertex has even degree.

Proof.
Exercise (see Problem Sheet 1). □

Read about the Seven Bridges of Königsberg on Wikipedia:
`http://en.wikipedia.org/wiki/Seven_Bridges_of_Konigsberg`

Fact: the number of walks of length k from v to w in G is $(A(G)^k)_{vw}$ (the (v, w) entry of the kth power of $A(G)$).

This can be proved by induction on k (exercise).

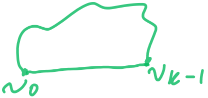
Fact: the number of walks of length k from v to w in G is $(A(G)^k)_{vw}$ (the (v, w) entry of the kth power of $A(G)$).

This can be proved by induction on k (exercise).

A walk is a path if it does not visit any vertex more than once. So a path is a sequence of distinct vertices, with subsequent vertices joined by an edge.

A path $v_0\, v_1 \cdots v_k$ with k edges is called a k-path, and has length k.

We usually think of a k-path as a graph with $k+1$ vertices and k edges.

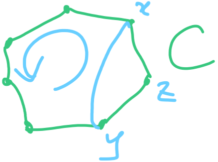


If $k \geq 3$ and $P = v_0\, v_1 \cdots v_{k-1}$ is a path of length $k-1$ then $C = P + v_0 v_{k-1}$ is a cycle of length k, also called a k-cycle.

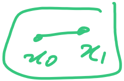
We may write $v_0\, v_1 \cdots v_{k-1}\, v_0$ to denote the cycle. It is a closed walk which visits no internal vertex more than once.

We usually think of a k-cycle as a graph with k vertices and k edges.

An edge which joins two vertices of a cycle C, but which is not an edge of C, is called a chord. An induced cycle is a cycle which has no chords.



$xy \notin E(C)$
$\Longrightarrow$ chord



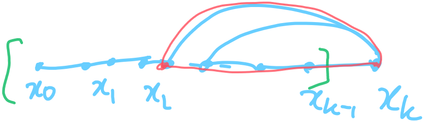
if $\delta(G) \geq 1$

Proposition 1.3.1

Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$, if $\delta(G) \geq 2$.

Proof.

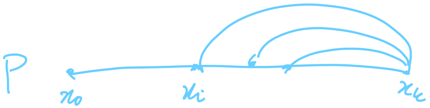
Let $P = x_0 x_1 \ldots x_k$ be a longest path in G.



□

By maximality of P, all neighbours of x_k lie on P.

Hence $\delta(G) \leq d(x_k) \leq k = |\{x_0, .., x_{k-1}\}|$, which proves the first statement.

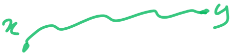


Let x_i be the smallest-indexed neighbour of x_k on P

Then $C = x_k x_i x_{i+1} \dots x_{k-1} x_k$ is a cycle of length $\geq \delta(G)+1$, because C contains the $d(x_k) \geq \delta(G)$ neighbours of x_k as well as x_k.

$\square$

Draw pictures!



The minimum length of a cycle in G is the girth of G, denoted by $g(G)$.

Given $x, y \in V$, let $d_G(x, y)$ be the length of a shortest path from x to y in G, called the distance from x to y in G.

Set $d_G(x, y) = \infty$ if no such path exists.

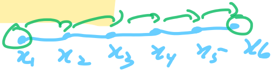
We say that G is connected if $d_G(x, y)$ is finite for all $x, y \in V$.

Check that $d_G(\cdot, \cdot)$ defines a metric on V if G is connected.

log n

Let the diameter of G be $\mathrm{diam}(G) = \max_{x,y \in V} d_G(x,y)$.

diam 5



Proposition 1.3.2

Every graph G which contains a cycle satisfies

$$g(G) \leq 2\,\mathrm{diam}(G) + 1.$$

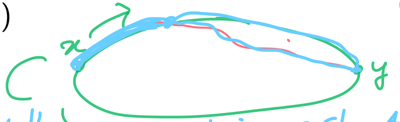
Proof. Let C be a shortest cycle in G. So $|C| = g(G)$

For a contradiction, assume $g(G) \geq 2\,\mathrm{diam}(G) + 2$

Choose vertices x, y on C with $d_C(x,y) \geq \mathrm{diam}(G) + 1$.

In G the distance $d_G(x,y)$ is strictly smaller, so any shortest path P from x to y in G is not a subgraph of C. But using P together with the shorter arc of C from x to y gives a closed walk of length $< |C|$. □

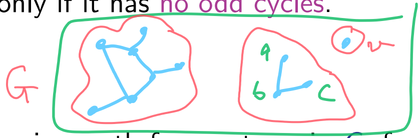
Graphs with $g(G) = 2\,\mathrm{diam}(G) + 1$ are called Moore graphs. (See lecture notes.)



This closed walk contains a shorter cycle than C, contradiction!! □

3 components

Fact: a graph is bipartite if and only if it has no odd cycles.
(Exercise: see Problem Sheet 1.)



1.4 Connectivity

Recall that G is connected if there is a path from v to w in G, for all $v, w \in V$.

A maximal connected subgraph of G is called a component (or connected component) of G.

Components are nonempty.

Maximal = with respect to the subgraph relation

Olive Nunn (2008)

Proposition 1.4.1

The vertices of a connected graph can be labelled $v_1, v_2, \ldots, v_n$ such that $G_n = G$ and $G_i = G[v_1, \ldots, v_i]$ is connected for all i.

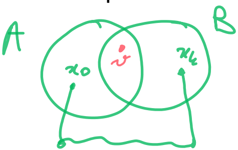
Proof. Choose v_1 arbitrarily. Now suppose that we have labelled $v_1, .., v_i$ such that $G_j = G[v_1, .., v_j]$ is connected for all $j = 1, .., i$

If $i < n$ then $G_i \neq G$, so there exists some $v_j \in \{v_1, , v_i\}$ with a neighbour $w \notin \{v_1, .., v_i\}$ in G (otherwise, $G_i \neq G$ is a component of G, impossible as G is connected.) Let $v_{i+1} = w$, then $G_{i+1} = G[v_1, .., \cancel{v_i}\ v_{i+1}]$ is connected

This completes the proof, by induction. $\square$

$\square$

Let $A, B \subseteq V$ be sets of vertices. An (A, B)-path in G is a path $P = x_0 \, x_1 \cdots x_k$ such that

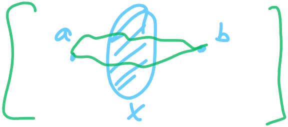


$$P \cap A = \{x_0\}, \qquad P \cap B = \{x_k\}.$$

(Note: Diestel writes "$A - B$ path" for this.)

Let $A, B \subseteq V$ and let $X \subseteq V \cup E$ be a set of vertices and edges. We say that X separates A and B in G if every (A, B)-path in G contains a vertex or edge from X.

We do not assume that A and B are disjoint.
If X separates A and B then $A \cap B \subseteq X$.



We say that X separates two vertices a, b if $a, b \notin X$ and X separates the sets $\{a\}$, $\{b\}$.

More generally, we say that X separates G, and call X a separating set for G, if X separates two vertices of G.

That is, X separates G if there exist distinct vertices $a, b \notin X$ such that X separates a and b.

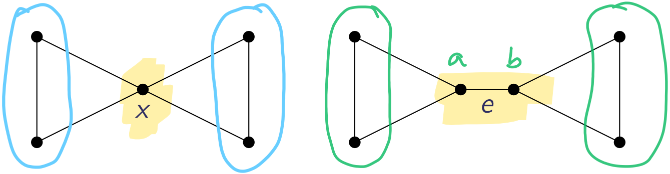
~~$a \in X$~~

~~$b \in X$~~

$a b \in X$

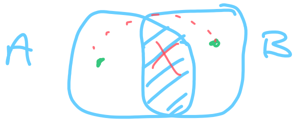
If $X = \{x\}$ is a separating set for G, where $x \in V$, then we say that x is a cut vertex. In this case, $G - x$ has more components than G.

If $e \in E$ and $G - e$ has more components than G then e is a bridge. In this case e separates its end vertices in G.



Fact: an edge is a bridge if and only if it does not lie on a cycle. (See Problem Sheet 1.)

Try Q1–Q7!



The unordered pair (A, B) is a separation of G if $A \cup B = V$ and G has no edge between $A - B$ and $B - A$.

The second condition says that $A \cap B$ separates A from B in G.

If both $A - B$ and $B - A$ are nonempty then the separation is proper. The order of the separation is $|A \cap B|$.

$\mathbb{N} = \{0, 1, 2, \dots\}$

Definition. Let $k \in \mathbb{N}$.
The graph G is k-connected if $\overbrace{|G| > k}^{*}$ and $G - X$ is connected for all subsets $X \subseteq V$ with $|X| < k$.

The connectivity $\kappa(G)$ of G is defined by

$$\kappa(G) = \max\{k : G \text{ is } k\text{-connected}\}.$$

It is impossible to disconnect a k-connected graph by deleting fewer than k vertices.

When is a graph 0-connected? When is a graph 1-connected?

$* \; |G| \geq 1$

$|G| \geq 2$ and G is connected

So, $\kappa(G) = 0$ if and only if G is trivial or G is disconnected.

Fact: $\kappa(K_n) = n - 1$ for all positive integers n.

Definition. Let $\ell \in \mathbb{N}$ and let G be a graph with $|G| \geq 2$. If $G - F$ is connected for all $F \subseteq E$ with $|F| < \ell$ then G is ℓ-edge-connected.

The edge connectivity $\lambda(G)$ is defined by

$$\lambda(G) = \max\{\ell : G \text{ is } \ell\text{-edge-connected}\}.$$

If G is disconnected then $\lambda(G) = 0$.

vertex connectivity

edge connectivity

min degree

Proposition 1.4.2
If $|G| \geq 2$ then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proof: Exercise (see Problem Sheet 1).

Hence high connectivity implies high minimum degree.

But large minimum degree is not enough to ensure high connectivity, or even high edge-connectivity: think of some examples.

The next result gives a link between connectivity and the average degree of a graph G, which equals $2|E(G)|/|V(G)|$.

Theorem 1.4.3 (Mader, 1973)
Let k be a positive integer. Every graph G with average degree at least $4k$ has a $(k+1)$-connected subgraph H with

$$\frac{|E(H)|}{|V(H)|} > \frac{|E(G)|}{|V(G)|} - k.$$

This result says that a graph G with sufficiently high average degree contains a subgraph H which is well-connected and almost as dense as G.

Proof.

Proof We write $|G|$ instead of $|V(G)|$.

Let $\gamma = \frac{|E(G)|}{|G|} \geq 2k$. Consider subgraphs G' of G which satisfy this condition:

$$\boxed{|G'| \geq 2k} \quad \text{and} \quad |E(G')| > \gamma(|G'| - k). \qquad — (*)$$

Such graphs G' exist as G satisfies (*):

[why? Average degree of G is $\frac{2|E(G)|}{|G|} \geq 4k$, so

$|G| \geq 4k$

and $\gamma(|G| - k) = \frac{|E(G)|(|G| - k)}{|G|} < |E(G)|$.

(definition of γ)]

Now let H be a subgraph of G of smallest order which satisfies (*). We continue the proof by proving three claims.

Claim 1 If G' satisfies (*) then $|G'| > 2k$.

Proof of Claim 1: If G' satisfies $(*)$ and $|G'| = 2k$ then

$$|E(G')| > \gamma(\underbrace{|G'| - k}_{=k}) \geq 2k^2 > \binom{|G'|}{2} = \frac{\cancel{2}k(2k-1)}{\cancel{2}},$$

contradiction! $\diamond$

Claim 2: $\delta(H) > \gamma$.

Proof of Claim 2: For a contradiction, suppose that $\delta(H) \leq \gamma$. Let G' be obtained from H by deleting a vertex of degree $\leq \gamma$. Then $|G'| < |H|$ and G' satisfies $(*)$, which is a contradiction. To see this, check:

$|G'| = |H| - 1 \geq 2k$, by Claim 1, and

$|E(G')| \geq |E(H)| - \gamma > \gamma(|H| - k - 1)$ as H satisfies $(*)$

$= \gamma(|G'| - k)$.

Hence $\delta(H) > \gamma$. $\diamond$ It follows that $|H| \geq \gamma$.

Hence $\dfrac{|E(H)|}{|H|} > \dfrac{\gamma(|H|-k)}{|H|}$ as H satisfies (*)

$$= \gamma - \frac{\gamma k}{|H|} \geq \gamma - k.$$

[This is half of what we want to prove!]

It remains to prove that H is $(k+1)$-connected.

<u>Claim 3</u>: H is $(k+1)$-connected.

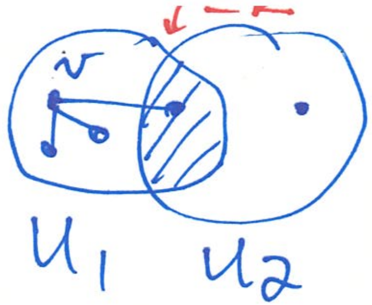
Proof of <u>Claim 3</u>: By Claim 1, $|H| \geq 2k+1 \geq k+2$ as $k \geq 1$.

So H is large enough.

For a contradiction, suppose that H is <u>not</u> $(k+1)$-connected.

Then H has a <u>proper separation</u> $\{U_1, U_2\}$ of order at most k.

That is: $U_1 \cup U_2 = V(H)$, $U_1 \cap U_2$ separates U_1 from U_2 in H, <u>$|U_1 \cap U_2| \leq k$,</u> and $\left.\begin{matrix} U_1 - U_2 \neq \emptyset, \\ U_2 - U_1 \neq \emptyset \end{matrix}\right\}$ <u>proper</u>



Let $H_i = H[U_i]$ for $i = 1, 2$.

Since any vertex $v \in U_1 - U_2$ has $d_H(v) \geq \delta(H) > \gamma$ (by Claim 2), and all neighbours of v in H belong to H_1, we have $|H_1| \geq \gamma \geq 2k$. Similarly, $|H_2| \geq 2k$.

By minimality of H, neither H_1 nor H_2 satisfies (*).

Hence $|E(H_i)| \leq \gamma(|H_i| - k)$ for $i = 1, 2$:

But then

$$\begin{aligned} |E(H)| &\leq |E(H_1)| + |E(H_2)| \\ &\leq \gamma(|H_1| + |H_2| - 2k) \\ &\leq \gamma(|H| - k) \text{ by inclusion-exclusion,} \end{aligned}$$

$|H| = |H_1| + |H_2| - |U_1 \cap U_2|$

since $|U_1 \cap U_2| \leq k$. This contradicts (*) for H!

So H is $(k+1)$-connected, completing the proof of Claim 3 and of the Theorem. $\square$

1.5 Trees and forests

A graph with no cycles is a forest (also called an acyclic graph).

A connected graph with no cycles is a tree.

The vertices of degree 1 in a tree or forest are called the leaves.

Fact: Every nontrivial tree has at least two leaves. (Consider a longest path.)

Theorem 1.5.1

The following are equivalent for a graph T:

(i) T is a tree;

(ii) Any two vertices of T are linked by a unique path in T;

(iii) T is minimally connected: that is, T is connected but $T - e$ is disconnected for every $e \in E(T)$;

(iv) T is maximally acyclic: that is, T is acyclic but $T + xy$ has a cycle for any two nonadjacent vertices x, y in T.

Exercise: see Problem Sheet 1.

A subgraph H of G is a spanning subgraph if $V(H) = V(G)$.

If H is a spanning subgraph which is a tree then H is a spanning tree.

Corollary.
If G is connected then G has a spanning tree.

Proof.

$\square$

Proof of Corollary: Let G be a connected graph and let H be a minimal connected spanning subgraph of G.

Note: H exists as G is a connected spanning subgraph of itself.

By Theorem 1.5.1, H is a tree. $\square$

Corollary 1.5.2

The vertices of a tree can be labelled as $v_1, \ldots, v_n$ so that for $i \geq 2$, vertex v_i has a unique neighbour in $\{v_1, \ldots, v_{i-1}\}$.

Proof. We use the labelling from Proposition 1.4.1

This labels the vertices of a given tree G as $v_1, .., v_n$ such that $G[v_1, .., v_i]$ is connected

Let $i \geq 1$.

Hence $G[v_1, .., v_i]$ is a tree.

Note $G[v_1, .., v_{i+1}]$ is connected, by Prop 1.4.1, so v_{i+1} has at least one neighbour in $G[v_1, .. v_i]$

For a contradiction, suppose that v_{i+1} has two neighbours z and w in $G[v_1, .., v_i]$. There is a (unique) path P in $G[v_1, .., v_i]$ between z and w, and this path does not visit v_{i+1} □

Hence $P \cup \{zv_{i+1}, wv_{i+1}\}$ is a cycle in G, contradiction.

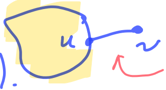
Then H has exactly $n-1$ edges, since it is a tree on n vertices.
Hence $H = G$, so G is a tree. $\square$

Corollary 1.5.3

A connected graph with n vertices is a tree if and only if it has $n-1$ edges.

Proof. Suppose that G is a tree on n vertices.
The result is true when $n = 1$.

Now suppose the result is true when $n = k$.
Let G be a tree on $k+1$ vertices. Let v be a leaf in G (eg take an endvertex of a longest path in G)



Then $G - v$ is a tree on k vertices,
so $G - v$ has $k-1$ edges (inductive hypothesis).
Therefore G has k edges as v has degree 1.
This concludes the proof, by induction.

Conversely, suppose that G is connected with n vertices and $n-1$ edges. Then G contains a spanning tree H, by an earlier corollary.

Corollary 1.5.4
If T is a tree and G is any graph with $\delta(G) \geq |T| - 1$ then G has a subgraph isomorphic to T.

Proof. Use the labelling of Proposition 1.4.1
Details are in Diestel. (SKIP)

$\square$

[End of Chapter 1. Try Problem Sheet 1.]