MATH5425 2019. ①

Exam draft [Q1] $G=(V,E)$ connected.

[5]

(a) $e\in E$, prove $\exists$ spanning tree containing e.

Proof Since G is connected it has a ~~max~~ spanning tree T. -1 If $e\in E(T)$ then we are done. -1

Otherwise, $T+e$ has a cycle C, since G is maximally acyclic. -1 Let $f\neq e$ be an edge of C. Then $(T+e)-f$ has -1 $n-1$ edges, n vertices and is connected, as f cannot be a bridge since it lies on a cycle. Hence $T'=(T+e)-f$ is a tree and $e\in E(T')$, as required. -1

(b) Let T, T' distinct spanning trees in G.

[5] Set

$T_0=T$ & suppose by induction that we have a sequence -1

$T=\underbrace{T_0, T_2, \ldots, T_i}_{}$ with T_{j+1} obtained from T_j by deleting an edge ~~of T_j~~ & inserting an edge ~~of $T_{j+1}-T_j$~~.

all distinct spanning trees of G.

If $T_i=T'$ then we are done. So suppose that $T_i\neq T'$.

Consider the set difference $E(T_i-T')\cup E(T'-T_i)$.

Since $|E(T_i)|=|E(T')|=n-1$ both $E(T_i-T')$ and $E(T'-T_i)$ -1 have $k\geq 1$ elements (same size).

-1

Let $f\in E(T'-T_i)$. Then T_i+f has a cycle, as T_i is a tree & so maximally acyclic. This cycle C must contain an edge $e\in E(T_i-T')$, -1 otherwise C is contained in T', contradicting the fact that T' is a tree.

Then $T_{i+1} = (T_i + f) - e$ is a tree,

since ~~f~~ e is not a bridge in $T_i + f$ (e belongs to cycle C)

so T_{i+1} is connected, n vertices, $n-1$ edges.

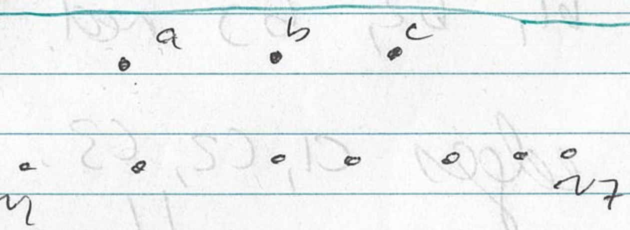
Also $|E(T_{i+1} - T')| = k-1 = |E(T' - T_{i+1})|$

so this process will terminate after $|E(T - T')|$ steps, a finite number. -1 $\square$

$\boxed{Q2}$

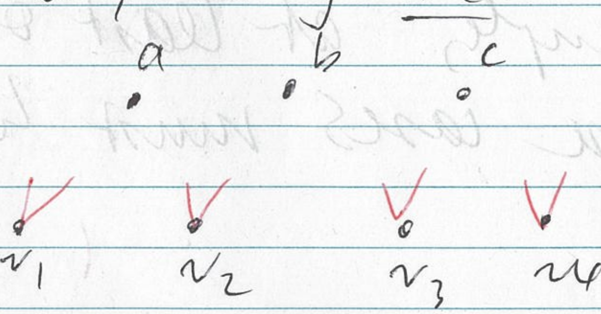
(a)

[5]



Call v_j red if $\geq$ two incident edges red, o'wise blue. $(d(v_j) = 3)$. -1

At least 4 vertices $\{v_1, .., v_7\}$ with same colour, wlog red + v_1, v_2, v_3, v_4 are red. -1

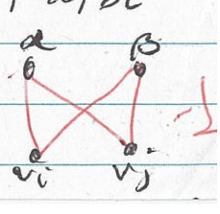


For each of $v_1, .., v_4$, specify a pair in $\{a, b, c\}$ st v_j is incident -1

via a red edge to (at least) these two vertices. If v_j incident with 3 red edges then wlog take $\{a, b\}$ as pair.

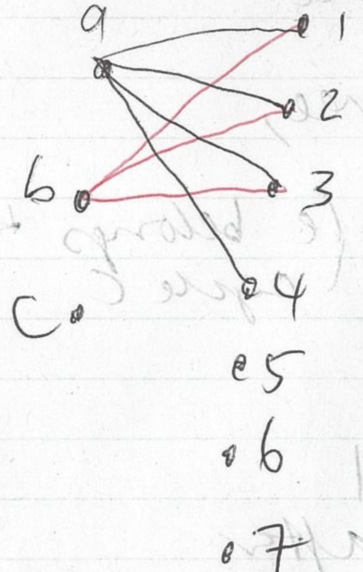
But only 3 options for this pair ab, ac, bc

So at least two of v_j with same pair, by pigeonhole principle. Hence -1 &



-1

Q2(a), Andrew Kaploun:



wlog a has ≥ 4 blue nbs 1, 2, 3, 4.

Of these, if ≥ 2 of 1, 2, 3, 4 have blue edges to b then we have blue $K_{2,2}$. So wlog b1, b2, b3 red.

□

Now consider edges c1, c2, c3.

If ≥ 2 blue then blue $K_{2,2}$ with a.

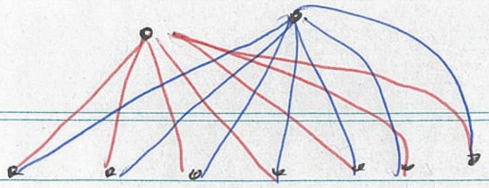
If ≥ 2 red then red $K_{2,2}$ with b.

By pigeonhole principle, at least one of these cases must hold.

See facing for alternative argument for Q2(a) ③

(b) [2]

$K_{2,7}$:

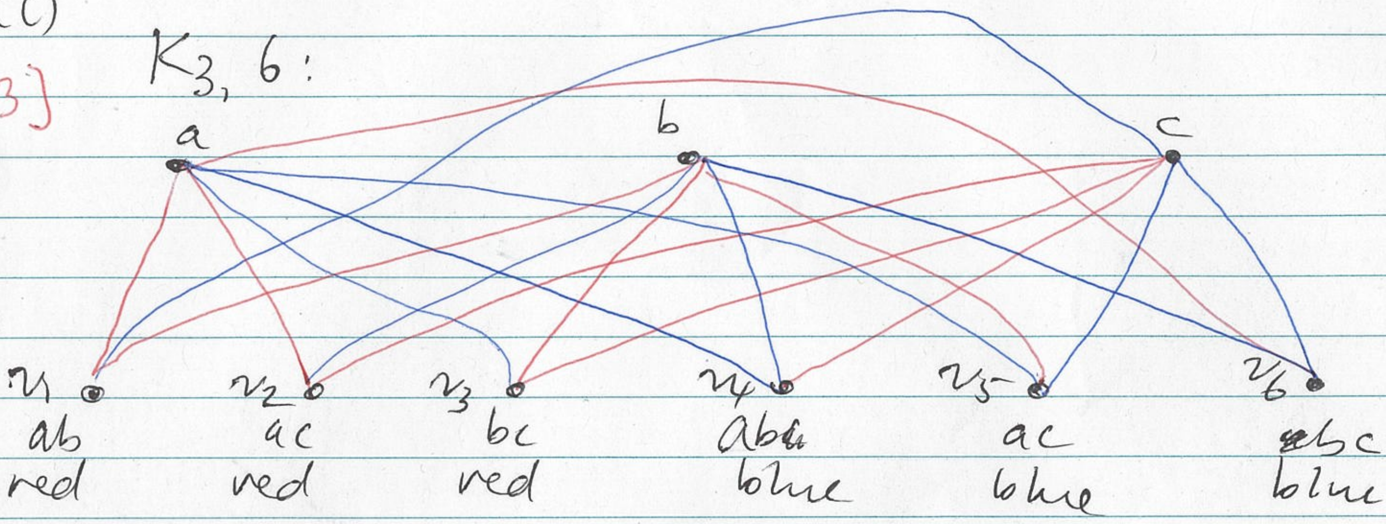


red graph $\cong$ blue graph $\cong K_{1,7}$, a tree.

$\Rightarrow$ no $K_{2,2}$.

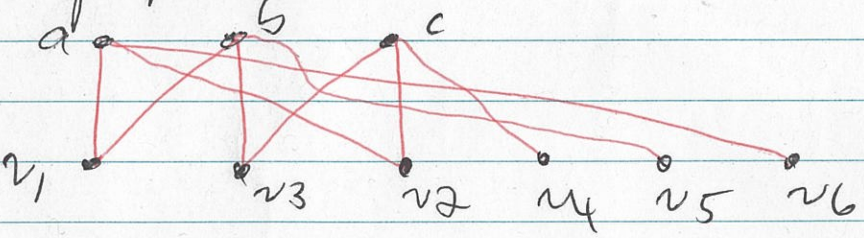
(c) [3]

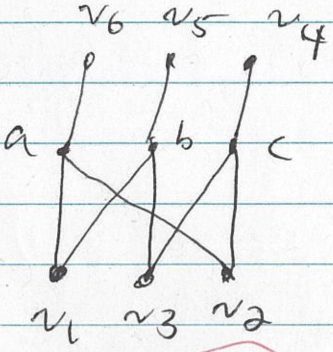
$K_{3,6}$:



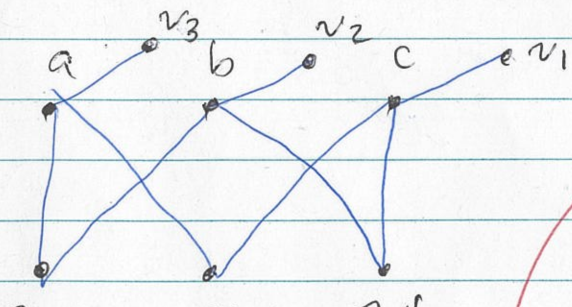
To avoid structures from (a), must be as above.

Red graph

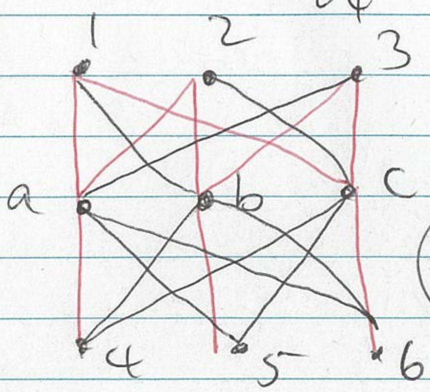


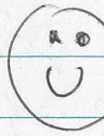


Blue graph



Both are a 6-cycle with 3 pendant edges, no 4-cycle ($K_{2,2}$) in sight.





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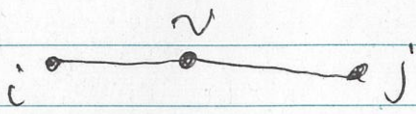
Q3 $G(n,p)$. $X = X(n,p)$ = # unordered pairs of distinct vertices with no common nbs.

(a) [4]

write $X = \sum'_{1 \le i < j \le n} X_{ij}$ where -1

$$X_{ij} = \begin{cases} 1 & \text{if } N(i) \cap N(j) = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Now



$\Pr(v \in N(i) \cap N(j)) = p^2$, -1 using independence, for all $v \notin \{i,j\}$.

So, ~~using independence again~~,

$\mathbb{E} X_{ij} = \Pr(X_{ij} = 1) = (1-p^2)^{n-2}$ -1 using independence again, as we have $n-2$ choices of v.

~~and~~ Then by linearity of expectation,

$$\mathbb{E} X = \binom{n}{2} (1-p^2)^{n-2} \quad -1$$

as there are $\binom{n}{2}$ choices for ~~i<j~~ $\{i,j\}$.

(b) [2] Hence, since $1 - x \le e^{-x}$ -1 and $\binom{n}{2} \le n^2$, -1

$$\mathbb{E} X \le n^2 e^{-p^2(n-2)} = \exp(2 \underline{\log n} - p^2(n-2)),$$ ($\ln n$)

as required.

(c) [2] well $2 \ln n \le p^2(n-2)$ iff

$$\frac{2 \ln n}{n-2} \le p^2, \quad \text{so } p \ge \sqrt{\frac{2 \ln n}{n-2}}.$$

We want strict inequality so take eg

$$p = p(n) = \sqrt{\frac{3 \ln n}{n-2}} \ . \quad -1$$

Note for $p \in (0,1)$ for n suff large; ($n \geq 5$ or so? guess).
NOTE: NO MARK $p \in (0,1)$ -1

Then with this function p, ~~value~~ ~~of~~ $\mathbb{E}X \leq \exp\left(2 \ln n - \frac{3 \ln n}{n-2}(n-2)\right)$

$$= \exp(2 \ln n - 3 \ln n) = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

(d) (i) [2] In $G(n,p)$, with this p, $Pr(X > 0) \leq \mathbb{E}X \to 0$.

Hence aas ~~[illegible]~~ $X = 0$, so aas:

given any $i<j$, i & j have a common nb in $G \in G(n, p(n))$.

Then $d(i,j) \leq 2$ so

$$\text{diam}(G(n,p)) \leq 2. \quad -1$$

(ii)

But diam $= 1$ iff $G(n,p) \cong K_n$, an event with probability

$$p^{\binom{n}{2}} \to 0 \quad -1 \text{ really fast!}$$

(since $p \to 0$ alone)

So aas $X = 0$ and $G(n,p) \not\cong K_n$, that is,

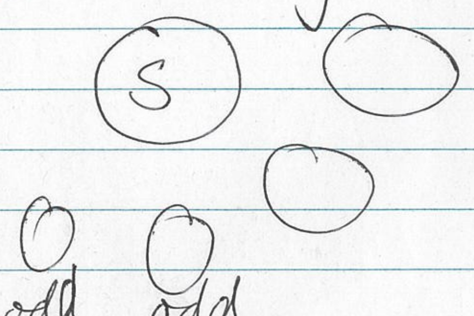
aas $\text{diam}(G) = 2$.

(6) (B)

$\boxed{Q4}$ Parts (a) + (b) are unrelated.

(a) $G=(V,E)$ n vertices, M a max matching.
[5] Fix $S \subseteq V$ + let $M' =$ edges in M which are disjoint from S.

(i) M' can be thought of as a
[2] matching in $G-S$. −1



The maximum number of vertices which can be matched by M' is −1

$|G-S| - q(G-S)$ since there must be at least one unmatched vertex in every odd component of G.

But $|G-S| = n - |S|$ so this gives the desired expression.

(ii) The maximum number of edges in $M-M'$
(3) is $|S|$, matching all elements of S to a vertex in $G-S$.
Otherwise, [Each edge in $M-M'$ matches at least one vertex of S, so $|M-M'| \le |S|$.]
if any edge of $M-M'$ is completely within S then it uses two vertices of S: −1
if number of such edges is i then

$|M-M'| \le i + (|S| - 2i) = |S| - i$,

maximised when $i=0$. −1

so

$$|M| = |M'| + |M-M'| \le \frac{1}{2}\left(n - |S| - q(G-S)\right) + |S|$$

$$= \frac{1}{2}\left(n + |S| - q(G-S)\right)$$ −1

as claimed. □

(b) Let $c: E(K_n) \to \{1,2,3\}$ with $c(uv)$
[5] assigned uar, & independently -1
for each edge $uv \in K_n$.

Let X = # subsets $U \subseteq [n]$ of size k st
$K_n[U]$ is monochromatic.

Write $X = \sum'_{\substack{U \subseteq [n] \\ |U|=k}} X_U$ where X_U is the
indicator variable
for the event -1
"$K_n[U]$ is monochromatic".

Then $\mathbb{E} X_U = \Pr(X_U = 1) = 3 \cdot \frac{1}{3^{\binom{k}{2}}} = 3^{1-\binom{k}{2}}$ -1

since there are 3 choices of colour
and then all $\binom{k}{2}$ edges must choose
this colour. This uses independence.

Hence, by linearity of expectation,
$\mathbb{E} X = \binom{n}{k} \cdot 3^{1-\binom{k}{2}}$ -1 as there are $\binom{n}{k}$ choices
for U.

By assumption, this gives
$P(X>0) \leq \mathbb{E} X = \binom{n}{k} 3^{1-\binom{k}{2}} < 1$. 4/6

So there is a particular colouring $c_0: E(K_n) \to \{1,2,3\}$
of the edges of K_n with no
monochromatic K_k. $\square$

-1

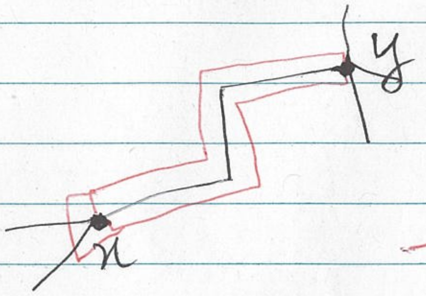
or: Q(a)(i) -2
(ii) -b2
(b) -6 ?

8 (22)

[3]

$\boxed{Q5}$ (a) If G is planar and e "xy" an edge of G

then take a plane embedding −1 of G: (which we also call G)



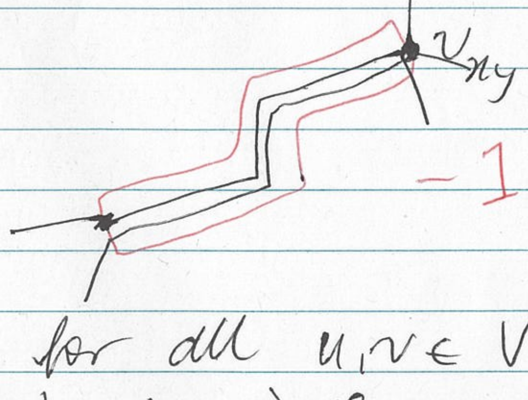
Take a thin "ribbon" of width $\varepsilon > 0$ around the edge

−1 from x to y, and around x with ε so small that this ribbon is disjoint from the rest of G.

Then we can embed G/e in the plane by, say, replacing $\bullet y$ by $\bullet v_{xy}$

and rerouting all edges incident with x (which are not already covered by $yw \mapsto v_{xy}w$, eg don't bother if common nb)

to stay within this ε-ribbon:



This is a plane embedding of G/e.

−1

So G/e is planar.

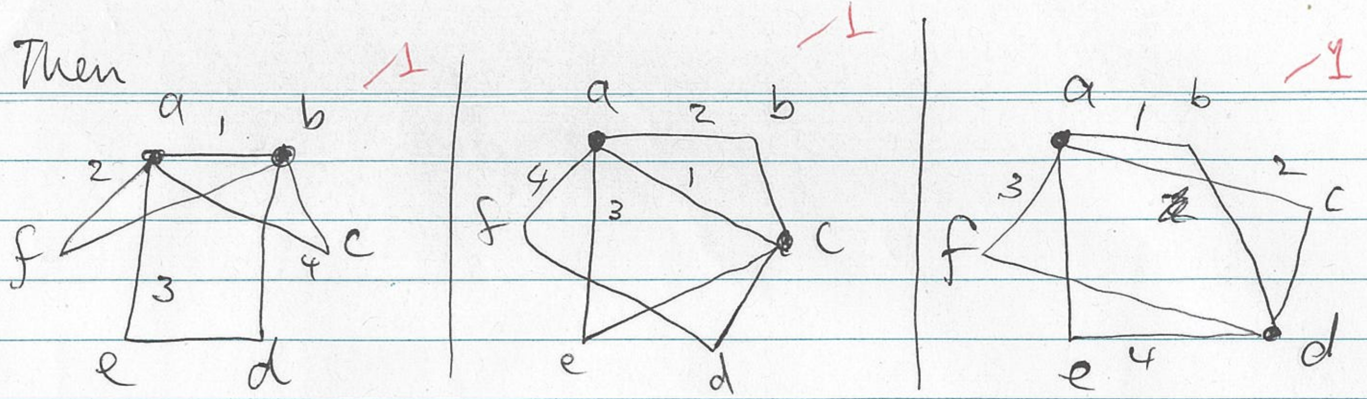
(i) [5]

(b) Rtp for all $u, v \in V(G)$, there are at least 4 indep paths from u to v. −1

Then Global Menger $\Rightarrow$ G is 4-conn.

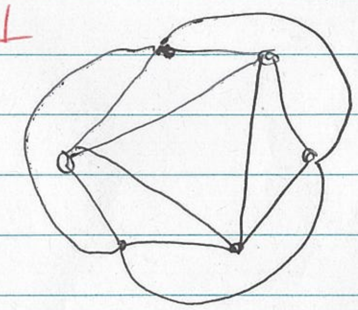
By symm, suff to check just for ab, ac, ad (rotate G to ~~port~~ cover all other ~~edge bits~~ pairs) −1

Then



as required.

(ii) Note, H is planar: −1
[2]



So if such a sequence

$G = G_0, G_1, \dots, G_r = H$

existed, with $G_0 = K_5$, then (a) implies that all G_i are planar, hence $G_0 = K_5$ is planar. −1

This is a contradiction.

OR: $|H| = 6$ so sequence would have $G_1 = H$.

Contract edge ab, for example, to give graph where c + f are still not adjacent. (an anti-matching)

This always happens: every vertex has a nonneighbour. Of the 4 non-contracted vertices in G_0, only 2 will have their non-neighbour contracted, so still two non-contracted nonneighbours. So $H/_{xy} \neq K_5$

for any $xy \in E(H)$.