9.4 Solutions to Chapter 4

Exercises 4.1

To prove Equation (4.4),

$$\begin{aligned}
\gamma(h) &= cov(Y_{t+h}, Y_t) \\
&= cov\left(\sum_{j=-\infty}^{\infty} \psi_j X_{t+h-j}, \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k cov\left(X_{t+h-j}, X_{t-k}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X\left(t+h-j-(t-k)\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_X\left(h+k-j\right).
\end{aligned}$$

To prove Equation (4.6),

$$\begin{aligned}
\gamma(h) &= cov(Y_{t+h}, Y_t) \\
&= cov\left(\mu + \sum_{j=-\infty}^{\infty} \psi_j Z_{t+h-j}, \mu + \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k cov\left(Z_{t+h-j}, Z_{t-k}\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z\left(t+h-j-(t-k)\right) \\
&= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z\left(h+k-j)\right).
\end{aligned}$$

Since

$$\gamma_Z(h) = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases}$$

we can easily see that, if $h+k-j \neq 0$ (or equivalently, if $j \neq h+k$), then $\gamma_Z(h+k-j) = 0$. Therefore,

$$\begin{aligned}
\gamma(h) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_Z\left(h+k-j)\right) \\
&= \sigma^2 \sum_{k=-\infty}^{\infty} \psi_{h+k} \psi_k
\end{aligned}$$

Exercises 4.2

We know that for a causal AR(p) process,

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + Z_t, \qquad Z_t \sim WN(0, \sigma^2).$$

Based on the definition, the PACF in lag h is defined as the correlation between the prediction errors

$$X_{h+1} - E(X_{h+1}|X_h, X_{h-1}, \ldots, X_3, X_2)$$

and

$$X_1 - E(X_1|X_h, X_{h-1}, \ldots, X_3, X_2).$$

For $h > p$, the best linear predictor of X_{h+1} is

$$\begin{aligned}\hat{X}_{h+1} &= E(X_{h+1}|X_h, X_{h-1}, \ldots, X_3, X_2) \\ &= \phi_1 X_h + \phi_2 X_{h-1} + \cdots + \phi_p X_{h-p+1}\end{aligned}$$

and, consequently,

$$\begin{aligned}X_{h+1} - E(X_{h+1}|X_h, X_{h-1}, \ldots, X_3, X_2) &= X_{h+1} - (\phi_1 X_h + \phi_2 X_{h-1} + \cdots + \phi_p X_{h-p+1}) \\ &= Z_{h+1}\end{aligned}$$

Besides, we know that $E(X_1|X_h, X_{h-1}, \ldots, X_3, X_2)$, whatever it is, is a linear function of $X_h, X_{h-1}, \ldots, X_3, X_2$. Now, using the fact that in AR processes $cov(Z_s, X_t) = 0$, for $s > t$, we have

$$corr(Z_{h+1}, X_t) = 0, \qquad t = h, h-1, \cdots, 2.$$

Therefore

$$\begin{aligned}&corr\left(X_{h+1} - E(X_{h+1}|X_h, X_{h-1}, \ldots, X_2), X_1 - E(X_1|X_h, X_{h-1}, \ldots, X_2)\right) \\ &= corr\left(Z_{h+1}, X_1 - E(X_1|X_h, X_{h-1}, \ldots, X_2)\right) \\ &= corr\left(Z_{h+1}, X_1\right) - corr\left(Z_{h+1}, E(X_1|X_h, X_{h-1}, \ldots, X_2)\right) \\ &= 0 + 0 = 0\end{aligned}$$

Exercises 4.3 (NOT Examinable)

Based on the definition of PACF, ϕ_{hh} is the last component of $\phi_h = \Gamma_h^{-1}\gamma_h$ or, equivalently, $\phi_h = R_h^{-1}\rho_h$, which ca be written also as $R_h\phi_h = \rho_h$. Using Cramer's rule, it can be shown that solving these equations for $h = 1, 2, 3, \cdots$ successively results in:

- For $h = 1$,

$$\phi_{1,1} = \rho(1) = \frac{\theta_1}{1 + \theta_1^2} \neq 0$$

- For $h = 2$,

$$\begin{pmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{2,1} \\ \phi_{2,2} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \end{pmatrix}$$

  consequently,

$$\Rightarrow \quad \phi_{2,2} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & \rho(2) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) \\ \rho(1) & 1 \end{vmatrix}} = \frac{-\rho(1)^2}{1 - \rho(1)^2} = \frac{-\theta_1^2}{1 + \theta_1^2 + \theta_1^4} \neq 0$$

- For $h = 3$,

$$\begin{pmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{pmatrix} \begin{pmatrix} \phi_{3,1} \\ \phi_{3,2} \\ \phi_{3,3} \end{pmatrix} = \begin{pmatrix} \phi(1) \\ \phi(2) \\ \phi(3) \end{pmatrix}$$

$$\Rightarrow \quad \phi_{3,3} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & \rho(2) \\ \rho(2) & \rho(1) & \rho(3) \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & \rho(2) \\ \rho(1) & 1 & \rho(1) \\ \rho(2) & \rho(1) & 1 \end{vmatrix}} = \frac{\begin{vmatrix} 1 & \rho(1) & \rho(1) \\ \rho(1) & 1 & 0 \\ 0 & \rho(1) & 0 \end{vmatrix}}{\begin{vmatrix} 1 & \rho(1) & 0 \\ \rho(1) & 1 & \rho(1) \\ 0 & \rho(1) & 1 \end{vmatrix}} = \frac{\rho(1)^3}{1 - 2\rho(1)^2} \neq 0$$

Generally, for ϕ_{hh} the determinant in the numerator is almost the same elements as the determinant in the denominator just with the last column relapced with ρ_h. Using the fact that $\rho_X(0) = 1$, $\rho_X(1) = \theta/(1+\theta^2)$ and $\rho_X(h) = 0$, for $|h| > 1$, we can show that

$$\phi_{hh} = \frac{-(-\theta)^h}{1 + \theta^2 + \cdots + \theta^{2h}}$$

To complete the proof note that

- $1 + \theta^2 + \cdots + \theta^{2h} = 1 - \theta^{2(h+1)}$
- For MA(1), due to the structure of the autocorrelation function, the matrix in the denominator, R_h, is called tri-diagonal matrix, where the elements on the main diagonal are 1 and the sub and sup diagonal elements are $\rho(1)$.
- For the determinant of this matrix refer to this link.

putting these points together, completes the proof.
Note: For another way to show this relationship refer to Brockwell and Davis, Introduction to time series and forecasting, Page 96, Exercise 3.12.

Exercises 4.4

Exercise 9.1 *Show that the autocovariance function for the above MA(q) process is given by*

$$\gamma(h) = \sigma^2 \sum_{j=0}^{q-h} \theta_j \theta_{j+h}, \ 0 \le h \le q$$

and zero otherwise.

Based on Exercise 4.1, we know that if

$$Y_t = \mu + \sum_{j=0}^{\infty} \psi_j Z_{t-j}, \tag{9.1}$$

then the ACVF is

$$\gamma_Y(h) = \sigma^2 \sum_{k=0}^{\infty} \psi_k \psi_{h+k}. \tag{9.2}$$

For an MA(q) time series, we have

$$Y_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}, \quad Z_t \sim WN(0, \sigma^2). \tag{9.3}$$

Comparing (9.1) and (9.3), we can coclude that

$$\psi_0 = 1, \;\; \psi_1 = \theta_1, \;\; \cdots, \;\; \psi_q = \theta_q, \;\; and \;\; \psi_j = 0 \;\; for \;\; j > q.$$

Therefore, to make sure that the elements in the sum are non-zero, we need θ_k and θ_{h+k} to be non-zero, which means that $0 \leq k \leq q$ and $0 \leq k + h \leq q$, which is equivalent $max\{0, -h\} \leq k \leq min\{q, q-h\}$. Thereore for $h > 0$, we have

$$\gamma_Y(h) = \sigma^2 \sum_{k=0}^{q-h} \theta_k \theta_{h+k}. \tag{9.4}$$

Exercises 4.5

Part 1:
We first consider the discrete time autoregressive signal plus noise process:

$$Y_t = X_t + U_t, \quad -\infty < t < \infty$$

in which X_t is a weakly stationary causal AR(1) process with parameters ϕ and σ^2 and U_t is an independent white noise process $WN(0, \tau^2)$. Note that Y_t, being a sum of two independent weakly stationary processes, is itself a weakly stationary process. Also, the autocovariance function for Y_t can be derived using the fact the autocovariance function for the sum of two independent processes is the sum of the autocovariance functions for the two processes. Therefore, the autocovariance function for Y_t is the sum of the autocovariances of X_t and U_t. From notes:

$$\gamma_Y(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} + \tau^2, & h = 0 \\ \frac{\sigma^2}{1-\phi^2} \phi^{|h|}, & h \neq 0 \end{cases}$$

Now consider the ARMA(1,1) process:

$$Y'_t = \phi' Y'_{t-1} + V_t + \theta V_{t-1}$$

We can look up the autocovariance function as:

$$\gamma_{Y'}(h) = \begin{cases} \left(1 + \frac{(\theta+\phi')^2}{1-\phi'^2}\right) \nu^2, & h = 0 \\ \left((\theta + \phi') + \frac{(\theta+\phi')^2 \phi'}{1-\phi'^2}\right) \nu^2, & h = 1 \\ \phi'^{h-1} \gamma_{Y'}(1), & h > 1 \end{cases}$$

Both autocovariances decay geometrically starting at lag 1.

Part 2:
To have identical autocovariances for all h, we must have the geometric rates the same in both autocovariance functions. This gives us the first equation:

$$\phi = \phi'$$

Two additional equations that relate the parameters (ϕ, σ^2, τ^2) and (ϕ, θ, ν^2) are given by equating $\gamma_Y(h) = \gamma_{Y'}(h)$ for $h = 0, 1$. It is slightly simpler to multiply these quantities by $(1 - \phi^2)$ to get:

$$(1 - \phi^2 + (\theta + \phi)^2)\nu^2 = \sigma^2 + (1 - \phi^2)\tau^2$$
$$[(\theta + \phi)(1 - \phi^2) + (\theta + \phi)^2\phi]\nu^2 = \sigma^2\phi$$

An alternative approach is to compute:

$$W_t = Y_t - \phi Y_{t-1} = U_t - \phi U_{t-1} + Z_t$$

where Z_t is $WN(0, \sigma^2)$. Then:

$$\gamma_W(h) = \begin{cases} \tau^2(1 + \phi^2) + \sigma^2, & h = 0 \\ -\phi\tau^2, & h = 1 \\ 0, & h > 1 \end{cases}$$

For the MA(1) part $W'_t = V_t + \theta V_{t-1}$:

$$\gamma_{W'}(h) = \begin{cases} \nu^2(1 + \theta^2), & h = 0 \\ \theta\nu^2, & h = 1 \\ 0, & h > 1 \end{cases}$$

Equating $\gamma_W(h) = \gamma_{W'}(h)$ for $h = 0, 1$ gives two equations relating (σ^2, τ^2) and (θ, ν^2) for ϕ fixed. Thus

$$\tau^2(1 + \phi^2) + \sigma^2 = \nu^2(1 + \theta^2)$$
$$-\phi\tau^2 = \theta\nu^2$$

Part 3:
Let $a = \tau^2(1 + \phi^2) + \sigma^2$, and $b = -\phi\tau^2$. Then solve:

$$\nu^2(1 + \theta^2) = a, \quad \theta\nu^2 = b$$

for θ and ν^2. Substituting $\nu^2 = b/\theta$ into the first equation:

$$\frac{b(1 + \theta^2)}{\theta} = a \Rightarrow b\theta^2 - a\theta + b = 0$$

This quadratic has the following solutions:

$$\theta = \frac{a \pm \sqrt{a^2 - 4b^2}}{2b}$$

both of which are real since $a^2 - 4b^2 = (a - 2b)(1 + 2b) = (\tau^2(1 - \phi)^2 + \sigma^2)(\tau^2(1 + \phi)^2 + \sigma^2) > 0$. Choose θ so that $\text{sign}(\theta)$ is opposite to ϕ to ensure $\nu^2/\tau^2 > 0$.

Exercises 4.6

Part 1

By assumption Y_n is a weakly stationary solution and hence $E(Y_n)$ and $var(Y_n)$ exist and are time invariant. Taking expectations of both sides of $Y_n = \phi_n Y_{n-1} + Z_n$ gives

$$E(Y_n) = E(\phi_n Y_{n-1} + Z_n) = E(\phi_n)E(Y_{n-1}) + E(Z_n)$$

using the fact that ϕ_n is independent of Y_{n-1}. Hence, using stationarity and the fact that $E(\phi_n) = \phi$ gives $E(Y_n) = \phi E(Y_n)$ from which it follows that $E(Y_n) = 0$. Next,

$$\begin{aligned} var(Y_n) &= E(Y_n^2) \\ &= E[(\phi_n Y_{n-1} + Z_n)^2] \\ &= E(\phi_n^2)E(Y_{n-1}^2) + 1, \qquad \text{(using independence)} \\ &= E[(\phi + \sigma\epsilon_n)^2]var(Y_n), \qquad \text{(stationarity of } Y_n) \\ &= (\phi^2 + \sigma^2)var(Y_n) + 1 \Rightarrow \mathrm{Var}(Y_n) = \frac{1}{1 - \phi^2 - \sigma^2} \end{aligned}$$

This is finite provided $\phi^2 + \sigma^2 < 1$.

Part 2

Substitute the expression for Y_n into both sides of the recurrence:

$$Y_n = Z_n + \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} \phi_{n-j} \right) Z_{n-i}$$

Shifting by one for Y_{n-1} and multiplying by ϕ_n gives:

$$\begin{aligned} \phi_n Y_{n-1} + Z_n &= \phi_n \left(Z_{n-1} + \sum_{i=1}^{\infty} \left(\prod_{j=0}^{i-1} \phi_{n-1-j} \right) Z_{n-1-i} \right) + Z_n \\ &= Z_n + \phi_n Z_{n-1} + \sum_{i=1}^{\infty} \left(\phi_n \prod_{j=0}^{i-1} \phi_{n-1-j} \right) Z_{n-1-i} \\ &= Z_n + \phi_n Z_{n-1} + \sum_{i=1}^{\infty} \left(\phi_n \prod_{j'=1}^{i} \phi_{n-j'} \right) Z_{n-1-i} \\ &= Z_n + \phi_n Z_{n-1} + \sum_{i=1}^{\infty} \left(\prod_{j'=0}^{i} \phi_{n-j'} \right) Z_{n-1-i} \\ &= Z_n + \phi_n Z_{n-1} + \sum_{i'=2}^{\infty} \left(\prod_{j'=0}^{i'-1} \phi_{n-j'} \right) Z_{n-i'} \\ &= Z_n + \sum_{i'=1}^{\infty} \left(\prod_{j'=0}^{i'-1} \phi_{n-j'} \right) Z_{n-i'} \end{aligned}$$

which matches the original expression for Y_n, hence proving the solution satisfies the recurrence.

Part 3

We could use the alternative form to calculate means and covariances at all lags to show these are time invariant in which case we have a weakly stationary solution. More straightforwardly we can claim that the given solution is strictly stationary which together with the above finite variance condition allows us to conclude the solution is also weakly stationary. To show strict stationarity we use the fact that a time invariant function of a strictly stationary process is strictly stationary. Now the joint process $\{\phi_n, Z_n\}$ is strictly stationary (in fact a jointly stationary Gaussian process of independent components and independent vectors between different times) and the mapping from this vector process to $\{Y_n\}$ is time invariant. Hence $\{Y_n\}$ is strictly stationary.