MATH5425 exam 2019

There were 5 questions worth 10 marks each.
Students were asked to complete 4 questions.

Question 1 (10 marks)

Let $G = (V, E)$ be a connected graph.

(a) Let $e \in E$ be an edge of G. Prove that there is a spanning tree in G which contains e.

(b) Let T, T' be distinct spanning trees in G. Prove that there is a finite sequence

$$T = T_0,\ T_1,\ \ldots,\ T_r = T'$$

such that $T_0, T_1, \ldots, T_r$ are distinct spanning trees in G, and for $i = 0, 1, \ldots, r-1$, the tree T_{i+1} is obtained from T_i by deleting one edge and inserting another.

Question 2 (10 marks)

(*We went through this in lectures at the end of Chapter* 7!)

Recall that $K_{a,b}$ is the complete bipartite graph with a vertices in one part and b vertices in the other.

(a) Prove that every red/blue-colouring of the edges of $K_{3,7}$ contains a monochromatic copy of $K_{2,2}$.

(b) Provide a red/blue-colouring of the edges of $K_{2,7}$ with no monochromatic copy of $K_{2,2}$. Explain why your colouring has the required property.

(c) Provide a red/blue-colouring of the edges of $K_{3,6}$ with no monochromatic copy of $K_{2,2}$. Explain why your colouring has the required property.

Question 3 (10 marks)
Consider the binomial random graph model $G(n,p)$. Let $X = X(n,p)$ be the number of unordered pairs of distinct vertices with no common neighbour. That is,

$$X = \left|\left\{\,\{i,j\} \,:\, 1 \le i < j \le n,\;\; N(i) \cap N(j) = \emptyset\,\right\}\right|$$

where $N(v)$ denotes the (random) neighbourhood of vertex v in $G \in G(n,p)$.

(a) Calculate $\mathbb{E}X$, with careful explanation.

(b) Prove that $\mathbb{E}X \le \exp\left(2\ln n - (n-2)p^2\right)$.

(c) Define a function $p : \mathbb{Z}^+ \to (0,1)$ such that

- $p(n) \to 0$ as $n \to \infty$, and
- in $G(n, p(n))$ we have $\mathbb{E}X \to 0$ as $n \to \infty$.

Justify your choice of p.

Question 3, continued...

(d) Let $p : \mathbb{Z}^+ \to (0, 1)$ be any function which satisfies the conditions required in part (c).

(i) Prove that a.a.s. the diameter of $G \in G(n, p(n))$ is at most 2.

(ii) Is it true that a.a.s., the diameter of $G \in G(n, p(n))$ is exactly 2? Explain your answer.

Question 4 (10 marks)

Parts (a) and (b) of this question are unrelated.

(a) Recall that $q(H)$ denotes the number of odd components in a graph H, that is, the number of components of odd order.

Let $G = (V, E)$ be a graph with n vertices. Suppose that M is a maximum matching of G.

Fix $S \subseteq V$ and let M' be the set of all edges in $M \cap E(G - S)$. That is, M' is the set of edges in M which are disjoint from S.

(i) Prove that at most $n - |S| - q(G - S)$ vertices of G are matched by M'.

(ii) Hence prove that $|M| \leq \frac{1}{2}\big(n + |S| - q(G - S)\big)$.

Question 4, continued...

(b) Prove that if

$$\binom{n}{k} 3^{1-\binom{k}{2}} < 1$$

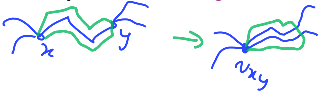
then there exists a colouring of the edges of K_n with 3 colours without a monochromatic copy of K_k. Justify each step carefully.

Question 5 (10 marks)

In this question we will show that Tutte's Theorem about 3-connected graphs cannot be extended to 4-connected graphs.

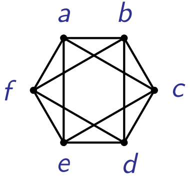
Prob Sheet 5!

(a) Suppose that G is a planar graph and e is an edge of G. Let $G' = G/e$ be the graph obtained from G by contracting the edge e. Prove that G' is planar.



(b) Consider the graph H shown below.

K_6 - perfect matching

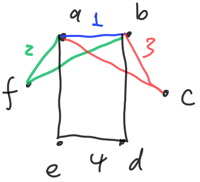


Enough to consider a+b; a+c, a+d (rotate picture)

(i) Using (the Global Version of) Menger's Theorem, or otherwise, prove that H is 4-connected.

Show ≥ 4 indep paths between any pair of vertices

$\Rightarrow$ 4-conn.



4 indep paths between a + b

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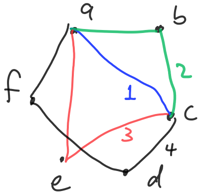
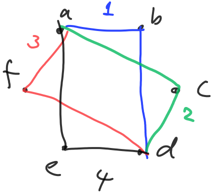


4 indep paths between a + c



4 indep paths between a + d

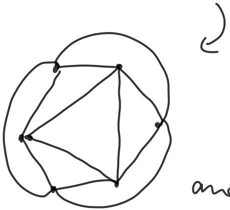
Hence by Global Menger's Th^{m},

$H = K_6 - \{ad, be, cf\}$

H is 4-connected.

solution

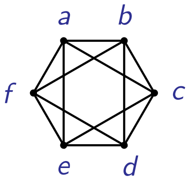
H is planar



and contraction preserves planarity.

Question 5(b), continued...

(b) Consider the graph H shown below.



(ii) Prove that there is **no** sequence of 4-connected graphs $G_0, G_1, \dots, G_r$ such that

- $G_0 = K_5$ and $G_r = H$,
- G_{i+1} has an edge e such that $G_i = G_{i+1}/e$, for $i = 0, \dots, r-1$.

$K_5 = G_0 \leftarrow G_1 = H$

So there is no sequence of contractions from H to K_5.

*** End of Exam ***