9.7 Solutions to Chapter 7

Exercise 7.1

Let us consider the following seasonal MA model:

$$X_t = Z_t + \Theta Z_{t-s}$$

The autocovariance function in lag h is

$$\begin{aligned}\gamma_X(h) &= cov(X_{t+h}, X_t)\\ &= cov(Z_{t+h} + \Theta Z_{t+h-s}, Z_t + \Theta Z_{t-s})\\ &= \gamma_Z(h) + \Theta\gamma_Z(h-s) + \Theta\gamma_Z(h+s) + \Theta^2\gamma_Z(h) \qquad (1)\end{aligned}$$

Note that

$$\gamma_Z(h) = \begin{cases}\sigma^2 & h = 0\\ 0 & h \neq 0\end{cases}$$

Therefore,

$$\begin{aligned}\gamma_X(h) &= \begin{cases}\gamma_Z(0) + \Theta^2\gamma_Z(0) & h = 0\\ \Theta\gamma_Z(0) & h = \pm s\\ 0 & h \neq 0, \pm s\end{cases}\\ &= \begin{cases}\sigma^2(1+\Theta^2) & h = 0\\ \Theta\sigma^2 & h = \pm s\\ 0 & h \neq 0, \pm s\end{cases}\end{aligned}$$

The auto-correlation function is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases}1 & h = 0\\ \frac{\Theta}{1+\Theta^2} & h = \pm s\\ 0 & h \neq 0, \pm s\end{cases}$$

Exercise 7.2

Let us consider the following seasonal AR model:

$$X_t = \Phi X_{t-s} + Z_t$$

For $h = 0$, we have

$$\begin{aligned}\gamma_X(0) &= var(X_t)\\ &= cov(X_t, X_t)\\ &= cov(\Phi X_{t-s} + Z_t, \Phi X_{t-s} + Z_t)\\ &= \Phi^2 cov(X_{t-s}, X_{t-s}) + 2\Phi cov(X_{t-s}, Z_t) + cov(Z_t, Z_t)\\ &= \Phi^2\gamma_X(0) + \sigma^2 \qquad \text{(since } Z_t,\ X_{t'} \text{ are uncorrelated for } t > t')\end{aligned}$$

Therefore

$$\gamma_X(0) = \frac{\sigma^2}{1-\Phi^2}$$

For $h > 0$,

$$\begin{aligned}\gamma_X(h) &= cov(X_{t+h}, X_t) \\ &= cov(\Phi X_{t+h-s} + Z_{t+h}, X_t) \\ &= \Phi cov(X_{t+h-s}, X_t) + cov(Z_{t+h}, X_t) \\ &= \Phi\gamma_X(h-s) \qquad \text{(since } Z_{t'} \text{ and } X_t \text{ are uncorrelated for } t' > t\text{)},\end{aligned} \tag{1}$$

Note that, since γ_X is an even function, for $s \neq 1$, we have

$$\begin{cases}\gamma(1) = \Phi\,\gamma_X(1-s) = \Phi\,\gamma(s-1) \\ \gamma(s-1) = \Phi\,\gamma(s-1-s) = \Phi\,\gamma(-1) = \Phi\,\gamma(1)\end{cases}$$

Therefore, it is easy to see that

$$\gamma(1) = \Phi\,\gamma(s-1) = \Phi^2\,\gamma(1)$$

and therefore,

$$(1-\Phi^2)\gamma(1) = 0$$

We know that $|\Phi| < 1$ and consequently $1 - \Phi^2 \neq 0$. So we can conclude that

$$\gamma(1) = 0$$

Using the same method, we can show that $\gamma(h) = 0$ for $h \neq ks, \quad k = 0, 1, 2, \ldots$.
From (1), we can show that, for $h = ks$:

$$\begin{aligned}\gamma_X(h) &= \Phi\,\gamma_X(h-s) \\ &= \Phi^2\,\gamma_X(h-2s) \\ &\vdots \\ &= \Phi^k\,\gamma_X(h-ks) \\ &= \Phi^k\,\gamma_X(0) \\ &= \Phi^k\,\frac{\sigma^2}{1-\Phi^2}\end{aligned}$$

Therefore,

$$\gamma_X(h) = \begin{cases}\dfrac{\sigma^2}{1-\Phi^2} & h = 0 \\ \dfrac{\Phi^k\sigma^2}{1-\Phi^2} & h = ks, \quad k = 1, 2, \ldots \\ 0 & h \neq ks, \quad k = 1, 2, \ldots\end{cases}$$

Exercise 7.3

$$\begin{aligned}
\gamma_X(h) &= cov(X_{t+h}, X_t) \\
&= cov\left(\Phi X_{t+h-12} + Z_{t+h} + \theta Z_{t+h-1}, X_t\right) \\
&= \Phi \underbrace{cov(X_{t+h-12}, X_t)}_{\gamma_X(h-12)} + cov(Z_{t+h}, X_t) + \theta cov(Z_{t+h-1}, X_t) \\
&= \begin{cases} \Phi\,\gamma_X(12) + cov(Z_t, X_t) + \theta\, cov(Z_{t-1}, X_t) & h = 0 \\ \Phi\,\gamma_X(11) + cov(Z_{t+1}, X_t) + \theta\, cov(Z_t, X_t) & h = 1 \\ \Phi\,\gamma_X(0) + cov(Z_{t+12}, X_t) + \theta\, cov(Z_{t+11}, X_t) & h = 12 \\ \Phi\,\gamma_X(h-12) & \text{otherwise} \end{cases}
\end{aligned} \tag{A}$$

Note that

- $cov(X_t, Z_{t'}) = 0 \quad \text{if } t' > t$ (1)
-
$$\begin{aligned}
cov(Z_t, X_t) &= cov\left(Z_t, \Phi X_{t-12} + Z_t + \theta Z_{t-1}\right) \\
&= 0 + cov(Z_t, Z_t) + 0 \quad \text{(using (1))} = var(Z_t) = \sigma^2
\end{aligned} \tag{2}$$
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$$\begin{aligned}
cov(Z_{t-1}, X_t) &= cov(Z_{t-1}, \Phi X_{t-12} + Z_t + \theta Z_{t-1}) \\
&= \Phi\, cov(Z_{t-1}, X_{t-12}) + 0 + \theta\, cov(Z_{t-1}, Z_{t-1}) \\
&= \theta\sigma^2 \quad \text{(since } Z_t \text{ are WN, using (1) and (2))}
\end{aligned} \tag{3}$$

Consequently, we can rewrite (A) as follows:

$$\begin{aligned}
\gamma_X(h) &= \begin{cases} \Phi\,\gamma_X(12) + cov(Z_t, X_t) + \theta\, cov(Z_{t-1}, X_t) & h = 0 \\ \Phi\,\gamma_X(11) + cov(Z_{t+1}, X_t) + \theta\, cov(Z_t, X_t) & h = 1 \\ \Phi\,\gamma_X(0) + cov(Z_{t+12}, X_t) + \theta\, cov(Z_{t+11}, X_t) & h = 12 \\ \Phi\,\gamma_X(h-12) & \text{otherwise} \end{cases} \\
&= \begin{cases} \Phi\,\gamma_X(12) + \sigma^2 + \theta \cdot \theta\sigma^2 = \Phi\gamma_X(12) + (1+\theta^2)\sigma^2 & h = 0 \\ \Phi\,\gamma_X(11) + 0 + \theta\sigma^2 & h = 1 \\ \Phi\,\gamma_X(0) + 0 + 0 & h = 12 \\ \Phi\,\gamma_X(h-12) & \text{otherwise} \end{cases}
\end{aligned}$$

Since

$$\gamma_X(0) = \Phi\,\gamma_X(12) + \sigma^2 + \theta^2\sigma^2$$

and, we have

$$\gamma_X(12) = \Phi\,\gamma_X(0)$$

Therefore, we can conclude that:

$$\gamma_X(0) = \Phi^2\gamma_X(0) + \sigma^2(1+\theta^2) \quad \Rightarrow \quad \gamma_X(0) = \frac{\sigma^2(1+\theta^2)}{1-\Phi^2}$$

Since $\gamma_X(h) = \Phi\gamma_X(h-12)$, we can easily show that

$$\gamma_X(12k) = \Phi^k \gamma_X(0)$$

Similarly, since

$$\gamma_X(1) = \Phi\gamma_X(11) + \theta\sigma^2 \quad \text{and} \quad \gamma_X(h) = \Phi\gamma_X(h-12) \quad \text{for } h \neq 0, 1, 12$$

we can conclude that

$$\gamma_X(1) = \frac{\theta}{1-\Phi^2}\sigma^2$$

and

$$\gamma_X(12k \pm 1) = \Phi^k \gamma_X(1) = \Phi^k \cdot \frac{\theta}{1-\Phi^2}\sigma^2 = \frac{\Phi^k \theta}{1+\theta^2}\gamma_X(0)$$

Therefore, for the auto-correlation function we have:

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \begin{cases} 1 & h = 0 \\ \Phi^k & h = 12k \\ \dfrac{\Phi^k \theta}{1+\theta^2} & h = 12k \pm 1 \\ 0 & \text{otherwise} \end{cases}$$