SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney

MATH5425 Graph Theory Term 1, 2025

Problem Sheet 7, Ramsey Theory

1. Prove that $R(3) = 6$ and that $R(2, s) = s$ for all $s \geq 2$.

2. Suppose that there exists $p \in [0, 1]$ such that

$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1.$$

   Prove that $R(s, t) > n$.

3. Prove that in any colouring of the edges of K_{17} using 3 colours (say red, blue, green) there exists a monochromatic triangle.

4. Prove that $R(C_4, C_4) = 6$, using the following steps. (Recall that C_4 is a 4-cycle.)

   (a) Use Bollobás Chapter 6 Theorem 11 (from lectures) to show that $R(C_4, C_4) \geq 5$.

   (b) Prove that $R(C_4, C_4) > 5$ by displaying a red-blue colouring of the edges of K_5 with no monochromatic 4-cycle.

   (c) Prove that in any red-blue colouring of the edges of K_6 there must be a monochromatic 4-cycle.

   Hint: For a contradiction, suppose that there is a red-blue colouring of the edges of K_6 with no monochromatic C_4. The fact that $R(3) = 6$ may help you to get started. Work out the colour of various edges and end up with a contradiction. You may have to analyse a couple of different cases.

5. **(Challenge!)** Prove that if both $R(s-1, t)$ and $R(s, t-1)$ are even then we have the slightly sharper bound $R(s, t) \leq R(s-1, t) + R(s, t-1) - 1$.

   Hint: Follow the proof by Erdős & Szekeres from the lectures and eventually consider the parity of $\sum_{v \in V} d_{blue}(v)$, where $d_{blue}(v)$ denotes the number of blue edges incident with v.

(... Please turn over for Questions 6 and 7)

6. We will prove that $R(3,4) = 9$.

   (a) Use Problem 5 to deduce that $R(3,4) \leq 9$.

   Now label the vertices of K_8 as $1, 2, \ldots, 8$ and for all $i \neq j$, colour the edge ij

$$\begin{cases} \text{red} & \text{if } |i-j| \text{ is congruent to 1 or 4 modulo 8,} \\ \text{blue} & \text{otherwise.} \end{cases}$$

   With respect to this colouring, write $N_{\text{red}}(v)$ for the set of vertices which are incident to vertex v by red edges, and similarly for $N_{\text{blue}}(v)$.

   (b) Prove that $N_{\text{red}}(1)$ is an independent set in the red subgraph (the subgraph formed by all the red edges), and that $N_{\text{blue}}(1)$ induces a blue path of length 3 in the blue subgraph (the graph formed by all the blue edges).

   (c) Hence, or otherwise, prove that $R(3,4) > 8$.
   (You may use the fact that the cyclic permutation $(1\,2\,3\,4\,5\,6\,7\,8)$ is an automorphism of both the red subgraph and the blue subgraph. This implies that each of these subgraphs is "vertex transitive", which means that the graph looks the same from the point of view of every vertex. In particular, we can replace vertex 1 by any vertex in part (b).)

7. In lectures we proved Erdős' 1947 result (Alon & Spencer Proposition 1.1.1) which shows that for $s \geq 3$, if

$$\binom{n}{s} 2^{1-\binom{s}{2}} < 1 \tag{$*$}$$

   then $R(s) > n$. From this we deduced that $R(s) > \lfloor 2^{s/2} \rfloor$ for $s \geq 3$. We now obtain a slightly improved lower bound using sharper analysis.

   (a) Show that if

$$n \leq \frac{s}{\sqrt{2}\, e}\, 2^{s/2}$$

   then

$$n^s < 2^{s(s-1)/2} \sqrt{\frac{\pi s}{2}} \left(\frac{s}{e}\right)^s. \tag{$**$}$$

   (b) Using one of Stirling's inequalities, explain why $(**)$ implies that Erdős' condition $(*)$ holds.

   (c) Hence conclude that $R(s) > \dfrac{s}{\sqrt{2e}}\, 2^{s/2}$ for $s \geq 3$.