$7 + 10 + 14 + 14$

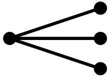
4 Qs, attempt all: 45 marks.

MATH5425 exam 2023

Question 1 (7 marks)

Recall that for fixed graphs H_1 and H_2, we define $R(H_1, H_2)$ to be the smallest positive integer n such that every red-blue colouring of the edges of K_n contains a red copy of H_1 or a blue copy of H_2.

The “claw” graph $K_{1,3}$ is shown below.

$K_{1,3}$: 

[2] (a) Prove that $R(K_{1,3}, K_{1,3}) > 5$.

[2] (b) Prove that $R(K_{1,3}, K_{1,3}) = 6$.

[3] (c) Decide whether the following statement is true or false:

$$R(K_3, K_{1,3}) = 6.$$

State and prove your answer.

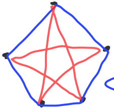
(a) Try Bollobás lower bound $R(H_1, H_2) \geq (\chi(H_1)-1)(|H_2|-1) + u(H_1)$

$H_1 = H_2 = K_{1,3}$ so $\chi(H_1) = 2$ as $K_{1,3}$ is bipartite,

$|H_2| = |K_{1,3}| = 4$, $u(H_1) = 1$.

Hence $R(K_{1,3}, K_{1,3}) \geq (2-1)(4-1)+1 = 4$.

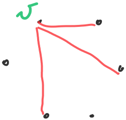
Not a great help!



The colouring of K_5 shown at left has both colour classes equal to a 5-cycle, hence no vertex is adjacent to 3 red edges or 3 blue edges. No mono $K_{1,3}$

Hence $R(K_{1,3}, K_{1,3}) > 5$.

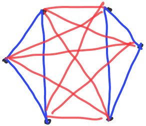
(b) Consider any red-blue colouring of the edges of K_6.



Choose any vertex v and consider the 5 edges incident with v. By the pigeonhole principle, some colour must appear on $\geq \lceil 5/2 \rceil = 3$ of these edges. This gives a monochromatic copy of $K_{1,3}$.

(c) We think it is <u>false</u>!

Counterexample:



want

$R(K_3, K_{1,3}) = 6$

no red K_3? no blue $K_{1,3}$

Blue edges form union of two 3-cycles, no vertex of degree ≥ 3
$\Rightarrow$ no blue $K_{1,3}$

Red edges form red $K_{3,3}$, bipartite, no red K_3.

OR use Bollobás: $R(K_3, K_{1,3}) \geq (\chi(K_3) - 1)(|K_{1,3}| - 1) + 1$

$$= 2 \times 3 + 1 = 7.$$

Hence $R(K_3, K_{1,3}) \geq 7$, so the statement is FALSE

Question 2 (10 marks) Consider the binomial random graph model $G(n,p)$. Recall the graph $K_{1,3}$ from Question 1.

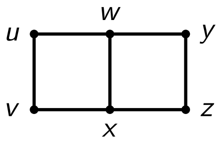
Let $X = X(n,p)$ be the number of copies of $K_{1,3}$ in $G \in G(n,p)$, and let $Y = Y(n,p)$ be the number of copies of L in $G \in G(n,p)$, where L is the "ladder graph" shown below:

Prob Sheet 8!

Markov's Lemma:

$\Pr(Y \geq 1) \leq \mathbb{E}Y$

L:



(a) Write down the set of automorphisms of L.

(b) Calculate $\mathbb{E}X$ and $\mathbb{E}Y$, with explanation.

(c) Define a function $p : \mathbb{Z}^+ \to \mathbb{R}$ such that $p(n) \in (0,1)$ for sufficiently large n, and when $p = p(n)$,

- $\mathbb{E}X \to \infty$,
- a.a.s. $Y = 0$. ie $\Pr(Y=0) \to 1$ ie $\underbrace{\Pr(Y>0)}_{}\overset{\Pr(Y\geq 1)}{} \to 0$

Justify your answer. ie $\mathbb{E}Y \to 0$

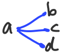
(a) The automorphisms of L are

$\mathrm{Aut}(L) = \{ \underset{\text{identity}}{()}, \quad \underset{\text{reflect in horizontal}}{(u\ v)(w\ x)(y\ z)}, \quad \underset{\text{reflect around } wx}{(u\ y)(v\ z)}, \quad \underset{\text{Do both}}{(u\ z)(w\ x)(y\ v)} \}$

$= \langle (uv)(wx)(yz), (uy)(vz) \rangle$ which has 4 elements

(b) $\boxed{\text{Prob Sheet}}$: Expected [# copies of H] as a subgraph (not neccessarily induced) of $G(n,p)$ is $\left(\dfrac{(n)_k}{|\mathrm{Aut}(H)|} p^{|E(H)|} \right)$.

Let $|H| = k$.

So 

$\mathbb{E}X = \dfrac{(n)_4}{6} p^3$

as $|K_{1,3}| = 4$, $|E(K_{1,3})| = 3$ and $K_{1,3}$ has 6 automorphisms given by permutations of the 3 leaves

$\mathbb{E}Y = \dfrac{(n)_6 p^7}{4}$. as L has 6 vertices, 7 edges and 4 automorphisms (by (a))

(c) Recall $\mathbb{E}X = \frac{(n)_4}{6} p^3 \sim \frac{n^4 p^3}{6}$, This tends to ∞ if and only if $n^4 p^3 \to \infty$.

Also $\mathbb{E}Y = \frac{(n)_6}{4} p^7 \sim \frac{n^6 p^7}{4}$, This tends to 0 if and only if $n^6 p^7 \to 0$

Try $p = \frac{1}{n^\alpha}$? want $n^{4-3\alpha} \to \infty$, so $4 > 3\alpha$, $\alpha < 4/3$

Want $n^{6-7\alpha} \to 0$, so $6 < 7\alpha$, $\alpha > \frac{6}{7}$

$\frac{6}{7} < \alpha < \frac{4}{3}$, can take $\underline{\alpha = 1}$.

Define $p = p(n) = \frac{1}{n}$, note $p \in (0,1)$ for all $n \geq 2$

Then $\mathbb{E}X \sim \frac{n^{4-3}}{6} = \frac{n}{6} \to \infty$ and $\mathbb{E}Y \sim \frac{n^{6-7}}{4} = \frac{1}{4n} \to 0$ as $n \to \infty$.

Therefore $\Pr(Y \geq 1) \leq \mathbb{E}Y \to 0$, using Markov's lemma,

so $\underline{\text{a.a.s. } Y = 0}$.

Question 3 (14 marks)

For any graph G and each nonnegative integer k, let $P_G(k)$ denote the number of k-colourings of G. Recall the following facts about P_G, which we proved in Problem Sheet 4, Question 3:

- If G has n vertices then P_G is a monic polynomial of degree n in k, called the chromatic polynomial of G.
- For any edge e of G,

$$P_G(k) = P_{G'}(k) - P_{G''}(k) \tag{1}$$

  where $G' = G - e$ (edge deletion) and $G'' = G/e$ (edge contraction).

(You do NOT need to prove the above two facts.) We can write

$$P_G(k) = k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0,$$

where $n = |G|$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$. Here a_0 is the constant coefficient of $P_G(k)$ and a_1 is the linear coefficient of $P_G(k)$.

(a) Prove that if T is a tree with $n \geq 1$ vertices then $P_T(k) = k(k-1)^{n-1}$.

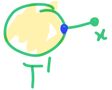
[3] (a) We prove using induction on n

Base case: When $n = 1$ then T is $\bullet$ and $P_T(k) = k$
$= k(k-1)^{1-1}$

The result ~~is~~ holds.

For the inductive step, suppose the result holds for trees with $n-1$ vertices and let T be a tree on n vertices.

We know that T contains a leaf x. Let $T' = T - x$ which has $n-1$ vertices. By the inductive hypothesis, $P_{T'}(k) = k(k-1)^{n-2}$.



Finally, for every k-colouring of T' there are exactly $k-1$ ways to extend this to a k-colouring c of T, since $c(x)$ must not be the colour of the unique neighbour of x in T'. Hence

$$P_T(k) = P_{T'}(k) \times (k-1) = k(k-1)^{n-1},$$ as required

This completes the proof, by induction.

(Question 3, continued)

[3] (b) Let C_4 be the cycle of length 4. Prove that the chromatic polynomial of C_4 is given by

$$P_{C_4}(k) = k(k-1)(k^2 - 3k + 3).$$

[1] (c) Suppose that G has r connected components $G_1, \ldots, G_r$. Briefly explain why

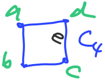
$$P_G(k) = \prod_{i=1}^{r} P_{G_i}(k).$$

[4] (d) Using induction on the number of edges, or otherwise, prove that for any connected graph G,

$$a_0 = 0 \quad \text{and} \quad a_1 = (-1)^{|G|-1}\,\ell \quad \text{for some integer } \ell \geq 1. \qquad (2)$$

(As defined above, a_0 and a_1 are the constant and linear coefficient of $P_G(k)$, respectively.)

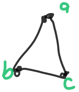
(b) We know that for any edge $e \in E(C_4)$, $P_{C_4}(k) = P_{C_4 - e}(k) - P_{C_4/e}(k)$.



Next, $C_4 - e = P_3$ path of 3 edges,

so $P_{P_3}(k) = k(k-1)^3$.

Also $C_4/e = K_3$, and $P_{K_3}(k) = k(k-1)(k-2)$.



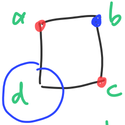
Substituting gives

$$P_{C_4}(k) = P_{P_3}(k) - P_{K_3}(k) = k(k-1)^3 - k(k-1)(k-2)$$
$$= k(k-1)\left((k-1)^2 - (k-2)\right)$$
$$= k(k-1)(k^2 - 3k + 3).$$

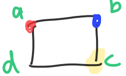
✓

Alternatively:

Colour C_4 directly. There are k choices for a, then $k-1$ choices for b.



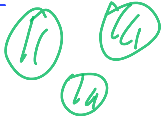
If c is coloured the same as a then there are $k-1$ choices for d



Otherwise, there are $k-2$ choices for d (and $k-2$ choices for c)

Putting this together. $P_{C_4}(k) = k(k-1)\big(\underset{[c\text{ red}]}{1.(k-1)} + \underset{[c \neq \text{red or blue}]}{(k-2)(k-2)}\big)$

$$= k(k-1)(k-1+k^2-4k+4)$$
$$= k(k-1)(k^2-3k+3)$$ ✓

(c) 

To colour G you must colour all the components $G_1, .., G_r$

So $c \longmapsto (c|_{G_1}, \ldots, c|_{G_r})$

gives a map from the set of k-colourings of G to the Cartesian product of the set of k-colourings of the G_i

This map has an inverse (colour the G_i independently), so

$$P_G(k) = \prod_{i=1}^{r} P_{G_i}(k).$$ ✓

(d) RTP: $a_0 = 0$, $a_1 = (-1)^{n-1} \ell$, for some $\ell \in \mathbb{Z}^+$

for all connected graphs.

(d) Check formula for trees, using (a) we have, using the Binomial Theorem.

Induction on # edges

$$P_G(T) = k(k-1)^{n-1} = k\left(k^{n-1} - (n-1)k^{n-2} + \dots + (-1)^{n-1}\right)$$

for any tree T with n vertices. This has constant coeff $a_0 = 0$ & linear coeff $(-1)^{n-1} = (-1)^{n-1} \times 1$ and $1 \in \mathbb{Z}^+$

$\Rightarrow$ result true for trees.

Now suppose that G is & $|G| = n$ connected but not a tree & result true if $< |E(G)|$ edges.

[$P_{C_4} = k(k-1)(k^2 - 3k + 3)$ ASIDE]

So G has an edge e which belongs to a cycle, So e is not a bridge & hence $G' = G - e$ is connected Also $G'' = G/e$ is connected

So by the inductive hypothesis, since G' has n vertices & $m-1$ edges and G'' has $n-1$ vertices & $\leq m-1$ edges

$$P_{G'}(k) = k^2 f(k) + (-1)^{n-1} l' k, \quad \text{where } l' \in \mathbb{Z}^+, \text{ and}$$

$$P_{G''}(k) = k^2 g(k) + (-1)^{n-2} l'' k, \quad \text{where } l'' \in \mathbb{Z}^+$$

and $f(k)$, $g(k)$ are some polynomials

Where $m = |E(G)|$

Then $P_G(k) = P_{G'}(k) - P_{G''}(k)$

$$= k^2 f(k) + (-1)^{n-1} l' k - \left(k^2 g(k) + (-1)^{n-2} l'' k\right)$$

$$= k^2 \left(f(k) + g(k)\right) + (-1)^{n-1} (l' + l'') k,$$

and $l' + l'' \in \mathbb{Z}^+$.

Hence $P_G(k)$ has constant coeff $a_0 = 0$ and linear coeff $a_1 = (-1)^{n-1} (l' + l'')$, $l' + l'' \in \mathbb{Z}^+$.

Proof is complete.

(f) G has 2 connected components & $P_{C_4}(k) = k(k-1)(k^2-3k+3)$, so $P_G(k) = P_{C_4}(k) \times k(k-1)^2$. So eg G is

or:

[check!]

(Question 3, continued)

[2] (e) Hence, or otherwise, prove that for any graph G with at least one vertex, the smallest power of k with nonzero coefficient in $P_G(k)$ is equal to the number of connected components in G.

[1] (f) Draw a graph with chromatic polynomial

$$k^2(k-1)^3(k^2-3k+3).$$

(You do not need to provide a proof, just the graph.)

(e) If G has components $G_1, \ldots, G_r$ then

$$P_{G_i}(k) = k f_i(k) \text{ for some polynomial } f_i(k), \text{ by (d).}$$

Note $f_i(k)$ has nonzero constant coefficient $(-1)^{|G_i|-1} l_i$ for some $l_i \in \mathbb{Z}^+$

Hence $P_G(k) = \prod_{i=1}^{r} P_{G_i}(k)$ by (c)

$= k^r \prod_{i=1}^{r} f_i(k)$. The smallest power of k with nonzero coeff is k^r as all f_i have nonzero constant coeff.

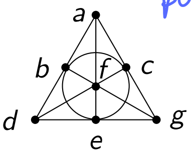
Question 4 (14 marks) The Fano plane F is a finite projective geometry with the following properties:

- F has 7 points and 7 lines.
- Each line contains exactly 3 points, and each point lies on exactly 3 lines.
- Each pair of distinct points lies on exactly one line.
- Each pair of distinct lines intersects in exactly one point.

We can draw the Fano plane as shown below, with set of points $U = \{a, b, c, d, e, f, g\}$ and set of lines

$$W = \{abd, \quad bce, \quad cdf, \quad deg, \quad aef, \quad bfg, \quad acg\}.$$

Note that the line bce is shown as a circle on the figure below.

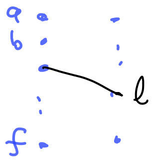


point a lies on lines abd
afe
acg

14

(Question 4, continued)

Now define the bipartite graph G with vertex bipartition $U \cup W$ and edge set



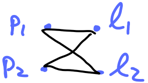
$$\{p\ell \mid p \in U,\ \ell \in W,\ \text{ line } \ell \text{ contains point } p\}.$$

[2]
(a) Write down the number of vertices and edges of G.
14 21

[3] (b) Prove that G has diameter 3.

[2] (c) Calculate the number of faces in any plane embedding of G. Justify your answer.

[2]
(d) Prove that G does not contain a 4-cycle.

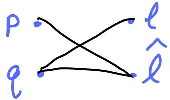


[1] (e) Write down a 6-cycle in G which contains the edge $\{a, abd\}$.

[4] (f) Without using Kuratowski's Theorem, prove that G is not planar. (Hence no plane embedding of G exists.)

Points p_1 + p_2 with $p_1 \neq p_2$ lie on a unique line. But a 4-cycle $p_1 \ell_1 p_2 \ell_2 p_1$ says that p_1, p_2 lie on 2 distinct lines ℓ_1, ℓ_2, contradiction

(b) If p, q are distinct points then they lie on a unique line ℓ.
So $p\ell q$ is a path of length 2 from p to q.



A similar argument shows that the distance between any two distinct lines is 2 (or use duality).

Now consider a point p and line ℓ.
If p lies on ℓ then $d_G(p, \ell) = 1$ as $p\ell \in E(G)$.
Otherwise, choose some point q which belongs to ℓ.
Then p and q lie on some line $\hat{\ell} \neq \ell$.
Then $d_G(p, \ell) = 3$ as $p\hat{\ell}q\ell$ is a path in G and $p\ell \notin E(G)$

Therefore $\operatorname{diam}(G) = 3$.

(c) Suppose that G has a plane embedding.

Note G is connected as it has finite diameter by (b).

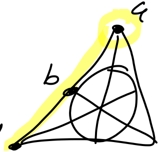
Hence Euler's formula applies and says that

$$14 - 21 + |F(G)| = 2.$$

So there would be $23 - 14 = \underline{9 \text{ faces}}$. ☆

(e) Find a 6-cycle containing $\{a, abd\}$.

[exercise]



(f) Prove G <u>not</u> planar (no Kuratowski)

Double-count incident (edge, face)-pairs in G:

Every edge lies in a 6-cycle, by symmetry & part (e).

$\Rightarrow$ Every edge of G is in a cycle so G is 2-conn.

$\Rightarrow$ 54 $= 6\ell \leq |\{(e,f)\}| = 2m = 42$, contradiction!

*** End of Exam ***