THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

Term 1, 2021

MATH5425
Graph Theory

(1) TIME ALLOWED – THREE (3) HOURS

(2) TOTAL NUMBER OF QUESTIONS – 4

(3) ANSWER ALL QUESTIONS

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) START EACH QUESTION ON A NEW PAGE

(6) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

YOU ARE TO COMPLETE THE TEST UNDER STANDARD EXAM CONDITIONS, WITH HANDWRITTEN SOLUTIONS.

DURING THE EXAM, YOU MAY CONSULT COURSE MATERIALS (e.g. LECTURE NOTES, TUTORIAL PROBLEMS) BUT YOU SHOULD NOT SEARCH THE INTERNET.

YOU WILL THEN SUBMIT ONE OR MORE FILES CONTAINING YOUR SOLUTIONS. **MAKE SURE YOU SUBMIT ALL YOUR ANSWERS.**

ONE OF THE SUBMITTED FILES MUST INCLUDE A PHOTOGRAPH OF YOUR **STUDENT ID CARD** WITH THE **SIGNED**, HANDWRITTEN STATEMENT:
"**I declare that this submission is entirely my own original work.**"

YOU CAN DELETE AND/OR RELOAD FILES UNTIL THE DEADLINE.

1. [**10 marks**]

Let $G = (V, E)$ be a graph with n vertices. Recall that $\Delta(G)$ denotes the maximum degree in G, and $\alpha(G)$ denotes the number of vertices in a largest independent set in G.

Also recall that a subset $U \subseteq V$ is a *cover* of G if every edge of G is incident with at least one vertex in U. Let $\tau(G)$ denote the number of vertices in a smallest cover of G.

a) Prove that if $A \subseteq V$ is an independent set in G then $U = V - A$ is a cover in G.

b) Hence or otherwise, prove that $\alpha(G) + \tau(G) \leq n$.

c) Now assume that G is triangle-free. That is, G contains no 3-cycles. The following inequality may be useful:

$$ab \leq \left(\frac{a+b}{2}\right)^2 \qquad \text{for all } a, b \in \mathbb{R}.$$

Do NOT use the result or proof method of Problem Sheet 1, Question 7, in your answer.

i) Prove that $\Delta(G) \leq \alpha(G)$.

ii) Hence or otherwise, prove that $|E(G)| \leq \alpha(G)\,\tau(G)$.

iii) **Without using the result of Problem Sheet 1, Question 7**, prove that $|E(G)| \leq n^2/4$.

(This gives a different proof of Mantel's Theorem: if G is a triangle-free graph with n vertices then $|E(G)| \leq n^2/4$.)

2. [10 marks]

a) Suppose that G is a plane graph with n vertices and ℓ faces, where ℓ is divisible by 7. Further suppose that exactly $3\ell/7$ faces are bounded by 4-cycles and the remaining faces are all bounded by 5-cycles.

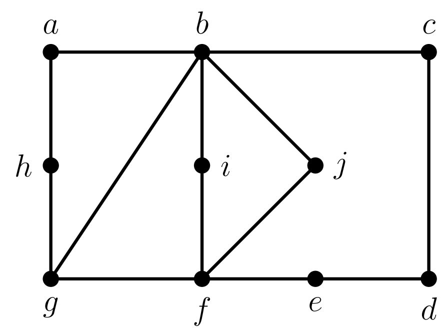
(Note: you do not need to prove that such a plane graph exists.)

i) Briefly explain why G is connected.

ii) Let e be an edge of G. Prove that e lies on the boundary of two distinct faces of G.

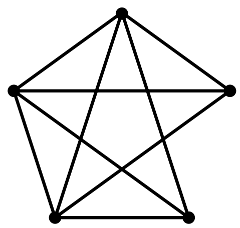
iii) Calculate the number of edges in G, as a function of n only.

b) Using Tutte's Theorem, or otherwise, show that the following graph has no perfect matching.



3. [10 marks]

a) Let H be the graph obtained from K_5 by deleting an edge, and let P_2 denote the path of length 2.

H 

 P_2

i) Prove that $R(H, P_2) \geq 7$.

ii) Prove that $R(H, P_2) = 7$.

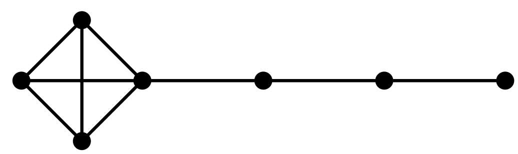
b) Now let H_1, H_2 be graphs, and suppose that H_1 is a subgraph of H_2. Is the following inequality true or false?

$$R(H_1, H_1) \leq R(H_2, H_2).$$

State your answer (true or false) and give a proof.

4. [10 marks]

Consider the binomial random graph model $G(n,p)$. Let $X = X(n,p)$ be the number of subgraphs of $G \in G(n,p)$ which are isomorphic to K_4, and let $Y = Y(n,p)$ be the number of subgraphs of $G \in G(n,p)$ which are isomorphic to the graph H shown below.



Recall that $(a)_b$ denotes the falling factorial $a(a-1)\cdots(a-b+1)$, and that an event holds asymptotically almost surely (a.a.s.) if the probability of the event tends to 1 as n tends to infinity.

a) Calculate $\mathbb{E}X$, with careful explanation.

b) Prove that $\mathbb{E}Y = \dfrac{1}{6}\,(n)_7\,p^9$.

c) Define a function $p : \mathbb{Z}^+ \to (0,1)$ such that

- $p(n) \to 0$ as $n \to \infty$,
- in $G(n,p(n))$ we have $\mathbb{E}X \to 0$ as $n \to \infty$,
- in $G(n,p(n))$ we have $\mathbb{E}Y \to \infty$ as $n \to \infty$.

Explain carefully why your choice of p satisfies these conditions.

d) Let $p : \mathbb{Z}^+ \to (0,1)$ be any function which satisfies the conditions required in part (c).

i) Prove that a.a.s. $X = 0$.

ii) Is it true that a.a.s. $Y > 0$? Justify your answer.

END OF EXAMINATION