UNSW Sydney

SCHOOL OF MATHEMATICS AND STATISTICS

Term 1, 2023   FINAL EXAMINATIONS

MATH5425 Graph Theory

(1) TIME ALLOWED – TWO (2) HOURS

(2) READING TIME – 10 MINUTES

(3) THIS EXAMINATION PAPER HAS 4 PAGES

(4) TOTAL NUMBER OF QUESTIONS – 4

(5) ATTEMPT ALL QUESTIONS. (THERE ARE 45 MARKS IN TOTAL.)

(6) THE QUESTIONS ARE **NOT** OF EQUAL VALUE: MARKS AVAILABLE FOR EACH QUESTION IS SHOWN IN THE QUESTION PAPER

(7) ANSWER **EACH** QUESTION IN A **SEPARATE** BOOK

(8) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(9) **ONLY** CALCULATORS WITH AN AFFIXED “UNSW APPROVED” STICKER MAY BE USED

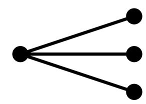
(10) STUDENTS MAY BRING IN ONE HANDWRITTEN DOUBLE-SIDED A4 PAGE OF NOTES. THIS MUST BE SUBMITTED WITH EXAM

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. **[7 marks]**

Recall that for fixed graphs H_1 and H_2, we define $R(H_1, H_2)$ to be the smallest positive integer n such that every red-blue colouring of the edges of K_n contains a red copy of H_1 or a blue copy of H_2.

The "claw" graph $K_{1,3}$ is shown below.

$K_{1,3}$: 

a) Prove that $R(K_{1,3}, K_{1,3}) > 5$.

b) Prove that $R(K_{1,3}, K_{1,3}) = 6$.

c) Decide whether the following statement is <u>true</u> or <u>false</u>:

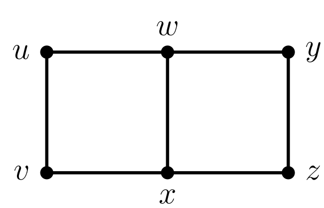
$$R(K_3, K_{1,3}) = 6.$$

State and prove your answer.

2. **[10 marks]**

Consider the binomial random graph model $G(n, p)$. Recall the graph $K_{1,3}$ from Question 1.

Let $X = X(n, p)$ be the number of copies of $K_{1,3}$ in $G \in G(n, p)$, and let $Y = Y(n, p)$ be the number of copies of L in $G \in G(n, p)$, where L is the "ladder graph" shown below:

L: 

a) Write down the set of automorphisms of L.

b) Calculate $\mathbb{E}X$ and $\mathbb{E}Y$, with explanation.

c) Define a function $p : \mathbb{Z}^+ \to \mathbb{R}$ such that $p(n) \in (0, 1)$ for sufficiently large n, and when $p = p(n)$,

- $\mathbb{E}X \to \infty$,
- a.a.s. $Y = 0$.

Justify your answer.

3. [14 marks]

For any graph G and each nonnegative integer k, let $P_G(k)$ denote the number of k-colourings of G. Recall the following facts about P_G, which we proved in Problem Sheet 4, Question 3:

- If G has n vertices then P_G is a monic polynomial of degree n in k, called the *chromatic polynomial* of G.
- For any edge e of G,

$$P_G(k) = P_{G'}(k) - P_{G''}(k) \tag{1}$$

  where $G' = G - e$ (edge deletion) and $G'' = G/e$ (edge contraction).

(You do NOT need to prove the above two facts.) We can write

$$P_G(k) = k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0,$$

where $n = |G|$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$. Here a_0 is the constant coefficient of $P_G(k)$ and a_1 is the linear coefficient of $P_G(k)$.

a) Prove that if T is a tree with $n \geq 1$ vertices then $P_T(k) = k(k-1)^{n-1}$.

b) Let C_4 be the cycle of length 4. Prove that the chromatic polynomial of C_4 is given by
$$P_{C_4}(k) = k(k-1)(k^2 - 3k + 3).$$

c) Suppose that G has r connected components $G_1, \ldots, G_r$. Briefly explain why
$$P_G(k) = \prod_{i=1}^{r} P_{G_i}(k).$$

d) Using induction on the number of edges, or otherwise, prove that for any connected graph G,

$$a_0 = 0 \quad \text{and} \quad a_1 = (-1)^{|G|-1}\,\ell \quad \text{for some integer } \ell \geq 1. \tag{2}$$

(As defined above, a_0 and a_1 are the constant and linear coefficient of $P_G(k)$, respectively.)

e) Hence, or otherwise, prove that for any graph G with at least one vertex, the smallest power of k with nonzero coefficient in $P_G(k)$ is equal to the number of connected components in G.

f) Draw a graph with chromatic polynomial $k^2(k-1)^3(k^2 - 3k + 3)$. (You do not need to provide a proof, just the graph.)

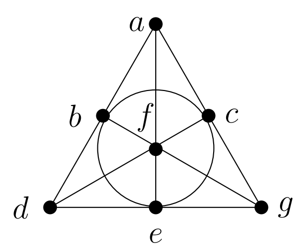
4. [14 marks]
The Fano plane F is a finite projective geometry with the following properties:

- F has 7 points and 7 lines.
- Each line contains exactly 3 points, and each point lies on exactly 3 lines.
- Each pair of distinct points lies on exactly one line.
- Each pair of distinct lines intersects in exactly one point.

We can draw the Fano plane as shown below, with set of points $U = \{a, b, c, d, e, f, g\}$ and set of lines

$$W = \{abd, \quad bce, \quad cdf, \quad deg, \quad aef, \quad bfg, \quad acg\}.$$

Note that the line bce is shown as a circle on the figure below.



Now define the bipartite graph G with vertex bipartition $U \cup W$ and edge set

$$\{p\ell \mid p \in U,\ \ell \in W,\ \text{ line } \ell \text{ contains point } p\}.$$

a) Write down the number of vertices and edges of G.

b) Prove that G has diameter 3.

c) Calculate the number of faces in any plane embedding of G. Justify your answer.

d) Prove that G does not contain a 4-cycle.

e) Write down a 6-cycle in G which contains the edge $\{a, abd\}$.

f) Without using Kuratowski's Theorem, prove that G is not planar. (Hence no plane embedding of G exists.)

END OF EXAMINATION