# SCHOOL OF MATHEMATICS AND STATISTICS

## UNSW Sydney

## MATH5425 Graph Theory        Term 1, 2025

## Assignment 1 (20%): due Thursday Week 3, 23:59

This assignment is designed to be attempted based on the material covered in the first two weeks of lectures. The number of marks awarded to each question is shown, to give you a rough idea of how much work is expected. (One mark roughly corresponds to one step or idea.) The assignment will be **marked out of 20** and is worth 20%.

Marks will be awarded for clear and logical explanations, as well as for correctness.

You will get the most out of this assignment by trying the questions yourself. You may discuss this assignment with other students, or look at textbooks if you wish. But it is very important that you write up your solution *on your own*. Do not copy work from other students, from textbooks or the internet.

You can use results from the lecture notes in your answers, as long as you clearly mention which result you are using. If a fellow student or textbook gave you an idea which helped you, please acknowledge this help in your solutions. (For example, write something like "Discussions with XXX were helpful" or "The main idea for part (b) came from XXX, page number XXX".) This is how professional mathematicians acknowledge assistance from their colleagues or from prior work.

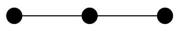
The use of generative AI is **not permitted** for this assignment, in accordance with School policies.

Use of LaTeX is optional and will not affect the mark awarded. The LaTeX file for the assignment will be made available on Moodle.

The assignment is due on Thursday of Week 3 at 23:59 (late evening Thursday 6 March). Please upload your assignment to the MATH5425 Moodle site.

If you think you will be unable to submit your assignment on time, please contact me in advance (say by the end of week 2).

1. **(11 marks)**
Let $P$ denote a set of two adjacent edges:

$P$ 

We say that the graph $G = (V, E)$ is $P$-*splittable* if there is a partition of $E$ into disjoint copies of $P$. That is,

$$E = \bigcup_{j=1}^{t} P_j$$

where each $P_j$ is a set of two adjacent edges, and $P_1, \ldots, P_t$ are pairwise disjoint. If $G$ is $P$-splittable as above, then we say that $\mathcal{P} = \{P_1, \ldots, P_t\}$ is a $P$-*decomposition* of $G$.

(a) Write down a $P$-decomposition of $K_4$. (Assume that the vertices of $K_4$ are labelled $1, 2, 3, 4$.)

(b) Let $G$ be a graph with connected components $G_1, \ldots, G_r$. Briefly explain why $G$ is $P$-splittable if and only if $G_1, \ldots, G_r$ are all $P$-splittable.

(c) Find a graph $G$ with an even number of edges which is not $P$-splittable. Briefly explain your answer.

Given a connected graph $G = (V, E)$ and distinct edges $e, f \in E$, define the *distance* between $e$ and $f$ in $G$, denoted $d_G(e, f)$, to be the minimum length of a path in $G$ from an endvertex of $e$ to an endvertex of $f$.

(d) Suppose $G = (V, E)$ is a connected graph with distinct edges $e, f \in E$ such that $G - \{e, f\}$ is $P$-splittable. Prove that if $d_G(e, f) > 0$ then there is an edge $h \in E - \{e, f\}$ such that $G - \{h, f\}$ is $P$-splittable and $d_G(h, f) < d_G(e, f)$.

(e) Hence, or otherwise, prove that if $G$ is a connected graph with an even number of edges then $G$ is $P$-splittable.

2. **(9 marks)**

(a) Let $G$ be a graph with $2r$ vertices and minimum degree $\delta(G) \geq r$, where $r \geq 1$. Prove that $G$ has a perfect matching.

(b) Let $G$ be a graph with minimum degree $\delta(G) \geq 1$ and maximum degree $\Delta = \Delta(G)$. Let $F$ be a maximum matching in $G$. Say that a vertex is *covered by* $F$ if it is the endvertex of an edge in $F$.

  (i) Let $x$ be a vertex not covered by $F$. Show that every neighbour of $x$ is covered by $F$.

  (ii) Let $x, y$ be two distinct vertices not covered by $F$. Show that if $xa, yb \in E$ then $ab \notin F$.

  (iii) Hence show that the number of vertices not covered by $F$ is at most $(\Delta - 1)\,|F|$.

  (iv) Prove that $|F| \geq n/(\Delta + 1)$, where $n = |G|$.