9.5 Solutions to Chapter 5

Exercises 5.1

We want to show that

$$E\left(X_{n+1} - \hat{X}_{n+1}\right)^2 = \gamma(n+1, n+1) - \boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n$$

where

$$\hat{X}_{n+1} = \sum_{j=1}^{n} a_{n,j} X_{n+1-j} = \boldsymbol{a}_n' \mathbf{X}_1^n, \quad \text{and} \quad \boldsymbol{a}_n = \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n.$$

$$\begin{aligned}
E\left(X_{n+1} - \hat{X}_{n+1}\right)^2 &= E\left(X_{n+1} - \boldsymbol{a}_n' \mathbf{X}_1^n\right)^2 \\
&= E\left(X_{n+1} - \boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \mathbf{X}_1^n\right)^2 \\
&= E\left(X_{n+1}^2\right) - 2E\left(\boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \mathbf{X}_1^n X_{n+1}\right) + E\left(\boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \mathbf{X}_1^n (\mathbf{X}_1^n)' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n\right) \\
&= \gamma(n+1, n+1) - 2\boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} E\left(\mathbf{X}_1^n X_{n+1}\right) + \boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} E\left(\mathbf{X}_1^n (\mathbf{X}_1^n)'\right) \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n \\
&= \gamma(n+1, n+1) - 2\boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n + \boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\Gamma}_n \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n \\
&= \gamma(n+1, n+1) - 2\boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n + \boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n \\
&= \gamma(n+1, n+1) - \boldsymbol{\gamma}_n' \boldsymbol{\Gamma}_n^{-1} \boldsymbol{\gamma}_n
\end{aligned}$$

Note that since the mean of $\mathbf{X}_n$ is zero,

$$E\left(\mathbf{X}_n X_{n+1}\right) = \mathrm{Cov}(\mathbf{X}_n, X_{n+1}) = \boldsymbol{\gamma}_n.$$

Exercises 5.2

Given $X_1, \ldots, X_n$, we can write the ARMA(1,1) model as

$$X_t = \phi X_{t-1} + Z_t + \theta Z_{t-1}, \quad t \in \{1, \ldots, n\}$$

As we are considering *truncated prediction*,

$$\tilde{X}_t^n = X_t, \quad t \in \{1, \ldots, n\}, \quad \text{and} \quad \tilde{X}_t^n = 0,\ t \le 0.$$

Besides, $\tilde{Z}_t^n = 0$ for $t \le 0$ and $t > n$. Therefore, the one-step ahead truncated prediction is

$$\tilde{X}_{n+1}^n = \phi \tilde{X}_n^n + \tilde{Z}_{n+1}^n + \theta \tilde{Z}_n^n = \phi X_n + \theta \tilde{Z}_n^n \qquad \text{(Future noise is set to zero)}$$

Similarly,

$$\begin{aligned}
\tilde{X}_{n+2}^n &= \phi \tilde{X}_{n+1}^n + \tilde{Z}_{n+2}^n + \theta \tilde{Z}_{n+1}^n = \phi \tilde{X}_{n+1}^n \\
&\vdots \\
\tilde{X}_{n+m}^n &= \phi \tilde{X}_{n+m-1}^n + \tilde{Z}_{n+m}^n + \theta \tilde{Z}_{n+m-1}^n = \phi \tilde{X}_{n+m-1}^n, \qquad m > 2 \qquad \text{(Future noise is set to zero)}
\end{aligned}$$

To find the *truncated forecast errors*, rewrite the model as

$$Z_t = X_t - \phi X_{t-1} - \theta Z_{t-1}, \quad t \in \{1, 2, \ldots, n\}$$

Set $\tilde{Z}_0^n = 0$ and $X_0 = 0$, then

$$\begin{aligned}
\tilde{Z}_t^n &= \tilde{X}_t^n - \phi \tilde{X}_{t-1}^n - \theta \tilde{Z}_{t-1}^n \\
&= X_t - \phi X_{t-1} - \theta \tilde{Z}_{t-1}^n
\end{aligned}$$

Note: we know the value of $X_1, \ldots, X_n$ and, therefore, $\tilde{X}_t^n = X_t$ for $t = 1, \ldots, n$.

Exercises 5.3

$p = 1,\ q = 0$ **(AR(1))**

$$\tilde{X}^n_{n+1} = \phi X_n$$
$$\tilde{X}^n_{n+m} = \phi \tilde{X}^n_{n+m-1}, \quad m \geq 2$$
$$\tilde{Z}^n_t = \begin{cases} 0 & t \leq 0 \\ X_t - \phi X_{t-1} & t = 1, \ldots, n \\ 0 & t > n \end{cases}$$

$p = 0,\ q = 1$ **(MA(1))**

$$\tilde{X}^n_{n+1} = \theta \tilde{Z}^n_n$$
$$\tilde{X}^n_{n+m} = 0, \quad m \geq 2$$
$$\tilde{Z}^n_t = \begin{cases} 0 & t \leq 0 \\ X_t - \theta \tilde{Z}^n_{t-1} & t = 1, \ldots, n \\ 0 & t > n \end{cases}$$

Forecast mean square error for ARMA(1,1) Using (5.20), we know that

$$E\left(X_{n+m} - \tilde{X}^n_{n+m}\right)^2 = \sigma^2 \sum_{j=0}^{m-1} \psi_j^2 \tag{1}$$

where $\psi_j = (\phi + \theta)\phi^{j-1},\ j \geq 1,\ \psi_0 = 1$.
Note: we know that in ARMA(1,1), $(1 - \phi B)X_t = (1 + \theta B)Z_t$ and consequently $X_t = (1 + \theta B)/(1 - \phi B)Z_t = (1 + \theta B)(1 + \phi B + \phi^2 B^2 + \ldots)Z_t$ and we can extract the values of ψ_j using this term.

Let's continue with (1):

$$\begin{aligned} E\left(X_{n+m} - \tilde{X}^n_{n+m}\right)^2 &= \sigma^2 \sum_{j=0}^{m-1} \psi_j^2 \\ &= \sigma^2 \left[1 + (\phi + \theta)^2 \sum_{j=1}^{m-1} \phi^{2(j-1)}\right] \qquad \text{geometric series} \\ &= \sigma^2 \left[1 + (\phi + \theta)^2 \left(\frac{1 - \phi^{2(m-1)}}{1 - \phi^2}\right)\right] \quad \checkmark \end{aligned}$$

For AR(1):

$$E\left(X_{n+m} - \tilde{X}^n_{n+m}\right)^2 = \begin{cases} \sigma^2 \left[1 + \phi^2 \left(\frac{1 - \phi^{2(m-1)}}{1 - \phi^2}\right)\right], & m \geq 2 \\ \sigma^2, & m = 1 \end{cases}$$

For MA(1):

$$E\left(X_{n+m}-\tilde{X}^n_{n+m}\right)^2=\begin{cases}\sigma^2(1+\theta^2), & m\geq 2\\ \sigma^2, & m=1\end{cases}$$

For $\mathbf{AR(1)}$, as $m\to\infty$ and $|\phi|<1$, we can conclude that

$$\tilde{X}^n_{n+m}\longrightarrow 0 \quad \text{and} \quad E\left(X_{n+m}-\tilde{X}^n_{n+m}\right)^2=\frac{\sigma^2}{1-\phi^2}=\gamma_X(0)$$

This means that if m is large, the forecasts are useless.

For $\mathbf{MA(1)}$, we have

$$\tilde{X}^n_{n+m}\longrightarrow 0 \quad \text{as} \quad m\to\infty$$

$$E\left(X_{n+m}-\tilde{X}^n_{n+m}\right)^2=\sigma^2(1+\theta^2)=\gamma_X(0)$$

Exercises 5.4

- **(a)** The best *linear predictor* for a weakly stationary AR(p) is

$$\hat{X}_{n+1|1,\dots,n}=\sum_{j=1}^{n}\phi_{n,j}X_{n+1-j}$$

where $\boldsymbol{\phi}_n=\boldsymbol{\Gamma}_n^{-1}\boldsymbol{\gamma}_n$ is completely determined by *autocovariances*.
If the process is *Gaussian*, then the best linear predictor is the best predictor in the mean square error sense.
The best predictor in the Gaussian case is the conditional expectation:

$$\begin{aligned}\tilde{X}_{n+1|1,\dots,n}&=E(X_{n+1}\mid X_n,\dots,X_1)\\&=E\left(\sum_{j=1}^{p}\phi_jX_{n+1-j}+Z_{n+1}\,\middle|\,X_n,\dots,X_1\right)\\&=\sum_{j=1}^{p}\phi_jXn+1-j\end{aligned}$$

Note that since we have a gaussian AR process, Z_r is independent from X_s for $r>s$ and consequently, $E(Z_{n+1}\mid X_n,\dots,X_1)=E(Z_{n+1})=0$.
Equating *coefficients* in $\tilde{X}_{n+1|1,\dots,n}$ and $\hat{X}_{n+1|1,\dots,n}$ results in

$$\phi_{n,j}=\begin{cases}\phi_j, & j=1,\dots,p\\ 0, & j>p\end{cases}$$

$\Rightarrow$ it is easy to see that $\phi_{h,h}=0$ if $h>p$.

- **(b)** Interpretation of partial autocorrelation as a partial correlation coefficient. We want to show that the forward and backward prediction errors have correlation equal to the partial autocorrelation ϕ_{nm}, thus:

$$\phi_{nm}=\text{cor}(X_1-\hat{X}_1|_{2,\dots,n},X_{n+1}-\hat{X}_{n+1}|_{m,\dots,2})$$

By the Durbin-Levinson recursion,

$$\begin{bmatrix} \phi_{n,1} \\ \phi_{n,2} \\ \vdots \\ \phi_{n,n-1} \end{bmatrix} = \begin{bmatrix} \phi_{n-1,1} \\ \phi_{n-1,2} \\ \vdots \\ \phi_{n-1,n-1} \end{bmatrix} - \phi_{nm} \begin{bmatrix} \phi_{n-1,n-1} \\ \phi_{n-1,n-2} \\ \vdots \\ \phi_{n-1,1} \end{bmatrix}$$

Multiply both sides by the row vector $[X_n, \ldots, X_2]$ to get

$$\begin{aligned} \sum_{j=1}^{n-1} \phi_{n,j} X_{n+1-j} &= \sum_{j=1}^{n-1} \phi_{n-1,j} X_{n+1-j} - \phi_{nm} \sum_{j=1}^{n-1} \phi_{n-1,j} X_{j+1} \\ &= \hat{X}_{n+1|2,\ldots,n} - \phi_{nm} \hat{X}_1|_{2,\ldots,n} \end{aligned}$$

and add $\phi_{nn} X_1$ to both sides to get

$$\sum_{j=1}^{n} \phi_{n,j} X_{n+1-j} = \hat{X}_{n+1|1,\ldots,n} = \hat{X}_{n+1|2,\ldots,n} + \phi_{nm}(X_1 - \hat{X}_1|_{2,\ldots,n})$$

then subtract both sides from X_{n+1} to get

$$X_{n+1} - \hat{X}_{n+1|1,\ldots,n} = X_{n+1} - \hat{X}_{n+1|2,\ldots,n} - \phi_{nn}(X_1 - \hat{X}_1|_{2,\ldots,n})$$

Now $X_1 - \hat{X}_1|_{2,\ldots,n}$ is in the space spanned by $X_1, \ldots, X_n$, and hence it is orthogonal to $X_{n+1} - \hat{X}_{n+1|1,\ldots,n}$ (being orthogonal to that space). Multiply through both sides by $X_1 - \hat{X}_1|_{2,\ldots,n}$ and take expectations to get

$$\begin{aligned} 0 &= E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})(X_{n+1} - \hat{X}_{n+1|1,\ldots,n})\right] \\ &= E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})(X_{n+1} - \hat{X}_{n+1|2,\ldots,n})\right] - \phi_{nm} E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})^2\right] \end{aligned}$$

and hence, solving for ϕ_{nm},

$$\begin{aligned} \phi_{nm} &= \frac{E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})(X_{n+1} - \hat{X}_{n+1|2,\ldots,n})\right]}{E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})^2\right]} \\ &= \frac{E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})(X_{n+1} - \hat{X}_{n+1|2,\ldots,n})\right]}{\sqrt{E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})^2\right] E\left[(X_{n+1} - \hat{X}_{n+1|2,\ldots,n})^2\right]}} \end{aligned}$$

The last line following from the fact that

$$E\left[(X_1 - \hat{X}_1|_{2,\ldots,n})^2\right] = E\left[(X_{n+1} - \hat{X}_{n+1|2,\ldots,n})^2\right]$$

as a result of the Toeplitz covariance structure for a stationary process.

Hence, ϕ_{nm} is a correlation as required.