University of New South Wales
School of Mathematics and Statistics

MATH5905 Statistical Inference
Term One 2025

Assignment Two

Given: Friday 4 April 2025 **Due date: Sunday, 20 April 2025**

Instructions: This assignment is to be completed **collaboratively** by a group of **at most** 3 students. The same mark will be awarded to each student within the group, unless I have good reasons to believe that a group member did not contribute appropriately. This assignment must be submitted no later than 11:59 pm on Sunday, 20 April 2025. You do not need to wait to the very last moment to submit your assignment. The first page of the submitted PDF should be **this page**. Only one of the group members should submit the PDF file on Moodle, with the names and signatures of the other students in the group clearly indicated in the document.

I/We declare that this assessment item is my/our own work, except where acknowledged, and has not been submitted for academic credit elsewhere. I/We acknowledge that the assessor of this item may, for the purpose of assessing this item reproduce this assessment item and provide a copy to another member of the University; and/or communicate a copy of this assessment item to a plagiarism checking service (which may then retain a copy of the assessment item on its database for the purpose of future plagiarism checking). I/We certify that I/We have read and understood the University Rules in respect of Student Academic Misconduct.

Name	Student No.	Signature	Date

Problem 1

Suppose $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are i.i.d. random variables, each with a density

$$f(x;\theta) = \begin{cases} 2\frac{x}{\theta}e^{-\frac{x^2}{\theta}}, x > 0, \theta > 0 \\ 0 \text{ elsewhere} \end{cases}$$

a) Find the maximum likelihood estimator of θ.

b) Using the density transformation formula, show that $Y = \frac{2}{\theta}X_1^2$ has a χ_2^2 distribution with a density $f_Y(y) = \frac{1}{2}exp(-\frac{y}{2}), y > 0$.

c) Do the MLE and the UMVUE of θ coincide for this family? Give reasons.

d) Prove that the family has a monotone likelihood ratio in the statistic $T = \sum_{i=1}^n X_i^2$.

e) Derive the uniformly most powerful $\alpha-$ size test φ^* of $H_0 : \theta \geq 2$ versus $H_1 : \theta < 2$. Explain all your steps.

f) Find the power function and sketch a graph of $E_\theta \varphi^*$.

Density transformation formula: For a known function $y = w(x)$ with uniquely defined inverse $x = w^{-1}(y)$ we have $f_Y(y) = f_X(w^{-1}(y))|\frac{dw^{-1}(y)}{dy}|$.

Problem 2

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a sample of n observations from the geometric distribution with

$$f(x,p) = \begin{cases} p(1-p)^{x-1}, x = 1, 2, \ldots \\ 0 \text{ else} \end{cases}$$

where $0 < p < 1$ is an unknown parameter. Denote the joint density by $L(X,p)$.

a) Find the maximum likelihood estimator of p.

b) Find the mean of this distribution and the maximum likelihood estimator of the mean.

c) Show that the family $\{L(X,p)\}, 0 < p < 1$ has a monotone likelihood ratio in the statistic $T = -\sum_{i=1}^n X_i$.

d) Using c) (or otherwise), determine the structure of the uniformly most powerful α-size test φ^* of $H_0 : p \leq 0.3$ versus $H_1 : p > 0.3$. (You are not asked to determine the threshold constant).

e) Using asymptotic arguments (CLT or GLRT), suggest a large sample test of $\tilde{H}_0 : p = 0.3$ versus $\tilde{H}_1 : p \neq 0.3$. If for $n = 30$ you had $\sum_{i=1}^{30} X_i = 120$, what is your decision when testing $\tilde{H}_0$ versus $\tilde{H}_1$ with $\alpha = 0.05$?

Problem 3
Suppose that X is a random variable with density function

$$f(x,\theta) = \frac{1}{\beta} e^{-\frac{(x-\theta)}{\beta}}, \qquad \theta < x < \infty,$$

and zero else. Here $\beta > 0$ is a known constant and θ is an unknown location parameter
Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a sample of n i.i.d. observations from this distribution.

i) Compute the cumulative distribution function and the density for $T = X_{(1)}$.

ii) Show that the family has the MLR property in $T = X_{(1)}$.

iii) Justify the existence of a uniformly most powerful (UMP) α-size test of

$$H_0 : \theta \geq \theta_0 \qquad \text{versus} \qquad H_1 : \theta < \theta_0.$$

When $\beta = 1$, determine this test completely by calculating the threshold constant for $n = 4$, $\theta_0 = 2$ and $\alpha = 0.05$.

iv) Determine the power function of the UMP $\alpha-$size test and sketch the graph of this function.

v) Suppose the following data was collected $\mathbf{x} = (1.1, 2, 1.3, 3.1)$ and that $\beta = 2$. Test the hypothesis that $H_0 : \theta \geq 1$ versus $\theta < 1$ with a significance level $\alpha = 0.05$.

vi) Let $Z_n = n\big(X_{(1)} - \theta\big)$. Show that the distribution of Z_n does not depend on n and recognize this distribution.

vii) Hence or otherwise justify that $X_{(1)}$ is a consistent estimator of θ.

Problem 4
Suppose $X_{(1)} < X_{(2)} < X_{(3)}$ are the order statistics based on a random sample of size $n = 3$ from the standard exponential density $f(x) = e^{-x}, x > 0$.

1. Find the density of the midrange $B = \frac{1}{2}(X_{(1)} + X_{(3)})$.
2. Using 1) (or otherwise) show that $P(B > 1) = 0.4687$.
3. Show that $Cov(X_{(1)}, X_{(3)}) = 1/9$.