THE UNIVERSITY OF NEW SOUTH WALES

SCHOOL OF MATHEMATICS AND STATISTICS

December 2019

MATH5425
Graph Theory

(1) TIME ALLOWED – 2 hours

(2) TOTAL NUMBER OF QUESTIONS – 5

(3) ATTEMPT FOUR QUESTIONS.

(4) THE QUESTIONS ARE OF EQUAL VALUE

(5) THIS PAPER MAY BE RETAINED BY THE CANDIDATE

(6) **ONLY** CALCULATORS WITH AN AFFIXED "UNSW APPROVED" STICKER MAY BE USED

(7) STUDENTS MAY BRING IN ONE HANDWRITTEN DOUBLE-SIDED A4 PAGE OF NOTES. THIS MUST BE SUBMITTED WITH EXAM

All answers must be written in ink. Except where they are expressly required pencils may only be used for drawing, sketching or graphical work.

1. [10 marks]

Let $G = (V, E)$ be a connected graph.

a) Let $e \in E$ be an edge of G. Prove that there is a spanning tree in G which contains e.

b) Let T, T' be distinct spanning trees in G. Prove that there is a finite sequence

$$T = T_0,\ T_1,\ \ldots, T_r = T'$$

such that $T_0, T_1, \ldots, T_r$ are distinct spanning trees in G, and for $i = 0, 1, \ldots, r-1$, the tree T_{i+1} is obtained from T_i by deleting one edge and inserting another.

2. [10 marks]

Recall that $K_{a,b}$ is the complete bipartite graph with a vertices in one part and b vertices in the other.

a) Prove that every red/blue-colouring of the edges of $K_{3,7}$ contains a monochromatic copy of $K_{2,2}$.

b) Provide a red/blue-colouring of the edges of $K_{2,7}$ with <u>no</u> monochromatic copy of $K_{2,2}$. Explain why your colouring has the required property.

c) Provide a red/blue-colouring of the edges of $K_{3,6}$ with <u>no</u> monochromatic copy of $K_{2,2}$. Explain why your colouring has the required property.

3. [10 marks]

Consider the binomial random graph model $G(n,p)$. Let $X = X(n,p)$ be the number of unordered pairs of distinct vertices with no common neighbour. That is,

$$X = \big|\{\, \{i,j\} \,:\, 1 \le i < j \le n,\ \ N(i) \cap N(j) = \emptyset \,\}\big|$$

where $N(v)$ denotes the (random) neighbourhood of vertex v in $G \in G(n,p)$.

a) Calculate $\mathbb{E}X$, with careful explanation.

b) Prove that $\mathbb{E}X \le \exp\big(2\ln n - (n-2)p^2\big)$.

c) Define a function $p : \mathbb{Z}^+ \to (0,1)$ such that

- $p(n) \to 0$ as $n \to \infty$, and
- in $G(n, p(n))$ we have $\mathbb{E}X \to 0$ as $n \to \infty$.

Justify your choice of p.

d) Let $p : \mathbb{Z}^+ \to (0,1)$ be any function which satisfies the conditions required in part (c).

i) Prove that a.a.s. the diameter of $G \in G(n, p(n))$ is at most 2.

ii) Is it true that a.a.s., the diameter of $G \in G(n, p(n))$ is exactly 2? Explain your answer.

4. [10 marks]

Parts (a) and (b) of this question are unrelated.

a) Recall that $q(H)$ denotes the number of odd components in a graph H, that is, the number of components of odd order.

Let $G = (V, E)$ be a graph with n vertices. Suppose that M is a maximum matching of G.

Fix $S \subseteq V$ and let M' be the set of all edges in $M \cap E(G - S)$. That is, M' is the set of edges in M which are disjoint from S.

i) Prove that at most $n - |S| - q(G - S)$ vertices of G are matched by M'.

ii) Hence prove that $|M| \le \frac{1}{2}\big(n + |S| - q(G - S)\big)$.

b) Prove that if

$$\binom{n}{k}\, 3^{1-\binom{k}{2}} < 1$$

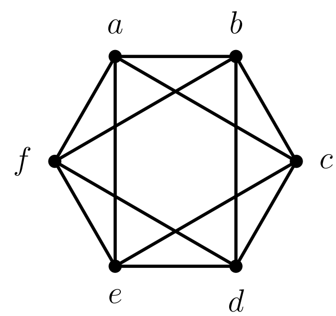
then there exists a colouring of the edges of K_n with 3 colours without a monochromatic copy of K_k. Justify each step carefully.

5. [10 marks]

In this question we will show that Tutte's Theorem about 3-connected graphs cannot be extended to 4-connected graphs.

a) Suppose that G is a planar graph and e is an edge of G. Let $G' = G/e$ be the graph obtained from G by contracting the edge e. Prove that G' is planar.

b) Consider the graph H shown below.



i) Using (the Global Version of) Menger's Theorem, or otherwise, prove that H is 4-connected.

ii) Prove that there is no sequence of 4-connected graphs $G_0, G_1, \ldots, G_r$ such that

- $G_0 = K_5$ and $G_r = H$,
- G_{i+1} has an edge e such that $G_i = G_{i+1}/e$, for $i = 0, \ldots, r-1$.