EXAM 2021

MATH5425

QUESTION 1 [10 marks]

(a) Let ~~U~~ $U = V - A$ where $A \subseteq V$ is an
[2] independent set. Suppose that edge
e of G is disjoint from U, $e = xy$. −1
[1] Then $\{x, y\} \subseteq A$ which contradicts the
assumption that A is an indep set.

(b) Let A be a maximum indep set,
[2] $|A| = \alpha(G)$. Then

$$n = |A| + |V - A| = \alpha(G) + |V - A| \quad -1$$

$$\geq \alpha(G) + \tau(G) \quad -1$$

since $V - G$ is a cover but might not be
minimal.

(c) Assume G is triangle-free.

[2] (i) If v has degree $\Delta(G)$ then $N(v)$ is
an indep set or else G has a
1 − triangle involving v & two adjacent
neighbours of v. Hence ~~AE~~

$$|N(v)| = \Delta(G) \leq \alpha(G). \quad -1$$

(ii) Every edge is incident with at least
[3] one vertex of U, where U is a
minimal cover. So

$$|E| \leq \sum_{u \in U} d_G(u) \overset{-1}{\leq} \Delta(G) \cdot |U| = \Delta(G)\,\tau(G). \quad -1$$

$\leq \alpha(G)\,\tau(G)$ by (i)

(ii) Hence
$|E| \le \Delta(G)\tau(G) \le \alpha(G)\tau(G)$
by (i). -1

QUESTION 1

(c) (iii) Hence, using fact $ab \le \left(\frac{a+b}{2}\right)^2$,

[2]

$$|E| \le \cancel{\Delta(G)\tau(G) \text{ by (ii)}}$$

$$\cancel{\le \left(\frac{\Delta(G)+\tau(G)}{2}\right)^2 \text{ by "fact"}}$$

$$\le \alpha(G)\tau(G) \text{ by (ii)}$$

$$\le \left(\frac{\alpha(G)+\tau(G)}{2}\right)^2 \text{ by "fact"} \quad -1$$

$$= \left(\frac{n}{2}\right)^2 = \frac{n^2}{4} \text{ by (b).} \quad -1$$

$\square$.

EXAM 2021

MATH 5425

QUESTION 2 (7) [10 marks overall]

(a) (i) If G was disconnected ↙it is not then at least [2] one face of G has a disconnected boundary. But every face of G is bounded by a cycle, which is connected. Hence G is connected. −1 −1

(ii) If $e \in E(G)$ then e lies on bdry of [2] at least one face, by Lemma 4.2.2. −1 Every face is bounded by a cycle, so e lies on a cycle. −1 Hence, by Lemma 4.2.2 again, e ~~belongs~~ lies on the boundary of exactly 2 faces.

(iii) As G is connected, Euler's formula [3] applies. Write 1− $m = |E(G)|$, $l = |F(G)|$ and note that by assumption and (ii),

$$2m = \frac{3l}{7}\cdot 4 + \frac{4l}{7}\cdot 5$$

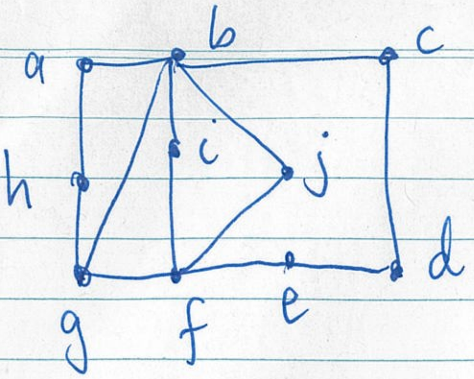
(double-counting the number of incident face-edge pairs).

So $$2m = \frac{12+20}{7} l = \frac{32}{7} l$$

and $l = m\frac{2\times 7}{32} = \frac{7}{16}m$. −1 Hence $n - m + \frac{7}{16}m = 2$ −1 by Euler's formula, so $\frac{9}{16}m = n-2$, $m = \frac{16}{9}(n-2)$.

QUESTION 2

(b) [3]

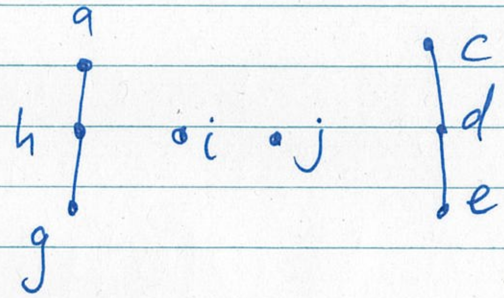


Show G has no perfect matching.

Enough to find ~~ALO~~ $S \subseteq V$ with

$q(G-S) \cancel{\geq} |S|$, so Tutte's condition fails.

eg $S = \{b, f\}$, $G-S$ is



1

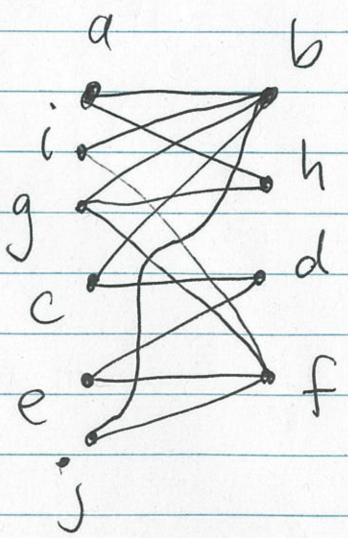
Then $q(G-S) = 4 > |S| = 2$ as all components of $G-S$ are odd.

1

$\square$

1 – state or use Tutte's condition correctly.

OR:



1 – choose S
1 – apply Hall's (or state)

b, f, h

let $S = \{i, j, g, a\}$

then $N(S) = \{b, f, h\}$

too small

Hall's Th^m fails.

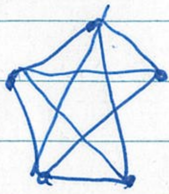
Also $S = \{a, c, e, i, g, j\}$
$N(S) = \{b, h, d, f\}$
ie $|LHS| \neq |RHS|$!!

BUT that only says no matching of LHS !!

1 – Also explain why this implies no perf. matching

MATHS425 Exam 2021

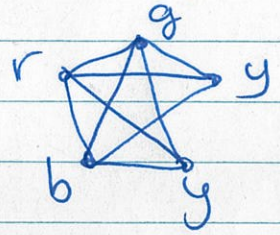
QUESTION 3. [10 marks] (6+4?)

(a) H:  P_2: 

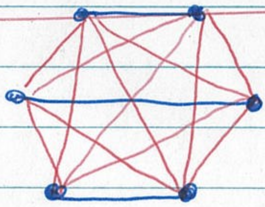
[2]

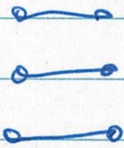
(i) $R(H, P_2) \geq 7$. Bollobás?

$$(\chi(H)-1)(|P_2|-1)+u(H) = 3\times 2+1 = 7$$ −1



$\chi(H)=4$, $u(H)=1$. −1

or: in K_6, 

blue matching  no blue P_2, −1

rest red: −1 any 5 vertices contain 2 blue edges & at most

But H has 9 edges.

$\binom{5}{2}$ ~~6~~ $-2 =$ ~~13~~ 8 red edges.

So, no red H.

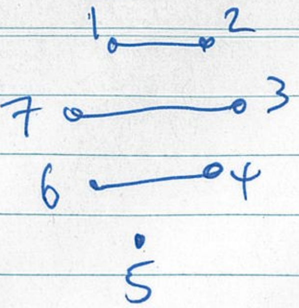
(ii) $R(H, P_2) \geq 7$. If 7 vertices, colour edges of K_7 blue or red, try to avoid a red H or blue P_2. Then blue edges ~~?~~ form a matching. If 3 blue edges (max possible)

[4] −1 idea

−1

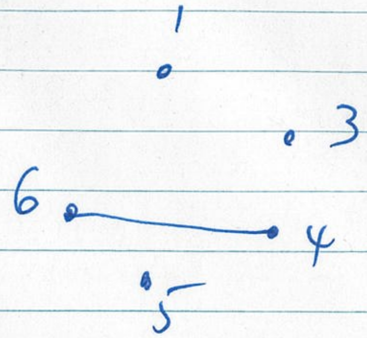
QUESTION 3 (a), continued

.. then



Take vertices 1, 3, 4, 5, 6, say:

−1



This includes exactly one blue edge & so all the rest are red, giving a red copy of H.

−1

Note, if less than 3 blue edges then we still get a red copy of H (even more easily).

[4]?

(b) $H_1 \subseteq H_2$. True or false? $R(H_1, H_1) \subseteq R(H_2, H_2)$

TRUE. −1 Proof let $n = R(H_2, H_2)$.

Then any red-blue colouring of the edges of K_n must contain a monochromatic H_2, by def^n of graph Ramsey numbers. −1

But $H_1 \subseteq H_2$, so we also have a monochromatic copy of H_1, always. −1

Therefore $R(H_1, H_1) \leq n = R(H_2, H_2)$. □

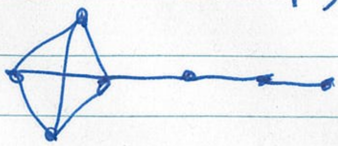
−1

MATH5425 Exam 2021

QUESTION 4 [10 marks]

In $G(n,p)$,

$X = \#K_4$'s, $Y = \#$ copies of H



(a) $\mathbb{E}X = \frac{(n)_4}{4!}p^6 = \frac{(n)_4}{24}p^6$ using result — 1

[2]

of Prob Sheet 8, as K_4 has 4 vertices, $\binom{4}{2} = 6$ edges

& $4! = 24$ automorphisms. — 1

~~quote Q5 of Prob Sheet~~

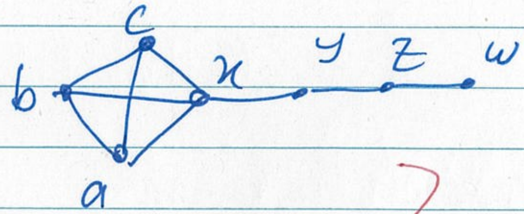
(b) $\mathbb{E}Y = \frac{1}{6}(n)_7 p^9$.

[2]

1

Well H has 7 vertices, 9 edges and

$3! = 6$ automorphisms,



as a, b, c can be permuted in all ways and no other automorphisms are present. 1

Hence by Prob Sheet 8 Q5 again,

$$\mathbb{E}Y = \frac{1}{|Aut(H)|}(n)_{|H|}\, p^{|E(H)|} = \frac{1}{6}(n)_7 p^9$$ as stated.

(c) Put $p = n^{-\alpha}$, $\alpha > 0$. Want

[2].

$\mathbb{E}X \approx \frac{1}{24} n^{4-6\alpha} \to 0$ so $4 - 6\alpha < 0$, $\frac{2}{3} < \alpha$.

Want

$\mathbb{E}Y = \frac{1}{6} n^{7-9\alpha} \to \infty$ so $7 - 9\alpha > 0$,

$\alpha < 7/9$.

eg $\frac{6}{9} = \frac{2}{3} < \alpha < 7/9$ ($\alpha = 3/4$ also works!)

so $\alpha = \frac{13}{18}$ would do. anything that works.

Note $p(n) = n^{-13/18}$ -1 $\in (0,1)$ for $n \geq 2$.

Take $p(1) = 1/2$, say. 1 - reason.

Other conditions hold by choice of α.

(d)(i) By Markov's lemma, for $p(n) = n^{-13/18}$,

[2]

$0 \leq P(X > 0) \leq \mathbb{E}X \to 0$ so -1

$P(X = 0) = 1 - P(X > 0) \to 1$.

Hence $X = 0$ aas. -1

(ii) No this is false. -1 If $Y > 0$ then $X > 0$

[2]

as H has a subgraph which is a copy of K_4. So $0 \leq \Pr(Y > 0) \leq \Pr(X > 0) \to 0$

& hence aas $Y = 0$ in fact. -1 reason