9.3 Solutions to Chapter 3

Exercises 3.1

Let's consider $X_t = Y_t - Y_{t-1}$, where Y_t is martingale. We want to show that

(i) $E(X_t) = 0$,

(ii) $E(X_t|X_{t-1}, \cdots, X_1) = 0$,

(iii) $cov(X_t, X_s) = 0, \quad t \neq s$.

Proof of (i)

$$\begin{aligned} E(X_t) &= E\left(E\left(X_t|X_{t-1}, \cdots, X_1\right)\right) \\ &= E\left(E\left(Y_t - Y_{t-1}|Y_{t-1}, \cdots, Y_1\right)\right) \\ &= E\left(E\left(Y_t|Y_{t-1}, \cdots, Y_1\right) - E\left(Y_{t-1}|Y_{t-1}, \cdots, Y_1\right)\right), \qquad (*) \\ &= E\left(Y_{t-1} - Y_{t-1}\right) = 0 \end{aligned}$$

Note that in (*), we use the fact that Y_t is martingale and consequently $E(Y_t||Y_{t-1}, \cdots, Y_1) = Y_{t-1}$, and, by the properties of conditional probability $E\left(Y_{t-1}|Y_{t-1}, \cdots, Y_1\right) = Y_{t-1}$.
Proof of (ii)

$$\begin{aligned} E\left(X_t|X_{t-1}, \cdots, X_1\right) &= E\left(Y_t - Y_{t-1}|Y_{t-1}, \cdots, Y_1\right) \\ &= E\left(Y_t|Y_{t-1}, \cdots, Y_1\right) - E\left(Y_{t-1}|Y_{t-1}, \cdots, Y_1\right) \\ &= Y_{t-1} - Y_{t-1} = 0 \end{aligned}$$

Proof of (iii)
Let $s > t$.

$$\begin{aligned} cov(X_t, X_s) &= E(X_t X_s) - E(X_t)E(X_s) \\ &= E(X_t X_s) \\ &= E\left(E\left(X_t X_s|X_{s-1}, \cdots, X_t, X_{t-1}, \cdots, X_1\right)\right) \\ &= E\left(X_t\ E\left(X_s|X_{s-1}, \cdots, X_t, X_{t-1}, \cdots, X_1\right)\right) \\ &= E\left(X_t \times 0\right) = 0 \end{aligned}$$

Exercises 3.2

To show that M_t is martingale, we need to show that $E(|M_t|) < \infty$, and $E(M_t|M_{t-1}, \cdots, M_1) = M_{t-1}$.

To show $E(|M_t|) < \infty$, note that

$$\begin{aligned}
E(|M_t|) &= E(|\prod_{j=1}^{t} Z_j|) \\
&= \prod_{j=1}^{t} E(|Z_j|), \qquad \text{by independency of } Z_t \\
&\leq \prod_{j=1}^{t} \sqrt{E(|Z_j|^2)} \qquad (*) \\
&= \prod_{j=1}^{t} \sqrt{\sigma^2 + 1} \qquad (**) \\
&= \left(\sigma^2 + 1\right)^{t/2} < \infty.
\end{aligned}$$

In (*) and (**), we use the fact that $\left(E(|Z_j|)\right)^2 \leq E\left(|Z_t|^2\right) = E(Z_t^2) = var(Z_t) + (E(Z_t))^2 = \sigma^2 + 1$. Besides, we need to show that $E(M_t|M_{t-1}, \cdots, M_1)$:

$$\begin{aligned}
E(M_t|M_{t-1}, \cdots, M_1) &= E(\prod_{j=1}^{t} Z_j|Z_{t-1}, \cdots, Z_1) \\
&= E(Z_t \prod_{j=1}^{t-1} Z_j|Z_{t-1}, \cdots, Z_1) \\
&= \prod_{j=1}^{t-1} Z_j \times E(Z_t|Z_{t-1}, \cdots, Z_1) \\
&= \prod_{j=1}^{t-1} Z_j \times E(Z_t) \\
&= \prod_{j=1}^{t-1} Z_j \\
&= M_{t-1}.
\end{aligned}$$

If $E(Z_t) = \mu \neq 1$, we need to redefine M_t as

$$M_t^* = \left(\frac{1}{\mu}\right)^t \prod_{j=1}^{t} Z_j.$$

It can be shown that M_t^* is martingale.

Exercises 3.3

Set $X_t = \theta Z_{t-1} + Z_t$, where $Z_t \sim WN(0, \sigma^2)$. For the autocovariance we have:

$$\begin{aligned}
\gamma_X(h) &= cov(X_{t+h}, X_t) \\
&= cov(\theta Z_{t+h-1} + Z_{t+h}, \theta Z_{t-1} + Z_t) \\
&= \theta^2 cov(Z_{t+h-1}, Z_{t-1}) + \theta\ cov(Z_{t+h-1}, Z_t) + \theta\ cov(Z_{t+h}, Z_{t-1}) + cov(Z_{t+h}, Z_t) \\
&= \theta^2\ \gamma_Z(h) + \theta\ \gamma_Z(h-1) + \theta\ \gamma_Z(h+1) + \gamma_Z(h)
\end{aligned}$$

We know that

$$\gamma_Z(h) = \begin{cases} \sigma^2 & h = 0 \\ 0 & h \neq 0 \end{cases} .$$

Therefore,

$$\begin{aligned}
\gamma_X(h) &= \theta^2\ \gamma_Z(h) + \theta\ \gamma_Z(h-1) + \theta\ \gamma_Z(h+1) + \gamma_Z(h) \\
&= \begin{cases} (1+\theta^2)\gamma_Z(0) & h = 0 \\ \theta\ \gamma_Z(0) & h = 1 \\ \theta\ \gamma_Z(0) & h = -1 \\ 0 & |h| > 1 \end{cases} \\
&= \begin{cases} (1+\theta^2)\gamma_Z(0) & h = 0 \\ \theta\ \gamma_Z(0) & |h| = 1 \\ 0 & |h| > 1 \end{cases} .
\end{aligned}$$

Finally we can conclude that

$$\begin{aligned}
\rho_X(h) &= \frac{\gamma_X(h)}{\gamma_X(0)} \\
&= \begin{cases} 1 & h = 0 \\ \frac{\theta}{(1+\theta^2)} & |h| = 1 \\ 0 & |h| > 1 \end{cases} .
\end{aligned}$$

To show that $\max_\theta(\rho_X(1)) = 0.5$, we need to take derivative from $\rho_X(1) = \frac{\theta}{(1+\theta^2)}$ with respect to θ :

$$\begin{aligned}
\frac{\partial \rho_X(1)}{\partial \theta} &= \frac{(1+\theta^2-2\theta^2)}{(1+\theta^2)^2} \\
&= \frac{(1-\theta^2)}{(1+\theta^2)^2}
\end{aligned}$$

By setting $\frac{\partial \rho_X(1)}{\partial \theta}$ to zero, we have

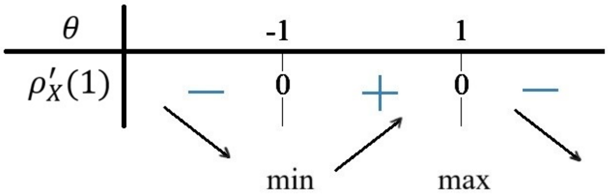
$$\begin{aligned}
\rho'_X(1) &= \frac{\partial \rho_X(1)}{\partial \theta} \\
&= \frac{(1-\theta^2)}{(1+\theta^2)^2} = 0
\end{aligned}$$

This equality holds if the numerator to be zero:

$$(1-\theta^2) = 0\ \Rightarrow \theta^2 = 1\ \Rightarrow \theta = \pm 1$$

It is easy to show that

$$\rho_X(1) = \begin{cases} 0.5 & \theta = 1 \\ -0.5 & \theta = -1 \end{cases} \quad \Rightarrow \quad \max_\theta(\rho_X(1)) = 0.5.$$



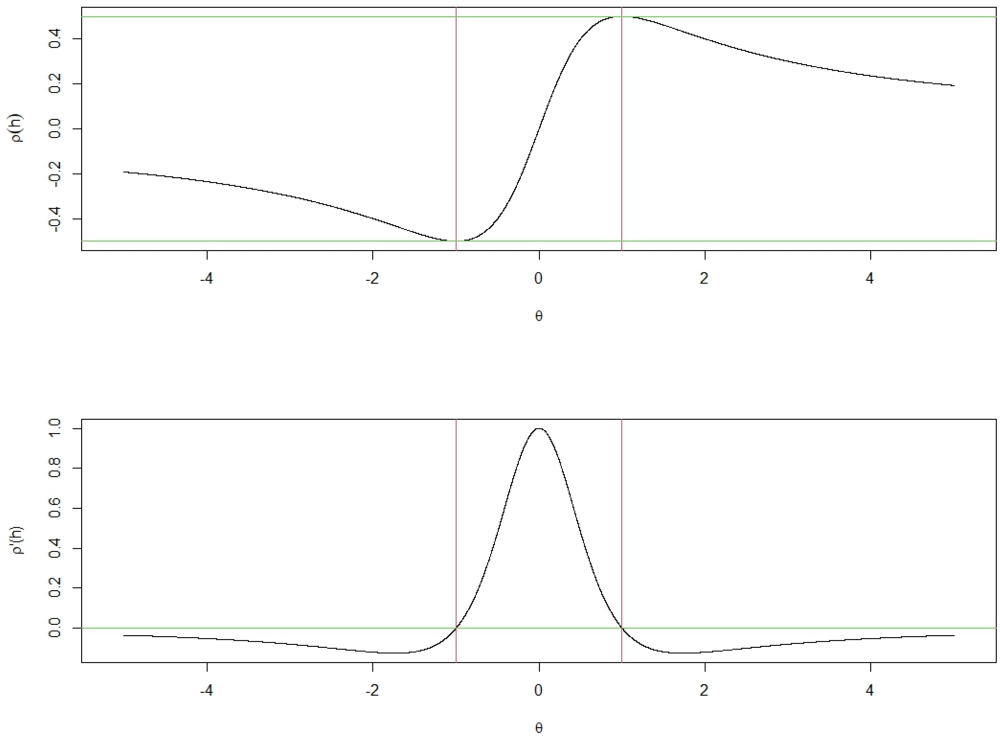


Figure 9.1: The plot of $\rho(h)$ and $\rho'(h)$ as a function of θ.

Exercises 3.4

Let $\{Z_t\} \sim$WN$(0, \sigma^2)$ and define, for $|\phi| < 1$,

$$X_t = \phi X_{t-1} + Z_t, \quad t = 0, \pm 1, \pm 2, \dots,$$

where Z_t is uncorrelated with X_s for each $s < t$.

$$\begin{aligned}
X_t &= \phi X_{t-1} + Z_t \\
&= \phi(\phi X_{t-2} + Z_{t-1}) + Z_t \\
&= \phi^2 X_{t-2} + \phi Z_{t-1} + Z_t \\
&= \phi^2(\phi X_{t-3} + Z_{t-2}) + \phi Z_{t-1} + Z_t \\
&= \phi^3 X_{t-3} + \phi^2 Z_{t-2} + \phi Z_{t-1} + Z_t \\
&\vdots \\
&= \phi^k X_{t-k} + \phi^{k-1} Z_{t-(k-1)} + \cdots + \phi^2 Z_{t-2} + \phi Z_{t-1} + Z_t \\
&= \phi^k X_{t-k} + \sum_{j=0}^{k-1} \phi^j Z_{t-j} \\
&\vdots \\
&= \sum_{j=0}^{\infty} \phi^j Z_{t-j} \qquad \text{as } n \to \infty
\end{aligned}$$

Note that $\phi^k \to 0$ as $n \to \infty$. Besides,

$$\begin{aligned}
\gamma_X(h) &= cov(X_{t+h}, X_t) \\
&= cov\left(\sum_{j=0}^{\infty} \phi^j Z_{t+h-j}, \sum_{k=0}^{\infty} \phi^k Z_{t-k}\right) \\
&= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi^j \phi^k cov\left(Z_{t+h-j}, Z_{t-k}\right)
\end{aligned}$$

We know that

$$cov\left(Z_{t+h-j}, Z_{t-k}\right) = \begin{cases} \sigma^2 & \text{if } \; t+h-j = t-k \quad \equiv \quad j = h+k \\ 0 & \text{if } \; t+h-j \neq t-k \quad \equiv \quad j \neq h+k \end{cases}$$

Therefore,

$$\begin{aligned}
\gamma_X(h) &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \phi^j \phi^k cov\left(Z_{t+h-j}, Z_{t-k}\right) \\
&= \sum_{k=0}^{\infty} \phi^{h+k} \phi^k cov\left(Z_{t+h-(h+k)}, Z_{t-k}\right) \\
&= \sum_{k=0}^{\infty} \phi^h \phi^{2k} cov\left(Z_{t-k)}, Z_{t-k}\right) \\
&= \sum_{k=0}^{\infty} \phi^h \phi^{2k} \gamma_Z(0) \\
&= \phi^h \sigma^2 \sum_{k=0}^{\infty} \phi^{2k}
\end{aligned}$$

Using the rules for the sum in geometric series, we have

$$\begin{aligned}\gamma_X(h) &= \phi^h\sigma^2\frac{1}{1-\phi^2}\\ &= \frac{\phi^h\sigma^2}{1-\phi^2}\end{aligned}$$

Exercises 3.5

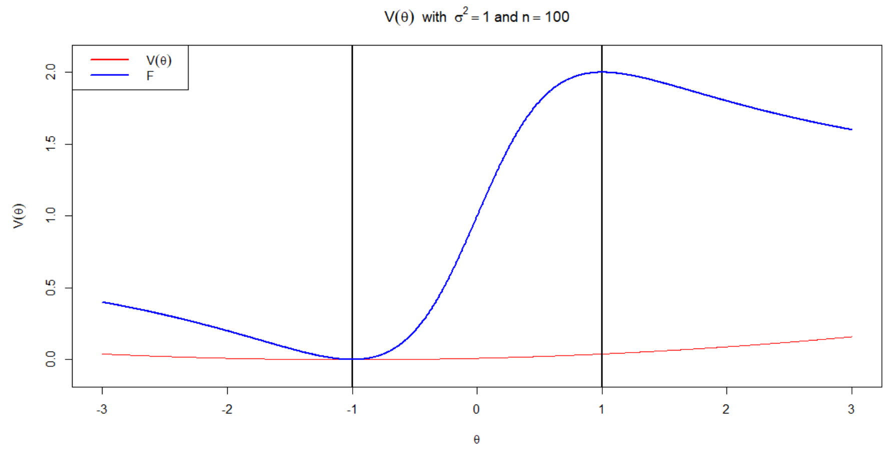


Figure 9.2: Asymptotic variance $V(\theta)$ as a function of θ.

Let

$$\begin{aligned}V(\theta) &= \frac{\gamma(0)}{n}\frac{(1+\theta)^2}{1+\theta^2}\\ &= \frac{\gamma(0)}{n}\,F.\end{aligned}$$

- For θ close to 1, $V(\theta) = \frac{2\gamma(0)}{n}$.
- For θ close to -1, $V(\theta) \to 0$.

Note that $\gamma(0)$ is estimated directly from the sample using

$$\gamma(0) = \frac{1}{n}\sum_{t=1}^{n-1}(X_t - \bar{X})^2$$

Exercises 3.6

Let

$$V(\phi) = \frac{\sigma^2}{(1-\phi)^2}.$$

- For ϕ close to 1, $V(\phi) \to 0$.
- For ϕ close to -1, $V(\phi) \to \frac{\sigma^2}{4}$.

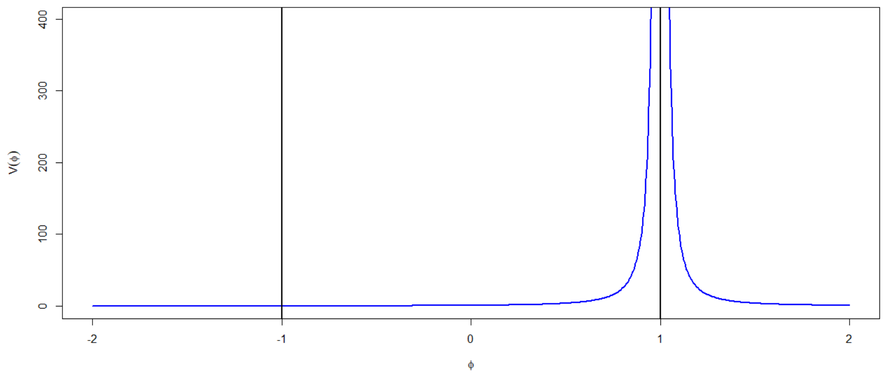


Figure 9.3: Asymptotic variance $V(\phi)$ as a function of ϕ.