SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney
MATH5425 Graph Theory   Term 1, 2025
Problem Sheet 3, Probabilistic Method

1. Consider the uniform model of random graphs on the vertex set $\{1, \dots, n\}$. Recall that $\Pr(ij \text{ is an edge}) = \frac{1}{2}$, *independently* for each $1 \le i < j \le n$.

   (a) For $1 \le i < j \le n$ let X_{ij} be the indicator variable for the event that $\{i, j\}$ is an edge in a uniformly random graph. Calculate $\mathbb{E}X_{ij}$.

   (b) What is counted by $\sum_{1 \le i < j \le n} X_{ij}$?

   (c) Let X be the number of edges in a uniformly random graph on n vertices. Using linearity of expectation, prove that $\mathbb{E}X = \frac{1}{2}\binom{n}{2}$.

   (Compare this proof with the direct proof given in lectures: which do you prefer?)

2. Let Ω be the set of all functions $g : S \to \{1, \dots, k\}$ for some finite set S and positive integer k. Form the random function $f \in \Omega$ using the following procedure: choose $f(a) \in \{1, \dots, k\}$ uniformly at random, *independently* for each $a \in S$.

   (Note: choosing an element *uniformly at random* from a set B means "according to the uniform probability distribution on B", with each element of B equally likely to be chosen.)

   (a) Calculate $|\Omega|$.

   (b) Let $f_0 \in \Omega$ be any fixed function in Ω. Prove that $\Pr(f = f_0) = 1/|\Omega|$.

3. Let Ω be the set of all subsets of a given finite set S, and $p \in \mathbb{R}$ with $0 \le p \le 1$. Define the map $\pi : \Omega \to [0, 1]$ by $\pi(X) = p^{|X|}(1 - p)^{|S|-|X|}$ for all $X \subseteq S$.

   (a) Show that π is a probability distribution (i.e., $\sum_{X \subseteq S} \pi(X) = 1$).

   (b) Now consider a random subset Y of S constructed by the following procedure: put $a \in Y$ with probability p, *independently* for each element $a \in S$.

   Prove that for any subset Y_0 of S we have $\Pr(Y = Y_0) = \pi(Y_0)$. (That is, the probability distribution on Ω which results from this procedure is exactly given by π.)

   (In (b), you can imagine flipping a biased coin which has $\Pr(\text{heads}) = p$, independently for each $a \in S$, and place $a \in Y$ if the coin comes up heads.)

(... Please turn over for Question 4)

4. A *dominating set* in a graph is a set of vertices $U \subseteq V$ such that every vertex which is not in U has a neighbour in U. Suppose that G has n vertices and minimum degree $\delta \geq 1$. We will prove that G has a dominating set of at most

$$n(1 + \ln(\delta + 1))/(\delta + 1)$$

vertices, using the probabilistic method.

(a) For a given $p \in [0, 1]$, choose a random subset X of V using the procedure of Question 3(b). Let $Y = Y_X$ be the random set of all vertices in $V - X$ that do not have a neighbour in X. Calculate $\mathbb{E}|X|$ and prove that $\mathbb{E}|Y| \leq n(1 - p)^{\delta+1}$.

(b) Hence show that G has a dominating set of size at most

$$np + n(1 - p)^{\delta+1}.$$

(c) Use the bound $1 - p \leq e^{-p}$ (which is a very good approximation when p is small) to (approximately) minimise the above expression, completing the proof.