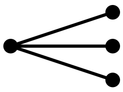
SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney

MATH5425 Graph Theory Term 1, 2023

Exam solutions and mark scheme

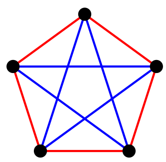
Question 1 (7 marks)
Recall that for fixed graphs H_1 and H_2, we define $R(H_1, H_2)$ to be the smallest positive integer n such that every red-blue colouring of the edges of K_n contains a red copy of H_1 or a blue copy of H_2.

The "claw" graph $K_{1,3}$ is shown below.

$K_{1,3}$: 

(a) Prove that $R(K_{1,3}, K_{1,3}) > 5$.

Solution: The following red-blue colouring of the edges of K_5 has no monochromatic $K_{1,3}$, since the set of red edges forms a 2-regular graph and so does the set of blue edges. (Both are Hamilton cycles.)



(b) Prove that $R(K_{1,3}, K_{1,3}) = 6$.

Solution: Consider any red-blue colouring of the edges of K_6 and let x be any vertex. Then x has 5 incident edges and by the pigeonhole principle, there are either 3 red edges or 3 blue edges incident with x. In either case this gives a monochromatic $K_{1,3}$.

(c) Decide whether the following statement is <u>true</u> or <u>false</u>:

$$R(K_3, K_{1,3}) = 6.$$

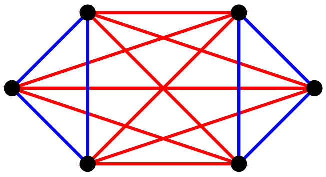
State and prove your answer.

Solution: The statement is false. We can use the Bollobás bound:

$$R(K_3, K_{1,3}) \geq (\chi(K_3) - 1)(|K_{1,3}| - 1) + 1 = 2 \times 3 + 1 = 7.$$

Hence $R(K_3, K_{1,3}) \neq 6$.

Alternative solution: The statement is false, as can be seen from the following red-blue colouring of the edges of K_6.



The red edges form $K_{3,3}$, which is bipartite and hence has no triangle. The blue edges form a 2-regular graph and hence contain no $K_{1,3}$.

Q1 Marking scheme:

(a) [**2 marks**]: 1 mark for a correct colouring, 1 mark for explanation. (Can use the Bollobás bound but it only proves $R(K_{1,3}, K_{1,3}) \geq 4$, so you still need a colouring of the edges of K_5.)

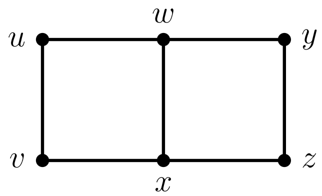
(b) [**2 marks**]: 1 mark for noting that an arbitrary vertex of K_5 has degree 5, 1 mark for explaining why this proves the result e.g. pigeonhole.

(c) [**3 marks**]: 1 mark for "FALSE" or for stating $R(K_3, K_{1,3}) \neq 6$ (or a strict inequality). Then EITHER 1 mark for correct use of the Bollobás bound and 1 mark for correct data for said bound, OR 1 mark for a correct colouring and 1 mark for explanation.

Question 2 (10 marks)
Consider the binomial random graph model $G(n,p)$. Recall the graph $K_{1,3}$ from Question 1.

Let $X = X(n,p)$ be the number of copies of $K_{1,3}$ in $G \in G(n,p)$, and let $Y = Y(n,p)$ be the number of copies of L in $G \in G(n,p)$, where L is the "ladder graph" shown below:

L:



(a) Write down the set of automorphisms of L.

Solution: There are four automorphisms of L, which we can write as

$$(\,), \qquad (u\,v)(w\,x)(y\,z), \qquad (u\,y)(v\,z), \qquad (u\,z)(w\,x)(v\,y).$$

That is: we can do nothing, we can reflect along the horizontal axis, we can reflect through the vertical axis, or we can perform both reflections (one after the other in any order).

(b) Calculate $\mathbb{E}X$ and $\mathbb{E}Y$, with explanation.

Solution: First note that $K_{1,3}$ has 4 vertices, 3 edges and has 6 automorphisms, as the leaves can be permuted arbitrarily. Hence by the result from the tutorial,

$$\mathbb{E}X = \frac{(n)_4}{6}\, p^3.$$

(Or, first choose the vertex of degree 3 in n ways, then choose a set of 3 leaves in $\binom{n-1}{3}$ ways, then require those 3 edges to be present, using independence.) Similarly, L has 6 vertices, 7 edges and 4 automorphisms, by (a). So

$$\mathbb{E}Y = \frac{(n)_6}{4}\, p^7.$$

(c) Define a function $p : \mathbb{Z}^+ \to \mathbb{R}$ such that $p(n) \in (0, 1)$ for sufficiently large n, and when $p = p(n)$,

- $\mathbb{E}X \to \infty$,
- a.a.s. $Y = 0$.

Justify your answer.

Solution: We want $n^4p^3 \to \infty$ and $n^6p^7 \to 0$. If we let $p = n^{-\alpha}$ for some $\alpha > 0$, then we want

$$4 - 3\alpha > 0 \qquad \text{and} \qquad 6 - 7\alpha < 0.$$

Rearranging gives $6/7 < \alpha < 4/3$, so for example we may choose $\alpha = 1$. That is, we define $p = p(n) = 1/n$ and we see that

$$\mathbb{E}X = \frac{(n)_4}{6\,n^3} \sim \frac{n}{6} \to \infty, \qquad \mathbb{E}Y = \frac{(n)_6}{4\,n^7} \sim \frac{1}{4n} \to 0.$$

Finally, by Markov's inequality we have

$$\Pr(Y \geq 1) \leq \mathbb{E}Y \to 0,$$

so $\Pr(Y = 0)$ tends to 1 as $n \to \infty$. In other words, a.a.s. $Y = 0$.

Q2 Marking scheme:

(a) **[2 marks]**: 1 mark for at least 2 correct automorphisms. The 2nd mark for 2 more correct automorphisms. Any clear explanation of the automorphisms is fine, e.g. cycle decomposition or description words.

(b) **[4 marks]**: 1 mark for correct data for $K_{1,3}$ (number of vertices, edges, automorphisms) and 1 mark for applying the formula. Same for L. Allow students to use their result from (a) if it is incorrect.

(c) **[4 marks]**: 1 for a clear definition of p, 1 mark for proving $\mathbb{E}X \to \infty$ (or calculations giving correct condition for this), 1 mark for proving $\mathbb{E}Y \to 0$ (or calculations giving correct condition for this), 1 mark for explaining why $\Pr(Y = 0) \to 1$ by e.g. Markov's inequality.

Question 3 (14 marks) For any graph G and each nonnegative integer k, let $P_G(k)$ denote the number of k-colourings of G. Recall the following facts about P_G, which we proved in Problem Sheet 4, Question 3:

- If G has n vertices then P_G is a monic polynomial of degree n in k, called the *chromatic polynomial* of G.
- For any edge e of G,
$$P_G(k) = P_{G'}(k) - P_{G''}(k) \tag{1}$$
where $G' = G - e$ (edge deletion) and $G'' = G/e$ (edge contraction).

(You do NOT need to prove the above two facts.) We can write

$$P_G(k) = k^n + a_{n-1}k^{n-1} + \cdots + a_1 k + a_0,$$

where $n = |G|$ and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$. Here a_0 is the constant coefficient of $P_G(k)$ and a_1 is the linear coefficient of $P_G(k)$.

(a) Prove that if T is a tree with $n \geq 1$ vertices then $P_T(k) = k(k-1)^{n-1}$.

Solution: We know that the vertices of T can be labelled as $v_1, \ldots, v_n$ so that v_j has exactly one neighbour in $\{v_1, \ldots, v_{j-1}$. Then there are k choices for the colour of v_1, and given a k-colouring of $G[\{v_1, \ldots, v_{j-1}\}]$ there are exactly $k-1$ choices for the colour of v_j, for $j = 2, \ldots, n$.

Alternative solution: We proceed by induction on n. The result is true when $n = 1$ as there are k ways to colour an isolated vertex. Now assume that the result is true for all graphs with at most n vertices, where $n \geq 1$. Let T be a tree with $n+1$ vertices and let v be a leaf in T. Then $T' = T - v$ is a tree with n vertices, so there are $k(k-1)^{n-1}$ distinct k-colourings of T', by assumption. Each such colouring can be extended to a k-colouring of T by choosing a colour for v, in $k-1$ ways (avoiding the colour of the unique neighbour of v). Hence

$$P_T(k) = (k-1)\, P_{T'}(k) = (k-1) \times k(k-1)^{n-1} = k(k-1)^n,$$

which proves the result for T as required.

(b) Let C_4 be the cycle of length 4. Prove that the chromatic polynomial of C_4 is given by

$$P_{C_4}(k) = k(k-1)(k^2 - 3k + 3).$$

Solution: We use (1). Let e be an edge of C_4. Then $C_4 - e$ is a path of length 3, and $P_{C_4-e}(k) = k(k-1)^3$ by part (a). Next, C_4/e is a complete graph on 3 vertices, and so $P_{C_4/e} = k(k-1)(k-2)$. Therefore

$$P_{C_4}(k) = k(k-1)^3 - k(k-1)(k-2) = k(k-1)\big((k-1)^2 - (k-2)\big) = k(k-1)(k^2-3k+3).$$

(A variation of this argument involves applying (1) again to calculate $P_{C_4/e}(k) = P_{K_3}(k)$.)

Alternative solution: Let the vertices of C_4 be a, b, c, d (in order as we walk around the cycle). There are k ways to colour a, then $k-1$ ways to colour b. If we colour c with the same colour as b then there are $k-1$ ways to colour d. Otherwise, there $k-2$ ways to colour c with a different colour from the colour b, and then $k-2$ ways to colour d. This gives

$$P_{C_4}(k) = k(k-1)\big(k - 1 + (k-2)^2\big) = k(k-1)(k^2 - 3k + 3).$$

(c) Suppose that G has r connected components $G_1, \ldots, G_r$. Briefly explain why

$$P_G(k) = \prod_{i=1}^{r} P_{G_i}(k).$$

Solution: We can colour each component independently, and there are $P_{G_i}(k)$ ways to k-colour the ith component, so there are $\prod_{i=1}^{r} P_{G_i}(k)$ ways to k-colour G.

(d) Using induction on the number of edges, or otherwise, prove that for any connected graph G,

$$a_0 = 0 \quad \text{and} \quad a_1 = (-1)^{|G|-1}\,\ell \quad \text{for some integer } \ell \geq 1. \tag{2}$$

(As defined above, a_0 and a_1 are the constant and linear coefficient of $P_G(k)$, respectively.)

Solution: Let m be the number of edges of G. If $m \leq n-1$ then $m = n-1$ and G is a tree, since G is connected. The result for trees holds by part (a), as the constant coefficient of $k(k-1)^{n-1}$ is zero and the linear coefficient is $(-1)^{n-1}$.

Now suppose that the result is true for any connected graph with at most m edges, where $m \leq n-1$, and suppose that G is a connected graph with $m+1$ edges. Since $m+1 > n-1$ we see that G is not a tree, so there is an edge e which belongs to a cycle. Let $G' = G - e$ and $G'' = G/e$ for this edge e. Then both G' and G'' are connected, so the induction hypothesis applies to G' and to G''. Also G' has n vertices and G'' has $n-1$ vertices, and so by the inductive hypothesis,

$$P_{G'}(k) = k^2\, f(k) + (-1)^{n-1}\,\ell' k, \qquad P_{G''}(k) = k^2\, g(k) + (-1)^{n-2}\ell''\, k$$

for some polynomials f, g and some positive integers ℓ', ℓ''. Finally, applying (1) implies that

$$\begin{aligned} P_G(k) = P_{G'}(k) - P_{G''}(k) &= \big(k^2 f(k) + (-1)^{n-1}\,\ell' k\big) - \big(k^2\, g(k) + (-1)^{n-2}\ell''\, k\big) \\ &= k^2\big(f(k) - g(k)\big) + \big((-1)^{n-1}\ell' - (-1)^{n-2}\ell''\big)k \\ &= k^2\big(f(k) - g(k)\big) + (-1)^{n-1}\big(\ell' + \ell''\big)k. \end{aligned}$$

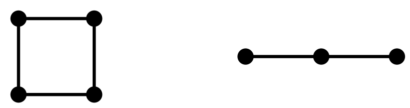
Hence the constant coefficient of $P_G(k)$ is zero and the linear coefficient is $(-1)^{n-1}\ell$ where $\ell = \ell' + \ell''$ is a positive integer.

(e) Hence, or otherwise, prove that for any graph G with at least one vertex, the smallest power of k with nonzero coefficient in $P_G(k)$ is equal to the number of connected components in G.

Solution: By (c), the smallest power of k with nonzero coefficient in $P_G(k)$ is given by the sum $d_1 + \cdots + d_r$, where d_j is the smallest power of k with nonzero coefficient in $P_{G_j}(k)$ for $j = 1, \ldots, r$. But (d) implies that this smallest power is always linear, as $a_0 = 0$ and $a_1 \neq 0$ in (d). Hence $d_1 = \cdots d_r = 1$ and the sum of these powers is exactly r, as required.

(f) Draw a graph with chromatic polynomial $k^2(k-1)^3(k^2 - 3k + 3)$.
(You do not need to provide a proof, just the graph.)

Solution: The following graph has the given chromatic polynomial:



(By (e) the smallest nonzero coefficient is the coefficient of k^2, so there are two components. By (b) and (c), we can take one to be C_4 and the other is $k(k-1)^2$, which is the chromatic polynomial of the path of length 2.)

Q3 Marking scheme:

(a) [**3 marks**]: 1 mark for base case. 1 mark for deleting a leaf. 1 mark for applying induction hypothesis. OR, 1 mark for explaining enumeration of vertices, 1 mark for colouring first vertex, 1 mark for stating $k-1$ choices for each remaining vertex.

(b) [**3 marks**]: 1 mark for applying deletion-contraction, 1 mark for chromatic polynomial of each of $C_4 - e$ and C_4/e. OR: 1 mark for colouring a and b, 1 mark for the case that c and b are coloured the same, 1 mark for the case that c and b are coloured differently.

(c) [**1 mark**]: 1 mark for mentioning that the components can be coloured *independently*, or that colouring of one component *does not affect* the other components, or words to that effect.

(d) **[4 marks]**: 1 mark for base case (which should really be trees, but some students did $m = 0, 1$). 1 mark to choose a non-bridge edge to delete. 1 mark for noting that G', G'' are connected. 1 mark for deletion-contraction using inductive hypothesis.

(e) **[2 marks]**: 1 mark for the fact that k^r is a factor, 1 mark for the fact that coefficient of k^r is nonzero.

(f) **[1 mark]**: 1 mark for a correct graph.

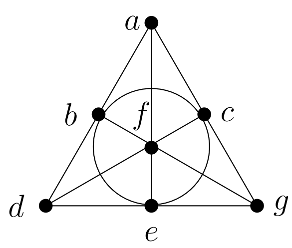
Question 4 (14 marks) The Fano plane F is a finite projective geometry with the following properties:

- F has 7 points and 7 lines.
- Each line contains exactly 3 points, and each point lies on exactly 3 lines.
- Each pair of distinct points lies on exactly one line.
- Each pair of distinct lines intersects in exactly one point.

We can draw the Fano plane as shown below, with set of points $U = \{a, b, c, d, e, f, g\}$ and set of lines

$$W = \{abd, \quad bce, \quad cdf, \quad deg, \quad aef, \quad bfg, \quad acg\}.$$

Note that the line bce is shown as a circle on the figure below.



Now define the bipartite graph G with vertex bipartition $U \cup W$ and edge set

$$\{p\ell \mid p \in U,\ \ell \in W,\ \text{ line } \ell \text{ contains point } p\}.$$

(a) Write down the number of vertices and edges of G.

Solution: There are $7 + 7 = 14$ vertices and $3 \times 7 = 21$ edges.

(b) Prove that G has diameter 3.

Solution: Since two points determine a line, any two distinct vertices in U are at distance 2. Similarly any two lines intersect in a point, so any two distinct vertices are at distance 2. Now suppose that $u \in U$ and $\ell \in W$. Then point u lies on line ℓ if and only if $d(u, \ell) = 1$. Suppose that point u does not lie on line ℓ and let x be any point on the line ℓ. Then there is a line ℓ' which contains u and x, and so $u\, \ell'\, x\, \ell$ is a path from u to ℓ of length 3. Hence the diameter of G is 3.

(c) Calculate the number of faces in any plane embedding of G. Justify your answer.

Solution: Since G has finite diameter we know that G is connected. Applying Euler's formula, the number of faces of G is $2 - 14 + 21 = 9$.

(d) Prove that G does not contain a 4-cycle.

Solution: If G has a 4-cycle then it is $u\ell v\ell' u$ for some $u, v \in U$ and $\ell, \ell' \in W$. But this implies that the distinct points u, v both belong to the distinct lines ℓ, ℓ', which contradicts the definition of the Fano plane. Hence G does not contain a 4-cycle.

(e) Write down a 6-cycle in G which contains the edge $\{a, abd\}$.

Solution: One such 6-cycle is

$$a,\ abd,\ b,\ bfg,\ g,\ acg,\ a.$$

(f) Without using Kuratowski's Theorem, prove that G is not planar.
(Hence no plane embedding of G exists.)

Solution: For a contradiction, suppose that G has a plane embedding. Arguing as in (e), by symmetry, we can show that every edge of G is contained in a 6-cycle. Hence every vertex of G is contained in a cycle. This implies that G is 2-connected (we already know that G is connected). Hence every face is bounded by a cycle, and this cycle has length at least 6 by (d). Furthermore, every edge lies on the boundary of at most two faces. Double-counting the number of incident (edge,face) pairs leads to

$$6 \times (\text{number of faces}) \leq 2 \times (\text{number of edges}).$$

But by (a) and (c), this gives $54 = 6 \times 9 \leq 2 \times 21 = 42$, which is a contradiction. Hence G has no plane embedding, so G is not planar.

(Note, it is not necessary for this question to prove that every edge lies on the boundary of exactly 2 faces, since an upper bound is enough.)

Q3 Marking scheme:

(a) **[2 marks]** 1 mark for correct number of vertices, 1 mark for correct number of edges.

(b) **[3 marks]** 1 mark for distance between vertices on same side of bipartition. 1 mark for dealing with incident points-lines. 1 mark for non-incident points-lines.

(c) **[2 marks]** 1 mark for a PROOF that G is connected. 1 mark for applying Euler's formula.

(d) **[2 marks]** 1 mark for noting what a 4-cycle means, 1 mark for saying this contradicts definition of Fano plane.

(e) **[1 marks]** 1 mark for a correct 6-cycle. (Do not penalise if final a ommitted, but everyting else should be correct.)

(f) **[4 marks]** 1 mark for CAREFUL PROOF of why each face boundary contains a cycle (either using 2-connectivity, or noting that G is not a forest by (e), so every face boundary contains a cycle). 1 mark for concluding (via (d)) that each face boundary has at least 6 edges. 1 mark for idea double counting incident edge-face pairs, 1 mark for using data from (a), (c) to get a contradiction. OR if students attempt a direct proof about embedding G, be cautious. The students might make assumptions about where the vertices are, e.g. with clear bipartition shown. This is not convincing: max 1. But one student argued carefully starting from two disjoint 6-cycles, a couple of extra edges, and noting no good choice for where to put a 13th vertex. This was convincing.