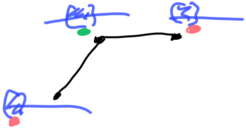
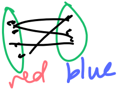


Chapter 4. Graph Colourings

The main reference for this section is Diestel Graph Theory, Chapter 5.

How many different radio frequencies do you need to be able to assign a radio frequency to each radio station in such a way that nearby stations do not interfere with each other's broadcasts? This question can be answered using graph colourings.



A vertex colouring (or colouring) of a graph $G = (V, E)$ is a function $c : V \to S$ such that $c(u) \neq c(v)$ whenever $uv \in E$.

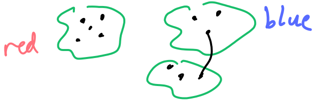
Here S is the set of available colours, usually $S = \{1, 2, \ldots, k\}$ for some positive integer k.

A k-colouring of G is a colouring $c : V \to \{1, 2, \ldots, k\}$.

Does not need to use all k colours!

Often we want the smallest value of k for which a k-colouring of G exists. This smallest value of k is called the chromatic number of G, denoted by $\chi(G)$. "chi"

If $\chi(G) = k$ then G is said to be k-chromatic.
If $\chi(G) \leq k$ then G is said to be k-colourable.

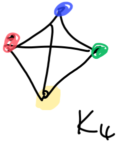


The set of all vertices in G with a given colour under c is called a colour class. Each colour class is an independent set. (Recall, an independent set is a set of vertices which contains no edge of G.)

So a k-colouring is a partition of $V(G)$ into k independent sets.
For example, G is 2-colourable if and only if G is bipartite.

The use of colouring terminology arises from the famous Four Colour Theorem for planar graphs: more later.

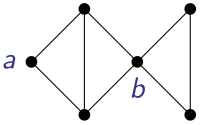
You can check that $\chi(K_r) = r$.



A clique in a graph G is a complete subgraph of G.
The order of the largest clique in G is called the clique number of G, denoted by $\omega(G)$.

$\omega(G) = 3$

$\omega(G+ab) = 4$

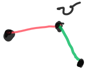


Then $\chi(G) \geq \omega(G)$.

Recall that $\alpha(G)$, the independence number of G, is the number of vertices in a largest independent set in G.

Fact: For any graph G we have $\chi(G) \geq n/\alpha(G)$.

Exercise: see Problem Sheet 4.



An edge colouring of G is a map $c : E \to S$ such that $c(e) \neq c(f)$ whenever e and f share an endvertex.

If $S = \{1, 2, \ldots, k\}$ then c is a k-edge-colouring and G is k-edge-colourable.

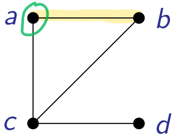
Let $\chi'(G)$ be the smallest positive integer k for which G is k-edge-colourable. We call $\chi'(G)$ the chromatic index (or edge-chromatic number) of G.

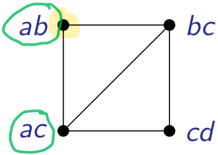
A colour class in an edge colouring is a matching of G. Hence an edge colouring displays $E(G)$ as a union of disjoint matchings.

colour class = set of all edges with the same colour

The line graph, denoted $L(G)$, has vertex set $E(G)$ and $e, f \in E(G)$ form an edge of $L(G)$ if and only if e, f share an endvertex in G.

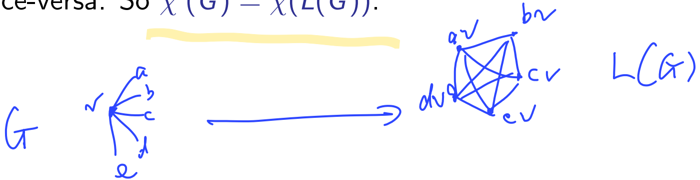
G

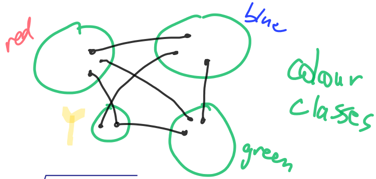




$L(G)$

Every edge-colouring of G is a vertex colouring of $L(G)$, and vice-versa. So $\chi'(G) = \chi(L(G))$.





5.2 Vertex colourings

Proposition 5.2.1

If graph G has m edges then $\chi(G) \leq \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$.

Proof. Fix a k-colouring of G with $k = \chi(G)$ colours. Then G has at least one edge between any two distinct colour classes, or we could merge them to give a colouring of G with $\leq k-1$ colours. ["Merge" means colour the union of these two colour classes with a single colour.]

Hence $m \geqslant \binom{k}{2} = \frac{1}{2}k(k-1)$.

Solve for k to complete the proof □

Greedy algorithm

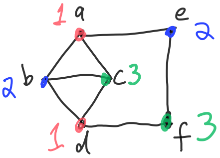
Given a graph G, fix an ordering $v_1, v_2, \ldots, v_n$ on the vertices of G, and colour them one by one in this order.

To start with, all vertices are uncoloured. At step i, where $i = 1, 2, \ldots, n$, colour vertex i with the first available colour: that is, the least positive integer which has not been used to colour any of the neighbours of v_i in $\{v_1, \ldots, v_{i-1}\}$.

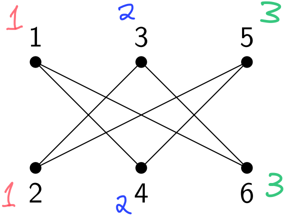
Since v_i has at most $\Delta(G)$ neighbours in $v_1, \ldots, v_{i-1}$, this produces a k-colouring of G with $k \leq \Delta(G) + 1$. That is,

$\Delta(G) = 3$

$$\chi(G) \leq \Delta(G) + 1.$$

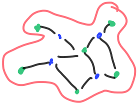


This greedy algorithm can be wasteful if your enemy provides a bad vertex labelling.



This example can be extended to produce a bipartite graph for which the greedy algorithm will use r colours, for any integer $r \geq 4$.

Fact: $\chi(G) = \Delta(G) + 1$ if G is a complete graph or an odd cycle. (See Problem Sheet 4).

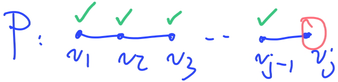


Theorem 5.2.4 (Brooks, 1941)
Let G be a connected graph. If G is neither complete nor an odd cycle then $\chi(G) \le \Delta(G)$.

Proof. (*Long!! Also, different to Diestel's proof.*)

We give a proof by Mariusz Zajac (2018)

First an observation. Let G be a graph with maximum degree $\Delta(G) \le k$, where $\{1,..,k\}$ will be our set of colours.
Suppose that G is partially coloured (some vertices are uncoloured, some coloured from $\{1,..,k\}$, no monochromatic edges).
Let $P = v_1 v_2 \dots v_j$ be a path in G such that all vertices of P are uncoloured.



Then we can colour vertices $v_1, v_2, \dots, v_{j-1}$ in this order, since at the moment that we colour v_i $(1 \le i \le j-1)$, we know that v_i has an uncoloured neighbour v_{i+1}, and hence at most $\Delta - 1$ coloured nbs.

Call this procedure PATHCOLOUR($v_1, v_2, \ldots, v_{j-1}; v_j$)
Note that this procedure colours $v_1, \ldots, v_{j-1}$ but it leaves v_j uncoloured

In particular, if $j=1$ then PATHCOLOUR(v_1) leaves the graph unchanged.

Theorem (Restatement of Brooks Theorem) "new version"
Let $k \geq 3$ be an integer and let G be a graph with $\Delta(G) \leq k$.
If G does not contain K_{k+1} as a subgraph then
G is k-colourable.

Before proving this new version, we show that it implies
Theorem 5.2.4 (Brooks Th^m, original) Assume that "new version" is true

Suppose that G is a connected graph and take $\Delta(G) = k \geq 3$ colours.
Hence G is not an odd cycle as $\Delta(G) \geq 3$
Also suppose that G is not complete, so $G \neq K_{\Delta(G)+1}$.
But since G has maximum degree $\Delta(G)$,

G contains $K_{\Delta(G)+1}$ as a subgraph $\Big\}$ if and only if $G = K_{\Delta(G)+1}$.

Therefore G satisfies the conditions of "new version" with $k = \Delta(G)$.

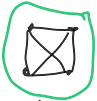
Applying this result, we conclude that G is $\Delta(G)$-colourable.
Therefore $\chi(G) \leq \Delta(G)$. Hence we obtain Brook's Th^m (Th^m 5.2.4) as a corollary to "new version".

(Also even length cycles are 2-colourable! Trees are 2-colourable!
Unicyclic graphs where the unique cycle has even length are 2-colourable!)

ie graphs with a unique cycle.

[All bipartite graphs are 2-colourable]

$\Delta = 3$



K_4

G not complete & max deg 3

Can G contain K_4 as a subgraph?

The K_4 would have to be a component of G

But G is connected & not complete $\square$

Proof of "new version": Proof by induction on $n = |G|$, where G is a graph with $\Delta(G) \leq k$ and $k \geq 3$.
If $n \leq k$ then we can k-colour G by giving each vertex a distinct colour.

CLAIM: If G has a vertex of degree $< k$ (ie G is not k-regular) then G is k-colourable.

Proof of claim: Let v be a vertex of degree $< k$ and let $G' = G - v$. By the inductive hypothesis we can k-colour G'. Fix one such colouring c. Then at most $k-1$ colours are used by c on neighbours of v, so we have an "available" colour which we can use to colour v.
[not used on a neighbour]

Now we assume that G is k-regular. Let v be a vertex of G and consider $G[\{v\} \cup N(v)]$. Since G has no subgraph isomorphic to K_{k+1}, we know that v has two neighbours x, y which are not adjacent.

Let $v_1 = x$, $v_2 = v$, $v_3 = y$ and extend the path $v_1 v_2 v_3$ to a maximal length path in G which starts with $v_1 v_2 v_3$: $P = v_1 v_2 v_3 \dots v_r$.

P.

Note: all neighbours of v_r lie on P since otherwise we could form a longer path.

Case 1: Suppose that $r = n$. This means that all vertices of G lie on P (we say that P is a "Hamilton path")

Let v_j be any neighbour of v_2 other than v_1 and v_3: since G is k-regular and $k \geq 3$ we can choose such a vertex v_j.

[Strategy: Colour $v_1 + v_3$ with the same colour; colour everything else somehow but leave v_2 uncoloured; finally colour v_2].

We now describe how to k-colour G.

- First, colour v_1 and v_3 with the same colour
- Next, apply PATHCOLOUR$(v_4, v_5, \dots, v_{j-1}; v_j)$ which colours $v_4, \dots, v_{j-1}$ and leaves v_j uncoloured.

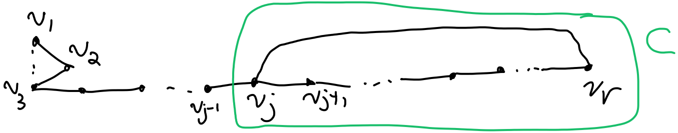
CASE 1

[all vertices of G lie on P] $r=n$

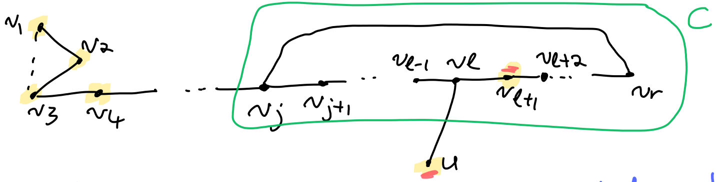
- Next, apply PATHCOLOUR($v_n, v_{n-1}, \ldots, v_j ; v_2$) which will colour all remaining vertices of G except v_2.
- Finally we have an available colour for v_2 since two of its neighbours (v_1 and v_3) have the same colour. Colour v_2 with an available colour.

<u>Case 2</u> Now suppose that $r<n$. Recall that all neighbours of v_r lie on P. Let v_j be the neighbour of v_r with the smallest index. Then $C = v_j v_{j+1} \cdots v_r v_j$ is a cycle in G

Let $G' = G - V(C)$. We can k-colour G', by induction. If there is no edge between G' and C then we can also k-colour $G[V(C)]$, by induction, and we are done. [This case needs $j=1$.]



Otherwise [$G[V(C)]$ is not a component of G]:
let v_ℓ be the vertex on C with largest index which has a neighbour in G', and let u be a neighbour of v_ℓ in G'.
Note, v_ℓ is well defined as v_j has a neighbour in G' if $j \geq 2$.



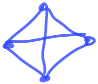
Note, $\ell \leq r-1$ since all neighbours of v_r belong to $V(C)$.
Also $v_{\ell+1}$ has no neighbours outside C, by choice of v_ℓ.
[Strategy: colour v_ℓ last!] We now describe how to k-colour vertices of C, giving a k-colouring of G

- First, colour $v_{\ell+1}$ with the colour assigned to u.
- Next, apply PATHCOLOUR$(v_{\ell+2}, \ldots, v_r, v_j, v_{j+1}, \ldots, v_{\ell-1}; v_\ell)$ which colours all remaining vertices of G except v_ℓ.
- Finally, colour v_ℓ with an available colour, which exists because

... v_l has two neighbours with the same colour

~~Hence in both cases, we created a k-colouring of G, so G is k-colourable.~~

This completes the proof in Case 2, by mathematical induction. $\square$



Calculating $\chi(G)$ (or more precisely, determining whether $\chi(G) \leq k$ for some fixed positive integer k) is an NP-complete problem.

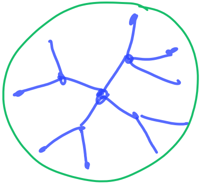
We know that if a graph G contains a copy of K_r then $\chi(G) \geq r$.

However, having large dense substructures is not a necessary condition for a high chromatic number.

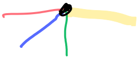
Using the probabilistic method, Erdős proved in 1959 that there exists graphs with arbitrarily high chromatic number and yet arbitrarily high girth.

We will prove this result at the end of this course.

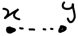
locally tree-like !



5.3 Edge colourings



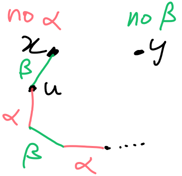
By considering a vertex of maximum degree, we see that the chromatic index $\chi'(G)$ satisfies $\chi'(G) \geq \Delta(G)$ for all graphs G.



Proposition 5.3.1 (König, 1916)
If G is bipartite then $\chi'(G) = \Delta(G)$.

Proof. We prove this by induction on $m = |E(G)|$.
If $m=0$ then the result is trivially true.
So, assume that $m \geqslant 1$ and that the result holds for all bipartite graphs with at most $m-1$ edges.
Let $\Delta = \Delta(G)$. Choose $xy \in E$ and let $G' = G - xy$.
By induction, we can fix a Δ-edge-colouring of G'.
We call edges coloured α, "α-edges", etc.
In G', vertices x, y both have degree $\leq \Delta - 1$.

So there are colours $\alpha, \beta \in \{1, 2, .., \Delta\}$ such that
x is not incident with an α-edge, and
y is not incident with a β-edge



If $\alpha = \beta$ then we can colour the edge xy with colour α to give a Δ-edge-colouring of G, and we are done

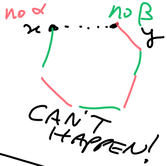
Now assume that $\alpha \neq \beta$.
Without loss of generality, we can assume that x is incident with a β-edge xu. [If not, we could colour xy with colour β in G.]

Extend the β-edge xu to a maximal walk W whose edges are coloured α, β alternately. Since no such walk can contain a vertex twice, W is a path.

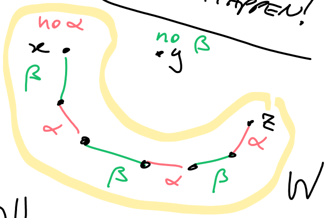
CLAIM: W does not contain y.

Proof of claim: For a contradiction, suppose that y lies on W. Then y must be an endvertex of W, and the edge of W incident with y must be an α-edge

Hence W has even length, and so $W + xy$ is an odd cycle in the bipartite graph G. This is a contradiction!



By maximality of W, we can swap the colours α and β on all edges of W. This gives a new Δ-edge-colouring of G' such that β does not appear on any edge incident with x.



Since y does not lie on W, there is still no β-edge incident with y.

Finally we can colour edge xy with colour β in G, giving a Δ-edge-colouring of G

This completes the proof, by induction $\square$

Theorem 5.3.2 (Vizing, 1964)
Every graph G satisfies

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

We omit the proof: it uses alternating path arguments very similar to proofs we've already seen.

Fact.
It is an NP-complete problem to decide whether an input graph G has $\chi'(G) = \Delta$ (G is "class 1") or $\chi'(G) = \Delta + 1$ (G is "class 2").

[End of Chapter 4. Try Problem Sheet 4.]