SCHOOL OF MATHEMATICS AND STATISTICS

UNSW Sydney

MATH5425 Graph Theory Term 1, 2025

Assignment 2 (30%): due Thursday Week 8, 23:59

This assignment is designed to be attempted based on the material covered in Chapters 3–5, covered in weeks 3–5 of lectures. The number of marks awarded to each question is shown, to give you a rough idea of how much work is expected. (One mark roughly corresponds to one step or idea.) The assignment will be **marked out of 30** and is worth 30%.

Marks will be awarded for clear and logical explanations, as well as for correctness. Your solutions should be written in complete sentences.

You will get the most out of this assignment by trying the questions yourself. You may discuss this assignment with other students, or look at textbooks if you wish. But it is very important that you write up your solution *on your own.* Do not copy work from other students, from textbooks or the internet.

You can use results from the lecture notes in your answers, as long as you clearly mention which result you are using. If a fellow student or textbook gave you an idea which helped you, please acknowledge this help in your solutions. (For example, write something like "Discussions with XXX were helpful" or "The main idea for part (b) came from XXX, page number XXX".) This is how professional mathematicians acknowledge assistance from their colleagues or from prior work.

The use of generative AI is **not permitted** for this assignment, in accordance with School policies.

Use of LaTeX is optional and will not affect the mark awarded. The LaTeX file for the assignment will be made available on Moodle.

The assignment is due on Thursday of Week 8 at 23:59 (late evening Thursday 10 April). Please upload your assignment to the MATH5425 Moodle site.

If you think you will be unable to submit your assignment on time, please contact me in advance (say by the end of week 7).

1. **(14 marks)**
Let $k \geq 2$ be an integer. Suppose that G is a graph with $\chi(G) \geq k+1$ such that any subgraph H of G with $|H| < |G|$ is k-colourable.

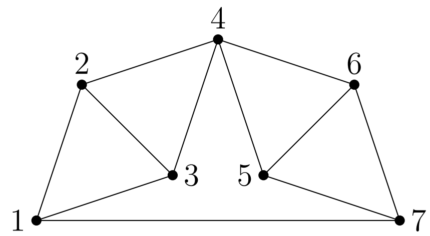
(a) Briefly explain why G is not bipartite.

(b) Prove that G is 2-connected.

(c) Let x, y be distinct vertices of G. Prove that there exists a path from x to y of odd length and a path from x to y of even length.

Hint: By (a) we know that G contains an odd cycle C. Menger's Theorem (Theorem 3.3.1) may help.

(d) Let G_0 be the following graph, which is called *Moser's spindle*:



Choose **one** of the following options:

Option 1: Prove that G_0 is not 3-connected, $\chi(G_0) = 4$, and every subgraph H of G_0 with $|H| < |G_0|$ is 3-colourable.

Option 2: If you prefer, instead of using Moser's spindle, define your own graph G_0 and prove that your graph G_0 has the above properties.

(*Remark: This example illustrates that part* (b) *cannot be strengthened.*)

2. **(7 marks)**
Let $G = (V, E)$ be a graph with $\chi(G) = k \geq 2$, and let $c : V \to \{1, 2, \ldots, k\}$ be a k-colouring of G. Define sets $S_1, S_2, \ldots, S_k \subseteq V$ as follows:

$$S_1 = \{u \in V : c(u) = 1\},$$
$$S_j = \{u \in V : c(u) = j \ \text{ and } \ uw \in E \text{ for some } w \in S_{j-1}\}$$

for $j = 2, \ldots, k$. That is, S_1 is the set of all vertices coloured 1 under c, and S_j is the set of all vertices coloured j with a neighbour in S_{j-1}, for $j = 2, \ldots, k$.

(a) Prove that the sets S_1, $S_2, \ldots, S_k$ are all non-empty.

(b) Prove that G contains a path P of length $k-1$ such that the k vertices of P are all coloured with distinct colours under c.

(c) Suppose that H is a graph such that the longest path in H has length $r \geq 1$. Prove that $\chi(H) \leq r + 1$.

3. **(9 marks)**
Fix integers $r \geq t \geq 4$ and let $\mathcal{U}$ be a set of subsets of $[n] = \{1, 2, \ldots, n\}$, such that each subset $S \in \mathcal{U}$ has size r.

Suppose that

$$|\mathcal{U}| \leq \frac{t^{r-1}}{(t-1)^r}.$$

Use the probabilistic method to prove that there exists a map $c : [n] \to \{1, 2, \ldots, t\}$ such that for every $S \in \mathcal{U}$,

$$\{c(u) \mid u \in S\} = \{1, 2, \ldots, t\}.$$