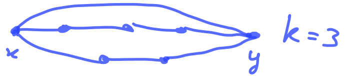


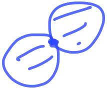
Chapter 5. Connectivity

The main reference for this section is Diestel Graph Theory, Chapter 3.

Recall that a graph G is k-connected if $|G| > k$ and G cannot be disconnected by deleting fewer than k vertices.

Soon, we will prove an alternative characterisation, called Menger's Theorem (1927): A graph is k-connected if and only if any two vertices can be joined by k independent paths (that is, paths with no common internal vertices).

$|G| \geq 3$ and we cannot disconnect G by deleting ≤ 1 vertex.
$\Rightarrow$ G is connected and no cutvertex



3.1 2-connected graphs

Let G be a graph. A maximal connected subgraph of G with no cut vertex is called a block.

Every block of G is either a maximal 2-connected subgraph of G or a bridge or an isolated vertex.

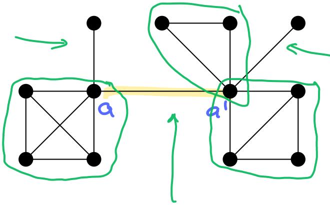
Conversely, every such subgraph of G is a block.

By maximality, different blocks of G overlap in at most one vertex, which must be a cut vertex in G.

Hence every edge of G lies in a unique block, and G is the union of its blocks.

Blocks are the 2-connected analogues of components.

How many blocks?



This graph has two cut-vertices a and a'

3 2-connected blocks

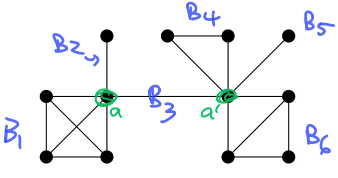
+ 3 bridges "→"

Let A be the set of cut vertices in G and let $\mathcal{B}$ be the set of blocks in G.

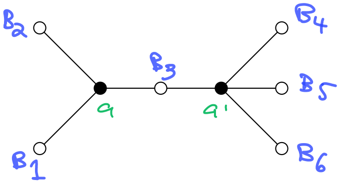
Form the bipartite graph on $A \cup \mathcal{B}$ with edge set

$$\{aB : a \in A,\ B \in \mathcal{B} \text{ and } a \in B\}.$$

This bipartite graph is the block graph of G.



G



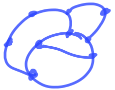
block graph of G

Always a tree!

Lemma 3.1.4
The block graph of a connected graph is a tree.

Proof: Exercise (see Problem Sheet 5).

Let H be a subgraph of a graph G. An H-path is a path in G which intersects H only in its endvertices.



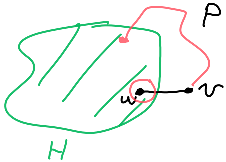
Proposition 3.1.1
A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths (or "ears") to graphs H already constructed.

Proof. Every graph constructed in this way is 2-connected [check!!]

Conversely, let G be 2-connected. Then $|G| \geqslant 3$ and G contains a cycle (as forests are NOT 2-connected)

Hence G has a maximal subgraph H which is constructible using the method described in the proposition statement

If $H = G$ then we are done. For a contradiction, suppose that $H \neq G$



Since any edge $xy \in E(G) - E(H)$ with $x, y \in H$ is an H-path, by maximality we see that every $xy \in E(G)$ with $x, y \in H$ must belong to $E(H)$. Hence H is an induced subgraph of G.

By the fact that G is connected, there is an edge vw with $v \in G - H$, $w \in H$. Since G is 2-connected we know that $G - w$ is connected. Let P be a shortest path from v to H in $G - w$.

Then wvP is an H-path in G, and $H \cup wvP$ is a larger constructible subgraph than H, contradicting the maximality of H

$\square$

$\square$

This result is useful in inductive proofs about 2-connected graphs.

3.2 3-connected graphs

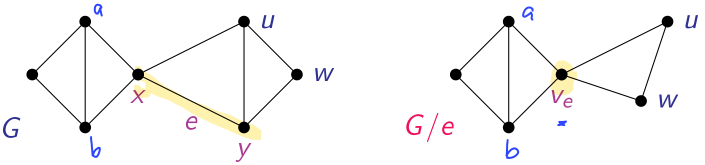
"G contract e"

Let $e = xy \in E(G)$. Define the graph $G/e = (V', E')$ where
$V' = (V - \{x, y\}) \cup \{v_e\}$,

keep all edges of G disjoint from e

$$E' = \{uw \in E(G) : \{u, w\} \cap \{x, y\} = \emptyset\}$$
$$\cup \{v_e w : xw \in E(G) \text{ or } yw \in E(G)\}.$$

$xu, yu \in E$



We say that G/e is formed by contracting the edge e in G.

This creates a new vertex v_e which replaces the endvertices of e.

Lemma.
Let G be a 3-connected graph with $|G| \geq 5$. Then G has an edge e such that G/e is 3-connected.

(LONG!)

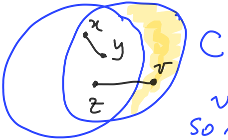
Proof. For a contradiction, suppose that no such edge exists.
For any $xy \in E(G)$, the graph G/xy is not 3-connected, but $|G/xy| = |G| - 1 \geq 4$ by assumption that $|G| \geq 5$. Hence G/xy has a separating set S with $|S| \leq 2$. Since G is 3-connected, the contracted vertex v_{xy} must belong to S, and $|S| = 2$, or we would have a separating set in G with ≤ 2 vertices. So there is some $z \in V(G)$, $z \notin \{x, y\}$ such that $S = \{v_{xy}, z\}$.
Any two vertices separated in G/xy by S are also separated in G by the set $T = \{x, y, z\}$

FACT: Since no proper subset of T separates G, by the 3-connectivity of G,

[every vertex in T has a neighbour in every component C of $G-T$.]

Exercise: see Problem Sheet 5.

✱ Choose the edge xy, vertex z, and component C of $G-\{x,y,z\}$ such that $|C|$ is as small as possible.

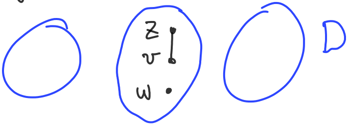


$v \in C$
so $v \notin \{x,y,z\}$

Let v be a neighbour of z in C, which we know must exist by our FACT.

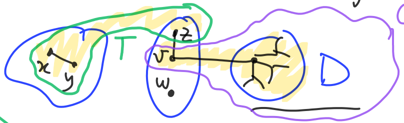
By assumption, G/zv is not 3-connected, and $|G/zv| = |G|-1 \geq 4$. Hence (by our earlier argument) there is a vertex $w \notin \{v,z\}$ such that $\{v,w,z\}$ separates G.

Also, by our FACT, every vertex in $\{v,w,z\}$ has a neighbour in every component of $G-\{v,w,z\}$



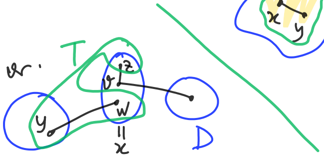
Since x and y are adjacent, $G-\{z,v,w\}$ has a component D such that $D \cap \{x,y\} = \emptyset$.

By our FACT we know that v has a neighbour in D.



Recall that $\boxed{v \in C}$, component in $G - \{x,y,z\}$

or:



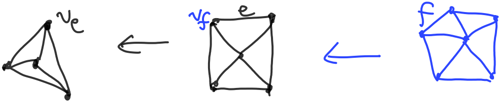
Since D is connected and $(\{v\} \cup V(D)) \cap \{x,y,z\}$, it follows that $\{v\} \cup V(D) \subseteq V(C)$.

Hence D is a proper subgraph of C, as $\boxed{v \notin V(D)}$.

Therefore $|D| < |C|$, contradicting the minimality of C !!

Hence G/e is 3-connected for some $e \in E(G)$. $\square$

Reversing this, we can construct all 3-connected graphs starting with K_4 and "uncontracting" edges.



etc

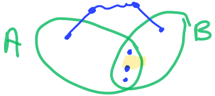
Theorem 3.2.3 (Tutte, 1961)

A graph G is 3-connected if and only if there exists a sequence $G_0, G_1, \ldots, G_r$ of graphs such that

(i) $G_0 = K_4$ and $G_r = G$,

(ii) G_{i+1} has an edge xy with degrees $d(x), d(y) \geq 3$ such that $G_i = G_{i+1}/xy$, for $i = 0, \ldots, r-1$.

The proof of this result is not too difficult, but we don't have time to cover it. See Diestel if interested.



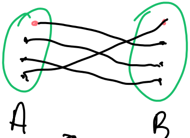
3.3 Menger's Theorem

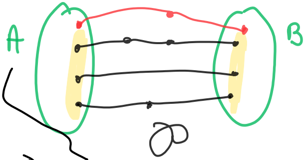
Recall that if $A, B \subseteq V$ then an (A, B)-path is a path $P = x_0 \ldots x_k$ such that $A \cap P = \{x_0\}$ and $B \cap P = \{x_k\}$.

Diestel writes "A-B path"

A set $S \subset V$ separating A from B in G is called an (A, B)-separator. This means that every (A, B)-path intersects S, and in particular $A \cap B \subseteq S$.

Let $\mathcal{P}$, $\mathcal{Q}$ be sets of disjoint (A, B)-paths in G. Say that $\mathcal{Q}$ exceeds $\mathcal{P}$ if the set of vertices in A which belong to paths in $\mathcal{P}$ is a proper subset of the set of vertices in A which belong to paths in $\mathcal{Q}$, and similarly for B.



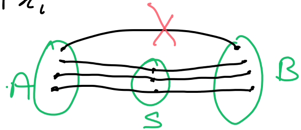


$\mathcal{Q}$ exceeds $\mathcal{P}$

$\mathcal{P} \cup$ new path $= \mathcal{Q}$

If $P = x_0 x_1 \cdots x_k$ then we write Px_i for the subpath $x_0 \cdots x_i$, and we write $x_i P$ for the subpath $x_i x_{i+1} \cdots x_k$.

Px_i



Theorem 3.3.1 (Menger's Theorem, 1927)

Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G equals the maximum number of disjoint (A, B)-paths in G.

(Here "disjoint" means *including their endpoints*.)

Proof. *(Long!!)* Let $k = k(G, A, B)$ be the minimum number of vertices separating A and B in G. (That is, $k = |S|$ where $S \subseteq V$ is a smallest (A, B)-separating set.) Then k is an upper bound on the maximum number of disjoint (A, B)-paths, or else we could not separate A and B by deleting any set of k vertices.

So it suffices to prove that a set of k disjoint (A,B)-paths exists. In fact, we will prove a stronger statement.

If $\mathcal{P}$ is any set of $< k$ disjoint (A,B)-paths, then there is a set $\mathcal{Q}$ of $|\mathcal{P}|+1$ disjoint (A,B)-paths in G which exceeds $\mathcal{P}$.

We will keep G and A fixed and let B vary, applying induction on the number of vertices in

$$\bigcup_{P \in \mathcal{P}} P.$$

Base case. If $\mathcal{P} = \emptyset$ then $\left|\bigcup_{P\in\mathcal{P}} P\right| = 0$. We can let $\mathcal{Q} = \{P\}$ for any (A,B)-path P. Then $\mathcal{Q}$ exceeds $\mathcal{P}$.

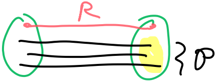
"Inductive step": Let $\mathcal{P}$ be a set of $< k$ disjoint (A,B)-paths, and $B_0 \subseteq B$ be the set of endvertices of paths in $\mathcal{P}$ ("start vertices" are in A, "endvertices" are in B).

Since $|B_0| \le k-1$, B_0 is not an (A,B)-separating set and hence there is an (A,B)-path in $G - B_0$

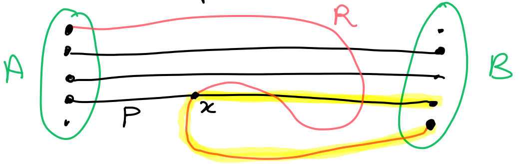
Call this (A,B)-path R. So R is disjoint from B_0.

If R avoids all vertices in $\bigcup_{P \in \mathcal{P}} P$ then $Q = \mathcal{P} \cup \{R\}$ exceeds $\mathcal{P}$, as required

[This is true when $\mathcal{P} = \emptyset$, so it covers the base case!]



Otherwise, let x be the last vertex of R (traversing R from A to B) that lies on some path $P \in \mathcal{P}$.



(Black paths $= \mathcal{P}$)

Note that $x \notin B$, by choice of R, so Px is shorter than P.

Let $B' = B \cup V(xP \cup xR)$ and let

$$\mathcal{P}' = (\mathcal{P} - \{P\}) \cup \{Px\}.$$

Then $\mathcal{P}'$ is a set of disjoint (A,B')-paths. Also $|\mathcal{P}'| = |\mathcal{P}|$, but the union of paths in $\mathcal{P}'$ is strictly smaller than $\left|\bigcup_{\hat{P} \in \mathcal{P}} \hat{P}\right|$.

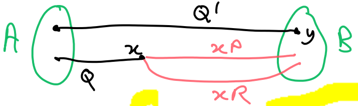
Also, $B \subseteq B'$, so an (A,B')-separating set is also an (A,B)-separating set. Hence $k(G,A,B') \geq k(G,A,B)$.

So $|\mathcal{P}'| \underset{\text{(assumption)}}{<} \underbrace{k(G,A,B)}_{=k} \leq k(G,A,B')$.

Applying the inductive hypothesis to $G, A, B', \mathcal{P}'$, we conclude that there is a set $\mathcal{Q}'$ of $|\mathcal{P}|+1$ disjoint (A,B')-paths in G which exceeds $\mathcal{P}'$!

Now $\mathcal{Q}'$ contains a path Q which ends in x, and a unique path Q' whose last vertex y is not among the last vertices of the paths in $\mathcal{P}$!

In particular, $y \neq x$.



Case 1: $y \in B$

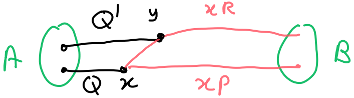
If $y \in B$ then define

$$\mathcal{Q} = (\mathcal{Q}' - \{Q\}) \cup \{QxP\}.$$

← unless y is endvertex of P!

If so, $\mathcal{Q} = (\mathcal{Q}' - \{Q\}) \cup \{QxR\}$ works.

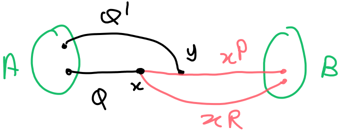
Case 2: $y \notin B$ and $y \in xR$.



If $y \in xR$ and $y \notin B$
then $y \notin xP$,
and we let

$$Q = (Q' - \{Q, Q'\}) \cup \{QxP,\ Q'yR\}$$

Case 3: $y \notin B$ and $y \in xP$.

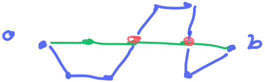


If $y \in xP$ and $y \notin B$
then $y \notin xR$, and we
define

$$Q = (Q' - \{Q, Q'\}) \cup \{QxR,\ Q'yP\}$$

In all cases, Q is a set of $|\mathcal{P}|+1$ disjoint (A,B)-paths, which exceeds $\mathcal{P}$, proving the inductive step.
Hence there is a set of k disjoint (A,B)-paths in G, as required. $\square$

Recall that two paths are independent if they do not share an internal vertex.



Corollary 3.3.5

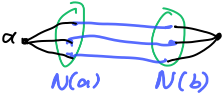
Let a, b be distinct vertices of G.

(i) If $ab \notin E$ then the minimum number of vertices (distinct from a and b) separating a from b is equal to the maximum number of independent (a, b)-paths in G.

(ii) The minimum number of edges separating a from b in G equals the maximum number of edge-disjoint (a, b)-paths in G.

(sketch)

Proof. (i) Apply Menger's theorem (Theorem 3.3.1) with $A = N(a)$, $B = N(b)$. Note that a set of k disjoint (A, B)-paths corresponds to a set of independent (a, b)-paths by adding vertex a at the start and vertex b at the end



[check remaining details: $S \subseteq V$ separates A from B iff it separates a from b?]

(ii) Apply Menger's Theorem (Theorem 3.3.1) to the line graph $L(G)$ of G, with
$A = E(a)$, the set of edges of G incident with a,
$B = E(b)$, the set of edges of G incident with b

[Check details!]

$\square$

Theorem 3.3.6 (Global version of Menger's Theorem) $|G| \geq 2$

(i) A graph is k-connected if and only if it has order at least 2 and there are k independent paths between any two distinct vertices.

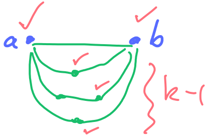
(ii) A graph is k-edge-connected if and only if it has at least two vertices and k edge-disjoint paths between any two distinct vertices.

Proof. Part (ii) follows immediately from Corollary 3.3.5 (ii). [Check!]

For (i), suppose that G is a graph and $|G| \geq 2$

First suppose that G has k independent paths between any two distinct vertices $a, b \in V$

Then $|G| > k$, as there are at least $k-1$ paths of length at least two between a and b. Also, G cannot be disconnected by deleting a set of $\leq k-1$ vertices



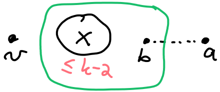
Hence G is k-connected

For the converse, suppose that G is k-connected, and assume for a contradiction that there are distinct vertices a, b with at most $k-1$ independent (a,b)-paths. Since G is k-connected we have $|G| \geq k+1$.

By Corollary 3.3.5(i), we must have $ab \in E$.

Let $G' = G - ab$. Then G' has at most $k-2$ independent (a,b)-paths.

Hence by Corollary 3.3.5(i), there is an (a,b)-separating set $X \subseteq V$ with $|X| \leq k-2$.



Since $|G| \geq k+1$, there is at least one more vertex $v \notin X \cup \{a,b\}$ in G. Now X separates v from at least one of a or b, say from a (since a, b lie in distinct components of $G' - X$).

But then $X \cup \{b\}$ is a set of at most $k-1$ vertices which separates v from a in G.

This contradicts the fact that G is k-connected

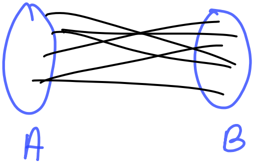
Hence G has at least k independent (a,b)-paths in G, completing the proof

$\square$

If you apply Menger's Theorem (Theorem 3.3.1) to a bipartite graph with bipartition A, B, you end up with König's Theorem (Theorem 2.1.1).

Why?

maximum matching ! = max set of disjoint (A,B)-paths



min separating set of vertices = minimum vertex cover

So König's Theorem is a special case of Menger's Theorem.

[End of Chapter 5. Try Problem Sheet 5.]

2-connectivity **Bonus Example**

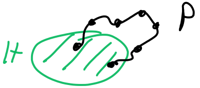
Lemma: Let G be a 2-connected graph with $|G| \geqslant 4$. Let e be an edge of G. Then either $G - e$ or G/e is 2-connected.

Proof The result is true if G is a cycle, since G/e is a cycle of length $|G| - 1 \geqslant 3$.

Next, by Proposition 3.1.1 we can assume that $G = H + P$ where H is a 2-connected graph and P is an H-path

Say $P = a \dots b$.

By induction, we can assume that for every edge e of H, either $H - e$ or H/e is 2-connected



Now choose $e \in G$.

- If e lies on P and P has at least two edges then $G/e = H + (P/e)$, which is 2-connected by Proposition 3.1.1, since P/e is an H-path of length $|P| - 1$

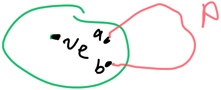
- If $P = e$ (length 1) then $G - e = H$ is 2-connected
- Now suppose that $e \in E(H)$

Then $G - e = (H - e) + P$, which is 2-connected by Proposition 3.1.1 (3.1.3?) whenever $H - e$ is 2-connected

This follows as P is an $(H-e)$-path

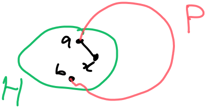
- Finally, suppose that $H - e$ is not 2-connected for some $e \in E(H)$. Then H/e is 2-connected, by induction.

Recall $P = a \dots b$



☆ If $\{a, b\} \cap e = \phi$ then P is an H/e-path, and $G/e = (H/e) + P$

Hence G/e is 2-connected



☆ If $|\{a, b\} \cap e| = 1$, say $e = ax$ for some $x \notin \{a, b\}$, then

$G/e = (H/e) + ((P - a) \cup \{v_e\})$ & G/e is 2-connected (Prop)

☆ Finally suppose $e = ab$. Note, $H - e$ is connected, since H is 2-connected.

<u>Claim</u>: $G - e = (H - e) + P$ is 2-connected.

[Exercise!!]

+ done $\square$