SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney
MATH5425 Graph Theory Term 1, 2025
Problem Sheet 6, Planar Graphs

1. Let G be a graph. Explain why it is possible to embed G in $\mathbb{R}^3$ with all edges straight and no edges crossing. (*Hint:* try induction on the number of vertices. You don't need to write a complete proof: we want to understand the main ideas.)

2. Let G be a plane graph. Prove that the boundary of any inner face of G contains a cycle.

3. Recall that the girth of a graph is the size of its smallest cycle. Let $g \geq 3$ be a fixed integer and let $\mathcal{H}$ be the set of all plane graphs with n vertices, minimum degree at least 1 and girth g. Say that a graph $G \in \mathcal{H}$ is *maximally $\mathcal{H}$-plane* if we cannot add an edge to G to give a new plane graph $G' \in \mathcal{H}$.

   (a) Suppose that G is a plane graph in $\mathcal{H}$ such that every face is bounded by a g-cycle. Prove that G is maximally $\mathcal{H}$-plane.

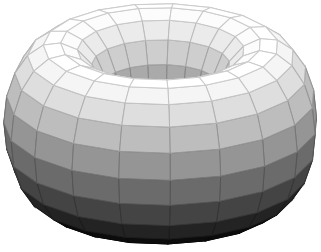
   (b) Suppose that G is a plane graph such that every face is bounded by a g-cycle. Further suppose that G has n vertices and m edges. Prove that
   $$(g-2)m = g(n-2).$$

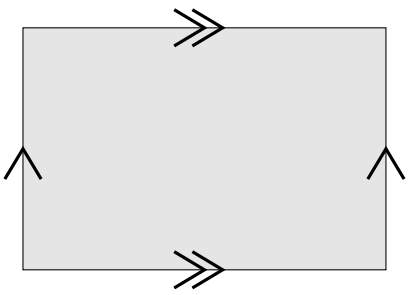
4. (a) Let G be a planar graph with edge e. Prove that G/e is also planar: that is, contraction of edges preserves planarity.

   (b) Without using Kuratowski's Theorem, prove that a planar graph does not contain any subdivision of $K_{3,3}$ or K_5 as a subgraph.

5. (a) Show that $K_{3,3}$ with one edge deleted is a planar graph, and similarly for K_5.

   (b) Show that you can draw $K_{3,3}$ and K_5 on the torus with no edges crossing. (A torus is shown on the left below. It may help to consider the torus as a rectangle with pairs of opposite sides identified, with no twists, as in the figure on the right.)





   (c) If necessary, adjust your embeddings of $K_{3,3}$ and K_5 such that every edge is incident with two distinct faces.

   (d) **Hard:** For $K_{3,3}$, can you find an embedding such that every face is bounded by a Hamilton cycle?