SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney

MATH5425 Graph Theory Term 1, 2025

Problem Sheet 8, Random Graphs

1. Prove that for $G \in G(n, p)$,
$$\Pr(\alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}.$$

2. Let $k \in \mathbb{Z}^+$. Calculate the expected number of k-paths in $G \in G(n, p)$.

3. Let H be a fixed graph on the vertex set $\{1, \ldots, k\}$. The *automorphism group* $\mathrm{Aut}(H)$ is the set of all graph isomorphisms from H to itself (equivalently, the set of all permutations of $V(H)$ which preserve the set $E(H)$).
   (a) Calculate the expected number of subgraphs of $G \in G(n, p)$ which are isomorphic to H, where $n \geq k$.
   (b) Check that you get the correct answer when H is a k-cycle or a k-path. (What is the order of the automorphism group of a k-cycle? What is the order of the automorphism group of a k-path?)
   (c) Now, calculate the expected number of *induced* subgraphs of $G \in G(n, p)$ which are isomorphic to H, where $n \geq k$.
   (d) Let H be a graph with 4 vertices and 5 edges. Find the expected number of induced subgraphs of $G \in G(n, p)$ which are isomorphic to H.

4. (a) Show that for $G \in G(n, p)$, the expected number of isolated vertices (that is, vertices with no neighbours) is $n(1-p)^{n-1}$.
   (b) For the rest of the question let $p = p(n) = \dfrac{2\ln n}{n}$. Show that the expected number of isolated vertices in $G \in G(n, p)$ is at most $n^{2/n-1}$. *Hint:* the inequality $1 - x \leq e^{-x}$ might be useful.
   (c) Prove that $\lim_{n\to\infty} n^{2/n-1} = 0$.
   (d) Hence show that a.a.s. $G \in G(n, p)$ has no isolated vertices.

5. Suppose that $p = p(n)$ is such that $\lim_{n\to\infty} n\, p(n) = 0$. Prove that a.a.s. $G \in G(n, p)$ is a forest.