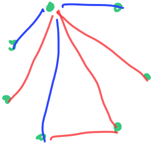
Chapter 7. Ramsey Theory

References: Diestel, Chapter 9, Bollobás, Chapter 6,
Alon & Spencer, Chapter 1

Q: How many people do need at a party before you are guaranteed to find either 3 people who already knew each other, or 3 people who were all strangers to one another?

If there are n people at the party, you can model this using the complete graph K_n. Each vertex represents a person, and the edge ij is coloured red if person i and person j already knew each other before the party, or coloured blue if they did not.

Note: These are not "proper" edge colourings.



Q: How large must n be before, in any red-blue colouring of the edges of K_n, you can find either a red triangle or a blue triangle?

More generally, for integers $s, t \geq 2$, let $R(s, t)$ be the least positive integer n such that any red-blue colouring of K_n has either a red copy of K_s or a blue copy of K_t.

Set $R(s, t) = \infty$ if no such n exists.

The numbers $R(s, t)$ are called Ramsey numbers.

Write $R(s)$ instead of $R(s, s)$ (this is the diagonal case).

Exercise: (See Problem Sheet 7)
Show that $R(3) = 6$ and that $R(s, 2) = R(2, s) = s$ for all $s \geq 2$.

$$\boxed{R(t,s) = R(s,t)}$$

Upper bounds

Theorem (Erdős & Szekeres, 1935)
For all integers $s, t \geq 2$, the Ramsey number $R(s,t)$ is finite. If $s > 2$ and $t > 2$ then

$$R(s,t) \leq R(s-1,t) + R(s,t-1) \tag{1}$$

and hence

$$R(s,t) \leq \binom{s+t-2}{s-1}. \tag{2}$$

Proof. We know that $R(s,2) = R(2,s) = s$ for all $s \geq 2$.
Assume by induction that $R(s-1,t)$ and $R(s,t-1)$ are both finite.
Let $n = R(s-1,t) + R(s,t-1)$. Consider any red-blue colouring of the edges of K_n. Let x be a vertex of K_n. Then x has degree

$$n-1 = R(s-1,t) + R(s,t-1) - 1.$$

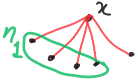
By the pigeonhole principle, either

- there are at least $n_1 = R(s-1,t)$ red edges incident with x

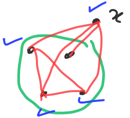
or

- there are at least $n_2 = R(s,t-1)$ blue edges incident with x.

Without loss of generality, assume the former.



Consider the subgraph K_{n_1} spanned by a set of n_1 vertices which are joined to x by red edges



* If K_{n_1} contains a blue copy of K_t then we are done.

* Otherwise, K_{n_1} contains a red copy of K_{s-1}, Since $n_1 = R(s-1,t)$.
Together with x this gives a red copy of K_s, completing the proof of (1). $\square$

In particular, $R(s,t)$ is finite

Exercise: Use induction on $s+t$ to prove (2).

$$\sqrt{2}^{\,s} \le R(s) \le \cancel{4} \, (4-\varepsilon)^s$$

Very few Ramsey numbers are known precisely:

$$\begin{array}{rrr} R(3) = 6, & R(3,4) = 9, & R(3,5) = 14, \\ R(3,6) = 18, & R(3,7) = 23, & R(3,8) = 28, \\ R(3,9) = 36, & R(4) = 18, & R(4,5) = 25. \end{array}$$

We know that $43 \le R(5) \le 46$ (lower bound: Exoo 1989; upper bound: Angeltveit & McKay 2024).

Since Ramsey numbers are hard to compute, our interest turns to asymptotics, especially for the diagonal Ramsey numbers $R(s)$.

We need Stirling's inequalities: for any positive integer n,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le e^{1/(12n)}\sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

$$e^{\frac{1}{12n}} = 1 + \frac{1}{12n} + O\left(\frac{1}{n^2}\right)$$

$k! \geq \left(\frac{k}{e}\right)^k$

$e^{\frac{1}{12n}} \leq 1.1$ if $n \geq 1$

Hence $n! \leq 1.1\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{(n)_k}{k!} \leq \frac{n^k}{k!}$

Also $\binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k}\right)^k$ for all positive integers k, n.

Asymptotic upper bound for $R(s)$: if $s \geq 2$ then

(Erd\H os)
↑ LaTeX

$R(s) \leq \binom{2s-2}{s-1}$ by Erdős + Szekeres

$$= \frac{(2s-2)!}{(s-1)!^2} \leq \frac{1.1\sqrt{\cancel{4}\pi(s-1)}\left(\frac{\cancel{2(s-1)}}{\cancel{e}}\right)^{2s-2}}{\cancel{2}\pi(s-1)\left(\frac{\cancel{s-1}}{\cancel{e}}\right)^{2(s-1)}}$$

using Stirlings bounds

$$= \frac{1.1 \qquad 2^{2s-2}}{\sqrt{\pi(s-1)}} \leq \frac{2^{2s}}{4\sqrt{s}} \quad \text{[check!]}$$

$$= \frac{4^s}{4\sqrt{s}}.$$

There have been some improvements in the upper bound:

Thomason, 1988: $R(s) \leq \frac{4^s}{s}$

Sah, 2020/2023, improving on Conlon, 2009: there is a positive constant $c > 0$ such that for any $s \geq 3$,

$$R(s) \leq e^{-c(\log s)^2} \binom{2s-2}{s-1} \leq \frac{4^s}{4s^{c\log(s)+1/2}}.$$

Then an exponential improvement:

Campos, Griffiths, Morris and Sahasrabudhe, arXiv preprint 2023:

$$R(s) \leq (4-\varepsilon)^s$$

with $\varepsilon = 2^{-7}$, say.

Lower bounds

We use the probabilistic method.

Theorem (Erdős, 1947) If $\overbrace{\binom{n}{s} 2^{1-\binom{s}{2}}} < 1$ then $R(s) > n$.

Hence $R(s) > \lfloor 2^{s/2} \rfloor$ for $s \geq 3$.

Proof. Take a random red-blue colouring of the edges of K_n, where each edge is coloured independently red or blue, each with probability $1/2$.

For any fixed set R of s vertices, let A_R be the event that the induced subgraph $K_n[R]$ is monochromatic [i.e. all edges red or all edges blue]

Then, using independence,

$$\Pr(A_R) = \underbrace{\left(\tfrac{1}{2}\right)^{\binom{s}{2}}}_{[\text{"all blue"}]} + \underbrace{\left(\tfrac{1}{2}\right)^{\binom{s}{2}}}_{[\text{"all red"}]} = \frac{2}{2^{\binom{s}{2}}},$$

since there are $\binom{s}{2}$ edges in $K_n[R]$ and the events "all red" and "all blue" on $K_n[R]$ are disjoint.

Let X be the number of monochromatic copies of K_s in the random red-blue colouring. Then $X = \sum_{\substack{R \subseteq [n] \\ |R| = s}} \mathbb{1}(A_R)$,

where $[n] = \{1, 2, .., n\} = V(K_n)$

and $\mathbb{1}(A_R)$ is the indicator variable for the event A_R.

Hence, by linearity of expectation,

$$\mathbb{E}X = \sum_{\substack{R \subseteq [n] \\ |R| = s}} \mathbb{E}(\mathbb{1}(A_R)) = \sum_{\substack{R \subseteq [n] \\ |R| = s}} \Pr(A_s) = \binom{n}{s} \frac{2}{2^{\binom{s}{2}}}$$

By the assumption we have

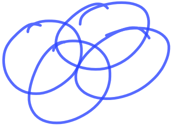
$$\mathbb{E}X = \binom{n}{s} 2^{1-\binom{s}{2}} < 1.$$

Therefore there is a fixed red-blue colouring of the edges of K_n with no monochromatic copy of K_s. Hence $R(s) > n$.

[Remark: we could have just done this:

$$\Pr(\text{there is a monochromatic } K_s) = \Pr\Big(\bigcup_{\substack{R\subseteq[n]\\ |R|=s}} A_R\Big)$$

"or"



$$\leq \sum_{\substack{R\subseteq[n]\\ |R|=s}} \Pr(A_R)$$

& then continue [union bound]

$\ldots < 1$ by assumption.

Hence $\Pr(\text{no mono } K_s) = 1 - \Pr(\text{there is a mono } K_s) > 0.$ etc

---

This proves the first statement. Now suppose that $s \geq 3$ and $n = \lfloor 2^{s/2} \rfloor = \lfloor \sqrt{2}^{\,s} \rfloor$. Then

$$\binom{n}{s} 2^{1-\binom{s}{2}} \leq \frac{2^{1+s/2-s^2/2}\, n^s}{s!} \leq \frac{2^{1+s/2-s^2/2}\, 2^{s^2/2}}{s!}$$

(as $n^s \leq 2^{s^2/2}$ by choice of n)

$$\leq \frac{2^{1+s/2}}{s!} < 1 \quad \text{as } s \geq 3. \qquad \square$$

For some constant $c>0$, $\frac{ct^3}{\log^4 t} \le R(4,t) \le (1+o(1))\frac{t^3}{\log^2 t}$

[Mattheus + Verstraete 2024] [Bohman + Keevash 2021]

Lower bounds

Note, it is possible to give a sharper analysis and obtain

(using finite geometry!!)

$$R(s) > \frac{s}{\sqrt{2e}}\,2^{s/2}$$

for $s \ge 3$. (See Problem Sheet 7.)

Huge gap between upper and lower bounds:

$$\log\sqrt{2} \le \lim_{s\to\infty}\frac{\log R(s)}{s} \le \log(4-2^{-7}).$$

Some good news: we know that

$$\left(\frac{1}{4}-o(1)\right)\frac{t^2}{\log t} < R(3,t) < (1+o(1))\frac{t^2}{\log t}.$$

$f(t)=o(1)$ if $\lim_{t\to\infty} f(t)=0$

Also good news $R(4,t)$: known up to polylog factors

Graph Ramsey Theory

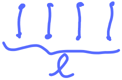
The main reference for this section is Bollobás, Chapter 6.

Let H_1, H_2 be fixed graphs with no isolated vertices, and let $R(H_1, H_2)$ be the least positive integer n such that in every red-blue colouring of the edges of K_n, there is either a red copy of H_1 or a blue copy of H_2.

Write $R(H) = R(H, H)$ and note that $R(K_s, K_t) = R(s, t)$, the Ramsey number.

So the graph Ramsey numbers $R(H_1, H_2)$ generalise the Ramsey numbers.

Write ℓK_2 for a set of ℓ independent edges.

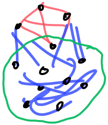


ie a matching of size ℓ

Theorem (Bollobás, Chapter 6, Theorem 10)
For $\ell \geq 1$ and $p \geq 2$,

$$R(\ell K_2, K_p) = 2\ell + p - 2.$$

Proof. First consider $K_{2\ell+p-3}$. We colour the edges of $K_{2\ell+p-3}$ so that there is a red $K_{2\ell-1}$ and all other edges are blue. Then we cannot find ℓ independent red edges



as this would require 2ℓ vertices which are incident with red edges, but we only have $2\ell-1$. That is, there is no red copy of ℓK_2.

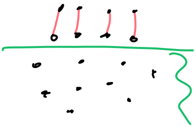
Next, the largest blue complete subgraph has

$2\ell+p-3-(2\ell-2) = p-1$ vertices, noting that we can keep exactly one vertex which is incident with a red edge.

Too small!

Hence there is no blue K_p, so $R(\ell K_2, K_p) \geq 2\ell + p - 2$.

Next, take any red-blue colouring of the edges of K_n, where $n = 2\ell + p - 2$. If we can find a red ℓK_2 then we are done. So suppose that there are at most s independent red edges, where $s \leq \ell - 1$.



Then the set of

$$n - 2s \geq 2\ell + p - 2 - 2(\ell - 1) = p$$

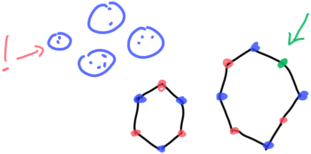
vertices which are not incident with these s red edges must span a blue complete subgraph: if not, we can find a larger red matching, contradicting the definition of s.

Hence $R(\ell K_2, K_p) \leq 2\ell + p - 2$, so $R(\ell K_2, K_p) = 2\ell + p - 2$, as claimed.

$\square$

For a graph G, let $c(G)$ be the number of vertices in the largest component of G, and let $u(G)$ be the chromatic surplus of G, which is the minimum size of the smallest colour class of G, taken over all $\chi(G)$-colourings of G.

Note that $u(C_{2k}) = k$ and $u(C_{2k+1}) = 1$.



Theorem (Bollobás, Chapter 6, Theorem 11)
For all graphs H_1, H_2 with no isolated vertices, we have

$R(H_2, H_1) \geq (\chi(H_2) - 1)(\quad) + \cdot$

$$R(H_1, H_2) \geq (\chi(H_1) - 1)(c(H_2) - 1) + u(H_1).$$

In particular, if H_2 is connected then

$$R(H_1, H_2) \geq (\chi(H_1) - 1)(|H_2| - 1) + 1.$$

Proof. Let $k = \chi(H_1)$, $u = u(H_1)$ and $c = c(H_2)$.
Then $R(H_1, H_2) \geq R(H_1, K_2) = |H_1| \geq \chi(H_1) u(H_1) = ku$.

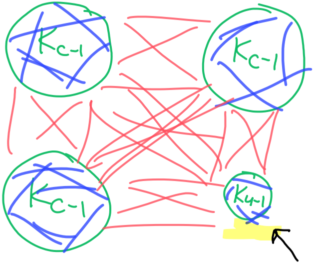
check!

Hence if $c \leq u$ then

$R(H_1, H_2) \geqslant ku \geqslant (k-1)c + u \geqslant (k-1)(c-1) + u$, as required

Now suppose that $c > u$ and let

$$n = (k-1)(c-1) + u - 1.$$



Partition the vertices of K_n into parts $A_1, A_2, .., A_{k-1}, B$

where $|A_j| = c-1$ for $j = 1, .., k-1$

and $|B| = u-1$. (eg A_j has $c-1$ vertices)

Let $K_n[A_i]$ be a blue K_{c-1} for all $i = 1, .., k-1$, and let $K_n[B]$ be a blue K_{u-1}. Colour all remaining edges red.

The largest component in H_2 has order c, but the largest component of the blue subgraph of K_n has order $c-1$, since $c > u$

Hence there is no blue copy of H_2

Next, if there is a red copy of H_1 then the k partite sets $A_1, .., A_{k-1}, B$ induce a k-colouring (proper vertex colouring) of H_1.

Furthermore, $k = \chi(H_1)$ and the smallest colour class in this vertex colouring contains $u-1$ vertices, as $u < c$. But this contradicts the definition of $u = u(H_1)$. Hence there is no red H_1 either, so $R(H_1, H_2) > n$.

So $R(H_1, H_2) \geq n+1 = (k-1)(c-1) + u$.

The second statement follows as $u(H_1) \geq 1$ for all graphs H_1 with no isolated vertices, and $c(H_2) = |H_2|$ if H_2 is connected.

$\square$

As a sample application, let $H_1 = K_s$, and let $H_2 = T$ be a tree with t vertices, where $s \geq 3$ and $t \geq 2$.

Then the Bollobás bound says that

$$R(K_s, T) \geqslant (\chi(K_s) - 1)(|T| - 1) + 1 \quad \text{as } T \text{ is connected}$$
$$= (s-1)(t-1) + 1$$

FACT: For any tree T this is equality:

$$R(K_s, T) = (s-1)(t-1) + 1.$$

Proof: omitted.

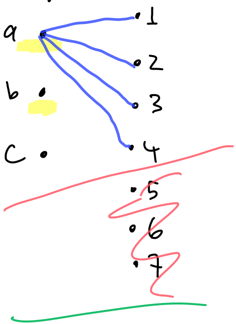
[End of Chapter 7. Try Problem Sheet 7.]

2019 Exam question: Bipartite Ramsey Theory!

(a) Prove that every red/blue colouring of the edges of $K_{3,7}$ has a monochromatic copy of $K_{2,2}$

4-cycle!

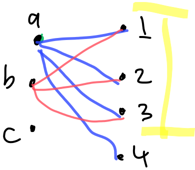
Proof Andrew Kaploun:



Without loss of generality, a has at least 4 blue incident edges, by the pigeonhole principle (as a has degree 7).

Assume without loss of generality that the edges a1, a2, a3, a4 are blue.

Next, if there are at least 2 blue edges from b into $\{1, 2, 3, 4\}$ then we have a blue $K_{2,2}$. So we can assume without loss of generality that edges b1, b2, b3 are red.



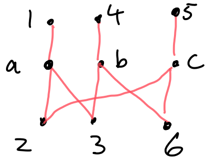
Now consider edges c1, c2, c3. If at least two of them are blue then we have a blue $K_{2,2}$ with a.

Otherwise, at least of these edges must be red and we have a red $K_{2,2}$ with b.

(b) Find red-blue col of edges of $K_{2,7}$ with no mono $K_{2,2}$ (Exercise!!)

Hint: let each colour class form a tree!

(c) Same for $K_{3,6}$:



Each vertex a, b, c should have 3 red edges + 3 blue edges incident.

Try to avoid red $K_{2,2}$.

Red edges no $K_{2,2}$ · complement is blue subgraph, iso.

Note: $K_{3,6}$ has 18 edges

So aim for $\frac{18}{2} = 9$ red edges?