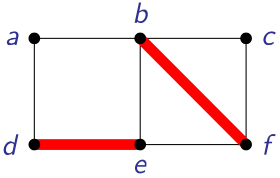
Chapter 2. Matchings and Hamilton Cycles

The main reference for this section is Diestel Graph Theory, Chapter 2 (matchings) and Chapter 10 (Hamilton cycles).

Two edges in a graph are called independent if they have no vertices in common.

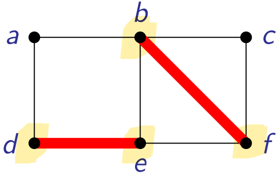
A set M of pairwise independent edges in a graph is called a matching.

$\emptyset$, $\{e\}$



Given $G = (V, E)$ and $U \subseteq V$, say that $M \subseteq E$ is a matching of U if M is a matching and every vertex in U is incident with an edge of M.

In the example, M is a matching of $\{b, d, e, f\}$, and it is also a matching of $\{b, d\}$.



We say that the vertices in U are matched by M, and that the vertices not incident with any edge of M are unmatched.

eg a, c

A matching M is a maximal matching of G if $M \cup \{e\}$ is not a matching for any $e \in E - M$.

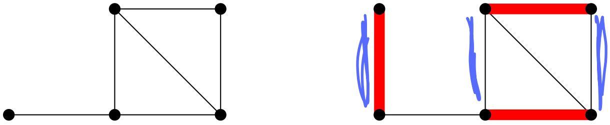
A maximum matching of G is a matching of G such that no set of edges with size greater than $|M|$ is a matching.

A perfect matching of G is a matching of G which matches every vertex of G.

Note: a perfect matching is a 1-regular spanning subgraph of G, also called a 1-factor of G.

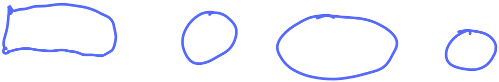
The graph on the left has no perfect matching, because ... $|G|$ is odd.

The graph on the right has a perfect matching as shown: any others?



A *k*-factor is a *k*-regular spanning subgraph.

A 2-factor in a graph is a union of disjoint cycles which covers all the vertices.



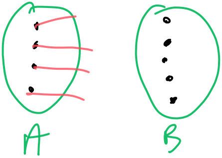
Q: When does a graph have a perfect matching?

2.1 Matchings in bipartite graphs

Let $G = (V, E)$ be a bipartite graph with vertex bipartition $V = A \cup B$. Here A, B are nonempty disjoint sets.

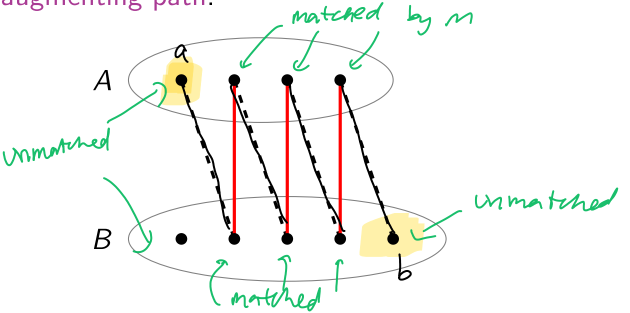
We use the convention that vertices called $a, a', a'', \ldots$ belong to A, while vertices called $b, b', b'', \ldots$ belong to B.

We want to find a matching in G which contains as many vertices of A as possible.

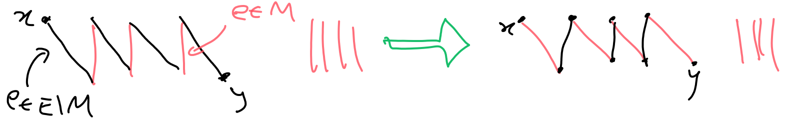


Let M be a matching in G. A path in G which starts at an unmatched vertex of A and contains, alternately, edges from $E - M$ and from M, is called an alternating path with respect to M.

If an alternating path P ends in an unmatched vertex of B then it is called an augmenting path.

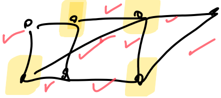


Here is an augmenting path P of length 7.



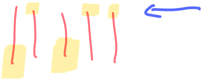
Fact: Starting from any matching, if you repeatedly "flip" augmenting paths in this manner until no more augmenting paths exist, then the result is a maximum matching of G.

More on augmenting paths on Problem Sheet 2.



Definition. A set $U \subseteq V$ is a cover (or vertex cover) of G if every edge of G is incident with a vertex in U.

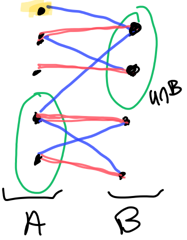
Theorem 2.1.1 (König, 1931)



Let G be a bipartite graph.
The size of a maximum matching in G is equal to the size of a minimum (i.e., smallest) vertex cover of G.

Proof. Let $\hat{U}$ be a cover in G and let M be a maximum matching. Then $|\hat{U}| \geq |M|$ as we must cover every edge of M. Hence it suffices to construct a cover U of G with $|U| = |M|$.
We build U by choosing one vertex from each edge of M to place into U, as follows:

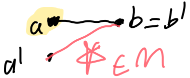
- If $ab \in M$ and some alternating path in G with respect to M ends in b
  then put b into $\mathcal{U}$
  otherwise put a into $\mathcal{U}$



We claim that $\mathcal{U}$ is a cover of G. Since $|\mathcal{U}| = |M|$ this will complete the proof.

Let $ab \in E$. If $ab \in M$ then $a \in \mathcal{U}$ or $b \in \mathcal{U}$, by definition of $\mathcal{U}$. Now assume $ab \notin M$. Since M is maximum, there exists $a'b' \in M$ with $a = a'$ or $b = b'$.

If a is unmatched in M then $b = b'$ for some $a'b' \in M$



Hence ab is an alternating path ending in $b = b'$, so we chose $b = b'$ to go into $\mathcal{U}$ from the edge $a'b' \in M$.

So the edge ab is covered by $\mathcal{U}$ in this case.

Hence we assume that $a = a'$ for some $a'b' \in M$.

If $a = a' \in \mathcal{U}$ then we are done. Otherwise, $b' \in \mathcal{U}$, so there is an alternating path P ending in b'

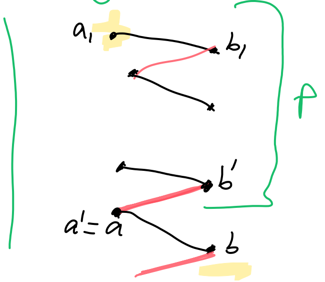
Then $P = a_1 b_1 a_2 b_2 \cdots b'$

Case (i): P does not include a or b.

Then $Pab = a_1 a_2 \dots b' a b$ is an alternating path in G with respect to M.

By maximality of M, b is matched or else we have an augmenting path.

Hence $b \in U$ as b is the chosen vertex from its matching edge



Case (ii): If b is on P before a, or $b \in P$ and $a \notin P$:

then $P = \underbrace{a_1 b_1 a_2 \cdots}_{\text{no } a} b \dots b'$. Then we let $P' = a_1 b_1 \dots b$.

This is an alternating path ending in b: finish proof as above.

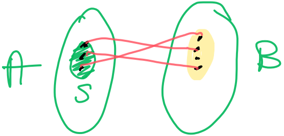
Case (iii): If a is on P before b, or $a \in P$ and $b \notin P$:

Then $P = a_1 b_1 \dots a_r b_r a \dots b'$ and we take $P' = a_1 b_1 \dots a b$

this is an alternating path ending in b: finish proof as above. $\square$

[restart 11:14]

For a subset $S \subseteq A$, let $N(S) = \cup_{v \in S} N(v)$ be the set of vertices in B which are neighbours of some vertex in S.

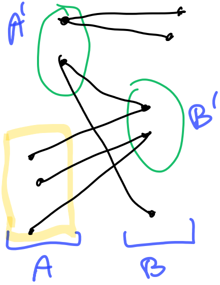


Theorem 2.1.2 (Hall 1935)
Let G be a bipartite graph.
Then G contains a matching of A if and only if

$$|N(S)| \geq |S| \qquad \text{for all } S \subseteq A. \tag{1}$$

(This condition is called "Hall's condition".)

Proof. We have that this condition is necessary.
Now suppose that (1) holds. For a contradiction, suppose that G has no matching of A
Then König's Theorem (Theorem 2.1.1) says that G has a cover $\mathcal{U}$ with $|\mathcal{U}| < |A|$ Suppose that $\mathcal{U} = A' \cup B'$ with $A' \subseteq A$ and $B' \subseteq B$



Then $|A'| + |B'| = |\mathcal{U}| < |A|$,
so $|B'| < |A| - |A'| = |A - A'|$

Since $\mathcal{U}$ is a cover, G has no edges from $A - A'$ to $B - B'$.
Hence $N(A - A') \subseteq B'$, and so

$|N(A - A')| \leq |B'| < |A - A'|$.

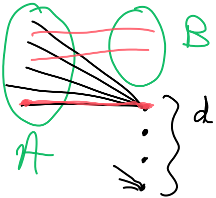
This contradicts Hall's condition (1) for $S = A - A'$. Hence G contains a matching of A

$\square$

Corollary

Let G be a bipartite graph and let $d \in \mathbb{N}$. If $|N(S)| \geq |S| - d$ for all $S \subseteq A$ then G has a matching of size $|A| - d$.

Proof. Add d new vertices to B and join each of them by an edge to each vertex of A



Then for all $S \subseteq A$, in the new graph G',

$$|N_{G'}(S)| \geqslant |S| - d + d = |S|$$

so Hall's condition is satisfied in G'.
Therefore there is a matching M in G' which matches all of A.
At least $|A| - d$ edges in M are edges of G.

$\square$

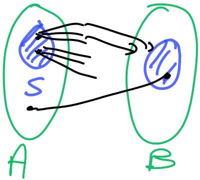
Assume $k \geq 1$

Corollary 2.1.3

If G is a k-regular bipartite graph then G has a perfect matching.

Proof. Since G is k-regular,

$|E(G)| = k|A| = k|B|$, so $|A| = |B|$.



Hence it suffices to prove that G contains a matching of A

Every set $S \subseteq A$ is joined to $N(S)$ by a total of $k|S|$ edges. These edges are a subset of the $k|N(S)|$ edges incident with $N(S)$. Hence

$$k|S| \leq k|N(S)|,$$ and dividing by k shows that Hall's condition holds

Hence G has a matching of A □

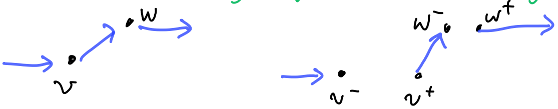
Hall's Theorem gives a very nice proof of the following:

Corollary 2.1.5 (Petersen, 1891)
Every regular graph of positive even degree has a 2-factor.

Proof. (not necessarily bipartite!)

Let G be any $2k$-regular graph, $k \geq 1$.
Without loss of generality, suppose that G is connected (or apply this argument to each component).
By Theorem 1.8.1, G has an Euler tour $v_0 v_1 \dots v_{\ell-1} v_\ell$ where $v_\ell = v_0$, $e_i = v_i v_{i+1} \in E(G)$ using each edge exactly once.

Replace each vertex $v \in V$ with a pair of vertices v^-, v^+, and replace every edge $e_i = v_i v_{i+1}$ by the edge $v_i^+ v_{i+1}^-$.

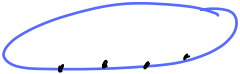


(directions inherited from Euler tour).

The resulting graph G' is a k-regular bipartite graph.
Hence by Corollary 2.1.3, G' has a perfect matching (1-factor).

Collapse every vertex pair (v^-, v^+) back into a single vertex v, for all $v \in V$
The 1-factor of G' becomes a 2-factor of G $\square$

$\square$



10. Hamilton cycles

A Hamilton cycle is a connected 2-factor.

That is, it is a cycle which includes every vertex.

Deciding whether or not a graph has a Hamilton cycle is an NP-complete problem.

Say G is Hamiltonian if it contains a Hamilton cycle.

A Hamiltonian graph G must be connected with minimum degree $\delta(G) \geq 2$.

Note: no constant lower bound on $\delta(G)$ is sufficient to guarantee a Hamilton cycle: exercise (see Problem Sheet 2).

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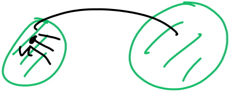
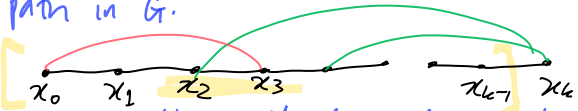


Theorem 10.1.1 (Dirac, 1952)

Every graph with $n \geq 3$ vertices and with minimum degree at least $n/2$ has a Hamilton cycle.

Proof. Let G be a graph with minimum degree $\geq n/2$ and $n \geq 3$ vertices. Then G is connected, as otherwise the degree of any vertex in the smaller component must be $< n/2$. Let $P = x_0 \dots x_k$ be a longest path in G.



By maximality, all neighbours of x_0 and x_k lie on P. So at least $n/2$ of the vertices $x_0, \dots, x_{k-1}$ are adjacent to x_k and at least $n/2$ of these same vertices satisfy $x_0 x_{i+1} \in E(G)$. By the pigeonhole principle, as $k < n$, there exists $i \in \{0, \dots, k-1\}$ with $x_0 x_{i+1}, x_i x_k \in E(G)$

This gives a cycle $x_0 x_1 \dots x_i x_k \dots x_{i+1} \dots x_0$

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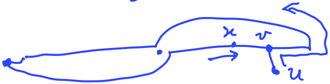
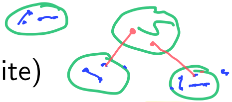


We claim this is a Hamilton cycle.

If not then, as G is connected, there is some $u \notin C$ with a neighbour $v \in C$. Then we can start at u, go to v then go around C (in some direction) & stop just before we reach v again (ie stop at $x \in N_C(v)$). This gives a path which is longer than P, contradiction!

$\square$



2.2 Matchings in *general* graphs (need not be bipartite)

Given a graph G, let C_G be the set of its components and let $q(G)$ denote the number of odd components (that is, the number of components of odd order).

Theorem 2.2.1 (Tutte, 1947)
A graph G has a perfect matching if and only if

$$q(G-S) \leq |S| \qquad \text{for all } S \subseteq V(G). \tag{2}$$

Proof. We have seen that condition (2) is necessary: If G has a p.m then (2) holds.
Now suppose that G has no p.m. We want to find a "bad" set S_0 which fails condition (2)

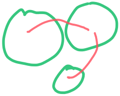
□

If $|G|$ is odd then $S_0 = \emptyset$ is bad.

So assume $|G|$ is even.

Claim 1: If G' is obtained from G by adding edges and $S_0 \subseteq V$ is bad for G' then S_0 is bad for G.

Proof: If S_0 bad for G' then $q(G'-S_0) > |S_0|$.

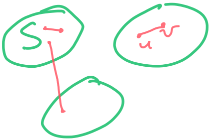


But each odd component of $G'-S$ is a disjoint union of components of $G-S$, at least one of which must be odd. So $q(G-S) \geq q(G'-S)$.

Hence by Claim 1, we can assume that G has no pm but adding any edge to G gives a graph G' which has a p.m.

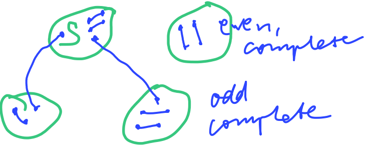
CLAIM 2: S is a bad set for G if and only if

(*) all components of $G-S$ are complete and every vertex in S is adjacent to all other vertices in G.



Proof if S is bad for G but does satisfy $(*)$ Then we can add an edge to G to get a graph G' with S still bad for G'. This contradicts our assumption on the maximality of G.

Conversely, suppose S satisfies $(*)$ but S is not bad.
$(q(G-S) \leq |S|)$



Then we can form a perfect matching (as shown), since $|G|$ is even. This is a contradiction as G has no perfect matching. Hence S is bad.

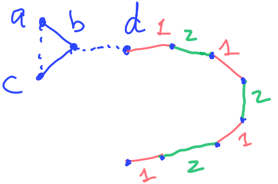
$\square$ [proof of Claim 2]

Define

$S_0 = \{v \in V : d_G(v) = n-1\}$ to be the set of all vertices v in G which are adjacent to every vertex $w \neq v$.

Claim 3 S_0 is bad.

Proof. We need to show that S_0 satisfies (*).
For a contradiction, suppose that S_0 does not satisfy (*).
Then $G - S_0$ has a component K which is not complete.
Let $a, a' \in V(K)$ with $aa' \notin E(G)$.
Fix a shortest path from a to a' in K which starts $abc \dots a'$. Such a path has length ≥ 2 and $ac \notin E(G)$.



Note $b \in K$, so $\boxed{b \notin S_0}$, so there is some $d \in V$ with $bd \notin E$.

By maximality of G, there is a p.m. M_1 in $G + ac$ and a p.m. M_2 in $G + bd$.

Take a maximal path P in G, starting at d with an edge M_1, and taking alternately edges from M_1 and M_2. Say $P = d \dots v$.

- If the last edge of P is in M_1 then $v = b$, or we could extend P.

FIXING TYPO FROM LECTURES

Let $C = P + bd$ (cycle in $G + bd$) in this case

- If the last edge of P is in M_2 then $v \in \{a, c\}$ as the M_1 edge incident with v must be ac.

Let C be the cycle $\underbrace{d \ldots v}_{P} b d$. [Can also write $dPvbd$.]

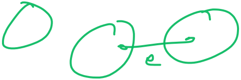
In each case, C is an alternating (even length) cycle in $G + bd$ which contains bd.

Form M_2' from M_2 by replacing $M_2 \cap C$ by $C - M_2$.

This gives a p.m of G, a contradiction.

Hence S_0 satisfies $(*)$, so Claim 3 holds & the proof is complete. □

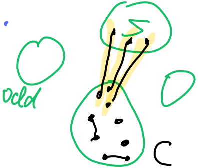
$\square$



We say a graph is cubic if it is 3-regular.

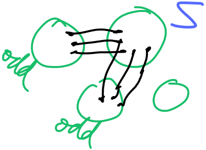
Corollary 2.2.2 (Petersen, 1891)
Every bridgeless cubic graph has a perfect matching.

Proof. Let G be a bridgeless cubic graph. We prove that G satisfies Tutte's condition. Let $S \subseteq V(G)$ be given and consider an odd component C of $G-S$.



The sum of the degrees of vertices in C is $3|C|$, which is an odd number. Every edge with both endvertices in C contributes an even number to this sum. Hence the number of edges from C to S is odd. As G has no bridge, there must be at least 3 edges from S to C.

Therefore the number of edges from S to $G-S$ is at least $3 \cdot q(G-S)$. But the number of edges from S to $G-S$ is bounded above by the sum of the degrees of vertices in S, which is $3|S|$ as G is cubic



Hence $\quad 3q(G-S) \leq \#\binom{\text{edges from}}{S \text{ to } G-S} \leq 3|S|$

and thus $\quad q(G-S) \leq |S|$.

Therefore by Tutte's Theorem (Theorem 2.2.1), G has a perfect matching

$\square$

[End of Chapter 2. Try Problem Sheet 2.]

Example (application of Hall's Theorem)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

Let m, n be positive integers with $m \le n$. An $m \times n$ matrix with entries from $\{1, 2, \ldots, n\}$ is a <u>Latin rectangle</u> if there are no repeated entries in any row or column.

Show that it is always possible to add rows to a Latin rectangle to produce an $n \times n$ Latin square.

<u>Proof</u> Let X be an $m \times n$ Latin rectangle, where $m < n$. Define the bipartite graph G with vertex set $\underbrace{\{c_1, \ldots, c_n\}}_{\text{[columns]}} \cup \underbrace{\{1, \ldots, n\}}_{\text{[symbols]}}$, and with edge set

$$\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \hline ? & ? & \cdots & ? \end{bmatrix}$$

$$\{(c_j, s) \in \{c_1, \ldots, c_n\} \times \{1, \ldots, n\} : \text{symbol } s \text{ does not appear in column } j \text{ of } X\}.$$

Observe that a matching of $\{c_1, \ldots, c_n\}$ in G is a perfect matching $M = \{(c_j, s_j) : j = 1, \ldots, n\}$ and it corresponds to a valid way to create the $(m+1)^{th}$ row: $[s_1 \; s_2 \ldots \; s_n]$

(You can $\boxed{\text{check}}$ that this gives a valid Latin rectangle of size $(m+1)\times n$, since M is a matching [row is OK] & by construction of G [all columns OK])

We claim that G is $(n-m)$-regular:

- Each column of X contains m distinct symbols, so c_j has degree $n-m$ in G, for $j = 1, .., n$
- Each symbol occurs in every row of X, and each occurrence is in a distinct column so symbol s has occurred in exactly m columns of X and hence s has degree $n-m$ in G, for all $s \in \{1, .., n\}$

By Corollary 2.1.3 to Hall's Theorem, since G is $(n-m)$-regular and bipartite with $n-m > 0$, G has a perfect matching. $\square$