**SCHOOL OF MATHEMATICS AND STATISTICS**
**UNSW Sydney**

## MATH5425 Graph Theory        Term 1, 2025

## Problem Sheet 4, Colourings

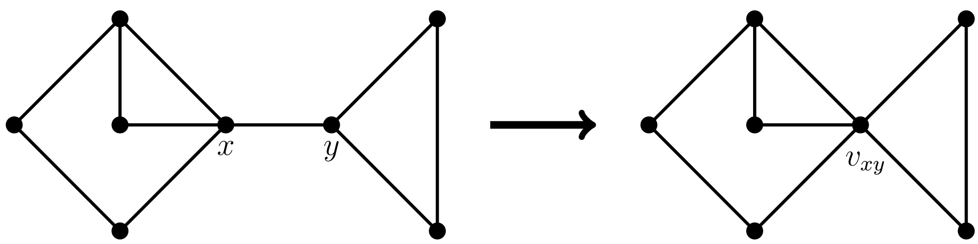
1. Check that $\chi(G) = \Delta(G) + 1$ if $G$ is a complete graph or an odd cycle.

2. Let $G$ be a graph with $n$ vertices. Prove that

$$n/\alpha(G) \leq \chi(G) \leq n + 1 - \alpha(G),$$

where $\alpha(G)$ is the independence number of $G$.

3. Let $G = (V, E)$ be a graph with $n$ vertices. Given $xy \in E$, let $G' = G - xy$ and form $G''$ from $G$ by *contracting* the edge $xy$ to a new vertex $v_{xy}$, as shown below:




Let $P_G(k)$ be the number of distinct $k$-colourings $c : V \to \{1, \ldots, k\}$ of $G$. (This means distinct as functions from $V \to \{1, \ldots, k\}$, taking the vertex labels into account.)

   (a) Find a formula relating $P_G(k)$ to $P_{G'}(k)$ and $P_{G''}(k)$.

   (b) By induction on $m = |E(G)|$, prove that $P_G(k)$ is a polynomial of degree $n$ in $k$, where the coefficient of $k^n$ is 1 and the coefficient of $k^{n-1}$ is $-m$. (We call $P_G(k)$ the *chromatic polynomial* of $G$.)

4. Let $G$ be a connected graph which is not regular. Suppose that any connected graph $H$ with fewer vertices than $G$ is $\Delta(H)$-colourable, unless $H$ is an odd cycle or a complete graph. Prove, without using Brooks Theorem, that $G$ is $\Delta(G)$-colourable.

(... Please turn over for Questions 5, 6 and 7)

5. A *latin square* is an $n \times n$ array such that each row and each column contain the integers $\{1, \ldots, n\}$ once each. Model the problem of constructing an $n \times n$ latin square into a graph colouring problem. How many vertices do you need? (Consider edge colourings as well as vertex colourings.)

6. Let $G$ be a cubic (3-regular) graph with a 3-edge-colouring

$$c : E(G) \to \{1, 2, 3\}.$$

   (a) Prove that each colour class of $c$ forms a perfect matching.

   (b) Now suppose that up to permutation of the colours $1, 2, 3$, the 3-edge-colouring $c$ is unique. (That is, any other 3-edge-colouring gives the same partition of the edge set of $G$ into colour classes.) Show that $G$ is Hamiltonian.

7. Without using Proposition 5.3.1 (König 1916), show that $\chi'(G) = k$ for all $k$-regular bipartite graphs $G$.