SCHOOL OF MATHEMATICS AND STATISTICS
UNSW Sydney
MATH5425 Graph Theory Term 1, 2025
Problem Sheet 1, Introduction

1. Find a graph which is isomorphic to its complement. (Is yours the smallest non-trivial such graph?)

2. The idea of this question is to get you thinking about diameter: you are not expected to supply any proofs.

   (a) Let G be a connected graph on n vertices. Find upper and lower bounds on the diameter of G, and find graphs which attain these bounds.

   (b) Now suppose that G is a connected graph with n vertices and maximum degree d. Find examples of graphs with high diameter and with low diameter, and express the diameter as a function of n and d.

3. Let G be a graph and let u, v be distinct vertices of G.

   (a) Prove that if G contains a walk from u to v then G contains a path from u to v.

   (b) Suppose that G contains a closed walk which starts and ends at v. Is it true that G must contain a cycle?

   (c) Suppose that P_1 and P_2 are distinct paths from u to v. Prove that the graph $P_1 \cup P_2$ contains a cycle.

4. We will prove a result called Mantel's theorem: any graph with n vertices and more than $\lfloor n^2/4 \rfloor$ edges contains a triangle (that is, a cycle of length 3). In fact, we prove the contrapositive.

   (a) Let $G = (V, E)$ be a graph on n vertices with no triangle. Prove that $d(x) + d(y) \leq n$ whenever $xy \in E$.

   (b) Hence prove that $\sum_{x \in V} d(x)^2 \leq n|E|$.

   (c) Using the Cauchy-Schwarz inequality and (b), show that $|E| \leq n^2/4$.

   (d) For any integer $n \geq 2$, describe a graph with n vertices, no triangle and exactly $\lfloor n^2/4 \rfloor$ edges. (*Hint: use Question 5.*)

   (*Recall that the Cauchy-Schwarz inequality for* $\mathbb{R}^N$ *says that*

$$\left(\sum_{j=1}^{N} a_j b_j \right)^2 \leq \left(\sum_{j=1}^{N} a_j^2 \right) \left(\sum_{j=1}^{N} b_j^2 \right).$$

   for any $(a_1, \ldots, a_N)$, $(b_1, \ldots, b_N) \in \mathbb{R}^N$.)

(Please turn over for Questions 5 – 9...)

5. Prove that a graph is bipartite if and only if it contains no odd cycles. (A cycle is *odd* if its length is odd.)

6. Prove Euler's theorem: a connected graph is Eulerian if and only if every vertex has even degree. (*Hint: consider a walk of maximal length which uses no edge more than once. Show it is closed and is an Euler tour.*) Try to avoid looking at Diestel's proof (Proposition 1.8.1).

7. Let $G = (V, E)$ be a graph and let $e \in E$. Prove that e is a bridge if and only if e does not belong to any cycle in G.

8. In this question we show that $\kappa(G) \leq \lambda(G) \leq \delta(G)$ for any graph G with at least two vertices. (That is, *connectivity* $\leq$ *edge-connectivity* $\leq$ *minimum degree.*) Our argument follows Diestel's proof (Proposition 1.4.2) but please try it yourself first!

   (i) First prove $\lambda(G) \leq \delta(G)$.

   (ii) Prove that $\kappa(G) \leq \lambda(G)$ when G is complete.

   Now suppose that G is not complete and that F is a set of $\lambda(G)$ edges such that $G - F$ is disconnected. That is, F is a minimal separating set of edges.

   (iii) Explain why no edge of F joins two vertices from the same component of $G - F$.

   We now split into two cases. For (iv), assume that some vertex v of G is not incident with any edge of F: call this Case 1. Let H be the connected component of $G - F$ which contains v.

   (iv) Find a set of at most $\lambda(G)$ vertices in H which separate v from $G - H$, and hence conclude that $\kappa(G) \leq \lambda(G)$ in Case 1.

   For the rest of the proof, assume that Case 1 fails. Then every vertex of G is incident with an edge of F. Call this Case 2. Choose a vertex v which has at least one non-neighbour w in G.

   (v) Prove that $d_G(v) \leq \lambda(G)$.

   (vi) Hence conclude that $\kappa(G) \leq \lambda(G)$ in Case 2, completing the proof.

9. Prove the theorem on trees stated in lectures (Theorem 1.5.1), namely that for any graph G the following are equivalent:

   (i) G is a tree.

   (ii) There exists a unique path between any two distinct vertices of G.

   (iii) G is minimally connected (that is, G is connected but if you delete any edge, it is not connected).

   (iv) G is maximally acyclic (that is, G is acyclic but adding any edge between two non-neighbours of G creates a cycle).

   (*Hint: prove that each condition is equivalent to* (ii).)