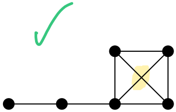
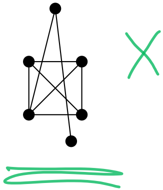
Chapter 6. Planar Graph

The main reference for this section is Diestel Graph Theory, Chapter 4.

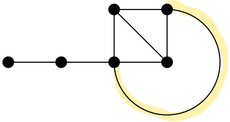
Q: What is the best way to draw a graph on a page?

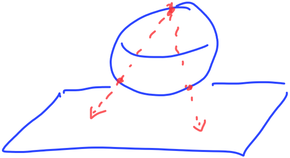




A graph which is drawn in the plane so that no two edges meet except at common endvertices is called a plane graph.
(We will define this concept more formally below.)

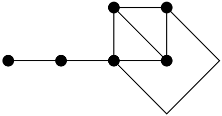
An abstract graph which can be drawn in this way is called planar.

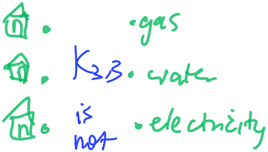




A graph is drawn in the Euclidean plane $\mathbb{R}^2$ by representing each vertex by a point and each edge by a curve between two distinct points.

To avoid complications we restrict to curves which are piecewise linear.





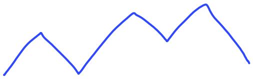
$K_{3,3}$ is not planar!

K_5 is not planar

4.2 Plane graphs

We skip the proofs of some topological statements that are proved in Diestel, Chapter 4.

An arc (or polygonal arc) is a subset of $\mathbb{R}^2$ composed of the union of finitely many straight line segments, which is homeomorphic to $[0, 1]$ (so it doesn't cross itself).



A plane graph is a pair (V, E) of finite sets (with elements of V called vertices and elements of E called edges) such that

(i) $V \subseteq \mathbb{R}^2$;

(ii) Every edge is an arc between two distinct vertices (that is, no loops);

(iii) Different edges have different sets of endvertices (that is, no repeated edges);

embedding of a simple graph

(iv) The interior of an edge contains no vertex and no point of any other edge.

planar!

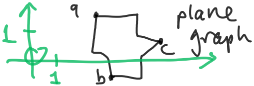
Here the interior of an edge/arc e, denoted by $\mathring{e}$, is the arc minus its endpoints: if e is the arc from x to y then $\mathring{e} = e - \{x, y\}$.

e 

closed arc

$\Rightarrow$ Interior $\mathring{e}$. 

open



Abstract graph: 

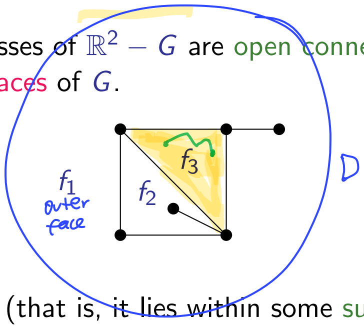
A plane graph defines a graph G in a natural way. We use the name G for the abstract graph, the plane graph and the point set

$$V \cup \left(\bigcup_{e \in E} e \right) \subseteq \mathbb{R}^2.$$

The point set of a plane graph G is a closed set in $\mathbb{R}^2$, and $\mathbb{R}^2 - G$ is open.

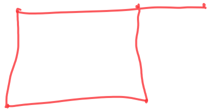
Two points in an open set O are equivalent if they are equal or they can be linked by an arc in O. This is an equivalence relation. ✓

The equivalence classes of $\mathbb{R}^2 - G$ are open connected regions, called the faces of G.



Since G is bounded (that is, it lies within some sufficiently large disc $D \subseteq \mathbb{R}^2$), exactly one face of G is unbounded: it is the face that contains $\mathbb{R}^2 - D$.

We call the unbounded face the outer face of G. All other faces of G are called inner faces.

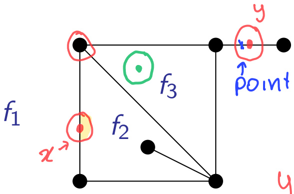


bdry on f_1

Let $F(G)$ be the set of faces of G.

(topological)

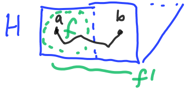
The boundary of a face f is called the frontier of f. It is the set of all points $y \in \mathbb{R}^2$ such that every neighbourhood of y meets both f and $\mathbb{R}^2 - f$.



point on arc: not in f_1

x lies on boundary of f_1 and the boundary of f_2

y lies on boundary of f_1



Lemma 4.2.1.

Let G be a plane graph with subgraph $H \subseteq G$ and face $f \in F(G)$.

(i) There is a face $f' \in F(H)$ which contains f (that is, $f \subseteq f'$).

(ii) If the frontier of f lies in H then $f' = f$.

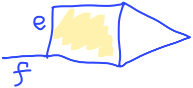
Proof. (i) Points in f are also equivalent in $\mathbb{R}^2 - H$, so they belong to an equivalence class f' of $\mathbb{R}^2 - H$. That is, $f \subseteq f'$ and $f' \in F(H)$.

(ii) We prove the contrapositive. Suppose that f is a proper subset of f' ($f \subsetneq f'$). Choose points $a \in f$ and $b \in f' - f$. Both a and b belong to f' in $\mathbb{R}^2 - H$, so there is an arc between them in $\mathbb{R}^2 - H$.

But a and b are not equivalent in $\mathbb{R}^2 - G$, as $a \in f$ and $b \notin f$. So the arc must meet a point x on the frontier X of f, and $x \notin H$ as $x \in f' \subseteq \mathbb{R}^2 - H$.

Therefore $X \notin H$. $\square$

$\square$



Lemma 4.2.2

Let G be a plane graph and let e be an edge of G.

(i) If X is the frontier of a face of G then either $e \subseteq X$ or $X \cap \mathring{e} = \emptyset$.

(ii) If e lies on a cycle $C \subseteq G$ then e lies on the frontier of exactly two faces of G, and these are contained in the distinct faces of C.

(iii) If e does not lie on a cycle then e lies on the frontier of exactly one face of G.

(Proof omitted.)

Corollary 4.2.3
The frontier of a face of a plane graph G is always the point set of a subgraph of G.

(Proof omitted.)

The subgraph of G whose point set is the frontier of a face f is said to bound f and is called the boundary of f. Denote this subgraph by $G[f]$.

A face is said to be incident with the vertices and edges of its boundary.

By Lemma 4.2.1(ii), every face of G is also a face of its boundary.

Proposition 4.2.4

A plane forest has exactly one face.

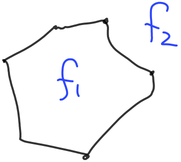
Proof. Exercise: Use induction on the number of edges.

Lemma 4.2.5

If a plane graph has two distinct faces with the same boundary then the graph is a cycle.

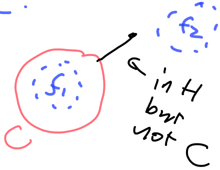
Proof. Let G be a plane graph and let f_1, f_2 be distinct faces of G with the same boundary

$$H = G[f_1] = G[f_2].$$



Since f_1, f_2 are also faces of H, Proposition 4.2.4 implies that H contains a cycle C. [H is not a forest.]

CLAIM 1: $H = C$. For a contradiction, suppose that H has a vertex or edge which is not in C. This additional vertex or edge of H lies in one of the faces of C and hence cannot lie on the boundary of whichever f_i is contained in the other face of C.



$\diamond$

Thus f_1 and f_2 are exactly the two distinct faces of C. Hence $f_1 \cup C \cup f_2 = \mathbb{R}^2$. But

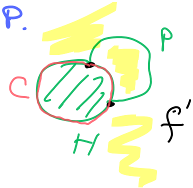
$$\mathbb{R}^2 = f_1 \cup C \cup f_2 \subseteq f_1 \cup G \cup f_2 \cup \{\text{other faces of } G\} = \mathbb{R}^2,$$

and therefore $G = C$. $\square$

Proposition 4.2.6

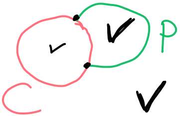
In a 2-connected plane graph, every face is bounded by a cycle.

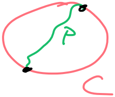
Proof. Let f be a face in a 2-connected plane graph G.
We proceed by induction using Proposition 3.1.1.
If G is a cycle then the result is true.
Now assume that G is not a cycle. Then, by Proposition 3.1.1, there is a 2-connected plane subgraph H of G and a plane H-path P such that $G = H \cup P$.
By the inductive hypothesis, every face of H is bounded by a cycle.



The interior of P lies in a face f' of H, and f' is bounded by a cycle C.
If f is a face of H then we are done ✓
Otherwise, the frontier of f intersects $P - H$, so $f \subseteq f'$.

Therefore f is a face of $C \cup P$ and hence f is bounded by a cycle, by observation $\square$





Note: the order is slightly different to Diestel for the next few results.

Theorem 4.2.9 (Euler's Formula, 1752)
Let G be a connected plane graph with n vertices, m edges and ℓ faces. Then

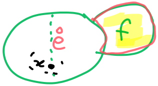
$$n - m + \ell = 2.$$

Proof. Fix n and apply induction on m. For $m \leq n-1$ then, as G is connected we must have $m = n-1$ and G is a tree. Then the result follows from Proposition 4.2.4 (check!).

Now suppose that $m \geqslant n$. Then G has an edge e which belongs to a cycle. Let $G' = G - e$ which is a connected plane graph

By Lemma 4.2.2 (ii), e lies on the boundary of exactly two distinct faces f_1 and f_2 of G.

There is a face f_e of G' which contains $\mathring{e}$, since all points of $\mathring{e}$ are equivalent in $\mathbb{R}^2 - G'$.



CLAIM:

$$F(G) - \{f_1, f_2\} = F(G') - \{f_e\}. \qquad -(*)$$

Note that if the claim holds then G' has exactly one less face and exactly one less edge than G. So the result for G follows by the formula for G', which holds by induction: $n - (m-1) + (\ell - 1) = 2$.]

Proof of Claim: First let $f \in F(G) - \{f_1, f_2\}$. By Lemma 4.2.2(i) $G[f] \subseteq G - \mathring{e} = G'$ and hence $f \in F(G')$, by Lemma 4.2.1(ii). Also $f \neq f_e$ as $\mathring{e} \subseteq f_e$ but $\mathring{e} \cap f = \emptyset$. So $f \in F(G') - \{f_e\}$, proving "$\subseteq$" part of $(*)$.

Next let $f' \in F(G') - \{f_e\}$. Then $f' \neq f_1, f_2$ (as open sets): for any $x \in \mathring{e}$, any open set around x intersects both f_1 and f_2, But there are open sets containing x which are disjoint from f', as $\mathring{e} \subseteq f_e$, f_e open, f' and f_e are disjoint.

Also $f' \cap \mathring{e} = \phi$ as $\mathring{e} \subseteq f_e$ and f_e is disjoint from f'

Hence every pair of points in f' belong to $\mathbb{R}^2 - G$, and they are equivalent in $\mathbb{R}^2 - G$.

Thus G has a face f which contains f'.

By Lemma 4.2.1(i), f is contained in a face f'' of G'

Hence

$$\underset{\in F(G')}{f'} \subseteq \underset{\in F(G)}{f} \subseteq \underset{\in F(G')}{f''}$$

Therefore $f' = f''$ (faces of G' which overlap must be equal)

and $f' = f \in F(G)$. So $f' \in F(G) - \{f_1, f_2\}$, completing the proof of the claim $\square$

$\square$

Warning: Euler's Formula only works for connected graphs.

Corollary. The graphs K_5, $K_{3,3}$ are not planar.

Proof. For a contradiction, suppose that K_5 is planar. Any planar embedding of K_5 must have ℓ faces where

$$5 - 10 + \ell = 2,$$

by Euler's Formula (note that K_5 is connected!) Rearranging gives $\ell = 7$.

But K_5 is 2-connected and hence every face is bounded by a cycle (of length at least 3), by Proposition 4.2.6. Also, every edge of G lies on the boundary of exactly two faces, as K_5 has no bridges and using Lemma 4.2.2(ii).

We will double count elements of the set

$$S = \{(e, f) : e \in E(K_5),\ f \in F(K_5),\ e \subseteq G[f]\}.$$

(incident edge-face pairs). We get

$$3\ell \leq |S| = 2 \times \underbrace{10}_{\text{number of edges}}.$$

Hence $\ell \leq 20/3 < 7$, contradiction!

So K_5 is not planar.

Similarly, as $K_{3,3}$ is connected, any planar embedding of $K_{3,3}$ would have ℓ faces, where

$$6 - 9 + \ell = 2 \quad \text{by Euler's formula (Theorem 4.2.9)}$$

So $\ell = 5$. Also, every face is bounded by a cycle of length at least 4, as $K_{3,3}$ is 2-connected and bipartite (using Proposition 4.2.6) and every edge is incident with exactly 2 faces, as above.

Double counting incident (edge, face) pairs gives

$$4\ell \leq 2 \times 9, \quad \text{so } \ell \leq \frac{9}{2} < 5.$$

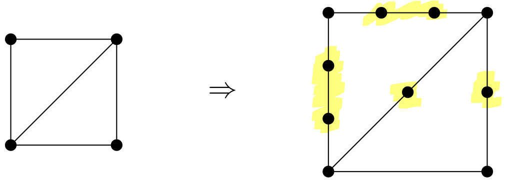
This contradiction shows that $K_{3,3}$ is not planar.

$\square$

$\square$

We now discuss Kuratowski's Theorem, one of the highlights of graph theory.

A subdivision of a graph G is obtained by replacing each edge of G by an independent path between its endvertices.



Kuratowski's Theorem (1930) says that a graph G is planar if and only if no subgraph of G is a subdivision of K_5 or $K_{3,3}$.

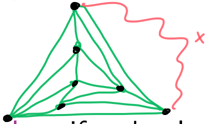
(Proof omitted: beyond the scope of the course.)

G → delete vertices
→ delete edges
→ contract edges
$\Longrightarrow$ minor of G.

Wagner: replace "subdivision" by "minor".

A plane graph G is maximally plane (or just maximal) if we cannot add a new edge to form a new plane graph G' with $V(G') = V(G)$ such that $E(G')$ strictly contains $E(G)$.

Call G a plane triangulation if every face of G (including the outer face) is bounded by a triangle.



Proposition 4.2.8

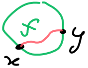
A plane graph of order at least 3 is maximally plane if and only if it is a plane triangulation.

Proof. [Different to Diestel's proof.] Let G be a plane graph with $|G| \geq 3$. First suppose that G is a plane triangulation. Then G is maximally plane. any additional edge e would have its interior completely within a face f of G, and the endvertices of e would lie on the boundary of f. But all these edges are already present as $G[f] \cong K_3$, which is complete, and repeated edges are not allowed.

For the converse, suppose that G is maximally plane. Let $f \in F(G)$ be a face and let $H = G[f]$.

Claim 1: The induced subgraph $G[H]$ is complete.

If not, say vertices x, y of $G[H]$ are not adjacent in G.



But we can add an edge through the face f between x and y, giving a plane graph with more edges than G. This contradicts maximality of G ◇

Hence $G[H] = K_r$ for some r. Then $r \le 4$ as K_5 is not planar. Note: H might not be complete (that is, H might not be an induced subgraph of G.)

Claim 2: H contains a cycle.

If not, then H is a forest. Either $r \geq 3$ and $H \subsetneq K_r = G[H] \subseteq G$, or $r = 2$ and $|G| \geq 3$ while $|H| = r = 2$.

In either case, $H \neq G$. But by Proposition 4.2.4, H has exactly one face f and hence $f \cup H = \mathbb{R}^2$

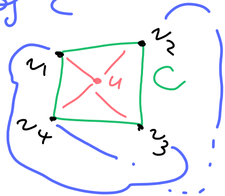
Therefore $G = H$, contradiction! $\mathbb{R}^2 = f \cup H \subseteq f \cup G \cup \{\text{all other faces of } G\} = \mathbb{R}^2$

CLAIM 3 $r=3$, and hence $H = K_3$ (and we are done!)

We know that $r \leq 4$, and by Claim 2 we have $r \geq 3$. So it is enough to rule out $r=4$.

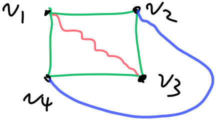
For a contradiction, suppose that $r=4$ and let $V(H) = \{v_1, v_2, v_3, v_4\}$. Without loss of generality let $C = v_1 v_2 v_3 v_4 v_1$ be a cycle in H (note, H contains a cycle by Claim 2: how do we know it is a 4-cycle?) Exercise

Since $C \subseteq G$, by Lemma 4.2.1 (i), The face f is contained within a face f_C of C. Let f_C' be the other face of C.



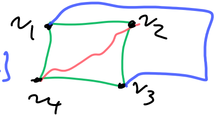
FACT: Edges v_1v_3 and v_2v_4 lie in different faces of C. If not, we can add a new vertex u in the face of C which does not contain these edges, and add edges uv_1, uv_2, uv_3, uv_4 giving a plane embedding of K_5, contradiction

But, since v_1 and v_3 lie on $G[f]$, they can be linked by an arc whose interior lies in f_C and which avoids G.



Hence the plane edge v_2v_4 of $G[H]$ goes through f_C', not f_C.

Similarly, since v_2 and v_4 lie on $G[f]$, they can be linked by an arc whose interior lies in f_C and which avoids G.

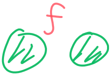


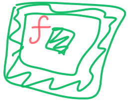
Hence the plane edge v_1v_3 of $G[H]$ runs through f_C', not f_C.

This contradicts our FACT. Hence $r \neq 4$, so $r = 3$ and Claim 3 holds.

So every face of G is bounded by a 3-cycle.

$\square$





Corollary 4.2.10

A plane graph with $n \geq 3$ vertices has at most $3n - 6$ edges.
Every plane triangulation has $3n - 6$ edges.

Proof. By Proposition 4.2.8 it suffices to prove the second statement. Let G be a plane triangulation. If G was disconnected then at least one face of G must have a disconnected boundary. But all faces of G are bounded by 3-cycles, so G is connected.

Next, every edge lies on the boundary of some face, which is a 3-cycle. So every edge of G belongs to a cycle and hence lies on the boundary of exactly two faces. Furthermore, every face boundary has exactly 3 edges. Let $n = |G|$, $m = |E(G)|$ and $\ell = |F(G)|$. Double-counting incident (edge-face) pairs gives $3\ell = 2m$.

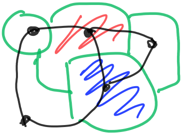
Thus $\ell = \frac{2m}{3}$. Substituting this into Euler's formula, as G is connected, gives

$$n - m + \frac{2m}{3} = 2.$$

Hence $m = 3(n-2)$
$= 3n-6$,

as required. $\square$

$\square$



5.1 Colouring maps

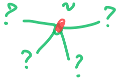
This is the famous Four Colour Theorem:

Theorem 5.1.1
Every planar graph is 4-colourable. (That is, there exists a proper 4-colouring of the vertices of any planar graph.)

This question arose in 1852.

In 1976, Appel and Haken gave a "computer-assisted" proof, with 1476 cases to check.

In 1996, Robertson, Sanders, Seymour and Thomas reduced the number of "unavoidable configurations" to be checked down to 633. This is still a computer-assisted proof.



Let's prove this one:

Proposition 5.1.2

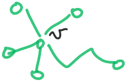
Every planar graph is 5-colourable.

Proof. Let G be a plane graph with n vertices and m edges.
If $n \le 5$ then 5-colouring is easy. So we assume that $n \geq 6$
Assume by induction that every plane graph with at most $n-1$ vertices can be 5-coloured
By Corollary 4.2.10, the average degree of G satisfies

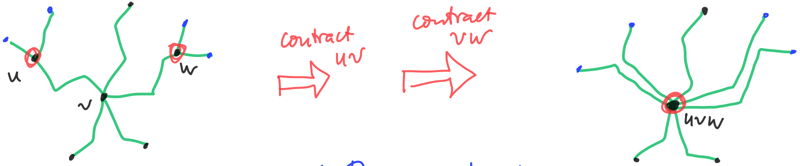
$$\bar{d}(G) = \frac{2m}{n} \le \frac{2(3n-6)}{n} < 6.$$

Hence G has at least one vertex of degree ≤ 5. Let v be a vertex of G with degree ≤ 5 If $d_G(v) \le 4$ then by induction we can 5-colour $G-v$ and extend this colouring to a 5-colouring of G by choosing a colour for v which does not appear on $N(v)$.

So we can assume that $d_G(v) = 5$.



Note, some pair of distinct neighbours $u, w \in N(v)$ must not be adjacent, as K_5 is not planar



Contract the edge uv and then contract the edge vw, preserving planarity (see Problem Sheet 6)

This gives a plane graph $\hat{G}$ with $n-2$ vertices.

By induction, $\hat{G}$ is 5-colourable. Let $\hat{c}$ be a 5-colouring of $\hat{G}$.

We can define a 5-colouring c of $G - v$ by

$$c(x) = \begin{cases} \hat{c}(x) & \text{if } x \notin \{u, w\}, \\ \hat{c}(uvw) & \text{if } x \in \{u, w\}. \end{cases}$$

Now at most 4 colours appear on $N(v)$ under c, so we can colour v with a missing colour to give a 5-colouring of G

This completes the proof, by induction. $\square$

Here's a nice result which we will not prove.

Theorem 5.1.3 (Grötzsch, 1959)
Every planar graph which does not contain a triangle is 3-colourable.

[End of Chapter 6. Try Problem Sheet 6.]