9.2 Solutions to Chapter 2

Exercises 2.1

1. $\gamma(0) \geq 0$, since
$$\gamma(0) = cov(X_t, X_t) = var(X_t) \geq 0.$$

2. We know that $|\rho(h)| \leq 1$ and, consequently, $\left|\frac{\gamma(h)}{\gamma(0)}\right| \leq 1$ or equivalently $|\gamma(h)| \leq |\gamma(0)|$. Using the result of Part (i), we can conclude that $|\gamma(h)| \leq \gamma(0)$.

3.
$$\begin{aligned}\gamma(h) &= cov(X_{t+h}, X_t)\\ &= cov(X_t, X_{t+h})\\ &= \gamma(-h)\end{aligned}$$

   Note that this equality does not hold in multivariate time series.

4. We know that variance is a non-negative term, therefore
$$\begin{aligned}0 \leq var\left(\sum_{j=1}^{n} a_j X_{t-j}\right) &= cov\left(\sum_{j=1}^{n} a_j X_{t-j}, \sum_{k=1}^{n} a_k X_{t-k}\right)\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j\ cov\left(X_{t-j}, X_{t-k}\right)\ a_k\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j\ \gamma_X(j-k)\ a_k\\ &= \sum_{j=1}^{n}\sum_{k=1}^{n} a_j\ a_k\ \gamma_X(j-k)\end{aligned}$$

Exercises 2.2

1. Based on the definition of $\rho(h)$, we have
$$\rho(0) = \frac{\gamma(0)}{\gamma(0)} = 1.$$

2. The absolute value of correlation is less than or equal to 1; i.e., $|\rho(h)| \leq 1$.

3. The autocorrelation function is even; i.e.,
$$\begin{aligned}\rho(h) &= \frac{\gamma(h)}{\gamma(0)}\\ &= \frac{\gamma(-h)}{\gamma(0)}\\ &= \rho(-h)\end{aligned}$$

Exercises 2.3

We know that

$$X_t = 2\cos\left(\frac{2\pi(t+15)}{50}\right) + W_t, \quad W_t \sim WN(0,\sigma^2).$$

Therefore,

$$\begin{aligned} cov(X_t, X_s) &= cov\left(2\cos\left(\frac{2\pi(t+15)}{50}\right) + W_t, 2\cos\left(\frac{2\pi(s+15)}{50}\right) + W_s\right) \\ &= cov\left(W_t, W_s\right) \\ &= \begin{cases} \sigma^2 & t = s \\ 0 & t \neq s \end{cases}. \end{aligned}$$

Exercises 2.4

$$\begin{aligned} var(\bar{X}) &= var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2} var\left(\sum_{i=1}^{n} X_i\right) \\ &= \frac{1}{n^2} cov\left(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} X_j\right) \\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} cov\left(X_i, X_j\right) \end{aligned}$$

By setting $cov\left(X_i, X_j\right) = \gamma(i-j)$, we have

$$var(\bar{X}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma(i-j)$$

Let $i - j = h$, consequently, we can rewrite the last equality as

$$\begin{aligned} var(\bar{X}) &= \frac{1}{n^2}\sum_{h=-(n-1)}^{n-1}(n-|h|)\gamma(h) \\ &= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\frac{n-|h|}{n}\gamma(h) \\ &= \frac{1}{n}\sum_{h=-(n-1)}^{n-1}\left(1 - \frac{|h|}{n}\right)\gamma(h) \end{aligned}$$

Exercises 2.5

(**i**) To establish weak stationarity of X_t, we need to calculate the mean, variance and covariance functions for this process.

$$\begin{aligned} E(X_t) &= E(Z_t Z_{t-1}) \\ &= E(Z_t)E(Z_{t-1}) \\ &= 0 \end{aligned}$$

Besides,

$$\begin{aligned} var(X_t) &= var(Z_t Z_{t-1}) \\ &= E(Z_t^2 Z_{t-1}^2) - E(Z_t Z_{t-1}) \\ &= E(Z_t^2 Z_{t-1}^2) \\ &= E(Z_t^2)E(Z_{t-1}^2) \\ &= 1 \end{aligned}$$

Finally,

$$\begin{aligned} cov(X_t, X_{t-h}) &= E(X_t X_{t-h}) - E(X_t)E(X_{t-h}) \\ &= E(X_t X_{t-h}) \\ &= E(Z_t Z_{t-1} Z_{t-h} Z_{t-h-1}) \\ &= \begin{cases} var(X_t) & h = 0 \\ E(Z_t)E(Z_{t-1}^2)E(Z_{t-2}) & h = 1 \\ E(Z_t)E(Z_{t-1})E(Z_{t-h})E(Z_{t-h-1}) & h > 1 \end{cases} \\ &= \begin{cases} 1 & h = 0 \\ 0 & h \geq 1 \end{cases} . \end{aligned}$$

Since mean and variance do not depend on t and covariance is a function of h, we can conclude that X_t is weakly stationary.

(**ii**)

$$\begin{aligned} (X_{t_1}, \cdots, X_{t_n}) &= f(Z_{t_1-1}, Z_{t_1}, \cdots, Z_{t_n-1}, Z_{t_n}) \\ &= f(Z_{t_1+h-1}, Z_{t_1+h}, \cdots, Z_{t_n+h-1}, Z_{t_n+h}) \qquad (*) \\ &= (X_{t_1+h}, \cdots, X_{t_n+h}) \end{aligned}$$

The equality $(*)$ holds since Z_t is strictly stationary.

Hint: Any time invariant function of a strictly stationary process is strictly stationary.

(**iii**) Since $E(X_t) = 0$ and

$$\gamma_X(h) = \begin{cases} 1 & h = 0 \\ 0 & h \neq 1 \end{cases}$$

Therefore , X_t is a white noise process.

(**iv**) Let $W_t = X_t^2$. Note that if $X \sim N(0,1)$, then $E(X) = 0$, $E(X^2) = 1$, $E(X^3) = 0$ and $E(X^4) = 3$. Therefore,

$$\begin{aligned} E(W_t) &= E(X_t^2) \\ &= E(Z_t^2 Z_{t-1}^2) \\ &= E(Z_t^2)E(Z_{t-1}^2) \\ &= 1 \times 1 = 1. \end{aligned}$$

Besides,

$$\begin{aligned} var(W_t) &= var(X_t^2) \\ &= E(X_t^4) - \big(E(X_t^2)\big)^2 \\ &= E(Z_t^4 Z_{t-1}^4) - \big(E(Z_t^2 Z_{t-1}^2)\big)^2 \\ &= E(Z_t^4)E(Z_{t-1}^4) - \big(E(Z_t^2)E(Z_{t-1}^2)\big)^2 \\ &= 3 \times 3 - (1 \times 1)^2 \\ &= 8 \end{aligned}$$

Moreover, for $h \geq 1$, we have

$$\begin{aligned} cov(W_t, W_{t-h}) &= cov(X_t^2, X_{t-h}^2) \\ &= E(X_t^2 X_{t-h}^2) - E(X_t^2)E(X_{t-h}^2) \\ &= E(Z_t^2 Z_{t-1}^2 Z_{t-h}^2 Z_{t-h-1}^2) - 1 \times 1 \\ &= \begin{cases} E(Z_t^2 Z_{t-1}^4 Z_{t-2}^2) - 1 \times 1 & h = 1 \\ E(Z_t^2 Z_{t-1}^2 Z_{t-h}^2 Z_{t-h-1}^2) - 1 \times 1 & h > 1 \end{cases} \\ &= \begin{cases} E(Z_t^2)E(Z_{t-1}^4)E(Z_{t-2}^2) - 1 \times 1 & h = 1 \\ E(Z_t^2)E(Z_{t-1}^2)E(Z_{t-h}^2)E(Z_{t-h-1}^2) - 1 \times 1 & h > 1 \end{cases} \\ &= \begin{cases} 1 \times 3 \times 1 - 1 \times 1 & h = 1 \\ 1 \times 1 \times 1 \times 1 - 1 \times 1 & h > 1 \end{cases} \\ &= \begin{cases} 2 & h = 1 \\ 0 & h > 1 \end{cases} \end{aligned}$$

(**v**) Based on Part (iv), it is easy to show that

$$\rho_W(h) = \begin{cases} 1 & h = 0 \\ \frac{1}{4} & h = 1 \\ 0 & h > 1 \end{cases}$$

(**vi**) When we talk about independence of a sequence, it means that we need pairwise independence along with independence in the whole sequence. In this question we know that $cov(X_t^2, X_{t+1}^2) = 2 \neq 0$, therefore we can conclude that $\{X_t^2\}$ and, consequently, $\{X_t\}$ are not independent.

Hint:

- If X and Y are independent $\Rightarrow cov(X,Y) = 0$.

- If $cov(X, Y) = 0 \nRightarrow X$ and Y are independent. (Except when you have the normality assumption).
- If $cov(X, Y) \neq 0 \Rightarrow X$ and Y are NOT independent.

(**vii**) In Part (iii), we show that X_t is white noise, while, in Part (vi), we show that the are not independent. This is a contradiction with normality assumption.