In [8]:
```
import torch
import torch.nn as nn
import torch.autograd as autograd
```

In [9]:
```
import numpy as np
import matplotlib.pyplot as plt
```

In [10]:
```
device_name = torch.device('cuda' if torch.cuda.is_available() else 'cpu')
```

In [11]:
```
def analytic_solution(t, x_0=1.0, v_0=0.0):
    return x_0 * np.cos(t) + v_0 * np.sin(t)
```

Architecture for Initial Conditions Fixed Nueral Network

In [21]:
```
class HarmonicModel(nn.Module):
    def __init__(self, x0, v0):
        super().__init__()
        self.network = nn.Sequential(
            nn.Linear(1, 64),
            nn.Tanh(),
            nn.Linear(64, 64),
            nn.Tanh(),
            nn.Linear(64, 1)
        )
        self.x0 = x0
        self.v0 = v0

    def forward(self, t):
        a = self.network(t)
        return self.x0 + self.v0 * t + a * t**2
```

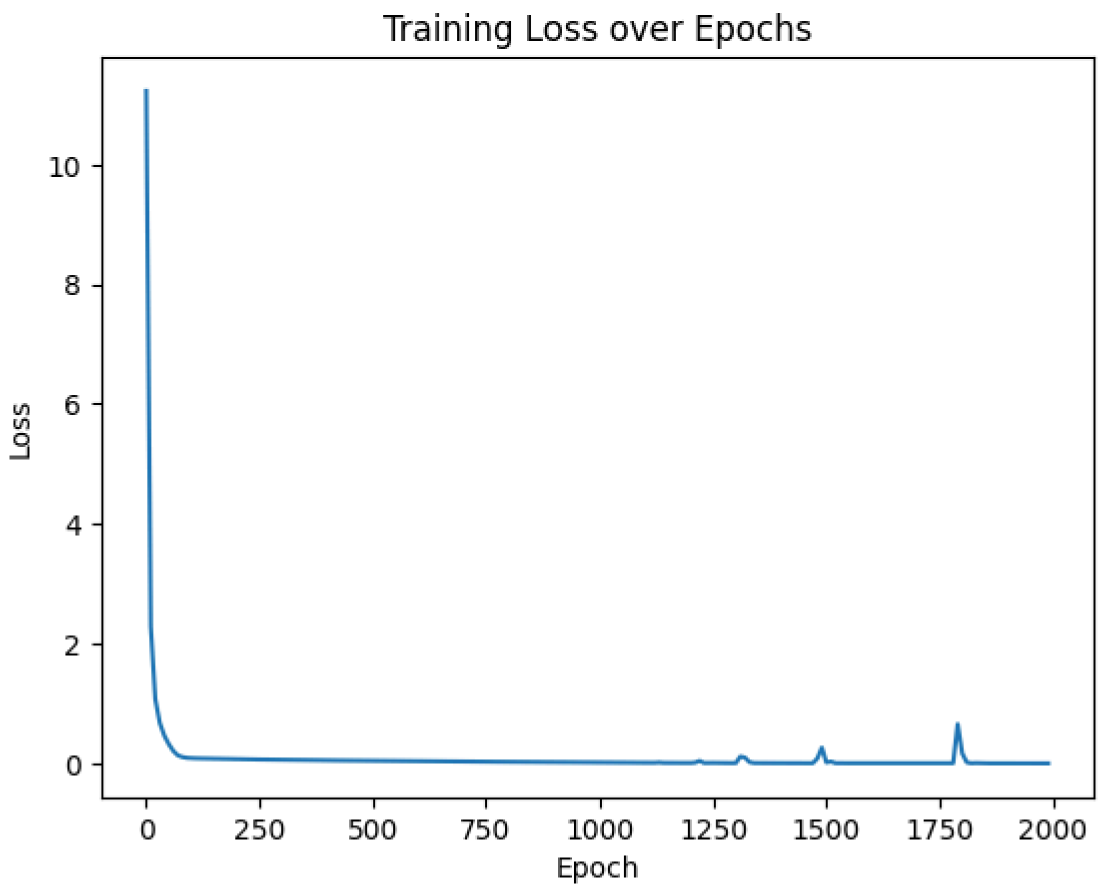
In [25]:
```
def train_model(model):
    num_samples = 200
    epochs = 2000
    lr = 1e-3
    optimizer = torch.optim.Adam(model.parameters(), lr=lr)

    values = torch.linspace(0, 2 * np.pi, num_samples, device=device_name).view(-1, 1).requires_grad_()
    loss_history = []
    for epoch in range(epochs):
        optimizer.zero_grad()
        x_predicted = model(values)

        dx = autograd.grad(x_predicted, values, torch.ones_like(x_predicted), create_graph=True)[0]
        d2x = autograd.grad(dx, values, torch.ones_like(dx), create_graph=True)[0]

        loss = torch.mean((d2x + x_predicted) ** 2)
        loss.backward()
        optimizer.step()

        if epoch % 500 == 0:
            print(f"Epoch {epoch}: Loss = {loss.item():.6f}")

        if epoch % 10 == 0:
            loss_history.append((epoch, loss.item()))

    return loss_history
```

In [26]:
```
x0 = 1.0
v0 = 1.0
model = HarmonicModel(x0, v0).to(device_name)
```

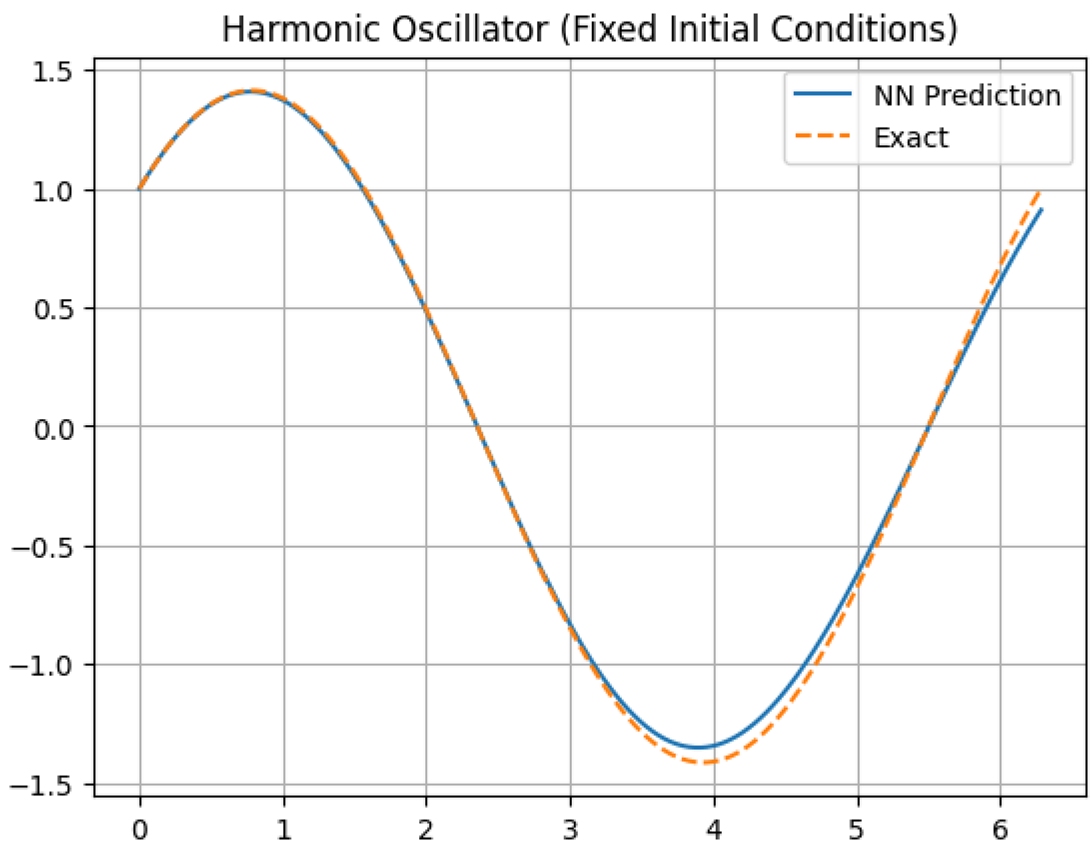
In [27]:
```
loss_history = train_model(model)
plt.plot(*zip(*loss_history), label='Training Loss')
plt.xlabel('Epoch')
plt.ylabel('Loss')
plt.title('Training Loss over Epochs')
```

```
Epoch 0: Loss = 11.233688
Epoch 500: Loss = 0.045333
Epoch 1000: Loss = 0.016686
Epoch 1500: Loss = 0.017797
```

Out[27]:
```
Text(0.5, 1.0, 'Training Loss over Epochs')
```



In [28]:
```
validation_values = torch.linspace(0, 2 * np.pi, 300, device=device_name).view(-1, 1)
with torch.no_grad():
    x_nn = model(validation_values).cpu().numpy()

x_true = analytic_solution(validation_values.cpu().numpy(), x_0=x0, v_0=v0)

plt.plot(validation_values.cpu(), x_nn, label='NN Prediction')
plt.plot(validation_values.cpu(), x_true, '--', label='Exact')
plt.title("Harmonic Oscillator (Fixed Initial Conditions)")
plt.legend()
plt.grid()
plt.show()
```



Architecture for Boundary Conditions Fixed Neural Network

In [48]:
```
class PeriodicResidualNet(nn.Module):
    def __init__(self, hidden_dim=32):
        super().__init__()
        self.base = nn.Sequential(
            nn.Linear(2, hidden_dim),
            nn.Tanh(),
            nn.Linear(hidden_dim, hidden_dim),
            nn.Tanh(),
            nn.Linear(hidden_dim, 1)
        )
        self.a0 = nn.Parameter(torch.tensor([0.0]))
        self.a1 = nn.Parameter(torch.tensor([1.0]))
        self.a2 = nn.Parameter(torch.tensor([0.0]))

    def forward(self, t):
        base_part = self.a0 + self.a1 * torch.cos(t) + self.a2 * torch.sin(t)
        trig_input = torch.cat([torch.sin(t), torch.cos(t)], dim=1)
```

```python
        residual = self.base(trig_input)
        return base_part + residual
```

In [49]:
```python
def second_derivative(model, t):
    t.requires_grad_(True)
    x = model(t)
    dx = torch.autograd.grad(x, t, torch.ones_like(x), create_graph=True)[0]
    d2x = torch.autograd.grad(dx, t, torch.ones_like(dx), create_graph=True)[0]
    return d2x

# Model and optimizer
model = PeriodicResidualNet().to(device_name)
optimizer = torch.optim.Adam(model.parameters(), lr=5e-4)
```

In [50]:
```python
epochs = 2000
N = 256
t_train = torch.linspace(0, 2 * np.pi, N, device=device).reshape(-1, 1)

for epoch in range(epochs):
    optimizer.zero_grad()
    d2x = second_derivative(model, t_train)
    x_hat = model(t_train)
    loss = torch.mean((d2x + x_hat)**2)  # ODE residual loss

    loss.backward()
    optimizer.step()

    if epoch % 500 == 0:
        print(f"Epoch {epoch} | Loss: {loss.item():.8f}")

# Evaluate
t_eval = torch.linspace(0, 2 * np.pi, 1000, device=device_name).reshape(-1, 1)
x_nn = model(t_eval).detach().cpu().numpy()
x_true = analytic_solution(t_eval.cpu().numpy())

# Plot
plt.figure(figsize=(10, 4))
plt.plot(t_eval.cpu(), x_true, '--', label='Exact Solution')
plt.plot(t_eval.cpu(), x_nn, label='NN Prediction')
plt.xlabel('t')
plt.ylabel('x(t)')
plt.title('Classical Harmonic Oscillator with Periodic Boundary Conditions')
plt.legend()
plt.grid(True)
plt.tight_layout()
plt.show()
```

```
Epoch 0 | Loss: 0.07109271
Epoch 500 | Loss: 0.00000445
Epoch 1000 | Loss: 0.00000115
Epoch 1500 | Loss: 0.00000046
```