MATHEMATICAL FOUNDATION-IV

Time Allowed : 3 Hours *Maximum Marks : 80*

Note : Attempt five questions in all, selecting ***one*** question from each unit in addition to compulsory **Question No. 9**. All questions carry equal marks.

UNIT-I

1. (a) If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

(b) Examine for maximum and minimum values of the function

$$xy + \frac{a^3}{x}+\frac{b^3}{y},\ a > 0,\ b > 0$$ 8,8

2 (a) If u and v are functions of x and y defined by $x = u + e^{-v} \sin u$ and $y = v + e^{-v} \cos u$, find

$$\frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}.$$

(d) Find the minimum value of the function $x^2 + y^2 + z^2$ subject to the condition $x + y + z = 3a$.

8,8

UNIT-II

3. (a) Find the intrinsic equaion of the curve $p = r \sin \alpha$.

(b) Find the whole leng:.i of the curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$ 8,8

4. (a) $u_n = \int_0^2 \theta \sin^n \theta d\theta (n > 1)$; prove that :

$$u_n = \frac{n-1}{n} u_{n-2} + \frac{1}{n^2}$$

and deduce that $u_5 = \frac{149}{225}$

(b) Rectify the loop of the curve $3ax^3 = y(y-a)^2$.

8,8

UNIT-III

5. (a) Evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y} dy\, dx$

(b) Find the surface area of sphere of radius 5.

8,8

6. (a) Find the common area to the parabola $y^2 = 4ax$ and $x^2 = 4ay$.

(b) Evaluate :

$$\iiint (z^5 + z)\, dx\, dy\, dz \text{ over } x^2 + y^2 + z^2 \leq 1.$$ 8,8

UNIT-IV

7. (a) If $\alpha^2 < 1$, prove that :

$$\int_0^{\pi/2} \log(1 - \alpha^2 \cos^2 \theta)\, d\theta = \pi \log\left(\frac{1 + \sqrt{1-\alpha^2}}{2}\right)$$

(b) Prove that

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m)$$ 8,8

8. (a) Find the equation of right circular cylinder, whose guiding circle is $x^2 + y^2 + z^2 = 4, x + y + z = 3$.

(b) Find the centre of the two spheres, which touch the plane $x + 2y + 2z - 5 = 0$ at the point (1, 1, 1) and the sphere $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$.
8,8

UNIT-V

9. (a) Prove that :

$$\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}.$$

(b) Define Beta and Gamma functions.

(c) Evaluate :

$$\int_0^1\int_0^1 e^{x+y}dx\,dy$$

(d) If $u = x(1 - y)$ and $v = xy$, find $\frac{\partial(x,y)}{\partial(u,v)}$

(e) If $u = xy\, f(y/x)$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial v}{\partial y}=2u$$

(f) Find the centre of the sectionof the sphere $x^2 + y^2 + z^2 = 49$ by the plane $2x + 3y + 6z = 4$.

(g) Show that the two spheres

$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$ and

$x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$

cur orthogonally.

(h) Find the equation to the cylinder with generator parallel to z-axis and passing through the curve

$x^2 + y^2 + 2z^2 = 12$ and

$x + y + z = 1$ $2 \times 8 = 16$