REAL ANALYSIS

SECTION - I

1. (a) Prove that the set of real numbers x such that $0 \leq x \leq 1$ is not countable. 10

   (b) Define Least upper bound of a set. Prove that least upper bound of a set, if exists, is unique. 2+8

2. (a) Prove that a set A is a neighbourhood of ξ iff there exists n $\in$ N such that

$$\left(\xi - \frac{1}{n}, \xi + \frac{1}{n}\right) \subseteq A$$ 10

   (b) Define Closed set. A is a closed set iff A contains all its limit points. 1+9

3. (a) If a sequence of non-negative terms converges to a, then prove that a is also non-negative. 10

   (b) Show that a sequence is convergent if and only if and only if it is a cauchy sequence. 10

4. (a) $\sum_{n=1}^{\infty} a_n$ is convergent, then prove that $\underset{n\to\infty}{Lt}\, a_n = 0$. Is the converse also true? 7+3

   (b) Test the series

$$1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \ldots\ldots\ldots$$

   for convergence for x>0. 10

SECTION - II

5. (a) If for $|x| < 1$, the series

   $a_0 + a_1x + a_2x^2 + \ldots\ldots. + a_n x^n + \ldots\ldots.$

   is absolutely convergent to A (x), then show that

$$(1-x)^{-1}(A(x)) = \sum_{n=0}^{\infty} S_n x^n$$

where $S_n = a_0 + a_1 + \ldots\ldots\ldots\ldots + a_n$. 10

(b) Show that every absolutely convergent infinite product is convergent. 10

6. (a) Examine the continuity of the function $f(x) = \begin{cases} \frac{x-|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

at the origin. 10

(b) Define uniform continuity of a function.

Prove that $f: R \to R$ defubed by $f(x) = x^2, \ \forall x \in R$ is uniform continuous.

7. (a) State and prove Rolle's theorem. 10

(b) Use Lagrangian mean value theorem to prove that

$$1 + x < e^x < 1 + e^x, \forall x > 0$$ 10

SECTION - III

8. (a) Show by Riemann integration that $\int_1^2 f(x)\,dx = 6$, where $f(x) = 2x + 3$

(b) State and prove Fundamental theorem of Integral calculus.

9. (a) Show that the integral $\int_0^\infty x^{n-1} e^{-x}\,dx$ is convergent if and only if $n > 0$. 10

(b) If a sequence of continuous functions $<f_n>$ is uniformaly convergent to a function f on [a,b] then show that f is continuous in [a,b]. 10

10. (a) Define Beta and Gamma functions. Establish the relation between these functions. 1+1+8

(b) Prove that

$$\int_0^{\pi/2} \log\left(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta\right) d\theta = \pi \log\left(\frac{\alpha+\beta}{2}\right), \alpha > 0, \beta > 0$$