REAL ANALYSIS

SECTION-A

1. (a) Prove that between two different real numbers there are infinitely many rational numbers. 10

   (b) Define (i) an interior point of a set (ii) open set (iii) closed set. Prove that interior of a set is an open set. 2+2+2+4

2. (a) Prove that every convergent sequence is bounded. Is the converse true? Give suitable example. 6+2+2

   (b) Define a Cauchy's sequence. Prove that $<(-1)^n>$ is not a Cauchy's sequence. 2+8

3. (a) State and prove Gauss test for convergence of series. 2+8

   (b) Show that $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$ is convergent. 10

4. (a) State only : Cauchy's General princip[le of convergence for sequences. 4

   (b) Prove that every real number is a limit point of Q, the set of rational numbers. 8

   (c) Discuss the convergence of series $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{\frac{1}{n}}$. 8

SECTION-II

5. (a) Find Cauchy product of series $\sum_{n=1}^{\infty} bn = 2+2+2^2+2^3+..... \text{ and } \sum_{n=1}^{x} bn = -1+1+1+....$

(b) Prove : If the product $\prod_{n=1}^{x}(1+an)$ is convergent then $\underset{n\to\infty}{Lt}\ a_n = 0$.

6. (a) Show that the function: $f(x)=\begin{cases} 1 \text{ if } x \text{ is rational} \\ -1 \text{ if } x \text{ is irrational} \end{cases}$ is discountionuous for every real number. 9

(b) Define uniform continuity of a function. 2

(c) Prove that every continuous function definded on a closed interval attains its bounds. 9

7. (a) State and prove Lagrange's Mean value theorem. 2+8

(b) Using Taylor's Theorem expand cos x as infinite series in x. 10

SECTION - III

8. (a) Prove that the function f given by

$$f(x)=\begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{cases}$$

is not integrable in [a, b]. 10

(b) Prove that every monotonic function defined on closed interval [a, b] is integrable. 10

9. (a) If p and q are positive show that

(i) $$\beta(p,q+1)=\frac{q}{p}\beta(p+1,q)$$ 6

(ii) $\beta(p,q)=\beta(q,p)$. 6

(b) Prove $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$. 8

10. (a) Test the convergence of imp[roper integral

$$\int_1^\infty \frac{dx}{x^{1/2}(5+x)^{1/3}}.$$ 10

(b) Using $\int_0^\infty \frac{dx}{x^2+a^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$ prove that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2}=\frac{1}{2a^3}\tan^{-1}\frac{x}{a}+\frac{x}{2a^2(x^2+a^2)}$$ 10