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Concept Review

Steady-State Error

Why Steady-State Error?

The Steady-State Error of a control system is an important metric which can be used to analyze the time domain properties of a stable control system. It can be calculated analytically by computing the relevant transfer function, or experimentally from a measured system response. It quantifies the difference between the desired value and the measured value of a variable at Steady-State.

1. Background

The Steady-State Error is the difference between the reference or desired value of a variable and the measured value or output of a system in the limit as time goes to infinity or when the system response has reached its steady-state. The steady-state error depends on the type of reference value that is provided to the system.

Consider a unity feedback loop as in Figure 1 where $r(t)$ is the reference value, $e(t)$ is the error between the measured value or output $y(t)$ and the reference value, $P(s)$ is the plant transfer function of the system and $C(s)$ is the controller that is used to control the system.

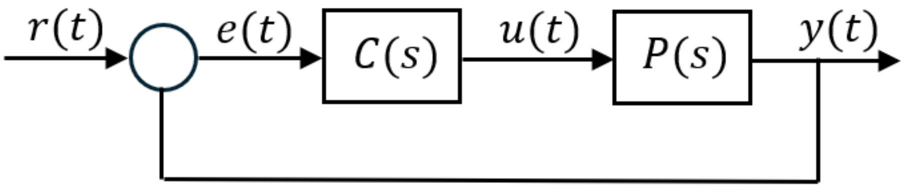

Figure 1. Unity feedback loop.

From the diagram above and using Laplace transforms of the time domain signals, we can deduce the following:

$$E(s) = R(s) - Y(s) \tag{1.1}$$

$$U(s) = C(s)E(s)$$

$$Y(s) = P(s)U(s) = P(s)C(s)E(s) \tag{1.2}$$

$$\text{Then, } E(s) = R(s) - P(s)C(s)E(s)$$
$$\big(1 + P(s)C(s)\big)E(s) = R(s)$$

$$\frac{E(s)}{R(s)} = \frac{1}{1 + P(s)C(s)} \tag{1.3}$$

Equation (1.3) is the closed-loop transfer function from the reference signal to the error of the system. The steady-state error of a system where the reference signal is a ramp is given in Figure 2.

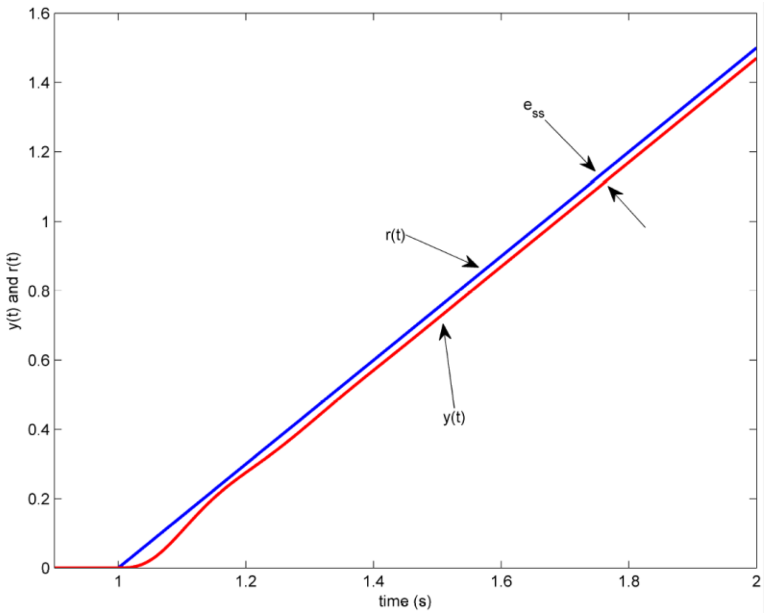

Figure 2. Steady-state Error in ramp response.

2. System Types

As mentioned previously, the steady-state error depends on the type of reference signal that is applied to the system. The concept of `System Types` classifies systems according to the type of reference signal that is applied to the system with specific importance on the degree of the polynomial of the reference signal.

3. Final Value Theorem

The Final Value Theorem can be used to determine the final or steady-state value of a variable in a control system. The final value theorem uses the Laplace transform of the given variable to determine its steady-state value. For an unstable system, the final value is unbounded. For a marginally stable system with complex conjugate poles on the imaginary axis, the final value is not defined. For a stable system, the final value is defined as follows:

$$y_{ss}(\infty) = \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \qquad 2.1$$

where $Y(s) = \mathcal{L}[y(t)]$ is the Laplace transform of $y(t)$. Using this information, let us classify the steady-state error (final value of the error) based on the type of reference signal applied to the system.

The steady-state error $e_{ss}(\infty)$ to a step reference input $R(s) = \frac{1}{s}$ is,

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+P(s)C(s)} R(s) = \lim_{s\to 0} \frac{1}{1+P(s)C(s)} = \frac{1}{1+\lim\limits_{s\to 0}[P(s)C(s)]} = \frac{1}{1+K_p} \qquad 2.2$$

where $K_p = \lim\limits_{s\to 0}[P(s)C(s)]$.

The steady-state error $e_{ss}(\infty)$ to a ramp reference input $R(s) = \frac{1}{s^2}$ is,

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+P(s)C(s)} R(s) = \lim_{s\to 0} \frac{1}{s+sP(s)C(s)} = \frac{1}{s+\lim_{s\to 0}[sP(s)C(s)]} = \frac{1}{K_v} \qquad 2.3$$

where $K_v = \lim_{s\to 0}[sP(s)C(s)]$.

And the steady-state error $e_{ss}(\infty)$ to a parabolic reference input $R(s) = \frac{1}{s^3}$ is,

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+P(s)C(s)} R(s) = \lim_{s\to 0} \frac{1}{s^2+s^2P(s)C(s)} = \frac{1}{s^2+\lim_{s\to 0}[s^2P(s)C(s)]} = \frac{1}{K_a} \qquad 2.4$$

where $K_a = \lim_{s\to 0}[s^2P(s)C(s)]$. The above results are summarized in the Table 1 below.

Type of Input	Step	Ramp	Parabolic
Type 0	$\frac{1}{1+K_p}$	∞	∞
Type 1	0	$\frac{1}{K_v}$	∞
Type 2	0	0	$\frac{1}{K_a}$

Table 1. System Types.

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