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Concept Review

Orientation and Position

How to Describe Orientation and Position?

Angular orientation or attitude of a vehicle in three-dimensional (3D) space is typically described using a body-fixed reference frame, which is attached to the vehicle body - translating and rotating with it. The orientation of that body-fixed reference frame relative to a globally fixed reference frame will represent the vehicle's attitude. There are multiple ways to represent or describe the orientation. Three common methods will be described in this document: Rotation Matrix, Euler Angle and Quaternions.

Finally, to capture both the position and rotation of a vehicle body in 3D space, the rotation matrix representation is expanded into a 4x4 Homogeneous Transformation matrix, which will also be introduced in this document.

Rotation Matrix

For three-dimensional (3D) motion, a rotation matrix R is a 3 x 3 matrix that represents the orientation (rotation) of a Cartesian reference frame with respect to the inertial reference frame. The 3-element column vectors of a rotation matrix can be interpreted as consisting of the Cartesian unit basis vectors $\vec{u}_x$, $\vec{u}_y$ and $\vec{u}_z$ expressed with respect to the inertial reference frame,

$$R \triangleq [\overrightarrow{u_x} \quad \overrightarrow{u_y} \quad \overrightarrow{u_z}] \tag{1}$$

This means the rotation matrix can be used to change the basis of a vector from being expressed with respect to the rotated reference frame $\vec{v}_1$ to being with respect to the inertial (original/unrotated) reference frame $\vec{v}_0$:

$$\overrightarrow{v_0} = R\,\overrightarrow{v_1} \tag{2}$$

For example, the x unit basis vector $\overrightarrow{u_x}$ of the rotated reference frame expressed with respect to the inertial frame $\overrightarrow{{}^0u_x}$ can be obtained by right multiplying the x unit vector to the rotation matrix R,

$$\overrightarrow{{}^0u_x} = R\begin{bmatrix}1\\0\\0\end{bmatrix} \tag{3}$$

To obtain the reverse, the rotation matrix can be inverted. For rotation matrices, which are orthogonal matrices with determinant equals to 1, its inverse is the same as its transpose, as such,

$$\overrightarrow{v_1} = R^{-1}\,\overrightarrow{v_0} = R^T\,\overrightarrow{v_0} \tag{4}$$

When the rotations are about the basis axes of the reference frame (i.e., x, y, or z axes), the resulting rotation matrices have the following basic forms:

$$R_x(\phi) = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi & \cos\phi\end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix} \tag{5}$$

$$R_z(\psi) = \begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

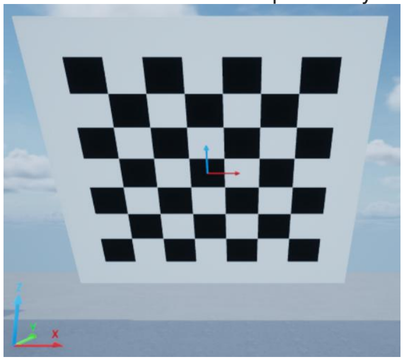
Rotation matrices for compound rotations, that is rotations obtained by successive rotations, can be obtained by multiplying the intermediate rotation matrices together. For example, the rotation matrix for a rotation sequence of first rotating about the z-axis by ψ, followed by a rotation about the resulting y-axis by θ can be obtained as follows:

$$R = R_z(\psi)R_y(\theta) \tag{6}$$

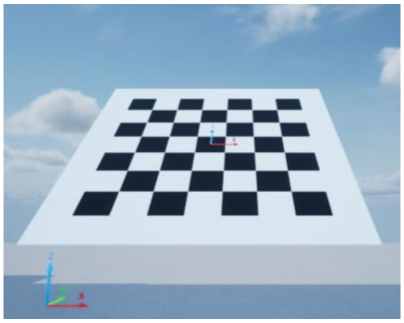
Euler Angles

Euler angles are a set of three (successive) rotation angles used to describe (parametrize) orientation of a rigid body. It was introduced by Leonhard Euler. There are many variations in the selection of the three angles based on the different combinations of 3 axes selected to describe the rotations. Also, the successive rotations can be described with respect to either fixed (extrinsic) or relative (intrinsic) axes/reference frames.

Rotations about each of the primary axis:

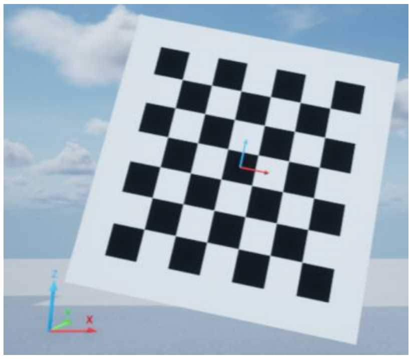


Positive rotation in x-axis

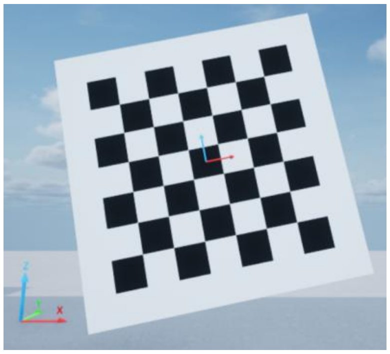


Negative Rotation in x-axis

Figure 1: Rotations about X (roll ϕ)

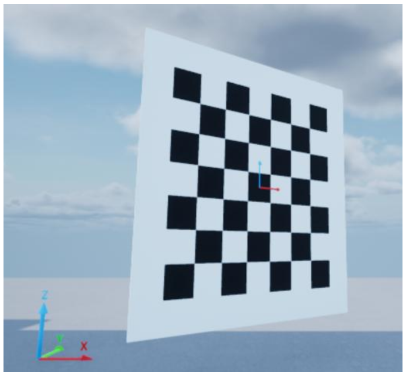


Positive rotation in y-axis

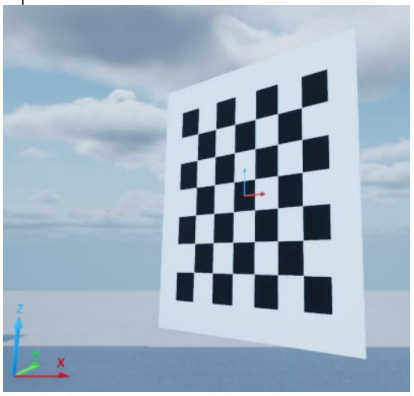


Negative rotation in y-axis

Figure 2: Rotations about y (pitch θ)



Positive rotation in z-axis



Negative rotation in z-axis

Figure 3: Rotations about z (yaw ψ)

For the selection of the combination of axes, they can generally be classified into two groups: Classic Euler and Roll-Pitch-Yaw Euler (also called Tait-Bryan) angles. In Classic Euler angles, the first and third rotation axes are the same, e.g., Z-X-Z rotation. In Roll-Pitch-Yaw Euler angles, the rotations are about different axes, e.g., Z-Y-X rotation, which is the common choice for Euler angles for unmanned vehicles,

$$R_{zyx} = R_z(\psi)R_y(\theta)R_x(\phi)$$

$$R_{zyx} = \begin{bmatrix} \cos\psi\cos\theta & cos\psi\sin\theta\sin\phi - \sin\psi\cos\phi & \cos\psi\sin\theta\cos\phi + \sin\psi\sin\phi \\ \sin\psi\cos\theta & \sin\psi\sin\theta\sin\phi + \cos\psi\cos\phi & \sin\psi\sin\theta\cos\phi - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{bmatrix} \tag{7}$$

Here, the rotation is obtained through relative (intrinsic) rotations - by first rotating about the z-axis by ψ, then about the resulting y-axis by θ and finally about the resulting x-axis by ϕ.

Quaternions

Quaternions are a set of four real numbers commonly used to represent rotations in three-dimensional (3D) space. It was first described by William Rowan Hamilton. A typical definition of a quaternion $\bar{q}$ for rotation has the following form:

$$\bar{q} = q_0 + q_1 i + q_2 j + q_3 k \tag{8}$$

The coefficients q_0, q_1, q_2, and q_3 are the quaternion parameters, and i, j and k are the quaternion basis/units with this property:

$$i^2 = j^2 = k^2 = ijk = -1 \tag{9}$$

A compact vector form is usually used to express the quaternion:

$$\bar{q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{10}$$

For rotation representation, the first quaternion parameter q_0 can be interpreted as the scalar part and the remaining 3 element parameters be interpreted as the vector part. A quaternion has only non-zero scalar part is called a scalar quaternion. Similarly, a vector quaternion has zero scalar part but non-zero vector part. With this interpretation, the conjugate $\bar{q}^*$ of the quaternion $\bar{q}$ can be expressed as having negative vector part of $\bar{q}$:

$$\bar{q}^* = \begin{bmatrix} q_0 \\ -q_1 \\ -q_2 \\ -q_3 \end{bmatrix} \tag{11}$$

Multiplication (product) between two quaternions $\bar{p} \otimes \bar{q}$ can be expressed as:

$$\bar{p} \otimes \bar{q} = (p_0 + p_1 i + p_2 j + p_3 k)(q_0 + q_1 i + q_2 j + q_3 k) \tag{12}$$

The right hand side can be expanded and simplified with the identity property in eq (9). This product is also referred to as the Hamilton product. Algebraically, this can be expressed in the form of a matrix multiplication between a matrix function $Q(\bar{p})$ of the first quaternion $\bar{p}$ and the second quaternion $\bar{q}$ expressed in vector form:

$$\bar{p} \otimes \bar{q} = Q(\bar{p})\bar{q} \tag{13}$$

Where the matrix function $Q(\bar{p})$ for a quaternion $\bar{p}$ is defined as:

$$Q(\bar{p}) = \begin{bmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \tag{14}$$

Notice that the quaternion product is non-commutative, i.e., $\bar{p} \otimes \bar{q} \neq \bar{q} \otimes \bar{p}$.

The norm $||\bar{q}||$ can be defined as the square root of the product between the quaternion and its conjugate:

$$||\bar{q}|| = \sqrt{\bar{q} \otimes \bar{q}^*} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \tag{15}$$

A quaternion with $||\bar{q}|| = 1$ is called an unit quaternion. To represent a rotation, a quaternion has to be an unit quaternion.

A quaternion inverse is defined as its conjugate divided by its norm squared:

$$\bar{q}^{-1} = \frac{\bar{q}^*}{||\bar{q}||^2} \tag{16}$$

Therefore, for unit quaternions, $\bar{q}^{-1} = \bar{q}^*$. The multiplication between a quaternion and its inverse would result in an unit scalar quaternion (with norm of 1) of the form:

$$\bar{q} \otimes \bar{q}^{-1} = \bar{q}^{-1} \otimes \bar{q} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

This means that when a quaternion $\bar{q}$ represents a rotation, its inverse $\bar{q}^{-1}$ represents the reverse rotation. When using to represent rotation or orientation in 3D space, the quaternion basis i, j and k can also be interpreted as the Cartesian basis vectors. This means a vector $\vec{v} = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T$ can be represented as a vector quaternion $\bar{q}_v = \begin{bmatrix} 0 & v_x & v_y & v_z \end{bmatrix}^T$. Then, a rotation represented by a quaternion $\bar{q}$ applying to a vector $\vec{v}_0$ will result in a rotated vector $\vec{v}_1$ given as follows:

$$\begin{bmatrix} 0 \\ \vec{v}_1 \end{bmatrix} = \bar{q} \otimes \begin{bmatrix} 0 \\ \vec{v}_0 \end{bmatrix} \otimes \bar{q}^{-1} \tag{18}$$

The above is referred to as the conjugate operation. Notice that both v_1 and v_0 are expressed with respect to the global inertial frame. To change the basis of a vector between rotated frames (represented by the quaternion $\bar{q}$), e.g., finding the global coordinate ${}^0\vec{r}$ of a vector expressed in the rotated frame ${}^1\vec{r}$, one would apply the reverse rotation $\bar{q}^{-1}$ to ${}^1\vec{r}$ to obtain ${}^0\vec{r}$:

$$\begin{bmatrix} 0 \\ {}^0\vec{r} \end{bmatrix} = \bar{q}^{-1} \otimes \begin{bmatrix} 0 \\ {}^1\vec{r} \end{bmatrix} \otimes \bar{q} \tag{19}$$

For successive rotations represented by $\bar{q}_1$, $\bar{q}_2$, ... $\bar{q}_n$, the resultant quaternion $\bar{q}$ can be expressed as:

$$\bar{q} = \bar{q}_n \otimes \cdots \otimes \bar{q}_2 \otimes \bar{q}_1 \tag{20}$$

A useful identity relationship for unit quaternion products, that is quaternion multiplication between unit quaternions $\bar{p}$ and $\bar{q}$ is:

$$(\bar{p} \otimes \bar{q})^* = \bar{q}^* \otimes \bar{p}^* \tag{21}$$

The rotation represented by the quaternion $\bar{q}$ can be related to the axis of rotation (represented by the unit vector $\hat{e}$) and the rotation angle θ, based on Euler's rotation theorem:

$$\bar{q} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \hat{e}\sin\frac{\theta}{2} \end{bmatrix} \tag{22}$$

Homogenous Transformation

A homogeneous transformation T is a 4×4 matrix encompassing both translational $\vec{p}$ and rotational R components. It has the following form:

$$T = \begin{bmatrix} R & \vec{p} \\ \vec{0}_{1\times 3} & 1 \end{bmatrix} \tag{23}$$

Where R is the rotation matrix and $\vec{p}$ is a position vector. The homogeneous transformation T represents the 6 degree-of-freedom (DOF) relative motion between two reference frame in 3D space. Similar to the rotation matrix, the homogeneous transformation matrix can be used to obtain the inertial coordinate ${}^0\vec{r}$ of a vector ${}^1\vec{r}$, expressed in a translated and rotated frame:

$$\begin{bmatrix} {}^0\vec{r} \\ 1 \end{bmatrix} = T \begin{bmatrix} {}^1\vec{r} \\ 1 \end{bmatrix} \tag{24}$$