

Concept Review

Routh-Hurwitz Criterion

Why Explore Routh-Hurwitz?

The Routh-Hurwitz method is a powerful mathematical tool that helps us determine if a control system is stable without having to actually solve for the roots of its characteristic equation. Instead of doing complex calculations to find all possible solutions, this method gives you a straightforward way to discover if any of those solutions would make your system unstable. It proves especially valuable when dealing with high-order systems where traditional root-solving approaches would be computationally demanding or impractical.

To determine the stability of a system, it is necessary to investigate:

- Whether or not all closed-loop poles are in the left half of the s -plane.
- If there is one or more poles in the right half plane, the system is unstable.
- If there is a pole on the imaginary axis, the system is marginally stable.

If and only if all poles are strictly in the left half plane the system is said to be stable.

Especially for higher order systems, it is not always feasible to find the exact locations of all system poles to determine the overall stability of the system. Instead, a classification of how many poles are in each side of the s -plane is sufficient.

1. Routh-Hurwitz Criterion

The Routh-Hurwitz Criterion is a two-step process to determine the system's stability *without* having to obtain exact pole locations. In the first step, a Routh table is created, which is then interpreted in the second step to find out how many poles are in the left half plane, on the imaginary axis, or in the right half plane. The major benefit of the Routh-Hurwitz Criterion is its inherent capability to determine the range for which unknown system parameters result in a stable closed-loop system response.

A unity feedback loop looks like the diagram in Figure 1, where r is the reference value, e is the error between the output y and the reference value. $P(s)$ is the plant of the system and $C(s)$ is the controller that will be applied to the system.

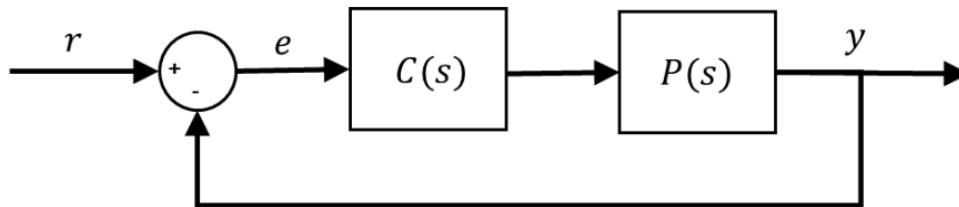


Figure 1. Unity feedback loop

The closed-loop system transfer function for unity feedback as shown in Figure 1 will always be given by,

$$\frac{Y(s)}{R(s)} = \frac{C(s) \cdot P(s)}{1 + C(s) \cdot P(s)} = \frac{N(s)}{D(s)} \quad 1.1$$

To determine the stability of equation 1.1, we are only interested in the pole locations of the closed-loop system, i.e. the roots of the closed-loop *denominator*, where $D(s)$ has the form

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_0 s^0 \quad 1.2$$

If and only if all poles are strictly in the left half plane, the system is stable.

For example, a fourth-order polynomial has the following form:

$$D(s) = a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0 \quad 1.3$$

To create a Routh Table, we add a row for each coefficient of $D(s)$ and add, starting from the highest coefficient, every other coefficient in decreasing order in the first line. Next, repeat the procedure for the second line, starting with the second highest coefficient and add every other coefficient in decreasing order similar to the first line. Both rows should be the same length. Continue until no coefficients are left and add zero as last coefficient if necessary. The Routh Table for equation 1.3 looks like table 1-1.

| | | | |
|-------|-------|-------|-------|
| s^4 | a_4 | a_2 | a_0 |
| s^3 | a_3 | a_1 | 0 |
| s^2 | | | |
| s^1 | | | |
| s^0 | | | |

1-1

The entry for row 3 is found as illustrated in Table 1-2 with explanation. The full table is shown in table 1-3 for the system given in equation 1.3.

Each entry is a negative determinant of a 2×2 matrix.

- The entries in the first column of the determinant are the first column value in the above two rows (blue rectangles in table below).
- The entries in the second column of the determinant are the values in the two rows above to the column to the right of the current value (dotted blue arrows and ovals).
- The matrix is divided by the first value in the previous row (red circles in table below).
- If the value doesn't exist, add 0's.

| | | | |
|-------|---|---|---|
| s^4 | a_4 | a_2 | a_0 |
| s^3 | a_3 | a_1 | 0 |
| s^2 | $-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$ | $-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$ | $-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$ |
| s^1 | | | |
| s^0 | | | |

1-2

Treat the rest of the rows the same as the 3rd row.

| | | | | |
|-------|---|---|---|-----|
| s^4 | a_4 | a_2 | a_0 | |
| s^3 | a_3 | a_1 | 0 | |
| s^2 | $-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$ | $-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$ | $-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$ | |
| s^1 | $-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$ | $-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$ | $-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$ | 1-3 |
| s^0 | $-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$ | $-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$ | $-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$ | |

Once the Routh Table is found, **the number of sign changes in the first column is equal to the number of right half plane (unstable) system poles.**

Case 1: Row is all zeroes

If an entire row in the Routh Table is zero, the denominator polynomial has a factor that is an even polynomial.

To complete the Routh Table for this case, move up one row, write out the corresponding polynomial of that row using the entries of the row as the coefficients. Then differentiate this polynomial and use the coefficients of the differentiated polynomial instead of the zeros previously obtained. For example, consider the polynomial

$$D(s) = s^5 + 7s^4 + 5s^3 + 35s^2 + 2s + 14 \quad 1.4$$

The corresponding table is

| | | | | |
|-------|--|--|---|-----|
| s^5 | 1 | 5 | 2 | |
| s^4 | 7 | 35 | 14 | |
| s^3 | $-\frac{\begin{vmatrix} 1 & 5 \\ 7 & 35 \end{vmatrix}}{1} = 0$ | $-\frac{\begin{vmatrix} 1 & 2 \\ 7 & 14 \end{vmatrix}}{1} = 0$ | $-\frac{\begin{vmatrix} 1 & 0 \\ 7 & 0 \end{vmatrix}}{1} = 0$ | 1-4 |
| s^2 | | | | |
| s^1 | | | | |
| s^0 | | | | |

Since the s^3 row has all 0 coefficients, thus the polynomial with coefficients one row above is

$$p(s) = 7s^4 + 35s^2 + 14 \quad 1.5$$

Differentiating $p(s)$ from equation 1.5 yields the following:

$$\frac{dp(s)}{ds} = 28s^3 + 70s^1 + 0 \quad 1.6$$

Which we use to replace s^3 instead of the zero row we had above. Using these coefficients instead of the zero row above and finding the remaining entries as outlined above yields the completed Routh Table

| | | | |
|-------|--|--|--|
| s^5 | 1 | 5 | 2 |
| s^4 | 7 | 35 | 14 |
| s^3 | 28 | 70 | 0 |
| s^2 | $-\frac{\begin{vmatrix} 7 & 35 \\ 28 & 70 \end{vmatrix}}{28} = 22.5$ | $-\frac{\begin{vmatrix} 7 & 14 \\ 28 & 0 \end{vmatrix}}{28} = 14$ | $-\frac{\begin{vmatrix} 7 & 0 \\ 28 & 0 \end{vmatrix}}{28} = 0$ |
| s^1 | $-\frac{\begin{vmatrix} 28 & 70 \\ 22.5 & 14 \end{vmatrix}}{22.5} = 52.58$ | $-\frac{\begin{vmatrix} 28 & 0 \\ 22.5 & 0 \end{vmatrix}}{22.5} = 0$ | $-\frac{\begin{vmatrix} 28 & 0 \\ 22.5 & 0 \end{vmatrix}}{22.5} = 0$ |
| s^0 | $-\frac{\begin{vmatrix} 22.5 & 14 \\ 52.58 & 0 \end{vmatrix}}{52.58} = 14$ | $-\frac{\begin{vmatrix} 22.5 & 0 \\ 52.58 & 0 \end{vmatrix}}{52.58} = 0$ | $-\frac{\begin{vmatrix} 22.5 & 0 \\ 52.58 & 0 \end{vmatrix}}{52.58} = 0$ |

| | | | |
|-------|-------|----|----|
| s^5 | 1 | 5 | 2 |
| s^4 | 7 | 35 | 14 |
| s^3 | 28 | 70 | 0 |
| s^2 | 22.5 | 14 | 0 |
| s^1 | 52.58 | 0 | 0 |
| s^0 | 14 | 0 | 0 |

1-5

Every even polynomial only has roots that are symmetrical about the origin. **Since there is no sign change in the first column of the Routh Table, there are no positive roots.** Therefore, the only possible symmetry is for the roots to be purely imaginary, thus the overall system is *marginally stable*.

Case 2: Zero in the first column

It may also happen that there is only a leading zero in the Routh Table and the remaining entries in that row are non-zero. In that case, replace the leading zero with ϵ and carry on deriving the remainder of the Routh Table as a function of ϵ . ϵ is infinitely small, so for example, $12 - 4\epsilon = 12$. Assume $\epsilon > 0$. Consult the literature for more details.

A step response test is a fundamental method for analyzing a system's dynamic behavior by observing how it responds to a sudden change in input. For DC motors, this typically involves instantly applying a fixed voltage and measuring how the motor's speed or position changes over time. This testing is crucial because it reveals essential system characteristics like response speed, stability, and accuracy, which are vital for control system design and optimization.

a. Routh-Hurwitz Stability on Qube-Servo

To investigate marginal stability using the Routh-Hurwitz Criterion, we will use a third-order transfer function using unity feedback with a compensator.

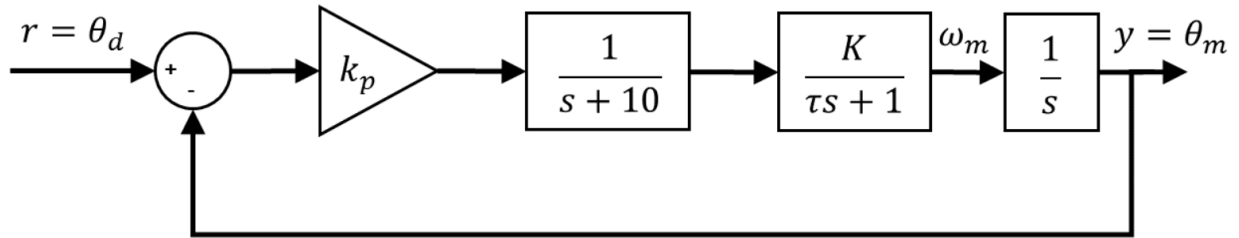


Figure 2. Unity feedback loop with compensator and proportional gain on Qube-Servo

Based on Figure 2, the compensator is:

$$C(s) = \frac{k_p}{s + 10} \quad 1.7$$

Where k_p is the proportional gain.

The plant voltage to position transfer function model of the DC motor:

$$P(s) = \frac{K}{s(\tau s + 1)} \quad 1.8$$

If no modeling lab has been done, for Qube-Servo 3, the steady state $K = 24$ and the time constant $\tau = 0.1$ are good defaults.

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