

# Differential Kinematics

## 1 FORWARD KINEMATIC EQUATIONS

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Recall the forward kinematics formulation  $f_{FPK}$

$$\begin{aligned} P &= f_{FPK}(\Theta) \\ p_x &= \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} \\ p_y &= \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} \\ p_z &= \lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23} \\ \gamma &= \phi_4 \end{aligned} \tag{1}$$

where  $\Theta$  is the vector of the joint states,

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \tag{2}$$

and  $\mathbf{p}$  is the position of the end-effector {4} (expressed in base frame {0})

$$\mathbf{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^0\mathbf{p}_4 \tag{3}$$

## 2 INVERSE KINEMATIC FORMULATION

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Differentiate the equation set 1, to yield,

$$\begin{aligned} v_x &= \frac{dp_x}{dt} = -\lambda_2 s_1 c_2 \dot{\theta}_1 - \lambda_2 c_1 s_2 \dot{\theta}_2 + \lambda_3 s_1 s_{23} \dot{\theta}_1 - \lambda_3 c_1 c_{23} \dot{\theta}_2 - \lambda_3 c_1 c_{23} \dot{\theta}_3 \\ v_y &= \frac{dp_y}{dt} = \lambda_2 c_1 c_2 \dot{\theta}_1 - \lambda_2 s_1 s_2 \dot{\theta}_2 - \lambda_3 c_1 s_{23} \dot{\theta}_1 - \lambda_3 s_1 c_{23} \dot{\theta}_2 - \lambda_3 s_1 c_{23} \dot{\theta}_3 \\ v_z &= \frac{dp_z}{dt} = \lambda_1 - \lambda_2 c_2 \dot{\theta}_2 + \lambda_3 s_{23} \dot{\theta}_2 + \lambda_3 s_{23} \dot{\theta}_3 \\ \dot{\gamma} &= \frac{d\gamma}{dt} = \dot{\theta}_4 \end{aligned} \tag{4}$$

This can be represented in matrix form,

$$\begin{aligned} \mathbf{V} &= {}^0\mathbf{J} \dot{\Theta} \\ \begin{bmatrix} v_x \\ v_y \\ v_z \\ \dot{\gamma} \end{bmatrix} &= \begin{bmatrix} -\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23} & -\lambda_2 c_1 s_2 - \lambda_3 c_1 c_{23} & -\lambda_3 c_1 c_{23} & 0 \\ \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} & -\lambda_2 s_1 s_2 - \lambda_3 s_1 c_{23} & -\lambda_3 s_1 c_{23} & 0 \\ 0 & -\lambda_2 c_2 + \lambda_3 s_{23} & \lambda_3 s_{23} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} \end{aligned} \tag{5}$$

where  ${}^0\mathbf{J}$  is the manipulator Jacobian expressed in base frame {0}.

## 2.1 SINGULARITY ANALYSIS

The determinant of the Jacobian is derived as,

$$\begin{aligned}\det^0 \mathbf{J} &= (-\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23})((- \lambda_2 s_1 s_2 - \lambda_3 s_1 c_{23})(\lambda_3 s_{23}) - (-\lambda_2 c_2 + \lambda_3 s_{23})(-\lambda_3 s_1 c_{23})) \\ &\quad + (\lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23})((- \lambda_2 c_2 + \lambda_3 s_{23})(-\lambda_3 c_1 c_{23}) - (-\lambda_2 c_1 s_2 - \lambda_3 c_1 c_{23})(\lambda_3 s_{23})) \\ \det^0 \mathbf{J} &= -\lambda_2 \lambda_3 (s_1 (-\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23})(s_2 s_{23} + c_2 c_{23}) - c_1 (\lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23})(c_2 c_{23} + s_2 s_{23})) \\ \det^0 \mathbf{J} &= \lambda_2 \lambda_3 (s_1^2 (\lambda_2 c_2 - \lambda_3 s_{23})(s_2 s_{23} + c_2 c_{23}) + c_1^2 (\lambda_2 c_2 - \lambda_3 s_{23})(c_2 c_{23} + s_2 s_{23})) \\ \det^0 \mathbf{J} &= \lambda_2 \lambda_3 (\lambda_2 c_2 - \lambda_3 s_{23})(s_2 s_{23} + c_2 c_{23})\end{aligned}\tag{6}$$

### 2.1.1 Singularity Condition 1:

$$\lambda_2 c_2 - \lambda_3 s_{23} = 0\tag{7}$$

Which corresponds to the end-effector directly above the base.

### 2.1.2 Singularity Condition 2:

$$\begin{aligned}s_2 s_{23} + c_2 c_{23} &= 0 \\ s_2 s_2 c_3 + s_2 c_2 s_3 + c_2 c_2 c_3 - c_2 s_2 s_3 &= 0 \\ c_3 &= 0\end{aligned}\tag{8}$$

Which corresponds to the arm either folder into itself or completely stretched out.