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Concept Review

Position Kinematics

Why Use Position Kinematics?

Complex robotic and mechatronic systems comprise of a variety of internal parameters representing the motion of the system as well as the forces acting on it. Position kinematics relate the internal configuration of the system to its external and high-level position. The internal parameters can be measured using a range of sensors, for example, encoders. This provides a straightforward means to estimate the position of the overall system. The inverse is also true. Sensors such as cameras, LiDAR, or GPS provide the high-level position of the overall system, and inverting the position kinematics provides a means to estimate the internal states. Position kinematics can also serve as the starting point for creating higher order relationships, such as velocity kinematics, which are out of the scope of this review.

General Definition

Consider a robotic system such as a mobile robot or manipulator, with n actuated or passive joints, represented by a series of joint parameters q_i for $i \in 1{:}n$. The joint parameters can be represented by a vector quantity $\vec{q}$ as,

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \tag{1}$$

The high level pose of the system, comprising of its position p and orientation R with respect to some inertial reference frame can be represented by,

$$\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \; R = R(\phi, \theta, \psi) \tag{2}$$

Where the orientation is represented as a rotation matrix, a function of the roll ϕ, pitch θ and yaw ψ angles. Note that since the rotation matrix R is 3x3, and the position vector $\vec{p}$ is a 3x1, we can collectively hold the pose of a system using a transformation matrix T that is 4x4.

$$T = \begin{bmatrix} R & \vec{p} \\ [0\ 0\ 0] & 1 \end{bmatrix} \tag{3}$$

Forward Position Kinematics

Often abbreviated simply as forward kinematics, this is a relationship that yields the transformation matrix T as a function of the joint parameters $\vec{q}$,

$$T = f_{fpk}(\vec{q}) \tag{4}$$

This relationship can be derived using a variety of methods. Geometric derivations are quite common for simple systems, for example, a 2 or 3 link manipulator. Advanced methods such as the Denavit-Hartenberg notation and system can also be used for sophisticated configurations or where the ease of use with a computational or numerical approach is required.

Inverse Position Kinematics

Once again, often abbreviated simply as inverse kinematics, this relationship yields the joint parameters $\vec{q}$ required to obtain the desired pose transformation matrix T,

$$\vec{q} = f_{ipk}(T) \tag{5}$$

Geometric methods can be used to invert the forward position kinematic equations. Unlike the forward position kinematics problem that can always be solved uniquely, the inverse position kinematics may yield either no solutions, a set of unique solutions, or infinite solutions.

Example

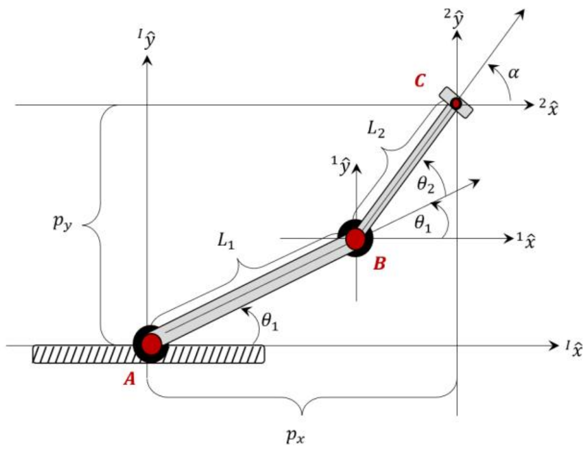

Figure 1 – 2-dof manipulator example diagram

Consider a 2-dof manipulator as shown in Figure 1. The manipulator is attached to the table at point A, where the inertial frame is set with axis ${}^{I}\hat{x}$ and ${}^{I}\hat{y}$. The manipulator has two links of length L_1 and L_2 actuated by two joints at positions A and B as well, with joint parameters θ_1 and θ_2. As these vary, the end-effector of the manipulator at point C moves around with respect to the inertial frame. We are interested in calculating the relationship between the end-effector's position (point C) given by $\vec{p} = \begin{bmatrix} p_x & p_y \end{bmatrix}^T$ as well as the end-effector orientation represented by an angle α. The z component of the position or other elements of the rotation matrix are not required as the manipulator is planar.

Forward Kinematics - Geometric Approach

The position of point B with respect to point A can be calculated with simple trigonometric relationships,

$$p_{BA,x} = L_1 \cos\theta_1 \quad \text{and} \quad p_{BA,y} = L_1 \sin\theta_1$$

Similarly, the position of point C with respect to point B can be calculated as,

$$p_{CB,x} = L_2 \cos(\theta_1 + \theta_2) \quad \text{and} \quad p_{CB,y} = L_2 \sin(\theta_1 + \theta_2)$$

Finally, the position of point C with respect to A is,

$$p_{CA,x} = p_{CB,x} + p_{BA,x} = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$$

$$p_{CA,y} = p_{CB,y} + p_{BA,y} = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$$

And the orientation angle $\alpha = \theta_1 + \theta_2$. The Combined forward position kinematics are then,

$$\begin{bmatrix} p_x \\ p_y \\ \alpha \end{bmatrix} = \begin{bmatrix} L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2) \\ L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{bmatrix} = f_{fpk}(\theta_1, \theta_2) \tag{6}$$

Forward Kinematics – DH Approach

The Denavit-Hartenberg notation and formulation can also be used to yield the same forward position kinematic equations. Please refer to the concept review for the DH Framework for more details on the schematic and DH-table. The manipulator schematic highlighting the selected z and x axis, as well as origins is also shown in Figure 2. Based on this schematic, the DH table is provided in Table 1. Note that the red dots in the schematic on the right represent a z axis coming out of the page. The choice for z_2 was made so as to place x_2 at the end effector and capture the length L_2 into the DH-table as well.

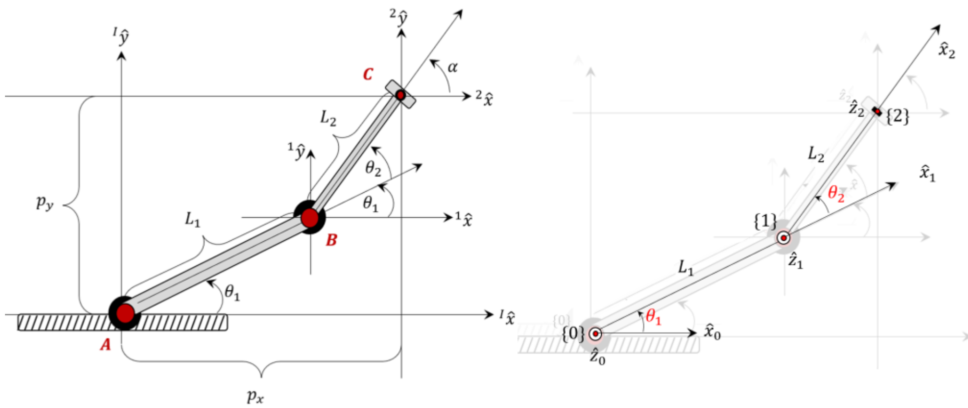

Figure 2. DH schematic of the 2-dof manipulator in Figure 1

i	a_n	α_n	d_n	θ_n	$\theta_n(0)$
1	L_1	0	0	θ_1	0
2	L_2	0	0	θ_2	0

Table 1 – DH parameters for the schematic in Figure 2

Using this schematic and the general transformation matrix,

$$^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\ \cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\ \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We get the transformations from frame $\{0\}$ to frame $\{1\}$, as well as frame $\{1\}$ to frame $\{2\}$.

$$^{0}T_1 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & L_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & L_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad ^{1}T_2 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & L_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & L_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}T_2 = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 & L_1\cos\theta_1 + L_2\cos(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 & L_1\sin\theta_1 + L_2\sin(\theta_1+\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the matrix $^{0}T_2$, the forward kinematics equations are found in the last coloumn,

$$\begin{bmatrix} p_x \\ p_y \\ \alpha \end{bmatrix} = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1 + \theta_2) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1 + \theta_2) \\ \theta_1 + \theta_2 \end{bmatrix} = f_{fpk}(\theta_1, \ \theta_2\ ) \tag{7}$$

Inverse Kinematics - Geometric Approach

A geometric approach is common for inverse kinematics, even when using the DH framework. There are iterative solutions for inverse kinematics solved using DH-based forward kinematics and differential kinematic formulations, but those are outside the scope of this document.

Geometrically, inverting the equations involves using α to simplify the position equations as,

$$L_1 \cos\theta_1 = p_x - L_2 \cos\alpha$$

$$L_1 \sin\theta_1 = p_y - L_2 \sin\alpha$$

This provides a cosine and sine term for θ_1 to solve using the arctangent. Subtracting the result from α also provides θ_2

$$\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix} \tan^{-1}\left(\dfrac{p_y - L_2 \sin\alpha / L_1}{p_x - L_2 \cos\alpha / L_1}\right) \\ \alpha - \tan^{-1}\left(\dfrac{p_y - L_2 \sin\alpha / L_1}{p_x - L_2 \cos\alpha / L_1}\right)\end{bmatrix} = f_{ipk}(x, y, \alpha) \tag{8}$$

There is a unique solution to the problem in this case given a desired end-effector angle α. If this is not provided, then the p_x and p_y equations can be squared and added to eliminate θ_1 and solve for θ_2. Plugging this back in the position equations will yield θ_1 using the arctangent. This will yield two unique solutions.

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