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Concept Review

Differential Kinematics

Why use Differential Kinematics?

Complex robotic and mechatronic systems comprise of a variety of internal parameters representing the motion of the system as well as the forces acting on it. While position kinematics can be used to relate the internal configuration of the system to its external and high-level position, and vice versa, it has limitations in modeling the system when its states are evolving. Differential kinematics relates the rate of change of internal states to the overall high-level motion of the system. This allows you to directly command velocity profiles or integrate to get position estimates with better accuracy when subject to sensor noise.

General Definition

Consider a robotic system such as a mobile robot or manipulator, with n actuated or passive joints, represented by a series of joint parameters q_i for $i \in 1{:}n$. The joint parameters can be represented by a vector quantity $\vec{q}$ as,

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} \tag{1}$$

The corresponding rates of change of the joint parameters can be represented by $\dot{\vec{q}}$,

$$\dot{\vec{q}} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} \tag{2}$$

The high-level pose of the system, comprising of it's position $\vec{p}$ and orientation $\vec{\theta}$ with respect to some inertial reference frame can be represented by,

$$\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}, \quad \vec{\theta} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} \tag{3}$$

Where the orientation is represented by the roll ϕ, pitch θ and yaw ψ angles. The corresponding rates can be expressed as,

$$\vec{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \quad \dot{\vec{\theta}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \tag{4}$$

Differential Kinematics

This is a relationship that yields the high-level system velocity as a function of the joint velocities $\dot{\vec{q}}$

$$\vec{v} = f_{fdk}(\dot{\vec{q}}) \tag{5}$$

This relationship can be derived using a variety of methods. A standard approach involves differentiating the position kinematic equations with respect to each joint parameter. Alternatively, kinematic constraints around the motion of a system might directly yield the kinematic equations. The provided examples cover a few cases.

Inverting these equations typically depends on the format of the equations. A formulation that provides the joint rates of change based on the system velocities is,

$$\dot{\vec{q}} = f_{idk}(\vec{v}) \tag{6}$$

Example 1 – Robotic Manipulator & the DH framework

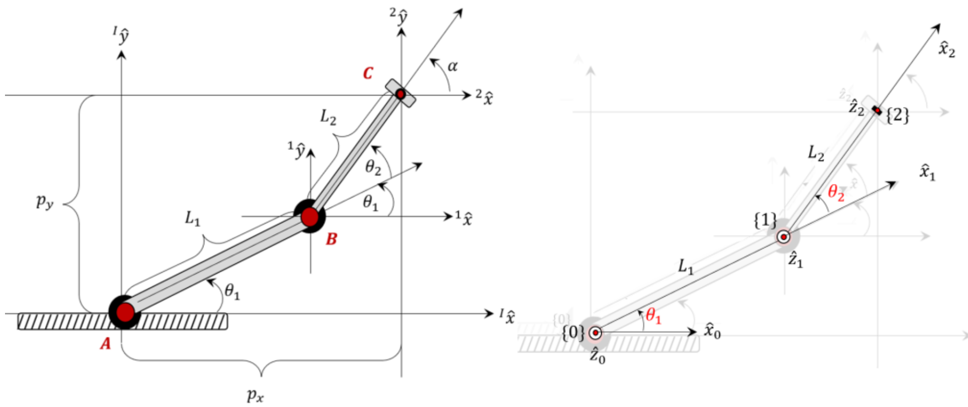

Figure 1 – DH schematic of a 2-dof manipulator

i	a_n	α_n	d_n	θ_n	$\theta_n(0)$
1	L_1	0	0	θ_1	0
2	L_2	0	0	θ_2	0

Table 1 – DH parameters for the schematic in Figure 1

Let's start with the forward kinematics equations derived for the manipulator in the Position Kinematics concept review,

$$\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \begin{bmatrix} L_1\cos(\theta_1) + L_2\cos(\theta_1 + \theta_2) \\ L_1\sin(\theta_1) + L_2\sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix} \tag{7}$$

One can differentiate to yield,

$$\vec{v} = \frac{d}{dt}\begin{bmatrix} p_x \\ p_y \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} f_1(\theta_1, \theta_2) \\ f_2(\theta_1, \theta_2) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} \\ \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J\,\dot{\vec{q}} \tag{8}$$

Where J is the Jacobian matrix. For the manipulator above, differentiation equations in 7 provide,

$$\vec{v} = \begin{bmatrix} -L_1\sin\theta_1 - L_2\sin(\theta_1 + \theta_2) & -L_2\sin(\theta_1 + \theta_2) \\ L_1\cos\theta_1 + L_2\cos(\theta_1 + \theta_2) & L_2\cos(\theta_1 + \theta_2) \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Inverting these equations is a matter of taking the inverse of the Jacobian matrix. Care must be taken to ensure that the Jacobian is not singular when doing so. Taking a determinant of the Jacobian and setting it to zero provides the singular condition, where the manipulator is either fully outstretched or folded onto itself via joint 2.

$$|\,J\,| = L_1 L_2 \sin\theta_2 = 0 \qquad \gg \qquad \theta_2 = 0, \pi$$

Example 2 – Differential Drive Robot

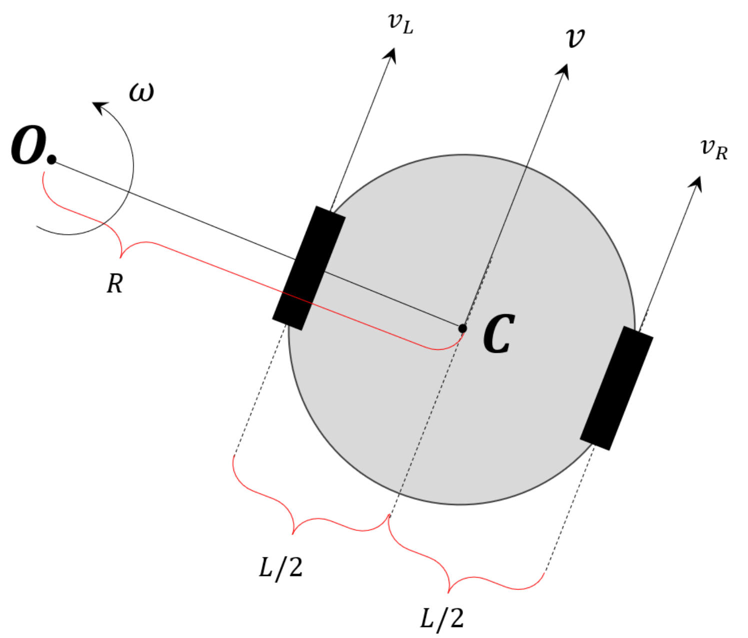

Figure 2 – A simple representation of a differential drive robot

Consider a differential drive robot turning around a point O, with the center of the robot at point C. As it turns around O, the left and right wheel speeds must vary to drive without slipping. However, their angular turning rate about the point O must be the same. That is,

$$\omega = \frac{V_L}{R - L/2} = \frac{V_R}{R + L/2} \tag{9}$$

In addition, forward velocity of the robot is equal to the average velocity of the left and right wheels,

$$v = \frac{v_L + v_R}{2} \tag{10}$$

From these two equations, one can get the combined differential kinematic equations of the robot,

$$\begin{bmatrix} v \\ \omega \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -2/L & 2/L \end{bmatrix} \begin{bmatrix} v_L \\ v_R \end{bmatrix} \tag{11}$$

The inverse of the matrix can also be calculated for all values as the determinant is never zero, yielding,

$$\begin{bmatrix} v_L \\ v_R \end{bmatrix} = \begin{bmatrix} 1 & -L/4 \\ 1 & L/4 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix} \tag{12}$$

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