QUANSER
INNOVATE · EDUCATE

Concept Review
Lead Compensators

Why Study Lead-Lag Compensators?

The classic PID control technique uses proportional, integration and derivative operations to design a compensator such that the overall system response tracks a reference signal or trajectory. The design of a compensator can be considered as a filter design problem. In this regard, a PI controller is a low-pass filter and because it adds a pole to the system, it introduces a negative phase (phase lag) over some frequency range. On the contrary, a PD controller is a high-pass filter and because it adds a zero to the system, it introduces a positive phase (phase lead) into the system.

1. Background

The general transfer function of a lead or lag compensator can be expressed as follows:

$$C(s) = K_c \frac{s+z}{s+p} = K_c \frac{\alpha T s + 1}{T s + 1} \quad (1.1)$$

where p is the pole of the compensator, z is the zero of the compensator, α and T are constant parameters to be designed. The equation in $C(s)$ becomes a phase lag compensator or low-pass filter when $\alpha < 1$ (or $p < z$) and a phase-lead compensator or high-pass filter when $\alpha > 1$ (or $p > z$).

The proportional gain of the lead compensator is used to attain a certain crossover frequency. In general, modifying the gain, and respectively the crossover frequency, essentially increases the bandwidth of the system thus also decreasing the system's peak time (speeding up the response). For example, a gain of $K_c > 1$ decreases the system's phase margin and if K_c is chosen too large, it will lead to large overshoots in the system response.

Even though lead-lag compensators work well in theory, they often suffer from saturation limits of other physical hardware and may not be able to achieve zero steady-state error specification. For this reason, it is beneficial to design the compensator in series with an integrator as in Figure 1 below which describes a control scheme for a lead compensator for speed control. Recall that integral action in a system eliminates steady-state error. The resulting controller has the form:

$$C(s) = K_c \frac{\alpha T s + 1}{(T s + 1) s}. \quad (1.2)$$

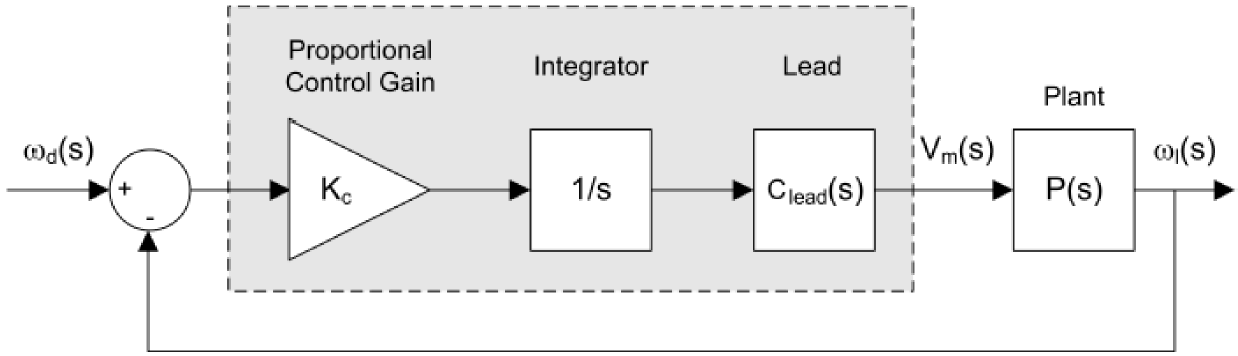

Figure 1: Closed-loop speed control with a lead compensator.

2. Lead Compensator Design Procedure

The two main design parameters for a Lead Compensator are the phase margin and the gain crossover frequency. The phase margin determines how much delay the system can withstand before becoming unstable and it tends to affect the overshoot of the response. A higher phase margin implies a more stable response with less overshoot. Generally speaking, the percentage overshoot, PO, of a system should not go beyond 5% for a phase margin of at least 75 degrees. This can be used as a guideline for the initial control design.

The gain crossover frequency ω_m is defined as the frequency where the gain of the system is 1 (or 0 decibels on a Bode plot). This parameter mainly affects the speed of the response and a larger ω_m implies a decrease in the peak time. In general, the peak time t_p should not exceed 0.05 seconds when the gain crossover frequency is at least 75 radians. This can be used as a guideline for the initial control design.

The lead compensator is a frequency domain design method and relies heavily on the Bode plot (frequency response) of the system. A typical Bode plot of a lead compensator is shown in Figure 2.

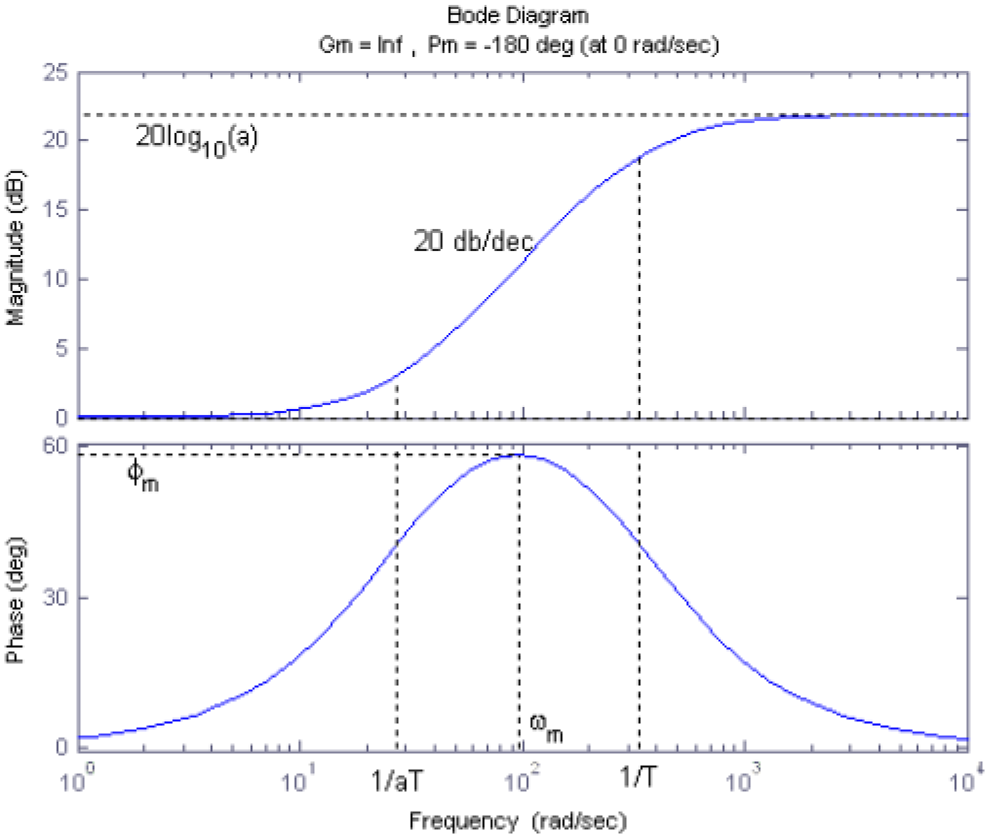

Figure 2: Bode plot of a typical lead compensator.

The design process for a lead compensator can be summarized as follows:

1. Generate the Bode plot of the open-loop uncompensated system.
2. The lead compensator itself will add some gain to the closed-loop system response. To make sure that the bandwidth requirement of the design is met, a proportional gain K_c needs to be added such that the open-loop crossover frequency is about a factor of two below the desired system bandwidth.
3. Determine the necessary additional phase lead ϕ_m for the plant with open-loop gain K_c. To do so, compute

$$\phi_m = PM_{des} - PM_{meas}. \tag{2.1}$$

4. Compute α. To attain the maximum phase ϕ_m at the frequency ω_m as shown in Figure 2, the compensator is required to add $20\log_{10}(\alpha)$ of gain. Here, ω_m is the geometric mean of the two corner frequencies from the zero and pole of the lead compensator, respectively. That is,

$$\log_{10}(\omega_m) = \frac{1}{2}\left(\log_{10}\left(\frac{1}{\alpha T}\right) - \log_{10}\left(\frac{1}{T}\right)\right). \tag{2.2}$$

Solving for ω_m yields,

$$\omega_m = \frac{1}{T\sqrt{\alpha}}. \tag{2.3}$$

The lead compensator is used to dampen the system to reduce the overshoot and increase the overall stability of the system by increasing the phase margin. The frequency response of the lead compensator from Equation 1.1 is obtained by substituting $s = j\omega$ as

$$C_{lead}(j\omega) = \frac{\alpha T j\omega + 1}{T j\omega + 1}, \tag{2.4}$$

with the corresponding magnitude and phase,

$$|C_{lead}(j\omega)| = \sqrt{\frac{\omega^2\alpha^2T^2 + 1}{\omega^2T^2 + 1}}, \tag{2.5}$$

$$\phi_m = \arctan(\omega\alpha T) - \arctan(\omega T). \tag{2.6}$$

Using the following trigonometric identity,

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) + \tan\,(\beta)}{1 + \tan(\alpha)\tan\,(\beta)}, \tag{2.7}$$

on Equation 2.6 yields,

$$\tan\left(\phi_m(j\omega)\right) = \frac{\omega\alpha T - \omega T}{1 + (\omega\alpha T)(\omega T)}. \qquad 2.8$$

Noting that,

$$\tan(\alpha) = \pm\frac{\sin(\alpha)}{\sqrt{1-\sin^2(\alpha)}}, \qquad 2.9$$

and using Equation 2.3, we find that,

$$\sin(\phi_m) = \frac{\alpha - 1}{\alpha + 1}. \qquad 2.10$$

Thus, if ϕ_m is known, α can be determined by solving,

$$\alpha = \frac{1 + \sin(\phi_m)}{1 - \sin(\phi_m)}. \qquad 2.11$$

5. Determine the value of T using Equation 2.3. To do so, place the corner frequencies of the lead compensator such that ϕ_m is located at ω_m, that is, the new gain crossover frequency (the geometric mean of $1/\alpha T$ and $1/T$) where the compensator has a gain of 10 dB. By design, ω_m has to be placed at the frequency where the magnitude of the uncompensated system is $C(j\omega) = -10\log_{10}\alpha\ \mathrm{dB}$. The gain crossover frequency ω_m is then obtained by finding the corresponding frequency in the uncompensated Bode plot.
6. Determine the pole and zero of the lead compensators.
7. Check whether or not the compensator fulfills the design requirements. To do so, draw the Bode plot of the compensated systema and check the resulting phase margin to see whether or not the system response meets the desired characteristics. Repeat the design steps for different ϕ_m if necessary.

Solutions for teaching and research. Made in Canada.