

Dynamics - Torques

1 LAGRANGIAN

Beginning with the Lagrangian,

$$\begin{aligned}
\mathcal{L} = & \left(\frac{1}{2} I_{1A} + \frac{1}{2} m_2 (\lambda_2 - \lambda_{c2})^2 c_2^2 + \frac{1}{2} I_{2A} s_2^2 + \frac{1}{2} I_{2L} c_2^2 + \frac{1}{2} m_3 \lambda_2^2 c_2^2 + \frac{1}{2} m_3 \lambda_{c3}^2 s_{23}^2 - m_3 \lambda_2 \lambda_{c3} c_2 s_{23} + \frac{1}{2} I_{3L} s_{23}^2 + \frac{1}{2} I_{3A} c_{23}^2 \right. \\
& + \frac{1}{2} m_4 (\lambda_3 - \lambda_{c4})^2 s_{23}^2 + \frac{1}{2} m_4 \lambda_2^2 c_2^2 - m_4 (\lambda_3 - \lambda_{c4}) \lambda_2 c_2 s_{23} + \frac{1}{2} I_{4L} s_{23}^2 + \frac{1}{2} I_{4A} c_{23}^2 + \frac{1}{2} m_L \lambda_3^2 s_{23}^2 \\
& \left. + \frac{1}{2} m_L \lambda_2^2 c_2^2 - m_L \lambda_2 \lambda_3 c_2 s_{23} \right) \dot{\theta}_1^2 \\
& + \left(\frac{1}{2} m_2 (\lambda_2 - \lambda_{c2})^2 + \frac{1}{2} I_{2L} + \frac{1}{2} m_3 \lambda_2^2 + \frac{1}{2} m_3 \lambda_{c3}^2 - m_3 \lambda_2 \lambda_{c3} s_3 + \frac{1}{2} I_{3L} + \frac{1}{2} m_4 (\lambda_2 + \lambda_3 - \lambda_{c4})^2 \right. \\
& \left. + \frac{1}{2} I_{4L} + \frac{1}{2} m_L \lambda_2^2 + \frac{1}{2} m_L \lambda_3^2 - m_L \lambda_2 \lambda_3 s_3 \right) \dot{\theta}_2^2 \\
& + \left(\frac{1}{2} m_3 \lambda_{c3}^2 + \frac{1}{2} I_{3L} + \frac{1}{2} m_4 (\lambda_{c4} - \lambda_3)^2 + \frac{1}{2} I_{4L} + \frac{1}{2} m_L \lambda_3^2 \right) \dot{\theta}_3^2 + \left(\frac{1}{2} I_{4A} \right) \dot{\theta}_4^2 - (I_{4A} c_{23}) \dot{\theta}_1 \dot{\theta}_4 \\
& + \left(m_3 \lambda_{c3}^2 - m_3 \lambda_{c3} \lambda_2 s_3 + I_{3L} + m_4 (\lambda_3 - \lambda_{c4} - \lambda_2 s_3) (\lambda_3 - \lambda_{c4}) + I_{4L} - m_L (\lambda_3^2 - \lambda_2 \lambda_3 s_3) \right) \dot{\theta}_2 \dot{\theta}_3 \\
& - m_1 g (\lambda_1 - \lambda_{c1}) - m_2 g (\lambda_1 - (\lambda_2 - \lambda_{c2}) s_2) - m_3 g (\lambda_1 - \lambda_2 s_2 - \lambda_{c3} c_{23}) \\
& - m_4 g (\lambda_1 - \lambda_2 s_2 - (\lambda_3 - \lambda_{c4}) c_{23}) - m_L g (\lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23})
\end{aligned} \tag{1}$$

We can use the following equation to derive the joint torques,

$$\tau_i = \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_i} \right) - \frac{d\mathcal{L}}{d\theta_i} \tag{2}$$

2 JOINT 4

The torque required at the fourth joint is then given by,

$$\tau_4 = \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_4} \right) - \frac{d\mathcal{L}}{d\theta_4} \tag{3a}$$

$$\frac{d\mathcal{L}}{d\dot{\theta}_4} = I_{4A} \ddot{\theta}_4 - I_{4A} c_{23} \dot{\theta}_1, \quad \frac{d\mathcal{L}}{d\theta_4} = 0 \tag{3b}$$

$$\frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_4} \right) = I_{4A} \ddot{\theta}_4 - I_{4A} c_{23} \ddot{\theta}_1 + I_{4A} s_{23} \dot{\theta}_1 \dot{\theta}_2 + I_{4A} s_{23} \dot{\theta}_1 \dot{\theta}_3 \tag{3c}$$

$$\tau_4 = -\{I_{4A} c_{23}\} \ddot{\theta}_1 + \{I_{4A}\} \ddot{\theta}_4 + \{I_{4A} s_{23}\} \dot{\theta}_1 \dot{\theta}_2 + \{I_{4A} s_{23}\} \dot{\theta}_1 \dot{\theta}_3 \tag{3d}$$

3 JOINT 3

The torque required at the fourth joint is then given by,

$$\tau_3 = \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_3} \right) - \frac{d\mathcal{L}}{d\theta_3} \quad (4a)$$

$$\begin{aligned} \frac{d\mathcal{L}}{d\dot{\theta}_3} = & (m_3\lambda_{c3}^2 + I_{3L} + m_4(\lambda_{c4} - \lambda_3)^2 + I_{4L} + m_L\lambda_3^2)\dot{\theta}_3 \\ & + (m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} \\ & - m_L(\lambda_3^2 - \lambda_2\lambda_3s_3))\dot{\theta}_2 \end{aligned} \quad (4b)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_3} \right) = & (m_3\lambda_{c3}^2 + I_{3L} + m_4(\lambda_{c4} - \lambda_3)^2 + I_{4L} + m_L\lambda_3^2)\ddot{\theta}_3 \\ & + (m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} \\ & - m_L(\lambda_3^2 - \lambda_2\lambda_3s_3))\ddot{\phi}_2 + (m_L\lambda_2\lambda_3c_3 - m_3\lambda_{c3}\lambda_2c_3 - m_4\lambda_2(\lambda_3 - \lambda_{c4})c_3)\dot{\theta}_2\dot{\theta}_3 \end{aligned} \quad (4c)$$

$$\begin{aligned} \frac{d\mathcal{L}}{d\theta_3} = & (m_3\lambda_{c3}^2s_{23}c_{23} - m_3\lambda_2\lambda_{c3}c_2c_{23} + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2s_{23}c_{23} \\ & - m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2s_{23}c_{23} - m_L\lambda_2\lambda_3c_2c_{23})\dot{\theta}_1^2 \\ & - (m_3\lambda_2\lambda_{c3}c_3 + m_L\lambda_2\lambda_3c_3)\dot{\theta}_2^2 + (I_{4A}s_{23})\dot{\theta}_1\dot{\theta}_4 \\ & - (m_3\lambda_{c3}\lambda_2c_3 + m_4\lambda_2c_3(\lambda_3 - \lambda_{c4}) - m_L(\lambda_2\lambda_3c_3))\dot{\theta}_2\dot{\theta}_3 - m_3g\lambda_{c3}s_{23} \\ & - m_4g(\lambda_3 - \lambda_{c4})s_{23} - m_Lg\lambda_3s_{23} \end{aligned}$$

$$\begin{aligned} \tau_3 = & \{m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3s_3)\}\ddot{\theta}_2 \\ & + \{m_3\lambda_{c3}^2 + I_{3L} + m_4(\lambda_{c4} - \lambda_3)^2 + I_{4L} + m_L\lambda_3^2\}\ddot{\theta}_3 \\ & - \{m_3\lambda_{c3}^2s_{23}c_{23} - m_3\lambda_2\lambda_{c3}c_2c_{23} + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2s_{23}c_{23} \\ & - m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2s_{23}c_{23} - m_L\lambda_2\lambda_3c_2c_{23}\}\dot{\theta}_1^2 \\ & + \{m_3\lambda_2\lambda_{c3}c_3 + m_L\lambda_2\lambda_3c_3\}\dot{\theta}_2^2 - \{I_{4A}s_{23}\}\dot{\theta}_1\dot{\theta}_4 \\ & + g\{m_3\lambda_{c3}s_{23} + m_4(\lambda_3 - \lambda_{c4})s_{23} + m_L\lambda_3s_{23}\} \end{aligned} \quad (4d)$$

4 JOINT 2

The torque required at the fourth joint is then given by,

$$\tau_2 = \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_2} \right) - \frac{d\mathcal{L}}{d\theta_2} \quad (5a)$$

$$\begin{aligned} \frac{d\mathcal{L}}{d\dot{\theta}_2} = & (m_2(\lambda_2 - \lambda_{c2})^2 + I_{2L} + m_3\lambda_2^2 + m_3\lambda_{c3}^2 - 2m_3\lambda_2\lambda_{c3}s_3 + I_{3L} + m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L} + m_L\lambda_2^2 + m_L\lambda_3^2 \\ & - 2m_L\lambda_2\lambda_3s_3)\dot{\theta}_2 \end{aligned} \quad (5b)$$

$$+ (m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3s_3))\dot{\theta}_3$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_2} \right) = & (m_2(\lambda_2 - \lambda_{c2})^2 + I_{2L} + m_3\lambda_2^2 + m_3\lambda_{c3}^2 - 2m_3\lambda_2\lambda_{c3}s_3 + I_{3L} + m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L} + m_L\lambda_2^2 \\ & + m_L\lambda_3^2 - 2m_L\lambda_2\lambda_3s_3)\ddot{\theta}_2 \\ & + (m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3s_3))\ddot{\theta}_3 \end{aligned} \quad (5c)$$

$$- (2m_3\lambda_2\lambda_{c3}c_3 + 2m_L\lambda_2\lambda_3c_3)\dot{\theta}_2\dot{\theta}_3 - (m_3\lambda_{c3}\lambda_2c_3 + m_4\lambda_2c_3(\lambda_3 - \lambda_{c4}) - m_L(\lambda_2\lambda_3c_3))\dot{\theta}_3^2$$

$$\begin{aligned} \frac{d\mathcal{L}}{d\theta_2} = & (-m_2(\lambda_2 - \lambda_{c2})^2 s_2 c_2 + I_{2A}s_2 c_2 - I_{2L}s_2 c_2 - m_3\lambda_2^2 s_2 c_2 + m_3\lambda_{c3}^2 s_{23} c_{23} + m_3\lambda_2\lambda_{c3}s_2 s_{23} - m_3\lambda_2\lambda_{c3}c_2 c_{23} \\ & + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - m_4\lambda_2^2 s_2 c_2 + m_4(\lambda_3 - \lambda_{c4})\lambda_2 s_2 s_{23} \\ & - m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2 s_{23}c_{23} - m_L\lambda_2^2 s_2 c_2 + m_L\lambda_2\lambda_3 s_2 s_{23} \\ & - m_L\lambda_2\lambda_3 c_2 c_{23})\dot{\theta}_1^2 + (I_{4A}s_{23})\dot{\theta}_1\dot{\theta}_4 + m_2 g (\lambda_2 - \lambda_{c2})c_2 + m_3 g (\lambda_2 c_2 - \lambda_{c3}s_{23}) \\ & + m_4 g (\lambda_2 c_2 - (\lambda_3 - \lambda_{c4})s_{23}) + m_L g (\lambda_2 c_2 - \lambda_3 s_{23}) \end{aligned}$$

$$\begin{aligned} \tau_2 = & \{m_2(\lambda_2 - \lambda_{c2})^2 + I_{2L} + m_3\lambda_2^2 + m_3\lambda_{c3}^2 - 2m_3\lambda_2\lambda_{c3}s_3 + I_{3L} + m_4(\lambda_2 + \lambda_3 - \lambda_{c4})^2 + I_{4L} + m_L\lambda_2^2 + m_L\lambda_3^2 \\ & - 2m_L\lambda_2\lambda_3s_3\} \ddot{\theta}_2 \\ & + \{m_3\lambda_{c3}^2 - m_3\lambda_{c3}\lambda_2s_3 + I_{3L} + m_4(\lambda_3 - \lambda_{c4} - \lambda_2s_3)(\lambda_3 - \lambda_{c4}) + I_{4L} - m_L(\lambda_3^2 - \lambda_2\lambda_3s_3)\} \ddot{\theta}_3 \\ & - \{-m_2(\lambda_2 - \lambda_{c2})^2 s_2 c_2 + I_{2A}s_2 c_2 - I_{2L}s_2 c_2 - m_3\lambda_2^2 s_2 c_2 + m_3\lambda_{c3}^2 s_{23} c_{23} + m_3\lambda_2\lambda_{c3}s_2 s_{23} \\ & - m_3\lambda_2\lambda_{c3}c_2 c_{23} + I_{3L}s_{23}c_{23} - I_{3A}s_{23}c_{23} + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - m_4\lambda_2^2 s_2 c_2 \\ & + m_4(\lambda_3 - \lambda_{c4})\lambda_2 s_2 s_{23} - m_4(\lambda_3 - \lambda_{c4})\lambda_2 c_2 c_{23} + I_{4L}s_{23}c_{23} - I_{4A}s_{23}c_{23} + m_L\lambda_3^2 s_{23}c_{23} \\ & - m_L\lambda_2^2 s_2 c_2 + m_L\lambda_2\lambda_3 s_2 s_{23} - m_L\lambda_2\lambda_3 c_2 c_{23}\} \dot{\theta}_1^2 \\ & - \{m_3\lambda_{c3}\lambda_2c_3 + m_4\lambda_2c_3(\lambda_3 - \lambda_{c4}) - m_L(\lambda_2\lambda_3c_3)\} \dot{\theta}_3^2 - \{I_{4A}s_{23}\} \dot{\theta}_1\dot{\theta}_4 \\ & - \{2m_3\lambda_2\lambda_{c3}c_3 + 2m_L\lambda_2\lambda_3c_3\} \dot{\theta}_2\dot{\theta}_3 \\ & - g \{m_2(\lambda_2 - \lambda_{c2})c_2 + m_3(\lambda_2 c_2 - \lambda_{c3}s_{23}) + m_4(\lambda_2 c_2 - (\lambda_3 - \lambda_{c4})s_{23}) + m_L(\lambda_2 c_2 - \lambda_3 s_{23})\} \end{aligned} \quad (5d)$$

5 JOINT 1

The torque required at the fourth joint is then given by,

$$\tau_1 = \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_1} \right) - \frac{d\mathcal{L}}{d\theta_1} \quad (5a)$$

$$\begin{aligned} \frac{d\mathcal{L}}{d\dot{\theta}_1} = & (I_{1A} + m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + I_{2A}s_2^2 + I_{2L}c_2^2 + m_3\lambda_2^2 c_2^2 + m_3\lambda_{c3}^2 s_{23}^2 - 2m_3\lambda_2\lambda_{c3}c_2s_{23} + I_{3L}s_{23}^2 + I_{3A}c_{23}^2 \\ & + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 + m_4\lambda_2^2 c_2^2 - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2s_{23} + I_{4L}s_{23}^2 + I_{4A}c_{23}^2 + m_L\lambda_3^2 s_{23}^2 + m_L\lambda_2^2 c_2^2 \\ & - 2m_L\lambda_2\lambda_3c_2s_{23})\dot{\theta}_1 - (I_{4A}c_{23})\dot{\theta}_4 \end{aligned} \quad (5b)$$

$$\begin{aligned} \frac{d}{dt} \left(\frac{d\mathcal{L}}{d\dot{\theta}_1} \right) = & (I_{1A} + m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + I_{2A}s_2^2 + I_{2L}c_2^2 + m_3\lambda_2^2 c_2^2 + m_3\lambda_{c3}^2 s_{23}^2 - 2m_3\lambda_2\lambda_{c3}c_2s_{23} + I_{3L}s_{23}^2 + I_{3A}c_{23}^2 \\ & + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 + m_4\lambda_2^2 c_2^2 - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2s_{23} + I_{4L}s_{23}^2 + I_{4A}c_{23}^2 + m_L\lambda_3^2 s_{23}^2 + m_L\lambda_2^2 c_2^2 \\ & - 2m_L\lambda_2\lambda_3c_2s_{23})\ddot{\theta}_1 - (I_{4A}c_{23})\ddot{\theta}_4 \\ & + (-2m_2(\lambda_2 - \lambda_{c2})^2 s_2c_2 + 2I_{2A}s_2c_2 - 2I_{2L}s_2c_2 - 2m_3\lambda_2^2 s_2c_2 + 2m_3\lambda_{c3}^2 s_{23}c_{23} + 2m_3\lambda_2\lambda_{c3}s_2s_{23} \\ & - 2m_3\lambda_2\lambda_{c3}c_2c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - 2m_4\lambda_2^2 s_2c_2 \\ & + 2m_4(\lambda_3 - \lambda_{c4})\lambda_2s_2s_{23} - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} \\ & - 2m_L\lambda_2^2 s_2c_2 + 2m_L\lambda_2\lambda_3s_2s_{23} - 2m_L\lambda_2\lambda_3c_2c_{23})\dot{\theta}_1\dot{\theta}_2 \\ & + (2m_3\lambda_{c3}^2 s_{23}c_{23} - 2m_3\lambda_2\lambda_{c3}c_2c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} \\ & - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} - 2m_L\lambda_2\lambda_3c_2c_{23})\dot{\theta}_1\dot{\theta}_3 \\ & + (I_{4A}s_{23})\dot{\theta}_2\dot{\theta}_4 + (I_{4A}s_{23})\dot{\theta}_3\dot{\theta}_4 \end{aligned} \quad (5c)$$

$$\frac{d\mathcal{L}}{d\theta_1} = 0$$

$$\begin{aligned} \tau_1 = & \{I_{1A} + m_2(\lambda_2 - \lambda_{c2})^2 c_2^2 + I_{2A}s_2^2 + I_{2L}c_2^2 + m_3\lambda_2^2 c_2^2 + m_3\lambda_{c3}^2 s_{23}^2 - 2m_3\lambda_2\lambda_{c3}c_2s_{23} + I_{3L}s_{23}^2 + I_{3A}c_{23}^2 + m_4(\lambda_3 - \lambda_{c4})^2 s_{23}^2 \\ & + m_4\lambda_2^2 c_2^2 - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2s_{23} + I_{4L}s_{23}^2 + I_{4A}c_{23}^2 + m_L\lambda_3^2 s_{23}^2 + m_L\lambda_2^2 c_2^2 - 2m_L\lambda_2\lambda_3c_2s_{23}\} \ddot{\theta}_1 \\ & - \{I_{4A}c_{23}\} \ddot{\theta}_4 \\ & + \{-2m_2(\lambda_2 - \lambda_{c2})^2 s_2c_2 + 2I_{2A}s_2c_2 - 2I_{2L}s_2c_2 - 2m_3\lambda_2^2 s_2c_2 + 2m_3\lambda_{c3}^2 s_{23}c_{23} + 2m_3\lambda_2\lambda_{c3}s_2s_{23} \\ & - 2m_3\lambda_2\lambda_{c3}c_2c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} - 2m_4\lambda_2^2 s_2c_2 \\ & + 2m_4(\lambda_3 - \lambda_{c4})\lambda_2s_2s_{23} - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} \\ & - 2m_L\lambda_2^2 s_2c_2 + 2m_L\lambda_2\lambda_3s_2s_{23} - 2m_L\lambda_2\lambda_3c_2c_{23}\} \dot{\theta}_1\dot{\theta}_2 \\ & + \{2m_3\lambda_{c3}^2 s_{23}c_{23} - 2m_3\lambda_2\lambda_{c3}c_2c_{23} + 2I_{3L}s_{23}c_{23} - 2I_{3A}s_{23}c_{23} + 2m_4(\lambda_3 - \lambda_{c4})^2 s_{23}c_{23} \\ & - 2m_4(\lambda_3 - \lambda_{c4})\lambda_2c_2c_{23} + 2I_{4L}s_{23}c_{23} - 2I_{4A}s_{23}c_{23} + 2m_L\lambda_3^2 s_{23}c_{23} - 2m_L\lambda_2\lambda_3c_2c_{23}\} \dot{\theta}_1\dot{\theta}_3 \\ & + \{I_{4A}s_{23}\} \dot{\theta}_2\dot{\theta}_4 + \{I_{4A}s_{23}\} \dot{\theta}_3\dot{\theta}_4 \end{aligned} \quad (5d)$$

6 NET DYNAMIC EQUATIONS

The equations can be represented in a matrix form as,

$$\tau = \mathbf{M}(\Theta)\ddot{\Theta} + \mathbf{B}(\Theta)\dot{\Theta}_i\dot{\Theta}_j + \mathbf{C}(\Theta)\dot{\Theta}_k^2 + \mathbf{G}(\Theta) \quad (6)$$

$$i, j \in \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}, \quad k \in \{1,2,3,4\}$$

where \mathbf{M} is the **Inertia** matrix, \mathbf{B} is the **Coriolis** matrix, \mathbf{C} is the **Centrifugal** matrix and \mathbf{G} is the **Gravity** matrix. In this format, also known as the Configuration-Space equation, the Coriolis and Centrifugal coefficient matrices are functions of the manipulator state Θ alone. Depending on the manipulator's configuration, some of the matrix coefficients are not applicable (or 0). For the Quanser Arm, these matrices are of the form,

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix} = \begin{bmatrix} M_{11} & 0 & 0 & M_{14} \\ 0 & M_{22} & M_{23} & 0 \\ 0 & M_{32} & M_{33} & 0 \\ M_{41} & 0 & 0 & M_{44} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \\ \ddot{\theta}_4 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} & 0 & 0 & B_{15} & B_{16} \\ 0 & 0 & B_{23} & B_{24} & 0 & 0 \\ 0 & 0 & B_{33} & 0 & 0 & 0 \\ B_{41} & B_{42} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_1\dot{\theta}_3 \\ \dot{\theta}_1\dot{\theta}_4 \\ \dot{\theta}_2\dot{\theta}_3 \\ \dot{\theta}_2\dot{\theta}_4 \\ \dot{\theta}_3\dot{\theta}_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ C_{21} & 0 & C_{23} & 0 \\ C_{31} & C_{32} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \\ \dot{\theta}_3^2 \\ \dot{\theta}_4^2 \end{bmatrix} + \mathbf{g} \begin{bmatrix} 0 \\ G_2 \\ G_3 \\ 0 \end{bmatrix} \quad (7)$$

For all the terms, a coefficient with subscript mn corresponds to the coefficients relating the torque on the m th joint to the corresponding n th kinematic term. For example, B_{24} relates the torque on joint 2 τ_2 to the corresponding 4th kinematic term $\dot{\theta}_2\dot{\theta}_3$.