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Concept Review

Denavit-Hartenberg Framework

Why Use the Denavit-Hartenberg Framework?

A general transformation between two frames can be represented with 6 parameters consisting of the position and orientation of a frame with respect to another. The Denavit-Hartenberg framework provide a reduced set of parameters that systematically transform the first frame into the other. This reduced set scales well for serial chains in manipulators that consist of many reference frames related by kinematic constraints (such as link lengths). The DH parameters are also a starting point on which the forward, inverse or differential kinematics, as well as dynamics can be formulaically built. It is a well accepted framework within the robotics community and a starting point for most explorations in this field.

Background

Consider two reference frames $\{A\}$ and $\{B\}$. The transformation matrix between the two can be represented by ${}^{A}T_B$,

$$ {}^{A}T_B = \begin{bmatrix} {}^{A}R_B & {}^{A}p_{AB} \\ [0\ 0\ 0] & 1 \end{bmatrix} \tag{1} $$

Where ${}^{A}R_B = R(\phi, \theta, \psi)$ is the rotation matrix from frame $\{B\}$ to frame $\{A\}$, and is a function of the roll ϕ, pitch θ and yaw ψ angles. The 3x1 matrix ${}^{A}p_{AB}$ is the matrix representation of the vector position of frame $\{B\}$ with respect to frame $\{A\}$, represented in frame $\{A\}$. This is shown in Figure 1.

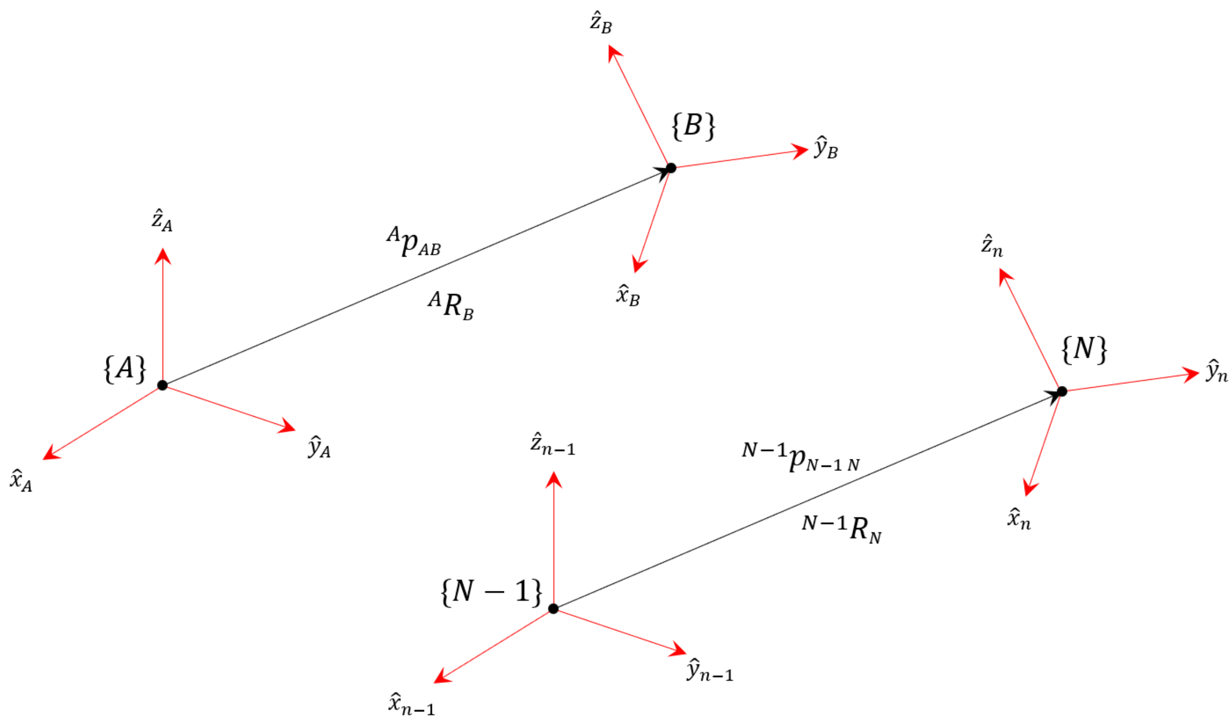


Figure 1. General frame transformation from frame $\{A\}$ to frame $\{B\}$

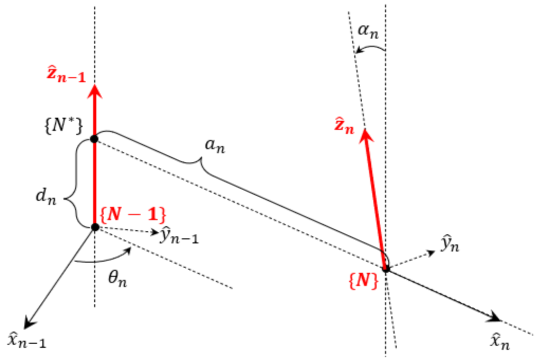
The DH framework provides a reduced set of 4 parameters to represent the position and orientation of frame $\{B\}$ with respect to frame $\{A\}$. For the rest of this document, we will focus on a particular frame of interest, frame $\{N\}$ instead of frame $\{B\}$. It's position and orientation will be in regard to the previous frame $\{N-1\}$ instead of frame $\{A\}$. This is also illustrated in Figure 1.

Keep in mind that the position of frames is not arbitrary with serial manipulators and is often the result of deliberate decision making. A clever choice of the frames of reference, as well as their relative position to each other contributes to the overall reduction in parameters.

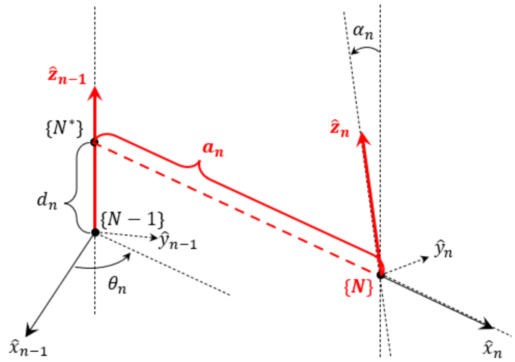
DH Framework

This reduction in parameters is made possible by making a few choices in regard to frame placement.

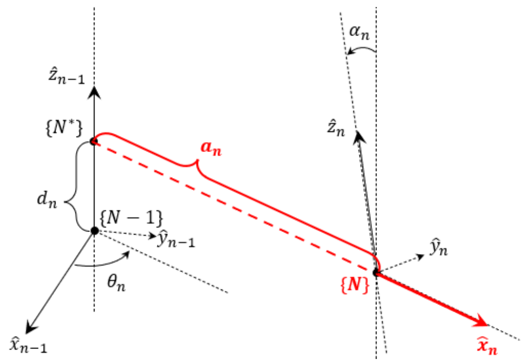
1. The joint axis at any given frame $\{N\}$ is the z-axis of frame $\{N\}$, that is $\hat{z}_n$. This implies that as the link connecting frames $\{N-1\}$ and $\{N\}$ moves with respect to frame $\{N-1\}$, it does so either rotating about $\hat{z}_{n-1}$ or translating along $\hat{z}_{n-1}$. Similarly, as the link connecting frames $\{N\}$ and $\{N+1\}$ moves with respect to frame $\{N\}$, it does so either rotating about or translating along $\hat{z}_n$.



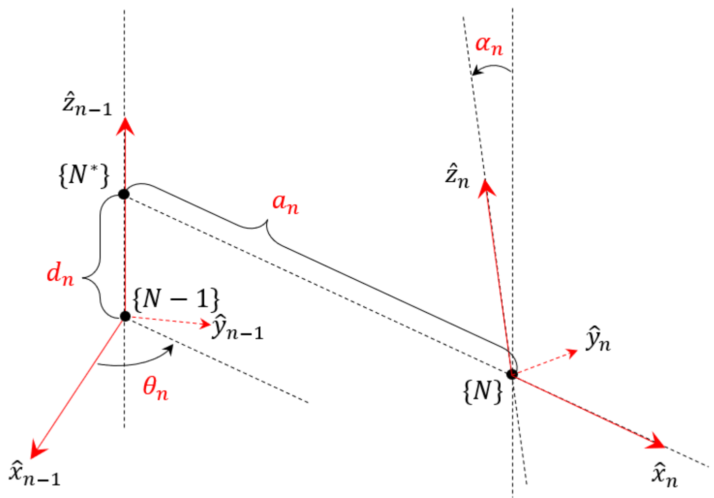
2. The position of the frame $\{N\}$ is chosen to lie at the intersection point of two axis. The first axis of these is the joint axis of frame $\{N\}$, or $\hat{z}_n$. The second axis is an axis along the common normal connecting the previous $\{N-1\}$ frame's joint axis ($\hat{z}_{n-1}$) to the current $\{N\}$ frame's joint axis ($\hat{z}_n$). For directionality, this second axis points along the common normal from the previous frame $\{N-1\}$ to the current frame $\{N\}$.



3. The $\hat{x}_n$ axis of the frame $\{N\}$ is picked to point along the common normal from the previous frame $\{N-1\}$ to the current frame $\{N\}$. This is the same common normal picked in point 2 above.



Note: There are a few variations of the DH notation arising from choices made in the two assumptions above. The version presented here is the standard DH notation by Spong and Vidyasagar [1].



Frames of interest, frame $\{N\}$ and frame $\{N-1\}$ connected through 4 DH parameters

We can make a few observations, as summarized below,

1. $\hat{z}_{n-1} \perp \hat{x}_{n-1}$ A natural consequence of any orthonormal frame
2. $\hat{z}_n \perp \hat{x}_n$ A natural consequence of any orthonormal frame
3. $\hat{z}_{n-1} \perp \hat{x}_n$ A consequence of picking $\hat{x}_n$ along the common normal from $\hat{z}_{n-1}$ to $\hat{z}_n$

These lead to interesting consequences,

1. A single rotation about $\hat{z}_{n-1}$ by an angle $\boldsymbol{\theta_n}$ gets us from $\hat{x}_{n-1}$ to $\hat{x}_n$
2. A single translation along $\hat{z}_{n-1}$ by a distance $\boldsymbol{d_n}$ gets us from the origin of $\{N-1\}$ to $\{N^*\}$
3. A single translation along $\hat{x}_n$ (now same angle as $\hat{x}_{n-1}$) by a distance $\boldsymbol{a_n}$ gets us from the origin of $\{N^*\}$ to $\{N\}$
4. A single rotation about $\hat{x}_n$ by an angle $\boldsymbol{\alpha_n}$ gets us from $\hat{z}_{n-1}$ to $\hat{z}_n$

Thus, one can get from frame $\{N-1\}$ to frame $\{N^*\}$ with 2 operations (step 1 and 2) both involving $\hat{z}_{n-1}$ that also help shape and define $\hat{x}_n$, and then from frame $\{N^*\}$ to frame $\{N\}$ with 2 operations (step 3 and 4) both involving $\hat{x}_n$.

The four parameters θ_n, d_n, a_n and α_n are collectively called the DH parameters and are summarized in Table 1 below. The parameters a and α represent structural parameters of the linkage and assembly, whereas θ and d represent joint parameters for revolute and prismatic joints, respectively.

parameter	type	from	to	about/along	category
θ_n	rotation	$\hat{x}_{n-1}$	$\hat{x}_n$	$\hat{z}_{n-1}$	joint
d_n	translation	$\hat{x}_{n-1}$	$\hat{x}_n$	$\hat{z}_{n-1}$	joint
a_n	translation	$\hat{z}_{n-1}$	$\hat{z}_n$	$\hat{x}_n$	structural
α_n	rotation	$\hat{z}_{n-1}$	$\hat{z}_n$	$\hat{x}_n$	structural

Table 1 – DH parameter summary

The process of assigning DH frames and drawing a schematic is presented then as,

1. Identify Joint axis J_1 through J_n for an N-joint manipulator. Assign axis $\hat{z}_{i-1}$ to joint axis J_i . This leads to standard and definite choices for $\hat{z}_0$ through to $\hat{z}_{n-1}$. Pick $\hat{z}_n$ as required.

2. Find the common normal from $\hat{z}_{i-1}$ to $\hat{z}_i$ for each set of joints and mark the intersection of this common normal and $\hat{z}_i$ as the origin of frame $\{i\}$. Pick the origin of frame $\{0\}$ as required.

3. Assign $\hat{x}_i$ along the common normal identified above, at the origins picked above, in the direction from $\hat{z}_{i-1}$ towards $\hat{z}_i$. Pick $\hat{x}_0$ as required.

4. Complete the DH table of parameters.

These steps are illustrated through an example in the next section. From the DH parameters, the general transformation from frame $\{i-1\}$ to frame $\{i\}$ can be written as,

$$
{}^{i-1}T_i = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} \cos\theta_i & -\sin\theta_i\cos\alpha_i & \sin\theta_i\sin\alpha_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\theta_i\cos\alpha_i & -\cos\theta_i\sin\alpha_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Example

Consider a 2-dof manipulator as shown in Figure 3. The manipulator is attached to the table at point A, where the inertial frame is set with axis ${}^{I}\hat{x}$ and ${}^{I}\hat{y}$. The manipulator has two links of length L_1 and L_2 actuated by two joints at positions A and B as well, with joint parameters θ_1 and θ_2. As these vary, the end-effector of the manipulator at point C moves around with respect to the inertial frame. We are interested in calculating the relationship between the end-effector's position (point C) given by $\vec{p} = \begin{bmatrix} p_x & p_y \end{bmatrix}^T$ as well as the end-effector orientation represented by an angle α. The z component of the position or other elements of the rotation matrix are not required as the manipulator is planar.

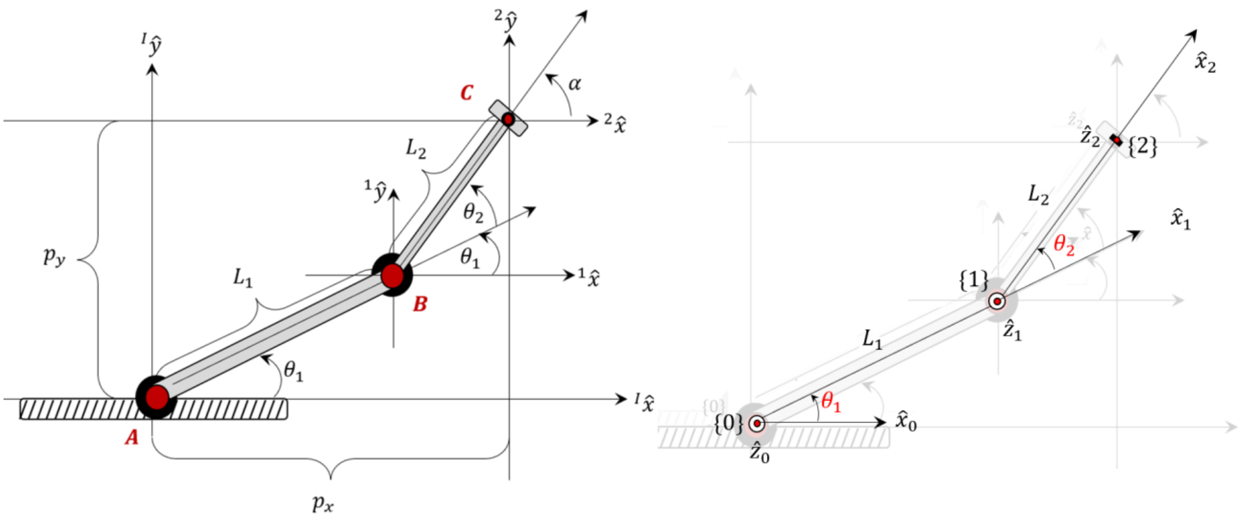


Figure 3. 2-dof manipulator example

The manipulator schematic highlighting the selected z and x axis, as well as origins is also shown in Figure 3. Based on this schematic, the DH table is provided in Table 2. Note that the red dots in the schematic on the right represent a z axis coming out of the page. The choice for z_2 was made so as to place x_2 at the end effector and capture the length L_2 into the DH-table as well.

i	a_n	α_n	d_n	θ_n	$\theta_n(0)$
1	L_1	0	0	θ_1	0
2	L_2	0	0	θ_2	0

Table 2 – DH parameters for the schematic in Figure 3

Note that the initial angles for the joints $\theta_i(0)$ were selected as 0 based on the definitions of the associated x axis. With this selection, the home position of zero joint angles leads to a manipulator that is completely horizontal and on the table.

[1] – Spong, M.W. and Vidyasagar, M. "Robot Dynamics and Control", Wiely, NY, 1989

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