

Differential Kinematics

1 FORWARD KINEMATIC EQUATIONS

Recall the forward kinematics formulation f_{FPK}

$$\begin{aligned} P &= f_{FPK}(\Theta) \\ p_x &= \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} \\ p_y &= \lambda_2 s_1 c_2 - \lambda_3 s_1 s_{23} \\ p_z &= \lambda_1 - \lambda_2 s_2 - \lambda_3 c_{23} \\ \gamma &= \phi_4 \end{aligned} \quad (1)$$

where Θ is the vector of the joint states,

$$\Theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{bmatrix} \quad (2)$$

and p is the position of the end-effector {4} (expressed in base frame {0})

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^0 p_4 \quad (3)$$

2 INVERSE KINEMATIC FORMULATION

Differentiate the equation set 1, to yield,

$$\begin{aligned} v_x &= \frac{dp_x}{dt} = -\lambda_2 s_1 c_2 \dot{\theta}_1 - \lambda_2 c_1 s_2 \dot{\theta}_2 + \lambda_3 s_1 s_{23} \dot{\theta}_1 - \lambda_3 c_1 c_{23} \dot{\theta}_2 - \lambda_3 c_1 c_{23} \dot{\theta}_3 \\ v_y &= \frac{dp_y}{dt} = \lambda_2 c_1 c_2 \dot{\theta}_1 - \lambda_2 s_1 s_2 \dot{\theta}_2 - \lambda_3 c_1 s_{23} \dot{\theta}_1 - \lambda_3 s_1 c_{23} \dot{\theta}_2 - \lambda_3 s_1 c_{23} \dot{\theta}_3 \\ v_z &= \frac{dp_z}{dt} = \lambda_1 - \lambda_2 c_2 \dot{\theta}_2 + \lambda_3 s_{23} \dot{\theta}_2 + \lambda_3 s_{23} \dot{\theta}_3 \\ \dot{\gamma} &= \frac{d\gamma}{dt} = \dot{\theta}_4 \end{aligned} \quad (4)$$

This can be represented in matrix form,

$$\begin{aligned} V &= {}^0 J \dot{\Theta} \\ \begin{bmatrix} v_x \\ v_y \\ v_z \\ \dot{\gamma} \end{bmatrix} &= \begin{bmatrix} -\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23} & -\lambda_2 c_1 s_2 - \lambda_3 c_1 c_{23} & -\lambda_3 c_1 c_{23} & 0 \\ \lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23} & -\lambda_2 s_1 s_2 - \lambda_3 s_1 c_{23} & -\lambda_3 s_1 c_{23} & 0 \\ 0 & -\lambda_2 c_2 + \lambda_3 s_{23} & \lambda_3 s_{23} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{\theta}_3 \\ \dot{\theta}_4 \end{bmatrix} \end{aligned} \quad (5)$$

where ${}^0 J$ is the manipulator Jacobian expressed in base frame {0}.

2.1 SINGULARITY ANALYSIS

The determinant of the Jacobian is derived as,

$$\begin{aligned}
 \det {}^0\mathbf{J} &= (-\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23})((- \lambda_2 s_1 s_2 - \lambda_3 s_1 c_{23})(\lambda_3 s_{23}) - (-\lambda_2 c_2 + \lambda_3 s_{23})(-\lambda_3 s_1 c_{23})) \\
 &\quad + (\lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23})((- \lambda_2 c_2 + \lambda_3 s_{23})(-\lambda_3 c_1 c_{23}) - (-\lambda_2 c_1 s_2 - \lambda_3 c_1 c_{23})(\lambda_3 s_{23})) \\
 \det {}^0\mathbf{J} &= -\lambda_2 \lambda_3 (s_1(-\lambda_2 s_1 c_2 + \lambda_3 s_1 s_{23})(s_2 s_{23} + c_2 c_{23}) - c_1(\lambda_2 c_1 c_2 - \lambda_3 c_1 s_{23})(c_2 c_{23} + s_2 s_{23})) \quad (6) \\
 \det {}^0\mathbf{J} &= \lambda_2 \lambda_3 (s_1^2(\lambda_2 c_2 - \lambda_3 s_{23})(s_2 s_{23} + c_2 c_{23}) + c_1^2(\lambda_2 c_2 - \lambda_3 s_{23})(c_2 c_{23} + s_2 s_{23})) \\
 \det {}^0\mathbf{J} &= \lambda_2 \lambda_3 (\lambda_2 c_2 - \lambda_3 s_{23})(s_2 s_{23} + c_2 c_{23})
 \end{aligned}$$

2.1.1 Singularity Condition 1:

$$\lambda_2 c_2 - \lambda_3 s_{23} = 0 \quad (7)$$

Which corresponds to the end-effector directly above the base.

2.1.2 Singularity Condition 2:

$$\begin{aligned}
 s_2 s_{23} + c_2 c_{23} &= 0 \\
 s_2 s_2 c_3 + s_2 c_2 s_3 + c_2 c_2 c_3 - c_2 s_2 s_3 &= 0 \\
 c_3 &= 0
 \end{aligned} \quad (8)$$

Which corresponds to the arm either folder into itself or completely stretched out.