NETWORK OPTIMIZATION AND GAME THEORY

IE532 –ANALYSIS OF NETWORK DATA

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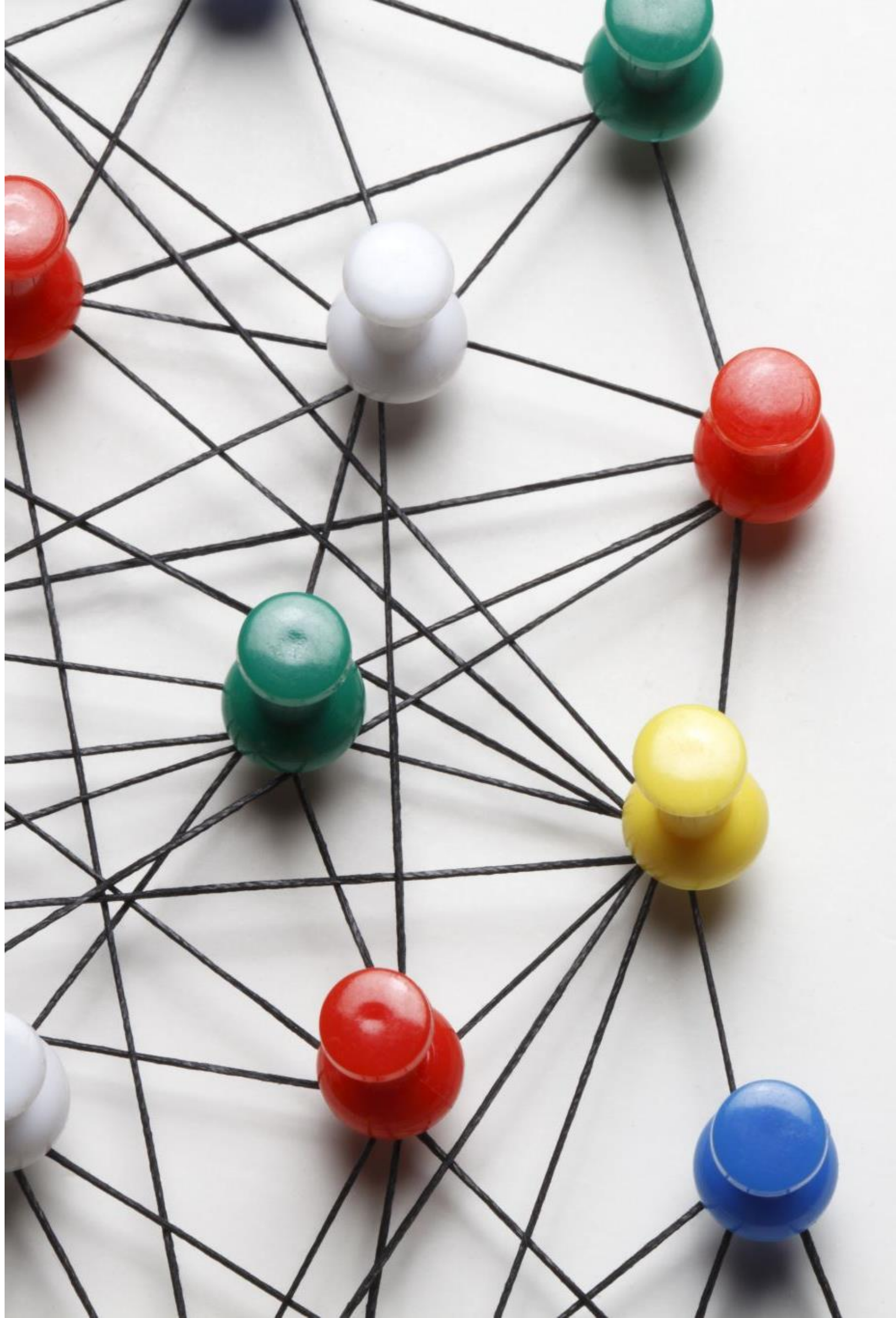
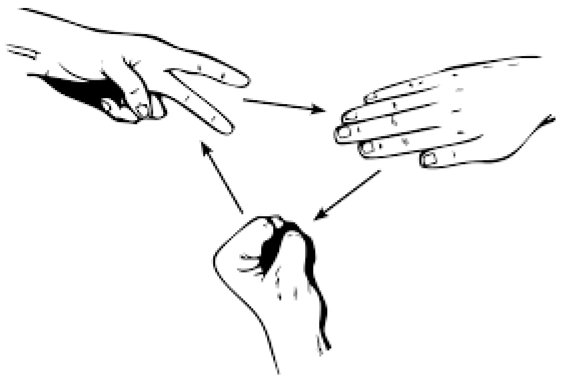


INTRODUCTION

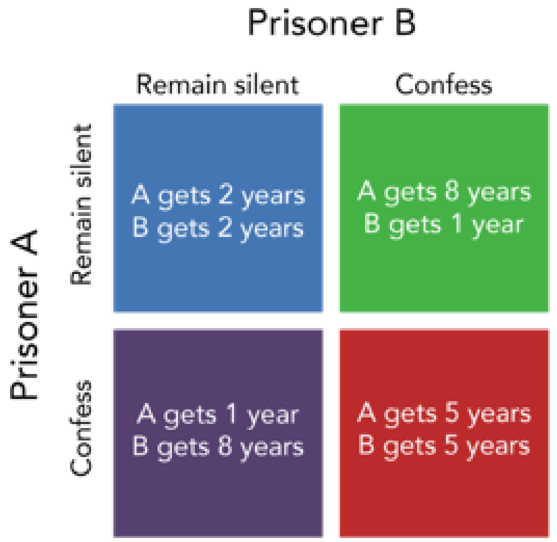
GAMES

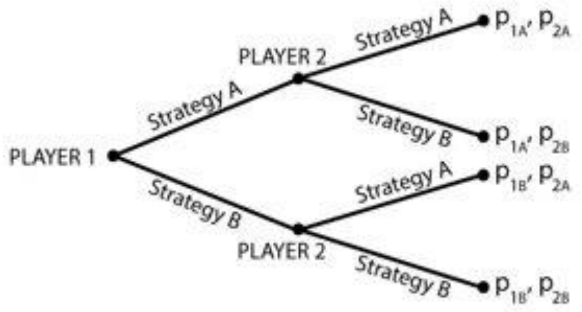
- A setting involving players/agents interacting with one another and each players actions affect/influence the actions of other players and hence, the total outcome from the game.
- Outcomes in games are termed as "payoffs" or "utility", defined as the total profit earned by an entity for a given set of actions corresponding to each player.
- Actions are termed as strategies. Given a set of possible strategies S_i for player i, a player chooses s_i . Strategy profile is the strategy (vector) chosen by each player $S = [s_1, s_2, \ldots s_n]$. (No strictly dominated strategies)
- Objective of Game Theory: Understanding the game => Understanding payoffs of each player => Forming your best responses



GAMES

- What is a best response? Given other players strategies S_{-i} , the strategy of player *i* from S_i that maximizes *i's* payoffs is the Best Response (BR) of *i*.
- Caveat? In most cases (non-sequential/simultaneous games) you are never aware of other players' responses. Your best response will hence be based on "rationality" and "knowledge" of others' expected response.
- This leads to the notion of an equilibrium/optimal response. Given each players payoffs corresponding to all possible strategy profiles one can define a notion of best response given every player has the same knowledge (common knowledge) and behaves rationally.
- Different types of games: coordination, competitive, repeated, zero-sum games, etc.
- Applications: Auctions, Voting, Evolution Theory





GAMES

- To demonstrate equilibrium, let's consider first a very simple 2 player game (P1, P2) with each player having 3 possible strategies (expressed in matrix form). Here, (0,4) in the first cell corresponds. 0 - $u_1(A,A)$ - utility of P1 if P1 chooses A and P2 chooses A.
- In this, if P2 chooses to play B, the P1's BR would be A since it offers the maximum utility for that strategy of P2.
- Any guesses for an equilibrium response? (C,C) i.e. (6,6) is a Nash Equilibrium (NE).
- What is Nash Equilibrium? A response such that no player can do strictly better on deviating from the NE strategy (. The NE strategy always involves strategies that are best responses to at least some response of other players.

		P2		
		A	**B**	**C**
P1	**A**	0,4	4,0	5,3
	B	4,0	0,4	5,3
	C	3,5	3,5	6,6

GAMES

- Mathematically formulation: Here s_i^* is the NE response of Player *i* and s_{-i}^* is the NE response of other players.
- When s_i and s_i^* exist and $\in S_i$, then we call it a pure strategy Nash Equilibrium.
- However, in many real applications, the NE response may not be pure but have a randomization or probability mix of multiple strategies. This is called Mixed Strategy Nash Equilibrium.
- Example: Battle of sexes, Rock Papers Scissors
- $u_1(A,p,1\text{-}p) = u_1(B,p,1\text{-}p)$, where LHS is the utility of P1 when P1 chooses *A* and P2 chooses *A* with probability *p* and *B* with probability *(1-p)*. This will give us the mixed strategy NE.

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \qquad \text{for all } s_i \in S_i$$

		WOMAN		
		Boxing	Shopping	
MAN	Boxing	2 , 1	0 , 0	r
	Shopping	0 , 0	1 , 2	1-r
		q	1-q	

NETWORKS

- Networks are a collection of nodes, and these nodes have some sort of interactions, which are represented by edges connecting the nodes.
- Examples: internet, social network
- Why and what about them is important? As a civilization, we are hard-wired to create links between things, be it physical, virtual or psychological. Studying them reveals how to better deal with them, since they are almost everywhere.
- Network Optimization is devising optimal techniques to carry out operations over these networks in order to achieve a desired outcome. Examples: Shortest Path, maximum flow.

WHY GAMES IN NETWORKS?

- What is common? Games involve people, interaction. So do a huge chunk of networks that we come across commonly.
- Our actions influence actions of people in our network with varying intensities depending upon different aspects of network structure such as popularity in the network, link strength, etc.
- What is not common? Games involve strategy, payoffs and competition. Network optimization techniques do not focus on the strategy and payoffs that motivate achieving a particular optimal outcome.
- Challenges? Decentralization, Fairness, Designing networks in a way such that agents in the network achieve the centralized (socially optimal) optimal solution while acting independently; Incorporating network effects in games.

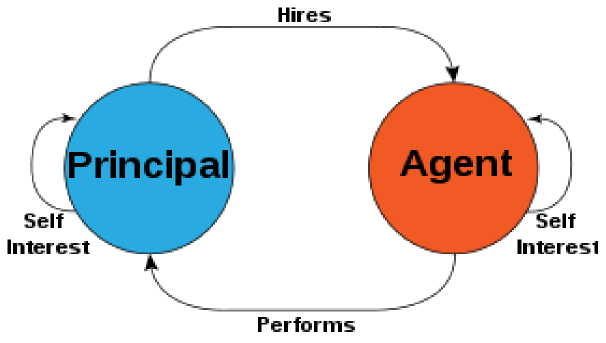
WHY GAMES IN NETWORKS?

- Principal-Agent Problem: Principal is the original owner and person managing an asset/task. He/She then delegates the task to an agent. Problem occurs when there is conflict in priorities of agent and principal.

- Actions of agent affect the value/utility derived by the principal from the asset. Common example: Promotors of a company assign a CEO. CEO decides to use majority of company's quarterly profit to give bonus to employees instead of investing in new projects.

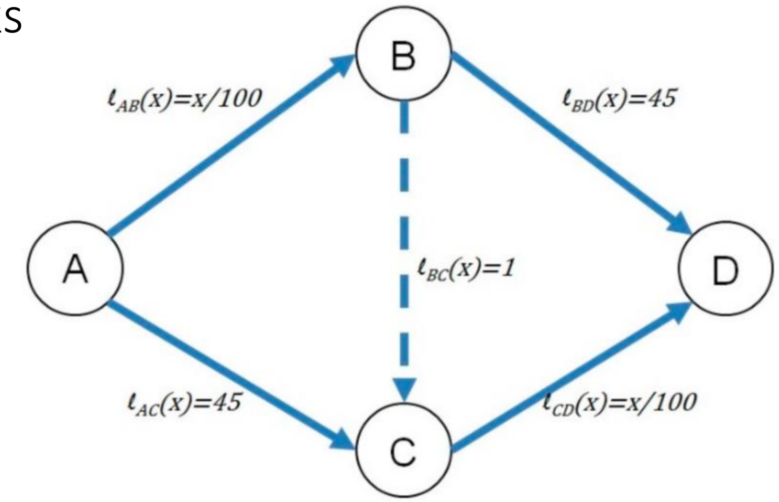
- Loss incurred to principal due to the agent acting against the principal's best interest is called Agency Cost.

- Why is this relevant? This gives rise to Mechanism Design Theory, an idea closely related to decentralization.



WHY GAMES IN NETWORKS?

- Game Theory helps us understand and design incentives for these autonomous players in the network (Mechanism Design Theory) since in most games, there is no coordination or cooperation ~ in some networks centralized control may not be possible.
- It is a major area of research at the intersection of Computer Science, Economics, and Operations Research.
- Some areas of research: Selfish routing, Social Networks-Dynamics, Transportation Networks.
- Braess' Paradox: Decentralized equilibrium
- Network optimization also does not focus on the fairness aspect of a solution.



OVERLAPPING PROBLEMS

SELFISH ROUTING (GAMES IN NETWORKS)

- Given a transport network, *G(V,E)* and latency/congestion of edge *i* as a function of flow x_i in the edge = $l(x_i)$. x_p = flow on path *p, where p is a path belonging to the set of paths P.*

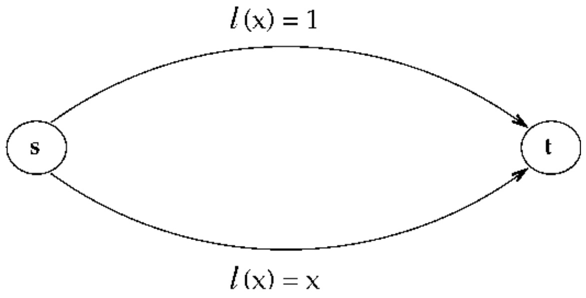
$$x_i = \sum_{\{p \in \mathcal{P} | i \in p\}} x_p$$

- The equation states that the sum of flows on the paths containing edge *i* is the total flow in edge *i*. The traditional routing problem can hence be formulated as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{i \in E} x_i l_i(x_i) \\ \text{subject to} \quad & \sum_{\{p \in \mathcal{P} | i \in p\}} x_p = x_i, \quad i \in E, \ x_p \geq 0, \quad p \in \mathcal{P} \end{aligned}$$

- This is also called the socially optimal solution, I.e. where everyone agrees owing to the social benefit of everyone. However, this is not possible to achieve when users act selfishly.

SELFISH ROUTING



- Wardrop Equilibrium (WE) is the solution for the routing problem that users achieve when they act selfishly. This is equivalent to a NE of a non-atomic (having large number of infinitesimal players) non-cooperative game i.e.

"It states that the journey times in all routes actually used are equal and less than those that would be experienced by a single vehicle on any unused route. The traffic flows that satisfy this principle are usually referred to as "user equilibrium" (UE) flows, since each user chooses the route that is the best."

- In this routing example where the latency of first link is constant and other is flow dependent, the NE or WE would converge to all traffic being through the *l(x)=1* path. (fair to all)
- The cost of selfish routing is always ≥ cost of social optimum routing.
- It should be noted that Nash equilibrium does not optimize any global criterion, and so not as such close to a solution of minimal total travel time, i.e., the system optimum. The degradation due to distributed control is measured using Price of Anarchy.

$$\frac{\text{Cost of Selfish Routing}}{\text{Cost of System Optimal}}$$

SELFISH ROUTING

- Assume for e.g. Internet, there are local network providers and end-users. If we wish to improve the network as well as allow decentralization, we need to shift the methodology. - Partially optimal Routing
- In POR, while the traffic routes selfishly across the whole network, it routes in a system optimal way in subnetworks of the graph operated by the local network providers.
- Question is how to design appropriate incentives?

Model:

- Assume I parallel links. Routing d units of inelastic traffic (traffic that cannot accommodate to changes in network fluctuations). Each link owned by a different service provider: charges a price p i per unit bandwidth on link I. Users have a reservation utility *R* i.e. they do not send their flow if the effective cost exceeds *R* [2].
- Wardrop's principle: Flows routed along paths with minimum "effective cost".

SELFISH ROUTING

Definition: Given $p \geq 0$, x^* is a *Wardrop Equilibrium* (WE) if

$$l_i(x_i^*) + p_i = \min_j \{l_j(x_j^*) + p_j\}, \qquad \text{for all } i \text{ with } x_i^* > 0,$$

$$l_i(x_i^*) + p_i \leq R, \qquad \text{for all } i \text{ with } x_i^* > 0,$$

and $\sum_{i \in \mathcal{I}} x_i^* \leq d$, with $\sum_{i \in \mathcal{I}} x_i^* = d$ if $\min_j \{l_j(x_j) + p_j\} < R$.

- Here the first equation essentially says that users will choose the route with min effective cost (latency + service provider bandwidth cost). $l_i = f(x_i)$ is convex, differentiable and non-decreasing.
- The second constraint says that the effective cost must be less than R for users to choose this route.
- Other than Wardrop's, there are other notions too such as Oligopoly Equilibrium that involve game-theoretic concepts like subgame perfect equilibrium. Independently, there are other ways too to create decentralized systems like closed Loop decentralized feedback system, as used in telecommunication networks [1]. But they have their own limitations, one if which is they do not seem to converge to a global solution.

GRAPHICAL GAMES (NETWORKS IN GAMES)

- The notion of the existence of equilibrium has been a prime focus of problems in Game Theory. This study presents economics presented as a graphical game, where the permitted interactions in the game are defined by the edges of the graph to emulate real networks and the network structure plays a role in establishing local equilibrium.
- First, we try to model our problem in an efficient way: Consider n consumers, *k* commodities, priced at p_h where $h = \{1,2,....k\}$ and exchange occurs only between agents adjacent to each other.
- There are also budget constraints and we define utility as the wealth earned by selling all initial amount of commodities held by each consumer or the wealth earned by consumption plan X_i.
- Applying adjacency constraints, price vector is defined locally for consumers adjacent to i

GRAPHICAL GAMES

$$p * e^i = \sum_{h \in K} p_h e^i_h$$

$$\text{Consumption Plan} = x_i \in R^k_+$$

$$\text{where } x^i_h = \text{amount of commodity h purchased by i}$$

$$\text{Initial Amount} = e_i \in R^+$$

$$\text{where } e^i_h = \text{initial amount of commodity h held by i}$$

$$\text{Budget Constraint} : p * x^i \leq p * e^i$$

$$\Downarrow$$

$$\text{Consumption Plan} = x_{ij} \in R^k_+$$

$$\text{where } x^{ij}_h = \text{amount of commodity h purchased by i from j}$$

$$\text{and } x^{ij}_h = 0 \quad \forall j \notin N(i) \quad \text{(neighbour of i)}$$

GRAPHICAL GAMES

- Here, rationality and market clearance are two ideas employed to attain the equilibrium. They must exist
- Rationality refers to the consumer choosing an optimal consumption plan that maximizes their utility under budget constraints.
- Market Clearance refers to the supply being equal to the demand

$$d^i(demand) = \sum_{j \in N(i)} x^{ij}$$

- *"A market or graphical equilibrium is a set of prices and plans in which all plans are optimal at the current prices and in which the market clears".*
- However, there are certain assumptions for the equilibrium to exist: First, the utility function is continuous, strictly monotonically increasing with commodities and quasi-concave. Second, the initial amount of commodities held by consumers is non-zero.

GRAPHICAL GAMES

- Given the model and conditions, the authors establish, through approximate algorithms, the existence of graphical equilibrium and also propose an efficient computation algorithm to compute the equilibrium.
- Since establishing the equilibrium is an NP-Hard problem, approximation algorithms are used.
- There can be multiple variations in this model:
  - Repeated Games – when the game is repeated over and over the same participants, the consumers gather some knowledge of other players which helps them evolve their strategies.
  - Preference of commodity and preferred neighbour to purchase commodity.
  - Modelling of prices

FUTURE WORK/CONCLUSION

FUTURE WORK

- There is ongoing work in the realm of online algorithms and dynamic algorithms.
- Considering different types of scenarios and network parameters like lower bounds on traffic network influence in choosing route, the routing problem can be modelled differently.
- Scalability to even larger networks.
- Stability of networks.

REFERENCES

- Algorithmic Game Theory - https://www.cs.cmu.edu/~sandholm/cs15-892F13/algorithmic-game-theory.pdf
- Networks' Challenge: Where Game Theory Meets Network Optimization - https://asu.mit.edu/sites/default/files/presentations/ISIT08-tutorial.pdf
- Games on Networks - https://web.stanford.edu/~jacksonm/GamesNetworks.pdf
- Graphical Economics - http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.115.5017&rep=rep1&type=pdf
- https://ocw.mit.edu/courses/sloan-school-of-management/15-094j-systems-optimization-models-and-computation-sma-5223-spring-2004/lecture-notes/lec24_soda_03.pdf

[1] https://ieeexplore.ieee.org/document/4048910

[2] https://www.researchgate.net/publication/2644855_Optimal_Routing_of_Traffic_Flows_with_Length_Restrictions_in_Networks_with_Congestion