LMAPR2020: Materials Selection

Second homework on influence of shape in material selection

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Question 1:

a. For the hollow rectangle we get:

$$I_{min} = \int_{-4t}^{4t} y^2 b(y) dy = \left[\frac{y^3}{3}\right]_{-4t}^{4t} 8t = (\frac{64t^3}{3} + \frac{64t^3}{3})8t = \frac{1024t^4}{3}$$

$$I_{max} = \int_{-5t}^{5t} y^2 b(y) dy = \left[\frac{y^3}{3}\right]_{-5t}^{5t} 10t = (\frac{125t^3}{3} + \frac{125t^3}{3})10t = \frac{2500t^4}{3}$$

$$I_{tot} = \frac{2500t^4}{3} - \frac{1024t^4}{3} = 492t^4$$

$$and$$

$$I_{top} = I_{bottom} = \frac{10t^4}{12} + 10t^2\left(\frac{9t}{2}\right)^2 = \frac{2440t^4}{12}$$

$$I_{left} = I_{right} = \frac{512t^4}{12}$$

$$I_{total} = I_{top} + I_{bottom} + I_{left} + I_{right} = \frac{4880t^4 + 1024t^4}{12} = 492t^4$$

For the I shape we get:

$$I_{top} = I' + Ad^2 = \frac{5t^4}{12} + 5t^2\left(\frac{9t}{2}\right)^2 = \frac{1220t^4}{12}$$

$$I_{center} = I' + Ad^2 = \frac{(8t)^3 t}{12} = \frac{524t^4}{12}$$

$$I_{bottom} = I' + Ad^2 = \frac{7t^4}{12} + 7t^2\left(\frac{9t}{2}\right)^2 = \frac{1708t^4}{12}$$

$$I_{total} = \frac{1220t^4}{12} + \frac{524t^4}{12} + \frac{1708t^4}{12} = \frac{863t^4}{3}$$

For the hollow circle we get:

$$I_{min} = \int_0^{2\pi}\int_0^{r} (r sin\theta)^2 r dr d\theta = \int_0^{2\pi} \frac{r^4}{4} sin^2\theta d\theta = \frac{r^4}{4}\left[\frac{1}{2}(\theta - sin\theta cos\theta)\right]_0^{2\pi} = \frac{\pi}{4}r^4$$

$$I_{max} = \int_0^{2\pi}\int_0^{R} (r sin\theta)^2 r dr d\theta = \int_0^{2\pi} \frac{r^4}{4} sin^2\theta d\theta = \frac{R^4}{4}\left[\frac{1}{2}(\theta - sin\theta cos\theta)\right]_0^{2\pi} = \frac{\pi}{4}R^4$$

$$I_{tot} = \frac{\pi}{4}R^4 - \frac{\pi}{4}r^4 = \frac{\pi}{4}((7t)^4 - (6t)^4) = \frac{\pi}{4}(2401t^4 - 1296t^4) = \frac{\pi}{4}1105t^4$$

b. In order to compute the shape factor, we compute the second moment of area for a square of the same cross section (same mass per unit of length) as the modified shape. (Note that we take the same material each time so the shape factor depends only on I and not E)

$$A_{hollowsquare} = 2(10t * t) + 2(8t * t) = 36t^2 \text{ and } I_{ref} = \frac{A^2}{12} = 108t^4$$

$$\text{For a hollow square shape}: \phi_B^e = \frac{I}{I_{ref}} = \frac{492}{108} = 4.56$$

$$A_{Ishape} = (5t * t) + (8t * t) + (7t * t) = 20t^2 \text{ and } I_{ref} = \frac{A^2}{12} = \frac{400t^4}{12}$$

$$\text{For a I shape}: \phi_B^e = \frac{I}{I_{ref}} = \frac{863/3}{400/12} = 8.63$$

$$A_{hollowcircle} = \pi(7^2 - 6^2)t^2 = 13\pi t^2 \text{ and } I_{ref} = \frac{A^2}{12} = \frac{169\pi^2 t^4}{12}$$

$$\text{For a hollow circle shape}: \phi_B^e = \frac{I}{I_{ref}} = \frac{1105\pi/4}{169\pi^2/12} = 6.24$$

c. As explained in Ashby's book, the shape factor is dimensionless. This means that it depends only on the shape and not on the scale. So even if the cross-section area is different for the three models, it does not change the result. Therefore, we can confirm that the I shape will always be more efficient in stiffness than a hollow circle and even more than a hollow square.

Question 2:

a.

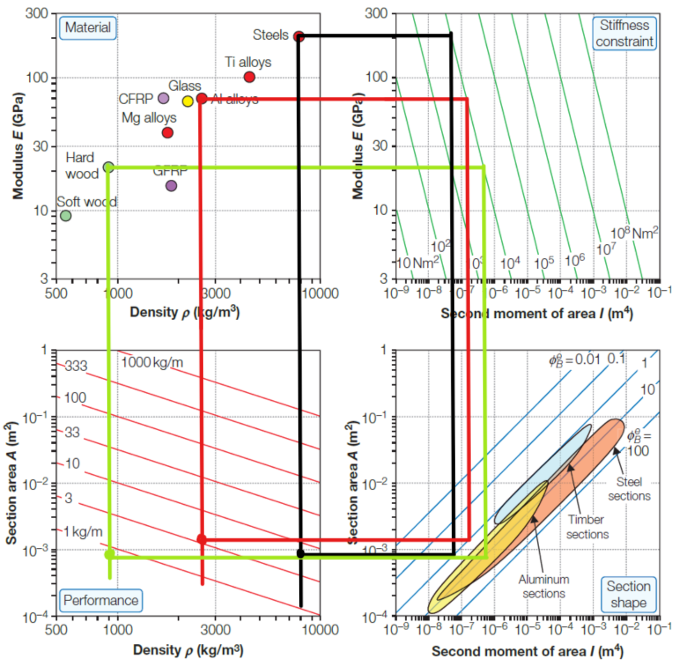

Figure 1: A comparison of steel, aluminum, and hard wood for a stiffness-limited design with $EI = 10^4 Nm2$.

We plot the various lines based on the data provided and the method used in Ashby's book. As we can see, steel weighs approximately 7 kg/m, aluminum 3.5 kg/m, and hardwood 0.8 kg/m.

b.

To compute the linear density $\frac{m}{L}$, we take the relation for stiffness-limited beam design:

$$EI = 10^4 \, \text{Nm}^2$$

$$I = \frac{EI}{E} \qquad A^2 = \frac{12I}{\phi_B^e} = \frac{12EI}{E \cdot \phi_B^e} \qquad \Rightarrow \qquad \frac{m}{L} = \rho A = \rho \sqrt{\frac{EI}{E \cdot \phi_B^e}}$$

Finally we get:

$$\frac{m}{L} = \rho \sqrt{\frac{12 \cdot 10^4}{E \cdot \phi_B^e}}$$

For the steel:

$$\frac{m}{L} = \rho \sqrt{\frac{12 \cdot 10^4}{E \cdot \phi_B^e}} = 7900 \sqrt{\frac{12 \cdot 10^4}{200 \times 10^9 \cdot 1}} = 6.12 \quad [kg/m]$$

For the aluminum:

$$\frac{m}{L} = \rho \sqrt{\frac{12 \cdot 10^4}{E \cdot \phi_B^e}} = 2700 \sqrt{\frac{12 \cdot 10^4}{70 \times 10^9 \cdot 1}} = 3.53 \quad [kg/m]$$

For the hard wood:

$$\frac{m}{L} = \rho \sqrt{\frac{12 \cdot 10^4}{E \cdot \phi_B^e}} = 900 \sqrt{\frac{12 \cdot 10^4}{23 \times 10^9 \cdot 10}} = 0.65 \quad [kg/m]$$

We note that we mathematically obtain values approximately similar to those obtained via the 4-quadrant chart.

Question 3:

a. Specifications:

- Function: Designing a stiff and light beam

- Constraint: Stiff beam that meets the required bending stiffness S^*

- Objective: minimize weight

- Free Variable: shape of the beam and material choice

b.

$$m = \rho.A.L \quad and \quad S = \frac{C_1 EI}{L^3} \quad with \quad \phi_B^e = \frac{12I}{A^2} \quad => \quad S = \frac{C_1}{12} \frac{E}{L^3} \phi_B^e A^2$$

$$m = \left(\frac{12S^*}{C_1} \right)^{1/2} L^{5/2} \left[\frac{\rho}{(\phi_B^e E)^{1/2}} \right]$$

So the material index is : $M_1 = \frac{(\phi_B^e E)^{1/2}}{\rho}$ and if the shape is the same $M_1 = \frac{E^{1/2}}{\rho}$

c. Regardless of shape, we have three different materials that maximize our material index: balsa wood, femur trabecular bone, and radius trabecular bone. Since two of these are bones, they are not relevant for our application. Therefore, the best choice seems to be balsa wood (ochroma spp.)(0.09-0.11)(L).

d. In order to create the pseudo materials, we need to redefine the material index to be able to display

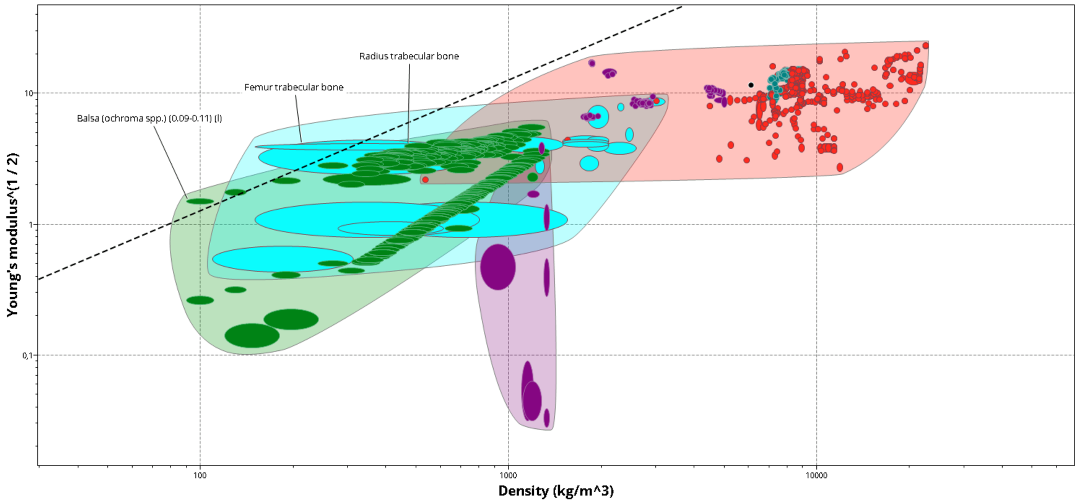

Figure 2: Graph of the Young's Modulus as a function of the density

them on Granta. We therefore have:

$$M_1 = \frac{(\phi_B^e E)^{1/2}}{\rho} = \frac{(\phi_B^e E)^{1/2}/\phi_B^e}{\rho/\phi_B^e} = \frac{(E/\phi_B^e)^{1/2}}{\rho/\phi_B^e} = \frac{(E^*)^{1/2}}{\rho^*}$$

	Density (kg/m^3)	Young's Modulus (GPa)	$\phi_B^e, 1$	$\frac{E^{1/2}}{\rho}$	$\frac{(E^*)^{1/2}}{\rho^*}$	$\phi_B^e, 2$	$\frac{(E^*)^{1/2}}{\rho^*}$
Balsa Wood	90 - 110	2,1 - 2,5	10	16,85	53,28	1	16,85
Al_8090_T851	$2,52 \times 10^3$ - $2,57 \times 10^3$	80 - 84	10	3,59	11,36	15	13,91
Berylium	$1,85 \times 10^3$ - $1,86 \times 10^3$	290 - 305	10	9,29	29,4	10	29,4
1020 Steel	$7,87 \times 10^3$ - $8,07 \times 10^3$	195 - 197	10	1,76	5,55	20	7,85
Oak	690 - 840	12,1 - 14,8	10	4,77	0,015	2	6,77

Table 1: Table with shape improvements for different materials

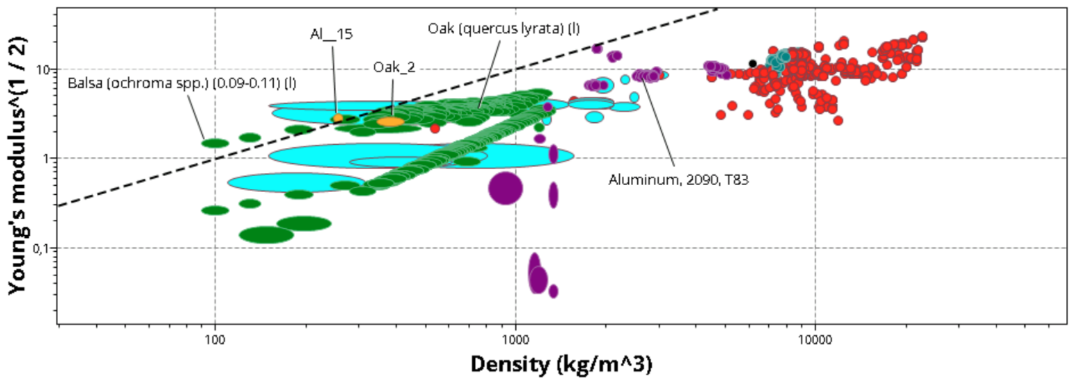

Figure 3: Graph of the Young's Modulus as a function of the density with pseudo material

When a shape factor of 10 is applied to all materials, balsa wood still maximize the material index, thanks to its extremely low density. However, this assumption does not reflect real-world design constraints. In practice, not all materials can be shaped with the same freedom. For example, balsa wood is very light but also anisotropic, brittle and difficult to machine. These properties limit its ability to be transformed into optimized structures such as I-beams. For this reason, in Ashby's book they take a maximum shape factor of 2 for woods, which significantly reduces their effective performance in structural applications. On the other hand, metallic materials such as steel or aluminum have an excellent suitability for manufacturing optimized shapes. Their typical shape factors range from 15 to 20, allowing them to take full advantage of their mechanical properties. As you can see Beryllium seems to be the best material but its cost and toxicity made of him a bad candidate. Finally, the best material for industrial application seems to be **Al_8090_T851**.

e. We want to achieve the same bending stiffness $S = \frac{C_1 EI}{L^3}$ for each material. Since $I = \frac{\phi_B^e A^2}{12}$, and by taking a reference area of $1m^2$ for aluminum we have:

$$S_1 = S_2 \quad \Rightarrow \quad \frac{E_1 \phi_{B,1}^e A_1^2}{12L^3} = \frac{E_2 \phi_{B,2}^e A_2^2}{12L^3} \quad \Rightarrow \quad A_1 = A_2 \sqrt{\frac{E_2 \phi_{B,2}^e}{E_1 \phi_{B,1}^e}}$$

Material	A (m^2)	Cost (EUR/kg)	Estimated Cost (EUR)
Balsa Wood	23.13	6.15 - 9.88	18,538.7
Al_8090_T851	1.00	22.7 - 24.7	60,316.5

Table 2: Cost comparison for equivalent stiffness

We observe that for the same stiffness, balsa is **3.3 times cheaper** than aluminum. However, this result has practical limitations. Indeed, Balsa wood is *anisotropic* and fragile which makes it unsuitable for some applications. Moreover, the required area of 23.13 m^2 is clearly unrealistic for a beam... Aluminum has a higher reliability in structural applications, it represents a better practical compromise between mechanical performance, machining and cost.

f. The selection of material doesn't only depend on mechanical properties or the cost but also on other aspects like:

- Aesthetics: The natural look of wood can be attractive as it has a natural look, while on the other hand steel beams symbolizing industrial revolution and modern design with its clean lines and shape.
- History: Timber frames has been used for over a thousand years, dating back to medieval European architecture, While steel I-beams became popular in the 19th century and the shows industrial innovation
- Region & accessibility: In forested remote areas, local timber is much cheaper and accessible due to abundance and reduced transport costs, since transporting heavy steel beams to isolated can be more difficult wood is mainly used in these regions.
- Machinability: wood is light weight and relatively easy to manufacture, but is affected by ambient conditions like moist and heat which significantly affect the final product accuracy and precision leading to irregular shapes.
- Durability & maintenance: Due to moist and heat wood deteriorate over time, it can wrap, rot, attract pests, so higher maintenance is needed compared to steel beams which have much higher durability and low maintenance as it can maintain its dimensional accuracy over time