LMAPR2020: Materials Selection

# First homework on multiple constraints and conflicting objectives

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# Question 1 :

## Why does the dependence on geometric constraints change?

When moving from a single-constraint problem to a multiple-constraint problem, the number of free variables gradually decreases. In other words, the constraints become interdependent, which limits the freedom to adjust the geometric parameters of the part. Initially, with a single constraint, it is possible to modify the geometry to optimize the set objective. However, adding new constraints restricts these adjustments by imposing limits on certain dimensions. Therefore, the optimal choice of material no longer depends only on a material performance index, but also on the geometric constraints imposed by the problem. This changes the optimization strategy and can lead to a different optimal material.

## Solution for a stiff and light tie of given length but free diameter

Minimize the mass given by:

$$m = AL\rho$$

With a minimum stiffness given by:

$$S^* \leq \frac{EA}{L}$$

With A the area of the tie so

$$S^* \leq \frac{E\pi d^2}{4L} \qquad => \qquad d_{min,stiff} = \sqrt{\frac{4LS^*}{E\pi}}$$

Finally by coupling the two equation with L the free parameter we find:

$$m_{1,min} = \frac{S^* L^2 \rho}{E} = S^* L^2 (\frac{\rho}{E}) => \text{So we need to minimize } M_1 = (\frac{\rho}{E})$$

## Solution by adding a second constraint while keeping the same problem

We add a constraint on tensile strength.

The mechanical stress undergone by the tie is:

$$\sigma = \frac{F}{A} = \frac{4F}{\pi d^2}$$

We impose that to avoid the break:

$$\frac{4F}{\pi d^2} \leq \sigma_{max} \qquad => \qquad d_{min,strength} = \sqrt{\frac{4F}{\pi \sigma_{max}}}$$

So we have:

$$m_{2,min} = \frac{FL\rho}{\sigma_{max}} = FL(\frac{\rho}{\sigma_{max}}) => \text{So we need to minimize } M_2 = (\frac{\rho}{\sigma_{max}})$$

### Finding the best material

We will use the same method as presented in Ashby's book. Using the graphical method, we equate the two indices $m_1$ and $m_2$.

$$L^2S^*(\frac{\rho}{E}) = LF^*(\frac{\rho}{\sigma_{max}}) \qquad => \qquad M_1 = \frac{F^*}{LS^*}M_2$$

Donc en prenant le logarithme on obtient l'équation finale:

$$\log(M_1) = \log(M_2) + \log(\frac{F^*}{LS^*}) \qquad \text{or} \qquad \log(M_2) = \log(M_1) + \log(\frac{LS^*}{F^*})$$

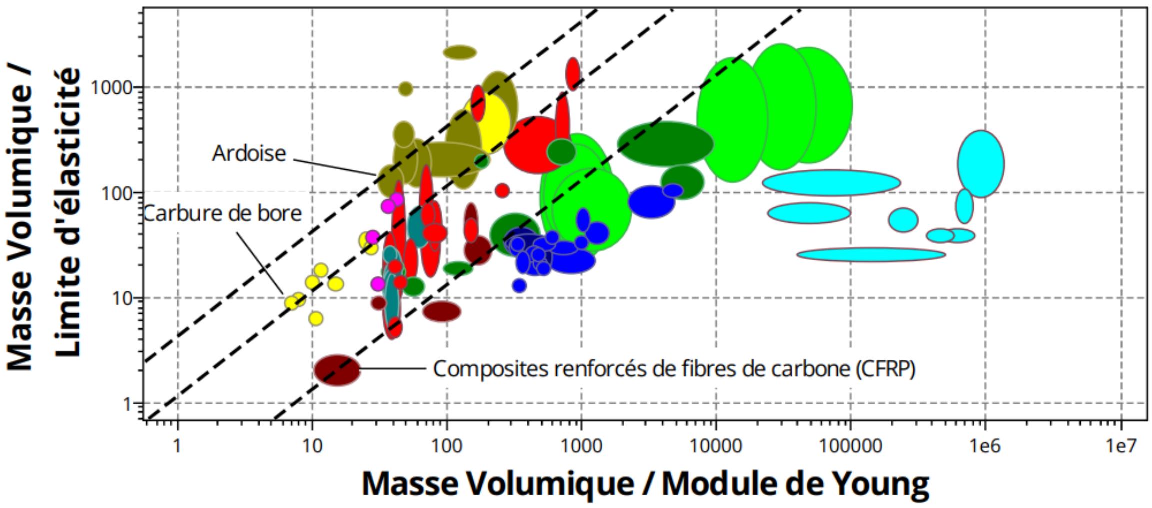


Figure 1: Graph of density / yield strength versus density / young's modulus

As can be seen from the graph the best material is either **carbon fiber reinforced composite** material or **boron carbide** or **slate** depending on the value of the coupling constant $(\frac{LS^*}{F^*})$. This clearly represents the importance of the L parameter in the choice of material which is no longer a free variable since it modifies the choice of the final material.

## Question 2 :

### a.

#### i) Expression for the material cost of the column

The total material cost is given by:

$$Cost = Mass * C_m$$

The total mass of the column is:

$$m = \rho * V = \rho * (\frac{\pi D^2}{4} H)$$

So the cost is equal to:

$$Cost = C_m \rho \frac{\pi D^2}{4} H$$

### ii) Expression of Buckling and Compression Stresses

For the compressive stress we have:

$$\frac{F}{A} \leq \sigma_c \quad => \quad \frac{4F}{\pi D^2} \leq \sigma_c \quad \text{so} \quad D \geq \sqrt{\frac{4F}{\pi \sigma_c}}$$

For the buckling stress we use the Euler formula with n = 1:

$$F_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

In our case K = 1 and L = H and replacing the moment of inertia we obtain:

$$F \leq \frac{\pi^3 E D^4}{64 H^2} \quad \text{so} \quad D \geq (\frac{64 F H^2}{\pi^3 E})^{1/4}$$

### iii) Material Index

For M1 we get:

$$Cost = C_m \rho \frac{F}{\sigma_c} H \quad \text{so} \quad M_1 = (\frac{C_m \rho}{\sigma_c})$$

For M2 we get:

$$Cost = C_m \rho (\frac{\pi H}{4} (\frac{64 F H^2}{\pi^3 E})^{1/2}) \quad => \quad Cost = C_m \rho H (\frac{2}{\pi^{1/2}} \frac{(FH^2)^{1/2}}{E^{1/2}}) \quad \text{so} \quad M_2 = (\frac{C_m \rho}{E^{1/2}})$$

### b.

If we calculate these indices for each material given in the table below:

| **Material** | $M_1$ | $M_2$ | Cost$_1$ | Cost$_2$ |
|---|---|---|---|---|
| Wood (spruce) | $\frac{0.5\times700}{25} = 14.0$ | $\frac{0.5\times700}{\sqrt{10000}} = 3.5$ | 4200000.0 | 11239.98 |
| Brick | $\frac{0.35\times2100}{95} = 7.74$ | $\frac{0.35\times2100}{\sqrt{22000}} = 4.96$ | 2321052.63 | 15913.79 |
| Granite | $\frac{0.6\times2600}{150} = 10.4$ | $\frac{0.6\times2600}{\sqrt{20000}} = 11.03$ | 3120000.0 | 35424.78 |
| Poured concrete | $\frac{0.08\times2300}{13} = 14.15$ | $\frac{0.08\times2300}{\sqrt{20000}} = 1.3$ | 4246153.85 | 4178.31 |
| Cast Iron | $\frac{0.25\times7150}{200} = 8.94$ | $\frac{0.25\times7150}{\sqrt{130000}} = 4.96$ | 2681250.0 | 15921.06 |
| Structural steel | $\frac{0.4\times7850}{300} = 10.47$ | $\frac{0.4\times7850}{\sqrt{210000}} = 6.85$ | 3140000.0 | 22004.81 |
| Al alloy 6061 | $\frac{1.2\times2700}{150} = 21.6$ | $\frac{1.2\times2700}{\sqrt{69000}} = 12.33$ | 6480000.0 | 39611.2 |

Table 1: Calculation of material performance indices

Materials with the lowest indices perform best, so we first take the maximum of Cost1 and Cost2 for each material, then the minimum among these. So **brick** seems to be the best choice.

**c.**

The coupling constant is obtained by equating the two cost functions as follows:

$$M_1 FH = M_2 H(\frac{2}{\pi^{1/2}}(FH^2)^{1/2}) \qquad => \qquad M_1 = M_2(\frac{2}{\pi^{1/2}}\frac{H}{F^{1/2}})$$

So the two coupling constant are :

$$M_2 = 93.41 \times M_1 \qquad \text{and} \qquad M_2 = 1401.25 \times M_1$$

Or

$$log(M_2) = log(M_1) + 1.97 \qquad \text{and} \qquad log(M_2) = log(M_1) + 3.15$$

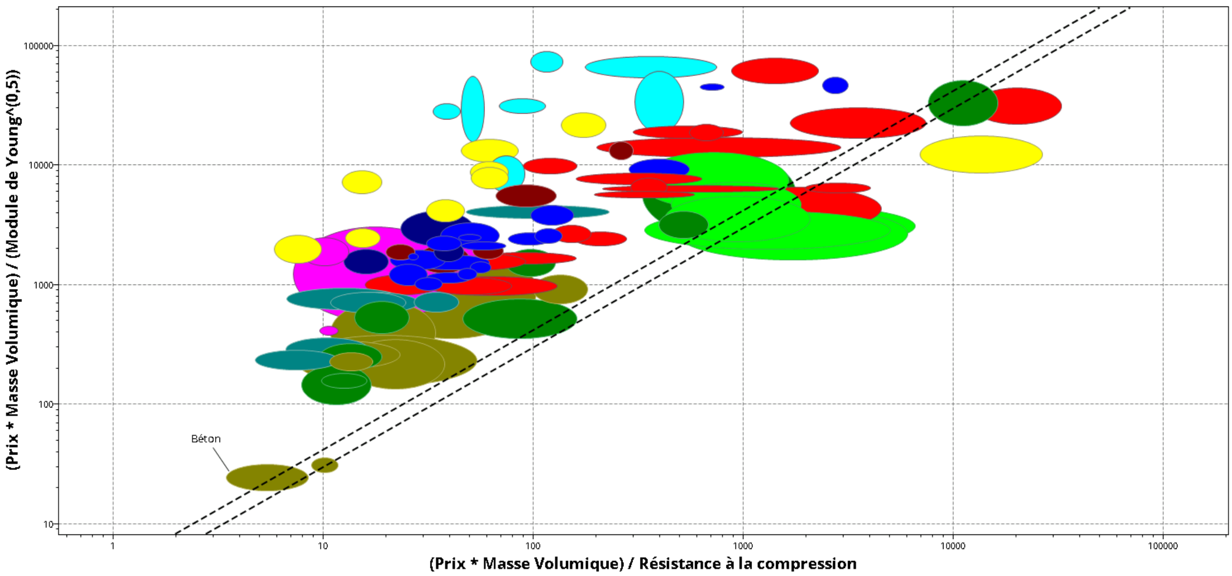


Figure 2: Graphical method selection

So here **concrete** seems to be the best material.

# Question 3 :

## $a)$Derivation of Material Indices

In order To design a lightweight and cost-effective solid shaft we have to ensure it does not plastically deform.

### Torsional Failure

From failure theory:

$$T_f = \frac{K\sigma_y}{d} \tag{1}$$

For a solid circular shaft:

$$K = \frac{\pi d^4}{32} \tag{2}$$

Substituting:

$$T_f = \frac{\pi d^3 \sigma_y}{32} \tag{3}$$

Solving for diameter:

$$d \propto \left(\frac{T}{\sigma_y}\right)^{1/3} \tag{4}$$

### Material Index for Mass Minimization

The mass of the shaft is:

$$m = \rho \frac{\pi d^2}{4} L \tag{5}$$

Substituting $d^2$:

$$m \propto \rho L \left(\frac{T}{\sigma_y}\right)^{2/3} \tag{6}$$

Minimizing mass requires maximizing:

$$M_1 = \frac{\sigma_y^{2/3}}{\rho} \tag{7}$$

### Material Index for Cost Minimization

The cost is:

$$\text{Cost} \propto C_m L \left(\frac{T}{\sigma_y}\right)^{2/3} \tag{8}$$

Minimizing cost requires maximizing:

$$M_2 = \frac{\sigma_y^{2/3}}{C_m} \tag{9}$$

## $b)$**Penalty Function Approach**

To balance mass and cost, we define the penalty function:

$$P = \alpha \frac{\rho}{\sigma_y^{2/3}} + (1-\alpha) \frac{C_m}{\sigma_y^{2/3}} \tag{10}$$

Minimizing $P$ is equivalent to maximizing:

$$M = \frac{\sigma_y^{2/3}}{\alpha \rho + (1-\alpha) C_m} \tag{11}$$

**Extreme Cases**

**1. Weight Priority** ($\alpha = 1$):

$$M_1 = \frac{\sigma_y^{2/3}}{\rho} \tag{12}$$

**2. Cost Priority** ($\alpha = 0$):

$$M_2 = \frac{\sigma_y^{2/3}}{C_m} \tag{13}$$

**Medium Case (Balanced)**

If weight and cost are equally important ($\alpha = 0.5$), we maximize:

$$M = \frac{\sigma_y^{2/3}}{\rho + C_m} \tag{14}$$

## $c)$Units and Physical Meaning of $\alpha$

Since the penalty function$P$ is dimensionless, the terms inside must also be dimensionless, meaning $\alpha$ is: Dimensionless

### Physical Meaning of $\alpha$

- $\alpha$ controls the trade-off between weight and cost:
- $\alpha = 1$ prioritizes lightweight materials (maximize $M_1$).
- $\alpha = 0$ prioritizes cost efficiency (maximize $M_2$).
- $\alpha = 0.5$ balances both factors.

A higher $\alpha$ favors strong, low-density materials, while a lower $\alpha$ favors cheaper materials.

## Question 4 :

In question 3 we only considered yield strength in order to fulfill the requirement which stated that there should be no plastic deformation, but at what point do we have to consider the stiffens?