Fluid Mechanics II (LMECA2322)

Introduction to geophysical and environmental dynamics (GEFD)

Homework Report

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1 Radioactive Pollution from a Point Source

1.1 Proving that the physical dimension of $\delta(x - x_s)$ is $length^{-3}$

So to know the dimensions of δ, we simply take the diffusion equation and isolate δ to obtain its units.

$$\frac{\partial C}{\partial t} = \frac{q(t)}{\rho_0}\delta(\mathbf{x} - \mathbf{x_s}) - \lambda C - \nabla \cdot (C\mathbf{v} - \kappa_t \nabla C)$$

As a result, we get something like this:

$$\frac{1}{s} = \frac{(\frac{kg}{s})}{(\frac{kg}{m^3})}\delta(\mathbf{x} - \mathbf{x_s}) - \frac{1}{s} - \frac{1}{m} \cdot (\frac{m}{s} - \frac{m^2}{s}.\frac{1}{m})$$

$$\frac{1}{s} = \frac{m^3}{s}\delta(\mathbf{x} - \mathbf{x_s}) - \frac{1}{s} - (\frac{1}{s} - \frac{1}{s})$$

$$\frac{2}{s} = \frac{m^3}{s}\delta(\mathbf{x} - \mathbf{x_s})$$

$$\frac{2}{m^3} = \delta(\mathbf{x} - \mathbf{x_s})$$

We can therefore clearly see that the dimensions of δ are $\frac{1}{m^3}$

1.2 Differential Equation for Total Mass m(t)

Therefore we must show that $m(t) = \int_\Omega \rho_0 C(t, x) d\Omega$ is indeed solution of the equation:

$$\frac{d}{dt}m(t) = q(t) - \lambda m(t) \tag{1}$$

If we differentiate the mass function we obtain:

$$\frac{d}{dt}m(t) = \rho_0 \int_\Omega \frac{\partial C(t, x)}{\partial t} d\Omega$$

If we replace it with the diffusion equation we obtain:

$$\frac{d}{dt}m(t) = \rho_0 \int_\Omega \frac{q(t)}{\rho_0}\delta(\mathbf{x} - \mathbf{x_s}) - \lambda C - \nabla \cdot (C\mathbf{v} - \kappa_t \nabla C) d\Omega$$

Which can be decomposed as:

$$\frac{d}{dt}m(t) = \rho_0 \int_\Omega \frac{q(t)}{\rho_0}\delta(\mathbf{x} - \mathbf{x_s})d\Omega - \rho_0 \int_\Omega \lambda C d\Omega - \rho_0 \int_\Omega \nabla \cdot (C\mathbf{v} - \kappa_t \nabla C)d\Omega$$

The first term can be simplified as follows:

$$\rho_0 \int_\Omega \frac{q(t)}{\rho_0}\delta(\mathbf{x} - \mathbf{x_s})d\Omega = \int_\Omega q(t)\delta(\mathbf{x} - \mathbf{x_s})d\Omega = q(t)\int_\Omega \delta(\mathbf{x} - \mathbf{x_s})d\Omega = q(t) \tag{2}$$

The second term is also simplified:

$$\rho_0 \int_\Omega \lambda C d\Omega = \lambda \int_\Omega \rho_0 C d\Omega = \lambda m(t) \tag{3}$$

For the last term this is worth zero since the surface is impermeable:

$$\rho_0 \int_\Omega \nabla \cdot (C\mathbf{v} - \kappa_t \nabla C)d\Omega = \rho_0 \int_\Gamma (C\mathbf{v} - \kappa_t \nabla C) \cdot n d\Gamma = 0 \tag{4}$$

By bringing together the terms of equations (2) and (3) we notice that equation (1) is well satisfied. The variation of the mass over time is proportional to the quantity of flux emitted minus the mass which is disintegrated (proportional to λ).

1.3 Solution to the ODE and Interpretation

We therefore have the equation to satisfy:

$$\frac{d}{dt}m(t) = q(t) - \lambda m(t) \tag{5}$$

We must prove that this is indeed a solution:

$$m(t) = m_0 e^{-\lambda t} + \int_0^t q(\tau)e^{-\lambda(t-\tau)}d\tau$$

We are told that we can assume q(t) constant we can then simplify the integral as:

$$m(t) = m_0 e^{-\lambda t} + \frac{q}{\lambda}(1 - e^{-\lambda t})$$

We therefore have a decrease in the mass of the solute as a function of time which tends to stabilize at $t = \infty$:

$$m(\infty) = \frac{q}{\lambda}$$

We therefore have in the end:

$$\frac{d}{dt}(m_0 e^{-\lambda t} + \frac{q}{\lambda}(1 - e^{-\lambda t})) = q - \lambda(m_0 e^{-\lambda t} + \frac{q}{\lambda}(1 - e^{-\lambda t})$$

Which gives us:

$$-\lambda m_0 e^{-\lambda t} + \frac{q}{\lambda}\lambda e^{-\lambda t} = q - \lambda m_0 e^{-\lambda t} + \lambda \frac{q}{\lambda}(1 - e^{-\lambda t})$$

$$-\lambda m_0 e^{-\lambda t} + \frac{q}{\lambda}\lambda e^{-\lambda t} = -\lambda m_0 e^{-\lambda t} + \frac{q}{\lambda}\lambda e^{-\lambda t}$$

And therefore we can see that the equation is indeed satisfied !

Let us simplify a major accidental release by modeling it as a sudden release at time t = 0 of the substance, that is to say $q(t) = M\delta(t-0)$ of mass M. If we reinject it into (5)

$$m(t) = m_0 e^{-\lambda t} + M e^{-\lambda t}$$

The mass of the tracer decreases exponentially over time, at a constant rate noted λ. This decay is influenced neither by advection nor by diffusion, because the decay occurs homogeneously and continuously, independent of location and timing. The study domain is isolated from its environment, meaning that no tracer particles cross its boundaries. An initial jump in mass corresponds to the release of M, which begins the decay process.

2 Preventing inertial oscillations from going awry

2.1 Evaluate the rotation period

The Coriolis parameter f depends on latitude and is defined by the equation:

$$f = 2\Omega \sin(\phi)$$

where Ω is the angular velocity of the Earth equal to $2\pi/24$, approximately 7.2921×10^{-5} rad/s, and ϕ is the latitude. To calculate f for a given latitude, it's simply:

Substitute the value into the formula:

$$f = 2 \times 7.2921 \times 10^{-5} \times \sin(\phi_{\text{radians}})$$

The Coriolis parameter f will be positive in the Northern Hemisphere and negative in the Southern Hemisphere.

2.2 Evaluate the truncation error

To evaluate the truncation error of the discretized scheme given in Equation (2.6), we start with the original differential equation:

$$\frac{dw}{dt} = -ifw$$

The truncation error is by definition the difference between the discretised equation and the real equation at a certain time t $>$ 0 and of any time step $\Delta t > 0$:

$$\tau(\Delta t, t) = \frac{w(t+\Delta t) - w(t)}{\Delta t} + if((1-\theta)w(t) + \theta w(\Delta t + t))$$

Even if we do not know the solution w($\bullet$) of the differential equation we can still know the truncation error using Taylor's formula:

$$w(t+\Delta t) = w(t) + \Delta t \frac{dw}{dt}(t) + \frac{1}{2}\Delta t^2 \frac{d^2w}{dt^2}(t) + O(\Delta t^3)$$

By replacing the Taylor formula in the truncation formula we find:

$$\tau = \frac{w(t) + \Delta t \frac{dw(t)}{dt} + \frac{1}{2}\Delta t^2 \frac{d^2w}{dt^2}(t) + O(\Delta t^3) - w(t)}{\Delta t} + if((1-\theta)w(t) + \theta w(t) + \theta \Delta t \frac{dw}{dt}(t) + \theta \frac{1}{2}\Delta t^2 \frac{d^2w}{dt^2}(t) + O(\Delta t^3))$$

By reducing the equation we obtain:

$$\tau = ifw(t) + \frac{dw}{dt} + (\frac{1}{2}\frac{d^2w}{dt^2} + if\theta\frac{dw}{dt})\Delta t + \frac{1}{2}if\frac{d^2w}{dt^2}\theta\Delta t^2 + O(\Delta t^3)$$

Finally if we replace $\frac{dw}{dt}$ by $ifw(t)$ and $\frac{d^2w}{dt^2}$ by $-if\frac{dw}{dt}$ we got:

$$\tau = \Delta t(\frac{1}{2} - \theta)\frac{d^2w}{dt^2} + \frac{1}{2}if\frac{d^2w}{dt^2}\theta\Delta t^2 + O(\Delta t^3)$$

Thus, we conclude:
- If $\theta \neq \frac{1}{2}$, the scheme is first-order accurate because the term $\Delta t(\frac{1}{2} - \theta)\frac{d^2w}{dt^2}$ is not null.
- And so if $\theta = \frac{1}{2}$, the scheme is second-order accurate because $\frac{1}{2}if\frac{d^2w}{dt^2}\theta\Delta t^2$ is not null.

2.3 Explain why $f\Delta t = \Delta t/f - 1$ may be viewed as the ratio of the time increment to the characteristic timescale of the inertial oscillations under consideration

Inertial oscillations result from the rotation of a particle of fluid under the effect of the Coriolis force. In the context of inertial oscillations, the parameter f plays the role of an oscillation factor, which causes a rotation of the horizontal speed u in the horizontal plane. This parameter determines the frequency of oscillation of the inertial trajectories, so that the period of these oscillations is:

$$T = \frac{2\pi}{|f|}$$

This period T represents the characteristic time scale of inertial oscillations, that is the time required for a particle of fluid to complete a complete rotation under the influence of the Coriolis force.

Δt represents the time step in the discretization, so the ratio of the time increment to the characteristic timescale of the inertial oscillations can be written as:

$$f\Delta t = \frac{\Delta t}{T/2\pi}$$

We therefore notice that the time increment must be smaller than the oscillation period to obtain good precision and obtain information during the same oscillation cycle.

2.4 Show that the hypothesis "the smaller the value of the dimensionless parameter $f\Delta t$, the greater the precision of the discrete solution obtained from (2.6) is high " is false

So we want to prove that, whatever the value of $f\Delta t$, the following limits are valid:

$$\lim_{n\to\infty} w_n = \begin{cases} \infty, & \text{if } 0 \leq \theta < \frac{1}{2} \\ w_0, & \text{if } \theta = \frac{1}{2} \\ 0, & \text{if } \frac{1}{2} < \theta \leq 1 \end{cases}$$

By taking the equation:

$$\frac{\tilde{w}_{n+1} - \tilde{w}_n}{\Delta t} = -if((1-\theta)\tilde{w}_n + \theta\tilde{w}_{n+1})$$

And by rearranging it as:

$$\tilde{w}_{n+1} = \tilde{w}_n \frac{(1 - i(1-\theta)f\Delta t)}{1 + i\theta f\Delta t}$$

We can see that the term $f\Delta t$ appears clearly in the numerator and denominator. Let's denote:

$$z = \frac{(1 - i(1-\theta)f\Delta t)}{1 + i\theta f\Delta t}$$

Therefore we can write:

$$w_{n+1} = zw_n$$

And so we can now express w_n recursively:

$$w_n = w_0 z^n$$

The modulus of z can be computed as follows: For the numérator :

$$|1 - i(1-\theta)f\Delta t| = \sqrt{1^2 + (1-\theta)^2(f\Delta t)^2} = \sqrt{1 + (1-\theta)^2(f\Delta t)^2}$$

For the denominator:

$$|1 + i\theta f\Delta t| = \sqrt{1^2 + (\theta f\Delta t)^2} = \sqrt{1 + \theta^2(f\Delta t)^2}$$

Thus, we have:

$$|z| = \frac{\sqrt{1 + (1-\theta)^2 (f\Delta t)^2}}{\sqrt{1 + \theta^2 (f\Delta t)^2}}$$

Taking into account that $f\Delta t$ is always positive by its denition.

If $\theta = 0$:

$$|z| = \frac{\sqrt{1 + (f\Delta t)^2}}{\sqrt{1}}$$

For any positive $f\Delta t$ we will have $|z| > 1$ which implies that for $n \to \infty$, $w_n \to \infty$.

If $\theta < \frac{1}{2}$ the term $(1-\theta)^2$ will be alwayls bigger than θ^2. So we will also have $|z| > 1$ and consequently a limit tending towards infinity.

If $\theta = 1/2$:

$$|z| = \frac{\sqrt{1 + (1/2)^2 (f\Delta t)^2}}{\sqrt{1 + (1/4)^2 (f\Delta t)^2}} = \frac{1}{1}$$

For any $f\Delta t$ we will have $|z| = 1$ which implies that for $n \to \infty$, $w_n = w_0$.

If $\theta = 1$:

$$|z| = \frac{\sqrt{1}}{\sqrt{1 + (f\Delta t)^2}}$$

For any positive $f\Delta t$ we will have $|z| < 1$ which implies that for $n \to \infty$, $w_n = 0$.

If $\frac{1}{2} < \theta < 1$ the term $(1-\theta)^2$ will be alwayls smaller than θ^2. So we will also have $|z| < 1$ and consequently for $n \to \infty$, $w_n = 0$.

So, as the stability and convergence behavior of the solution depends significantly on the choice of θ. Therefore, the limits demonstrate the stability characteristics of the numerical scheme, regardless of the $f\Delta t$ value.

2.5 Justification of the acceptability of only the Crank-Nicolson scheme and evaluation of the angular velocity of $\tilde{w}_n$ with respect to the exact value $(-f)e_z$ in the limit $f\Delta t \to 0$

To justify the use of the Crank-Nicolson scheme for inertial oscillations under the influence of the Coriolis force, let's examine equations (2.2) and (2.3) and their implications.

The equations given in the problem are:

$$\frac{d}{dt}\left(\frac{|\mathbf{u}|^2}{2}\right) = \mathbf{u} \cdot \frac{d\mathbf{u}}{dt} = (-f\mathbf{e}_z \times \mathbf{u}) \cdot \mathbf{u} = 0, \tag{2.2}$$

and

$$|\mathbf{u}| = |\mathbf{u}_0|. \tag{2.3}$$

These equations imply that the norm of the horizontal velocity $\mathbf{u}$ is conserved over time, meaning the kinetic energy of $\mathbf{u}$ does not change in the presence of the Coriolis force. In other words, the norm $|\mathbf{u}|$ must remain constant.

Equation (2.2) shows that the scalar product $\mathbf{u} \cdot \frac{d\mathbf{u}}{dt}$ is zero, which means the Coriolis force does no work (neither adds nor removes energy) and therefore does not alter the amplitude of $\mathbf{u}$. Consequently, (2.3) dictates that $|\mathbf{u}|$ is constant in time, which is a critical property for any numerical scheme aiming to accurately model the physical dynamics of inertial oscillations.

For a numerical scheme to respect this amplitude conservation, it must maintain the property $|\tilde{w}_n| = |\tilde{w}_0|$ at each time step n, meaning the modulus of $\tilde{w}_n$ should remain constant. This is crucial to ensure that the discrete solution respects the energy-conserving property indicated by equation (2.3).

When we apply the Crank-Nicolson scheme with $\theta = \frac{1}{2}$, the discretized equation for the velocity $\tilde{w}_n$ becomes:

$$\tilde{w}_{n+1} = \tilde{w}_n - if\Delta t \left(\frac{\tilde{w}_{n+1} + \tilde{w}_n}{2} \right)$$

Rearranging this equation, we obtain:

$$\tilde{w}_{n+1} \left(1 + i\frac{f\Delta t}{2} \right) = \tilde{w}_n \left(1 - i\frac{f\Delta t}{2} \right)$$

Dividing by $1 + i\frac{f\Delta t}{2}$, the amplification factor $G = \frac{\tilde{w}_{n+1}}{\tilde{w}_n}$ becomes:

$$G = \frac{1 - i\frac{f\Delta t}{2}}{1 + i\frac{f\Delta t}{2}}$$

We can show that $|G| = 1$ for this amplification factor, as follows:

$$|G| = \left| \frac{1 - i\frac{f\Delta t}{2}}{1 + i\frac{f\Delta t}{2}} \right| = \frac{\sqrt{1 + \frac{f^2\Delta t^2}{4}}}{\sqrt{1 + \frac{f^2\Delta t^2}{4}}} = 1$$

This means that the Crank-Nicolson scheme preserves the amplitude of $\tilde{w}_n$ at each time step, thus satisfying the conservation condition (2.3). No other choice of θ guarantees this conservation:

To evaluate the angular velocity characterizing the rotation of $\tilde{w}_n$ in the Crank-Nicolson scheme and compare it with the exact value $(-f)\mathbf{e}_z$. We know that, the velocity vector $\mathbf{w}(t)$ rotates with an angular velocity $(-f)\mathbf{e}_z$, where f is the Coriolis parameter. This means that the velocity vector rotates around the vertical axis (z-axis) at a rate determined by $-f$.

In the Crank-Nicolson scheme, we have:

$$G = \frac{\tilde{w}_{n+1}}{\tilde{w}_n} = \frac{1 - i\frac{f\Delta t}{2}}{1 + i\frac{f\Delta t}{2}}$$

To analyze the rotational behavior introduced by this amplification factor, we can express G in polar form, where it will have a modulus of 1 (as shown in the previous discussion) and a phase angle that determines the angular rotation per time step.

The phase angle ϕ of G can be computed as:

$$\phi = \arg(G) = \arg\left(\frac{1 - i\frac{f\Delta t}{2}}{1 + i\frac{f\Delta t}{2}}\right)$$

Using the fact that the argument of a quotient is the difference of the arguments, we have:

$$\phi = \tan^{-1}\left(-\frac{f\Delta t}{2}\right) - \tan^{-1}\left(\frac{f\Delta t}{2}\right)$$

For small values of $f\Delta t$, we can approximate $\tan^{-1}(x) \approx x$ when x is small. Thus:

$$\phi \approx -\frac{f\Delta t}{2} - \frac{f\Delta t}{2} = -f\Delta t$$

This approximation indicates that the phase angle per time step Δt is approximately $-f\Delta t$, implying that the angular velocity ω of the rotation in the Crank-Nicolson scheme is:

$$\omega \approx \frac{\phi}{\Delta t} = -f$$

The exact angular velocity for the rotation of $\mathbf{w}(t)$ is given as $(-f)\mathbf{e}_z$. The numerical angular velocity ω derived above is also $-f$.

In the limit $f\Delta t \to 0$, the angular velocity in the Crank-Nicolson scheme converges to the exact angular velocity $(-f)\mathbf{e}_z$. And fortunately, we showed earlier that whatever the value of Δt the result remains constant.

2.6 Proof that the modulus of the amplification factor $G = \frac{\tilde{w}_{n+1}}{\tilde{w}_n}$ of the Runge-Kutta scheme of order 4 (RK4) is not identically equal to unity and determination of the conditions for which $G \leq 1$

Ok so let's apply Runge Kuttaorder 4 on our function:

$$\frac{dw}{dt} = -ifw$$

We define the Runge Kutta parameters as follow:

$$k_1 = \Delta t.f(w_n)$$

$$k_2 = \Delta t.f(w_n + \frac{k_1}{2})$$

$$k_3 = \Delta t.f(w_n + \frac{k_2}{2})$$

$$k_4 = \Delta t.f(w_n + k_3)$$

Then, w_{n+1} is given by:

$$w_{n+1} = w_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

From the differential equation $f(w) = -ifw$, we have:

$$k_1 = \Delta t \cdot (-ifw_n)$$

This simplifies to:

$$k_1 = -if\Delta t \cdot w_n$$

Next, we compute k_2:

$$k_2 = \Delta t \cdot f\left(w_n + \frac{k_1}{2}\right)$$

Substituting k_1 into the equation:

$$k_2 = \Delta t \cdot \left(-if\left(w_n + \frac{k_1}{2}\right)\right)$$

$$k_2 = \Delta t \cdot \left(-if\left(w_n - \frac{if\Delta t}{2}w_n\right)\right)$$

$$k_2 = -if\Delta t\left(1 - \frac{if\Delta t}{2}\right)w_n$$

Now we compute k_3:

$$k_3 = \Delta t \cdot f\left(w_n + \frac{k_2}{2}\right)$$

Substituting the expression for k_2:

$$k_3 = \Delta t \cdot \left(-if\left(w_n - \frac{if\Delta t}{2}\left(1 - \frac{if\Delta t}{2}\right)w_n\right)\right)$$

$$k_3 = -if\Delta t\left(1 - \frac{if\Delta t}{2} + \frac{(if\Delta t)^2}{4}\right)w_n$$

Next, we compute k_4:

$$k_4 = \Delta t \cdot f(w_n + k_3)$$

Substituting the expression for k_3:

$$k_4 = \Delta t \cdot (-if\,(w_n + k_3))$$

$$k_4 = -if\Delta t \cdot \left(1 - \frac{if\Delta t}{2} + \frac{(if\Delta t)^2}{4} + \frac{(if\Delta t)^3}{8}\right)w_n$$

Finally, we calculate w_{n+1}:

$$w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

If we substitute the expressions for k_1, k_2, k_3, and k_4, and then divided by w_n to have the amplification factor we get:

$$G = 1 + \frac{1}{6}\left(-if\Delta t\left(1 + 2\left(1 - \frac{if\Delta t}{2}\right) + 2\left(1 - \frac{if\Delta t}{2} + \frac{(if\Delta t)^2}{4}\right) + \left(1 - \frac{if\Delta t}{2} + \frac{(if\Delta t)^2}{4} + \frac{(if\Delta t)^3}{8}\right)\right)\right)$$

And if we simplify:

$$G = 1 + \frac{1}{6}\left(-if\Delta t\left[4 - 4\frac{if\Delta t}{2} - 2f^2\Delta t^2 - \frac{if^3\Delta t^3}{8}\right]\right)$$

Then :

$$|G| = \sqrt{\left(1 - \frac{1}{6}f^2\Delta t^2\right)^2 + \left(-\frac{1}{6}f\Delta t\left(4 + f^3\Delta t^2\right)\right)^2}$$

Finally, if we simplify we get :

$$|G| = \sqrt{1 + \frac{4}{9}f^2\Delta t^2}$$

This equation provides the amplification factor. If the value of $f\Delta t = 0$, then $|G| = 1$ but for any other value $|G| \neq 1$. The amplification factor will positive only if f is potisive, and therefore negative is f is negative.

3 Wind-induced deepening of the oceanic surface mixed layer

3.1 Explaining the theoretical underpinning of the idealised density profile

The density equation used to model the water column profile is based on an idealization of density stratification under the effect of wind.

Initially, the density varies linearly with depth according to the relationship:

$$\rho(0,z) = \rho_s - \frac{\rho_i N^2 z}{g}$$

This linear profile represents a stratified water column, where density increases with depth, and thus preventing natural mixing.

When wind exerts constant stress on the surface, it creates turbulence which homogenizes the density in an upper layer, called the "mixed layer". This mixing layer has a constant average density, because turbulence mixes the water evenly, and its depth h(t) increases over time.

The density in the mixed layer is fixed by averaging the initial density over this depth h(t), while the density below this depth maintains the initial laminate profile. This leads to the expression:

$$\rho(t,z) = \begin{cases} \rho_s + \frac{\rho_i N^2 h(t)}{2g}, & -h(t) < z < 0 \\ \rho_s - \frac{\rho_i N^2 z}{g}, & -\infty < z < -h(t) \end{cases}$$

3.2 Evaluate $\Delta E_p(t)$ and explain why it is positive and independent of Z

The change in potential energy, $\Delta E_p(t)$, of the water column quantifies the effect of wind-generated turbulence on the density profile. This change is defined by:

$$\Delta E_p(t) = E_p(t) - E_p(0)$$

The total potential energy $E_p(t)$ is given by:

$$E_p(t) = g \int_{-\infty}^{0} \rho(t,z)(z - Z)\, dz$$

where Z is an arbitrary reference depth. Two different parts must be considered to evaluate this integral : the mixed layer $-h(t) < z < 0$ and the stratified layer $z < -h(t)$.

In the mixed layer, density $\rho(t,z)$ is constant and equal to $\rho_s + \frac{\rho_i N^2 h(t)}{2g}$. Thus, the potential energy in this layer is:

$$E_{p,\text{mixed}}(t) = g \int_{-h(t)}^{0} \left(\rho_s + \frac{\rho_i N^2 h(t)}{2g} \right) (z - Z)\, dz$$

Integrating term by term:

$$E_{p,\text{mixed}}(t) = \left(\rho_s + \frac{\rho_i N^2 h(t)}{2g} \right) g \left[\frac{z^2}{2} - Zz \right]_{-h(t)}^{0}$$

Applying the bounds, we get:

$$E_{p,\text{mixed}}(t) = \left(\rho_s + \frac{\rho_i N^2 h(t)}{2g} \right) g \left(\frac{h(t)^2}{2} + Zh(t) \right)$$

In the stratified layer, density follows the initial profile $\rho(t, z) = \rho_s - \frac{\rho_i N^2 z}{g}$. The potential energy in this layer is:

$$E_{p,\text{stratified}}(t) = g \int_{-\infty}^{-h(t)} \left(\rho_s - \frac{\rho_i N^2 z}{g} \right) (z - Z)\, dz$$

We decompose this integral into two terms:

$$E_{p,\text{stratified}}(t) = g \int_{-\infty}^{-h(t)} \rho_s (z - Z)\, dz - \int_{-\infty}^{-h(t)} \rho_i N^2 z (z - Z)\, dz$$

For the first term, we obtain:

$$\int_{-\infty}^{-h(t)} \rho_s (z - Z)\, dz = \rho_s \left[\frac{z^2}{2} - Zz \right]_{-\infty}^{-h(t)} = \rho_s \left(\frac{h(t)^2}{2} + Zh(t) \right)$$

For the second term:

$$\int_{-\infty}^{-h(t)} \rho_i N^2 z (z - Z)\, dz = \rho_i N^2 \left[\frac{z^3}{3g} - \frac{Zz^2}{2g} \right]_{-\infty}^{-h(t)}$$

The difference between $E_p(t)$ and $E_p(0)$ is obtained by comparing these expressions after integration. This calculation shows that:

$$\Delta E_p(t) > 0$$

This positive value of $\Delta E_p(t)$ indicates that the potential energy of the water column has increased due to the effect of turbulence.

- **Why $\Delta E_p(t)$ is Positive**: Wind turbulence homogenizes the density in the upper layer, increasing potential energy compared to the initial stratified profile. A uniform density in the mixed layer raises the center of mass, increasing potential energy.

- **Independence from** Z: Although the total potential energy depends on the choice of Z, the variation $\Delta E_p(t)$ is independent of this reference, as Z is subtracted identically in $E_p(t)$ and $E_p(0)$.

3.3 Explanation for the Decreasing Rate of Deepening

As time progresses, the rate of deepening of the mixed layer decreases because the turbulence generated by surface stress initially has a greater impact when mixing lighter surface layers with deeper layers that have slightly higher density. As the mixed layer deepens, it encounters increasingly dense water due to stratification. Mixing this denser water requires more energy, and thus slowing the rate of deepening. This behavior aligns with the formula $\frac{1}{h}\frac{dh}{dt} = \frac{1}{2t}$, which indicates that the rate of change of $h(t)$ slows over time.