Stochastic Differential Equations and Applications

Vanshika Gupta, Dr. Mrinmay Biswas

Department of Mathematics and Statistics, Indian Institute of Technology Kanpur

Preliminaries

σ-algebra

Let Ω be a non-empty set. A σ-algebra on Ω is a collection $\mathcal{F}$ of subsets of Ω satisfying the following properties:

- $\Omega \in \mathcal{F}$.
- $A \subseteq \Omega$ and $A \in \mathcal{F}, \Rightarrow A^C \in \mathcal{F}$.
- If $A_1, A_2, \cdots \in \mathcal{F}, \Rightarrow \bigcup_{i=1}^{\infty} A_k \in \mathcal{F}$.

Probability Function

Given a σ-algebra $\mathcal{F}$ on Ω. A real valued set function $P\colon \mathcal{F} \to [0,1]$ defined on $\mathcal{F}$ is said to be a probability function/measure if

- $P(\Omega) = 1$.
- $P(\emptyset) = 0$.
- **Countable Additivity**: Given a sequence of events $A_1, A_2, \dots$ which are pairwise disjoint ($A_i \cap A_j \ \forall\, i \neq j$) then

$$P\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} P(A_k)$$

Probability Space

Probability space or a probability triple $(\Omega, \mathcal{F}, P)$ is a mathematical construct that provides a formal model of a random process or "experiment" where Ω is sample space, $\mathcal{F}$ is σ-algebra (event space) and P is probability function as defined above.

Random Variable

A random variable X is a $\{\mathcal{F}, \mathcal{B}\}$ measurable function $X\ :\ \Omega \to \mathbb{R}$ from a sample space Ω as a set of possible outcomes. The technical axiomatic definition requires the sample space Ω to be a sample space of a probability triple (Ω,$\mathcal{F}$,P).

Stochastic Process

A collection $\{X(t) | t \geq 0\}$ of random variables is called a Stochastic process. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A stochastic process is a measurable function $X(t, \omega)$ defined on the product space $[0, \infty) \times \Omega$.

Brownian Motion

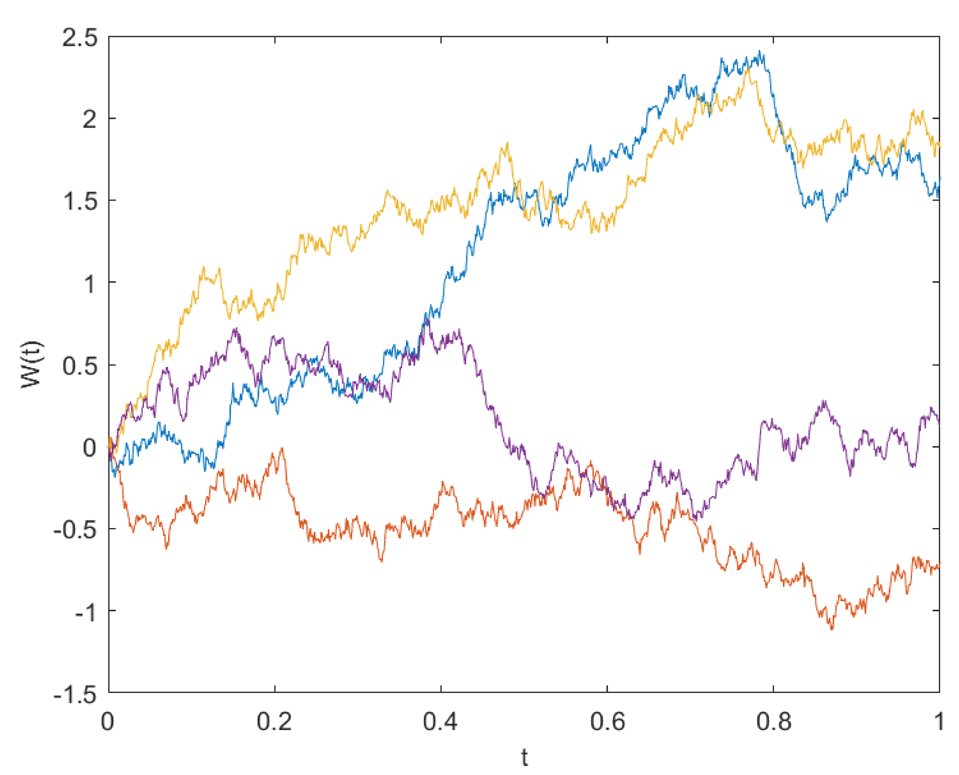
It is a **continuous time-space** Stochastic process.

- $(W_0) = 0$ almost surely in P.
- **Independent increments:** The random variable $(W_v) - (W_u)$ and $(W_t) - (W_s)$ are independent whenever $u \leq v \leq s \leq t$. (u, v) and (s, t) are disjoint random variable.
- **Normal increments:** $(W_{t+s}) - (W_s) \sim N(0, t)$ or $(W_t) - (W_0) \sim N(0, t)$ where $t > 0, x \in \mathbb{R}$

$$f(x; 0, t) = \frac{1}{(2\pi t)^{1/2}} e^{\frac{-(x)^2}{2t}}$$

- **Continuous Sample space:** With probability 1, the function $t \mapsto W(t, \omega)$ is continuous almost sure and it doesn't have any jumps or discontinuities.
- **Markov property:** This property of Brownian motion states that the future behavior of the process depends only on its current state and is independent of past history.
- **Self-similarity:** It is a property of Brownian motion where the statistical properties of the process are similar at different time scales.

Graph of Brownian Motion



Stochastic Integrals

Filtration

Increasing family of sub σ-algebras $\{\mathcal{F}_t\}$ of $\mathcal{F}$
$\mathcal{F}_0 \subseteq \mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ if $0 \leq s \leq t$.

Wiener Integral

- Suppose f is a step function given by $f = \sum_{i=1}^{n} a_i \cdot \mathbf{1}_{[t_{i-1}, t_i]}$, where $t_0 = 0$ and $t_n = T$. In this case, define

$$\mathbf{I}(f) = \int_0^T f(t)\, dW_t = \sum_{i=1}^{n} a_i \cdot (W(t_i) - W(t_{i-1}))$$

where $f \in L^2[0, T]$ is a step process.

- Let $f \in L^2[0, T]$. The limit $\mathbf{I}(f) = \lim_{n\to\infty} \mathbf{I}(f_n)$ in $L^2(\Omega)$ is called the Wiener integral of f. The Wiener integral $\mathbf{I}(f)$ of f will be denoted by:

$$\mathbf{I}(f)(\omega) = \int_0^T f(t)\, dW(t)(\omega)$$

where $f_n \in L^2[0, T]$ is a sequence of step process.

Properties of Wiener Integral

$\mathbf{I}(f) = \int_0^T f(t)\, dW_t$ is a **Normal Random variable** with mean 0 and variance $\int_0^T f(t)^2\, dt$.

1. $E[\int_0^T f(t)\, dW_t] = 0$.
2. $E[(\int_0^T f(t)\, dW_t)^2] = \int_0^T f^2(t)\, dt$.
3. Let $f \in L^2[0, T]$.Then the stochastic process

$$M_t = \int_0^t f(s)\, dW(s),\ 0 \leq t \leq T,$$

is a **Martingale** with respect to $\mathcal{F}_t$, where $\mathcal{F}_t = \sigma\{W_s : 0 \leq s \leq t\}$.

Itô Integral

- Fix a Brownian motion $W(t)$ and filtration $\{\mathcal{F}_t;\ 0 \leq t \leq T\}$ satisfying the following conditions:
  1. For each t, $W(t)$ is $\mathcal{F}_t$-measurable.
  2. For any $s \leq t$, the random variable $W(t) - W(s)$ is independent of the σ-algebra $\mathcal{F}_s$.
- We will use $L^2_{\text{ad}}([0, T] \times \Omega)$ to denote the space of all stochastic process $f(t, \omega),\ 0 \leq t \leq T,\ \omega \in \Omega$ satisfying the following conditions:
  1. $f(t, \omega)$ is adapted to the filtration $\{\mathcal{F}_t\}$.
  2. $\int_0^T E(|f(t)|^2)\, dt < \infty$.
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$$f(t, \omega) = \sum_{i=1}^{n} a_{i-1}(\omega) \mathbf{1}_{[t_{i-1}, t_i]}(t)$$

$$\mathbf{I}(f) = \int_0^T f(t, \omega)\, dW_t(\omega) = \sum_{i=1}^{n} a_{i-1}(W(t_i) - W(t_{i-1}))$$

Properties of Itô Integral

$f \in L^2_{\text{ad}}([0, T] \times \Omega)$. $\mathbf{I}(f) = \lim_{n\to\infty} \mathbf{I}(f_n)$ where $\{f_n(t, \omega);\ n \geq 1\}$ is a sequence of adapted step stochastic process.

1. $E[\int_0^T f(t)\, dW] = 0$.
2. $E[(\int_0^T f(t)\, dW_t)^2] = E[\int_0^T f^2(t)\, dt]$
3. If $f \in L^2_{\text{ad}}[(0, T) \times \Omega]$, then the indefinite integral $\mathbf{I}(\cdot)$ is a **Martingale**. Furthermore, $\mathbf{I}(\cdot)$ has a version with continuous sample paths a.s.

Itô formula

- $X(\cdot)$ is a real-valued stochastic process satisfying

$$X(r) = X(s) + \int_s^r F\, dt + \int_s^r G\, dW$$

for some $F \in L^1_{\text{ad}}[(0, T) \times \Omega], G \in L^2_{\text{ad}}[(0, T) \times \Omega]$ and all times $0 \leq s \leq r \leq T$.We say that $X(\cdot)$ has the stochastic differential

$$dX = F dt + G\, dW$$

for $0 \leq t \leq T$

- $X(\cdot)$ has a stochastic differential

$$dX = F dt + G\, dW$$

for $F \in L^1_{\text{ad}}[(0, T) \times \Omega], G \in L^2_{\text{ad}}[(0, T) \times \Omega]$.Assume $u\colon \mathbb{R} \times [0, T] \to \mathbb{R}, u = u(x, t)$ is continuous and that its partial derivatives $u_t = \frac{\partial u}{\partial t}, u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ exist and are continuous.
Then $Y(t) := u(X(t), t)$ has the stochastic differential

$$du(X, t) = u_t\, dt + u_x\, dX + \frac{1}{2} u_{xx} G^2\, dt$$
$$= (u_t + u_x F + \frac{1}{2} u_{xx} G^2)\, dt + u_x G\, dW$$

Stochastic Differential Equation

$$dX(t) = f(t, X(t))\, dt + \sigma(t, X(t))\, dW_t, \quad t \in (0, T)$$
$$X(0) = X_0 \quad \dots (1)$$
$$X(t) = X_0 + \int_0^t f(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW_s \quad \text{almost surely in } P.$$

Definition of solution of SDE

A stochastic process $X\colon [0, T] \times \Omega \to \mathbb{R}$ is called a solution of the SDE (1) if

- $X(\cdot)$ is progressively measurable.(i.e. $X\colon [0, t] \times \Omega \to \mathbb{R}$ is $\mathcal{B}[0, t] \times \mathcal{F}_t$ measurable for all $t \geq 0$).
- $f(\cdot, X(\cdot)) \in L^1_{\text{ad}}([0, T] \times \Omega)$.
- $\sigma(\cdot, X(\cdot)) \in L^2_{\text{ad}}([0, T] \times \Omega)$.
- $\forall t \in [0, T], X(t)$ satisfies

$$X(t) = X_0 + \int_0^t f(s, X(s))\, ds + \int_0^t \sigma(s, X(s))\, dW_s \quad \text{almost surely in } P.$$

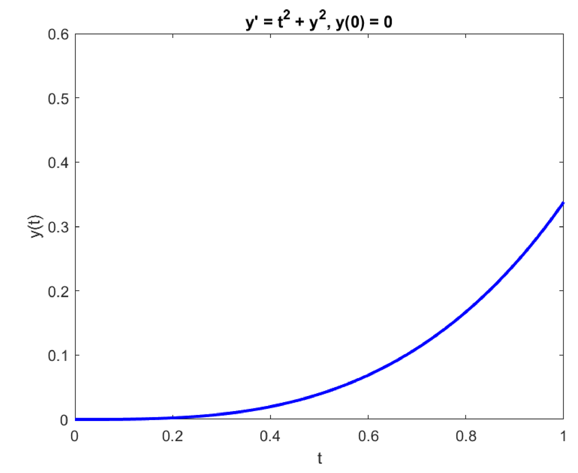
Existence and Uniqueness Theorem

Let $f\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma\colon [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous and for some $L > 0$ satisfies

$$|f(t, x) - f(t, y)| \leq L\, |x - y|\,,\ |\sigma(t, x) - \sigma(t, y)| \leq L\, |x - y|$$
$$|f(t, x)| \leq L(1 + |x|), |\sigma(t, x)| \leq L(1 + |x|)\ \forall t \in [0, T], x \in \mathbb{R}$$

Let $X_0\colon \Omega \to \mathbb{R}$ be random variable, $E\, |X_0|^2 < \infty$. Then, there exists a unique solution $X \in L^2_{\text{ad}}([0, T] \times \Omega)$ of SDE(1).

Deterministic vs Stochastic Differential Equation



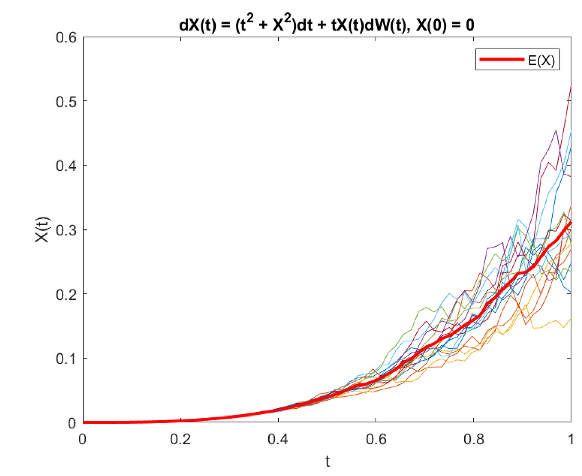
(a) $dy(t) = (t^2 + y^2)\, dt$
Euler's scheme to solve the ODE:

$$\frac{dy}{dt} = f(t, y(t)),\ 0 \leq t \leq T$$
$$y(0) = y_0$$

is given by:

$$\omega_0 = y_0$$
$$\omega_{n+1} = \omega_n + h f(t_n, \omega_n),$$
$$n = 0, 1, \dots, N - 1$$

where $h = \frac{T}{N},\ t_n = nh$



(b) $dX(t) = (t^2 + X^2)\, dt + tX(t)\, dW(t)$
Euler-Maruyama scheme to solve the SODE:

$$dX(t) = f(t, X(t))\, dt + \sigma(t, X(t))\, dW_t,\ 0 \leq t \leq T$$
$$X(0) = X_0$$

is given by:

$$X_{n+1} = X_n + h f(t_n, X_n) + \sigma(t_n, X_n) \Delta dW_n,$$
$$n = 0, 1, \dots, N - 1$$

where $h = \frac{T}{N},\ t_n = nh,$
$\Delta W_n = W(t_{n+1}) - W(t_n) \approx \sqrt{h} N(0, 1)$

Examples

- **Stock Price:** Let $S(t)$ denote the stock prices at time t. The evolution of $S(t)$ is given by the SDE:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW_t$$
$$t > 0,\ S(0) = S_0,\ \mu > 0,\ \sigma \in \mathbb{R}$$

- **Brownian Bridge:**

$$dB(t) = \frac{-B}{1 - t} dt + dW_t$$
$$0 < t < 1,\ B(0) = 0$$

- **Langevin's Equation:**

$$dX(t) = -bX(t)\, dt + \sigma\, dW_t$$
$$t > 0,\ X(0) = X_0$$

References

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- Oksendal B-Stochastic Differential Equations-Springer (2000)
- Evans L. C. - An Introduction to Stochastic Differential Equations (2014)