Monte Carlo

Wiener Process

Z : Markov process with mean change of 0 and variance rate of 1.0 per year

ΔZ : Change in a small interval of time Δt

Z follows a Wiener process if :

a) $\boxed{\Delta Z = \varepsilon \sqrt{\Delta t}}$, where $\varepsilon \sim N(0,1)$ (1)

b) Values of ΔZ for any 2 different (Non-overlapping) periods of time are independent

* ΔZ over a small time interval Δt:

μ of $\Delta Z = 0$

σ^2 of $\Delta Z = \Delta t$

σ of $\Delta Z = \sqrt{\Delta t}$

* ΔZ over a long period of time T:

μ of $[Z(T) - Z(0)] = 0$

σ^2 of $[Z(T) - Z(0)] = T$

σ " " $= \sqrt{T}$

→ Wiene Process has a <u>drift rate</u> (average change per unit time) of zero and variance rate of 1.

→ In a generalized Wiener process, the drift rate and the variance rate can be any chosen constants.

→ A variable X follows a generalized Wiener process with a drift rate of "a" and a variance rate of b^2 if:

$$\boxed{dx = a\,dt + b\,dz} \qquad (2)$$

Combining (1) & (2):

Generalized Wiener Process

$$\Delta X = \underbrace{a}_{\text{drift rate}} \Delta t + \underbrace{b}_{\text{std}} \varepsilon \sqrt{\Delta t}$$

- $a\Delta t$: Mean change in X in time Δt
- $b^2 \Delta t$: Variance of change in X " " " "
- $b\sqrt{\Delta t}$: St dv " " " " " "

*Not appropriate for Stocks:

- Stock prices never fall bellow zero
- For stock prices we can assume that its expected % change in a short period of time remains constant, but not its expected absolute change.
- We can assume that our uncertainty about the size of future stock price movements is proportional to the level of the stock price

Itô Process

* Drift rate and variance are functions of time:

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

$$X(t) = X_0 + \int_0^t a(x,t)\,ds + \int_0^t b(x,t)\,dz$$

Discrete version:

$$\Delta x = a(x,t)\Delta t + b(x,t)\,\varepsilon\sqrt{\Delta t}$$

In <u>Stock Price</u>:

μ: Expected Return
σ: Volatility

$$dS = \mu S\,dt + \sigma S\,dz$$

Discrete version:

$$\boxed{\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}} \quad (3)$$

P.S.: We can sample random paths for the stock price by sampling values for ε

Note: Itô's Lemma

If we know the stochastic process followed by x, Itô's Lemma tells us the stochastic process followed by some function $G(x,t)$

Since $G()$ is a function of x and t:
From Taylor's expansion

$$\underbrace{\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{\partial^2 G}{\partial x^2}\Delta x^2 \cdot \frac{1}{2}}_{(4)} + \frac{\partial^2 G}{\partial x \partial t}\Delta x \Delta t + \cdots$$

$$dx = a(x,t)dt + b(x,t)dz$$
$$\Delta x = a\Delta t + b\,\varepsilon\sqrt{\Delta t} \quad (5)$$

(4) & (5):

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}\left(b^2\varepsilon^2\Delta t + \cancel{a^2\Delta t^2}\right)$$

* Ignore terms of higher order than Δt:

Since $\varepsilon \sim N(0,1)$
$E(\varepsilon^2 \Delta t) = \Delta t$
Variance of Δt is proportional to Δt^2 and can be ignored!

Take Limits : $dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt$

Substitute : $dx = a\,dt + b\,dz$

$$\Rightarrow \boxed{dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} . b^2 \right) dt + \frac{\partial G}{\partial x} b\, dz} \quad (6)$$

Itô's Lemma

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Stock price process : $dS = \mu S dt + \sigma S dz$ (3)

Applying (6) :

If $G = \ln S$

$$dG = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz$$

Discrete Version:

$$\ln S(t + \Delta t) - \ln S(t) = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \varepsilon \sqrt{\Delta t}$$

$$\ln S(t + \Delta t) = \ln S(t) + \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \varepsilon \sqrt{\Delta t}$$

$$\boxed{S(t + \Delta t) = S(t) \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma \varepsilon \sqrt{\Delta t} \right)} \quad (7)$$

↳ Follows a log Normal Distribution

Assuming ↳ Geometric Brownian Motion

- $\Delta t = 1$, Each step into the future
- Python generate values for ε as Normal Distribution, 25 simulations for an interval of 30 days

Code

```
intervals = 30
iteration = 25

# Get data for closing   price of ticker
ticker = 'TSLA'
tesla = yf.Ticker('TSLA')
df = tesla.history(period='Max')

# Rename & Select col
df = df.rename(columns={"Close":ticker})
df = df[['TSLA']].

# Take Log
Log-df = np.log(1+ data.pct_change())

# Calculate components of equation (7)
u = log_df.mean()
var = log_df.var()
drift = u - (0.5 · var)
stdv = log_df.std()

# Calculate returns

returns = np.exp(drift.values + stdv.values ·
            norm.ppf(np.random.rand(intervals, iterations)))
```

*Remember:

$S_t = S_{t-1} \cdot$ Returns

```
# Start Point
S_zero = df.iloc[-1]

# Create a list for the predictions, same size as the returns,
# full of zeros, first value is the last collect prices

list_pred = np.zeros_like(returns)
list_pred[0] = S_zero

# Apply MC simulation by filling the list
# while following the rule: S_t = S_{t-1} · returns -> Eq. 7!!!

for t in range(1, intervals):
    list_pred[t] = list_pred[t-1] · returns[t]
```