

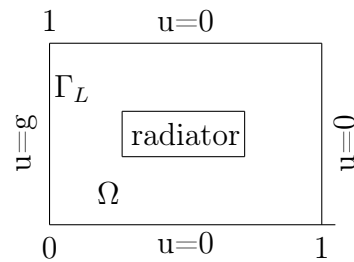
Exercise session I

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Goal of the exercise session: In this practical session, you will implement some classical domain decomposition methods to solve a model problem. The goal is to deepen the understanding of how these methods work, analyze their dependence on some parameters (e.g. overlap or Robin/relaxation parameters), verify the convergence results seen in the first lecture, and see how a Krylov method can accelerate convergence.

Model problem: Consider a heat diffusion problem posed on a square domain Ω . The adimensional temperature is equal to a step function g on the left edge, which assumes values $g = 0.3$ if $0 \leq y \leq 0.5 \wedge 0.9 \leq y \leq 1$ and $g = 1$ if $0.5 < y < 0.9$. On the rest of the boundary the temperature is fixed to zero. Inside the room there is a radiator whose temperature is equal to 50. The radiator is modeled by a source term $f(x, y) = 50$ if $(x, y) \in [0.4, 0.6] \times [0.4, 0.6]$ and zero otherwise. We want to find the temperature distribution inside the room described by the equation

$$\begin{aligned} -\Delta u &= f, & \text{in } \Omega, \\ u &= g, & \text{on } \Gamma_L, \\ u &= 0, & \text{on } \partial\Omega \setminus \Gamma_L. \end{aligned}$$



You are provided with Matlab/Octave scripts which you are partially required to complete.

1. Exercise 1 [Alternating Schwarz Method]:

- Run the script `Exercise1.m` and observe the behaviour of the iterative approximations as they converge to the correct solution. You can skip from one iteration to the next one by typing `enter` (in Octave set `pause(0.5)` in the script).
- Disable the plot by setting the variable `plt=0` in `room_data.m`. Run the script with different sizes of the overlap and positions of the interface. What is the influence of the overlap on the convergence?
- Check that, asymptotically, the convergence rate is

$$\rho_{\text{AltS}} := \max_{k \in [\pi, J\pi]} \frac{\sinh(k\beta)}{\sinh(k(1-\beta))} \frac{\sinh(k(1-\alpha))}{\sinh(k\alpha)},$$

where α and β are the location of Γ_1 and Γ_2 respectively.

2. Exercise 2 [Dirichlet-Neumann Method]:

- (a) Run the script `Exercise2.m` and observe the behaviour of the iterative approximations as they converge to the correct solution. Verify the asymptotic convergence rate

$$\rho_{\text{DN}} = \max_{k \in [\pi, J\pi]} \theta - (1 - \theta) \frac{\tanh(k\beta)}{\tanh(k\alpha)},$$

where α and β are the length of the first and second subdomains.

- (b) Vary the relaxation parameter θ and find heuristically an optimal value.
Hint: Set J odd to have a symmetric decomposition.
NB: To simplify the code we use a Robin solver for Ω_2 and impose Dirichlet boundary conditions through penalization.

3. Exercise 3 [Optimized Schwarz Method]:

- (a) Complete the script `Exercise3.m` and observe the behaviour of the iterative approximations as they converge to the correct solution. Verify the convergence rate

$$\rho_{\text{OSM}} = \max_{k \in [\pi, J\pi]} \frac{k \cosh(k(1 - \alpha)) - p \sinh(k(1 - \alpha))}{k \cosh(k\alpha) + p \sinh(k\alpha)} \frac{k \cosh(\beta) - p \sinh(k\beta)}{k \cosh(k(1 - \beta)) + p \sinh(k(1 - \beta))},$$

where α and β are the location of Γ_1 and Γ_2 respectively.

- (b) Run the script with different sizes of the overlap and parameter p . What is the influence of the overlap on the convergence? Does the convergence speed depend on the parameter p ?
- (c) Check that the Optimized Schwarz Method converges without overlap.
- (d) Compare the convergence speed between the non overlapping and the minimal overlapping cases both with the respective optimized parameters. Which one is the fastest?
4. Exercise 4 [Substructured Parallel Schwarz Method]: In this exercise, you will implement a Parallel Schwarz method in a substructured form and apply GMRES to accelerate convergence. First, realize that the iteration

```
for i=1:maxiter
    u1n=Solve2d(A1,f1,h,Nx1,Ny1,gg,u2(:,d+1));
    u2n=Solve2d(A2,f2,h,Nx2,Ny2,u1(:,end-d),gd);
    u1=u1n;
    u2=u2n;
    ....
end
```

can be written mathematically as

$$\begin{aligned} g_1^n &= \mathcal{L}_1^{-1}(f_1, g_2^{n-1}), \\ g_2^n &= \mathcal{L}_2^{-1}(f_2, g_1^{n-1}), \end{aligned} \tag{1}$$

where $g_1^n := u_1^n(:, \text{end} - d)$ and $g_2^n := u_2^n(:, d + 1)$. Using linearity and take the limit $n \rightarrow \infty$, we obtain the substructured system

$$\begin{pmatrix} I & \mathcal{L}_1^{-1}(0, \cdot) \\ \mathcal{L}_2^{-1}(0, \cdot) & I \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1^{-1}(f_1, 0) \\ \mathcal{L}_2^{-1}(f_2, 0) \end{pmatrix}. \tag{2}$$

- (a) Run the script `Exercise4.m` and compare the convergence behaviour of both the iterative Parallel Schwarz method and GMRES applied to the substructured system for different values of the overlap.

5. Homework [Neumann-Neumann Method]:

- (a) Implement a Neumann-Neumann method and observe the behaviour of the iterative approximations as they converge to the correct solution. Verify the convergence rate

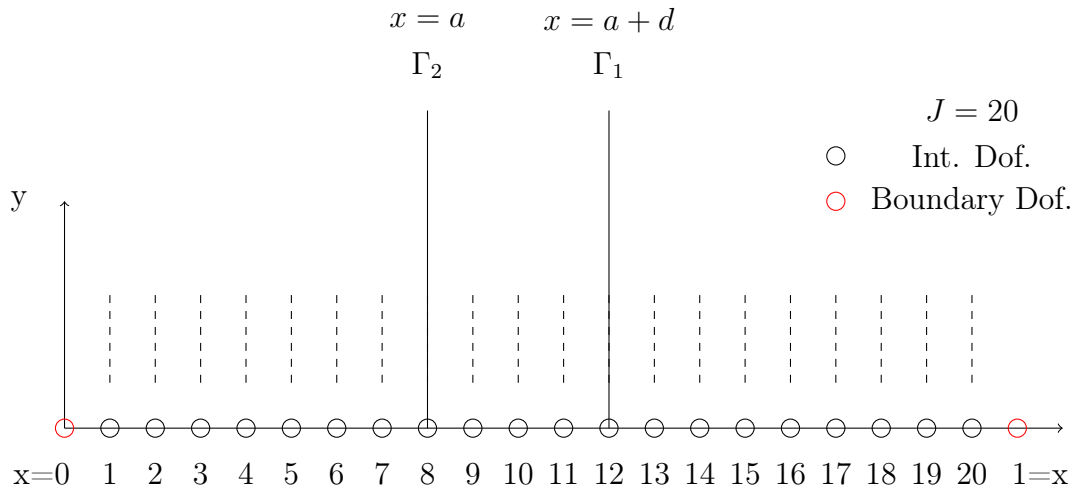
$$\rho_{\text{NN}} = \max_{k \in [\pi, J\pi]} 1 - \theta (\tanh(k\alpha) + \tanh(k\beta)) (\coth(ka) + \coth(kb)),$$

where α and β are the length of the first and second subdomains.

- (b) Vary the relaxation parameter θ . Does the method always converge?

Examples of discretizations

Overlapping decomposition of Ω for the alternating Schwarz method.



Nonoverlapping decomposition of Ω for the Dirichlet-Neumann method.

