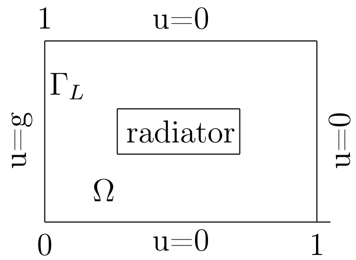
Exercise session I

Instructor: Tommaso Vanzan.
email:tommaso.vanzan@epfl.ch

Goal of the exercise session: In this practical session, you will implement some classical domain decomposition methods to solve a model problem. The goal is to deepen the understanding of how these methods work, analyze their dependence on some parameters (e.g. overlap or Robin/relaxation parameters), verify the convergence results seen in the first lecture, and see how a Krylov method can accelerate convergence.

Model problem: Consider a heat diffusion problem posed on a square domain Ω. The adimensional temperature is equal to a step function g on the left edge, which assumes values $g = 0.3$ if $0 \leq y \leq 0.5 \wedge 0.9 \leq y \leq 1$ and $g = 1$ if $0.5 < y < 0.9$. On the rest of the boundary the temperature is fixed to zero. Inside the room there is a radiator whose temperature is equal to 50. The radiator is modeled by a source term $f(x, y) = 50$ if $(x, y) \in [0.4, 0.6] \times [0.4, 0.6]$ and zero otherwise. We want to find the temperature distribution inside the room described by the equation

$$
\begin{aligned}
-\Delta u &= f, && \text{in } \Omega, \\
u &= g, && \text{on } \Gamma_L, \\
u &= 0, && \text{on } \partial\Omega \setminus \Gamma_L.
\end{aligned}
$$



You are provided with Matlab/Octave scripts which you are partially required to complete.

1. Exercise 1 [Alternating Schwarz Method]:

   (a) Run the script `Exercise1.m` and observe the behaviour of the iterative approximations as they converge to the correct solution. You can skip from one iteration to the next one by typing `enter` (in Octave set pause(0.5) in the script).

   (b) Disable the plot by setting the variable plt=0 in `room_data.m` . Run the script with different sizes of the overlap and positions of the interface. What is the influence of the overlap on the convergence?

   (c) Check that, asymptotically, the convergence factor is

   $$\rho_{\text{AltS}} := \max_{k \in [\pi, J\pi]} \frac{\sinh(k\beta)}{\sinh(k(1-\beta)))} \frac{\sinh(k(1-\alpha))}{\sinh(k\alpha)},$$

   where α and β are the location of Γ_1 and Γ_2 respectively.

2. Exercise 2 [Dirichlet-Neumann Method]:

(a) Run the script `Exercise2.m` and observe the behaviour of the iterative approximations as they converge to the correct solution. Verify the asymptotic convergence factor

$$\rho_{\text{DN}} = \max_{k\in[\pi,J\pi]} \theta - (1-\theta)\frac{\tanh(k\beta)}{\tanh(k\alpha)},$$

where α and β are the length of the first and second subdomains.

(b) Vary the relaxation parameter θ and find heuristically an optimal value.
Hint: Set J odd to have a symmetric decomposition.
NB: To simplify the code we use a Robin solver for Ω_2 and impose Dirichlet boundary conditions through penalization.

3. Exercise 3 [Optimized Schwarz Method]:

(a) Complete the script `Exercise3.m` and observe the behaviour of the iterative approximations as they converge to the correct solution. Verify the convergence factor

$$\rho_{\text{OSM}} = \max_{k\in[\pi,J\pi]} \frac{k\cosh(k(1-\alpha)) - p\sinh(k(1-\alpha))}{k\cosh(k\alpha) + p\sinh(k\alpha)} \frac{k\cosh(\beta) - p\sinh(k\beta)}{k\cosh(k(1-\beta)) + p\sinh(k(1-\beta))},$$

where α and β are the location of Γ_1 and Γ_2 respectively.

(b) Run the script with different sizes of the overlap and parameter p. What is the influence of the overlap on the convergence? Does the convergence speed depend on the parameter p?

(c) Check that the Optimized Schwarz Method converges without overlap.

(d) Compare the convergence speed between the non overlapping and the minimal overlapping cases both with the respective optimized parameters. Which one is the fastest?

4. Exercise 4 [Substructured Parallel Schwarz Method]: In this exercise, you will implement a Parallel Schwarz method in a substructured form and apply GMRES to accelerate convergence. First, realize that the iteration

```
for i=1:maxiter
    u1n=Solve2d(A1,f1,h,Nx1,Ny1,gg,u2(:,d+1));
    u2n=Solve2d(A2,f2,h,Nx2,Ny2,u1(:,end-d),gd);
    u1=u1n;
    u2=u2n;
    ....
end
```

can be written mathematically as

$$\begin{aligned} g_1^n &= \mathcal{L}_1^{-1}(f_1, g_2^{n-1}), \\ g_2^n &= \mathcal{L}_2^{-1}(f_2, g_1^{n-1}), \end{aligned} \tag{1}$$

where $g_1^n := u_1^n(:, end-d)$ and $g_2^n := u_2^n(:, d+1)$. Using linearity and take the limit $n \to \infty$, we obtain the substructured system

$$\begin{pmatrix} I & \mathcal{L}_1^{-1}(0,\cdot) \\ \mathcal{L}_2^{-1}(0,\cdot) & I \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \mathcal{L}_1^{-1}(f_1,0) \\ \mathcal{L}_2^{-1}(f_1,0) \end{pmatrix}. \tag{2}$$

(a) Run the script `Exercise4.m` and compare the convergence behaviour of both the iterative Parallel Schwarz method and GMRES applied to the substructured system for different values of the overlap.

5. Homework [Neumann-Neumann Method]:

(a) Implement a Neumann-Neumann method and observe the behaviour of the iterative approximations as they converge to the correct solution. Verify the convergence factor

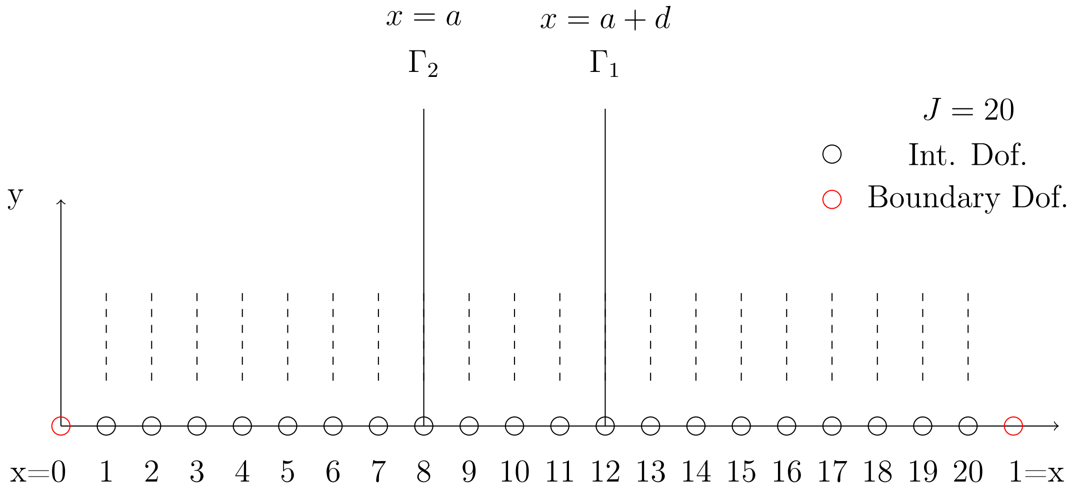
$$\rho_{\text{NN}} = \max_{k\in[\pi,J\pi]} 1 - \theta\left(\tanh(k\alpha) + \tanh(k\beta)\right)\left(\coth(ka) + \coth(kb)\right),$$

where α and β are the length of the first and second subdomains.

(b) Vary the relaxation parameter θ. Does the method always converge?

Examples of discretizations

Overlapping decomposition of Ω for the alternating Schwarz method.



Nonoverlapping decomposition of Ω for the Dirichlet-Neumann method.