Compact operators

Stefan Ehin

February 15, 2026

Let X, Y be Banach spaces. An operator

$$T \in \mathcal{L}(X, Y)$$

is called compact when $T(B_X)$ is relatively compact. The space of compact operators is denoted

$$\mathcal{K}(X, Y).$$

Every finite rank operator is compact. In general,

$$\overline{\mathcal{F}(X, Y)} \subset \mathcal{K}(X, Y)$$

and equality holds whenever X has the approximation property (in particular if X is a Hilbert space).

In other words compactness means that the image of the unit ball has vanishing Kolmogorov widths:

$$d_n(T) \to 0.$$

Hence the geometry of $T(B_X)$ admits low-dimensional approximations. In Hilbert spaces this is equivalent to decay of singular values

$$d_n(T) = \sigma_{n+1}(T).$$

Thus compactness encodes the intrinsic dimensionality of the solution manifold.

Insights for Operator Learning

We know that

$$T \in \mathcal{F}(X, Y) \Leftrightarrow \exists \varphi_j \in X^* \; T(x) = \sum_{j=1}^{k} \varphi_j(x) y^j.$$

Hence, to construct a finite rank approximation $T_n = P_{Y_n} T$ of a compact operator T between Hilbert spaces one has to choose $Y_n \subset Y$ such that $\dim Y_n = n$ and

$$d(Y_n, T(B_X)) \approx d_n(T),$$

an orthonormal basis $y^j \in Y_n$ and compute

$$\varphi_i(x) = \langle T_n(x), y^i \rangle = \langle T(x), y^i \rangle.$$

In conclusion, instead of looking for an exact compact solution operator T one could look for $P_{Y_n} T$ where $d(T(B_X), Y_n) \approx 0$.