Hamiltonian mechanics

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In standard mechanics, Hamiltonian $H : T^*Q \times \mathbb{R} \to \mathbb{R}$ coincides with total energy $T + V$. In coordinates $(p, q) \in T^*Q$ it induces

$$\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}} \quad \dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}.$$

Hence given H we obtain an evolution vector field $X_H \in \Gamma(TT^*Q)$. In a coordinate-free manner we can state the dynamics more concisely

$$\iota_{X_H}\omega = dH$$

where $\omega \in \Omega^2(T^*Q)$ is some symplectic form. Note that T^*Q has a tautological 1-form

$$\theta \coloneqq p_i dq^i \in \Omega(T^*Q)$$

and hence also a canonical symplectic form

$$-d\theta = dq^i \wedge dp_i.$$

Symplectic structure induces a Poisson bracket

$$\{F, G\} \coloneqq \omega(X_F, X_G)$$

that is handy for instance for calculating the total derivative of an observable

$$\frac{dF}{dt} = \{F, H\} + \frac{\partial F}{\partial t}.$$

This implies that if $\{F, H\} = 0$ and $\frac{\partial F}{\partial t} = 0$ then an observable F is conserved along the flow.