Lorentzian manifold

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February 14, 2026

The motion of a particle can be described by a smooth curve on a Lorentzian spacetime manifold (M, g). Such curve

$$\gamma : I \to M$$

is called a **worldline** and is physically admissible for a massive particle iff in $(-+++)$ signature convention

$$\forall \tau \in I \;\; g(\dot{\gamma}(\tau), \dot{\gamma}(\tau)) < 0$$

and $\dot{\gamma}$ lies in the chosen future light cone at each point.

A spacetime curve is more fundamental than a spatial trajectory parameterized by time, since a spacetime point has no distinguished time coordinate. To represent motion as x(t), one must choose a foliation of spacetime by spacelike hypersurfaces and thereby introduce a time function.

Lorentz transformations relate different inertial coordinate systems arising from different choices of spacetime splitting. In Minkowski spacetime, a Lorentz transformation is an isometry of the Lorentzian metric, and therefore preserves the light cone structure.

Insights for Operator Learning

Consider a problem

$$\partial_t u = Lu.$$

If L is a constant-coefficient, translation-invariant differential operator on $\mathbb{R}^d$, then the solution operator satisfies

$$S(t)u_0 = G_t \star u_0.$$

For hyperbolic equations, causality implies finite propagation speed

$$\operatorname{supp} G_t \subset B_{ct}(0).$$

This is exemplified by d'Alembert's formula for the 1D wave equation. It is also fascinating that from semigroup theory

$$S(t) = e^{tL}$$

and hence equivarience with L leads to equivarience with $S(t)$:

$$L(g \cdot u) = g \cdot (Lu) \Rightarrow S(t)(g \cdot u_0) = g \cdot (S(t)u_0)$$

where $g \cdot u(x) \coloneqq u(g^{-1}x)$ denotes the pullback action of g on functions.

Note also that the spacetime formulation replaces the evolution problem with a geometric problem of finding integral curves of a vector field on an appropriate configuration manifold.