Nonlinear Systems and Control

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January 21, 2024

Lecture notes from the 2024 graduate course Nonlinear Systems and Control, given by professor Pavankumar Tallapragada at IISc Banglore in the academic year 2023-23. Credit for the material in these notes is due to professor Pavankumar The credit for the typesetting is my own, along with any contributions from the PRs, of the notes hosted on GitHub.

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Spring Term: 2023 – 24,
Last Update: January 21, 2024,

Contents

4 Jan 2024

1 Lecture 1 — Introduction to Nonlinear Systems

In general, a nonlinear system can be defined with the following:

$$
\begin{aligned}
f &: \mathbb{R}^n \to \mathbb{R}^n \\
\frac{dx}{dt} &= \dot{x} = f(t, x, u, d) \\
y &= g(t, x, u, d)
\end{aligned}
$$

with the initial condition,

$$x(t_0) = x_0$$

where, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, and $d \in \mathbb{R}^q$ is the disturbance.

We can also write the control signal u as a function of the state x and some parameter vector θ. Thus, using this "state feedback" we have:

$$
\begin{aligned}
u &= h(t, x, \theta) \\
\Rightarrow \dot{x} &= \bar{f}(t, x, \theta)
\end{aligned}
$$

This free choice of θ allows us to design the system to meet certain requirements, or to optimize some performance metric etc.

One of the special case of nonlinear systems is called as autonomous systems, where the system does not depend on time explicitly. Thus, we have:

$$\dot{x} = f(x)$$

It is possible to convert a non-autonomous system to an autonomous system by adding a new state variable z, such that:

$$\dot{z} = 1$$

and incorporating the new variable z, in the state vector, increasing the order of the system.

Note:-

Note that, the state z and hence the state x, is unbounded with time in this conversion.

In the case of the linear systems, we have the following:

$$
\begin{aligned}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x + D(t)u
\end{aligned}
$$

with the solution of the above system being:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

with the state transition matrix $\Phi(t, \tau)$. The state transition matrix simplifies even further under the case of LTI systems as:

$$\Phi(t, \tau) = e^{A(t-\tau)}$$

In genreal the solution of the nonlinear system cannot be wriiten in a closed form. However, there are certain observations that are present in the nonlinear systems, that are not present in the linear systems.

1.1 Observations in Nonlinear Systems (vs Linear Systems)

1.1.1 Equillibrium

In general the eqillibrium of any system $\dot{x} = f(x)$ is defined as the point $x^\star$ where:

$$f(x)|_{x^\star} = 0 \Leftrightarrow \dot{x}|_{x^\star} = 0$$

Example (Linear Systems)**.** Consider the case of the linear system,

$$\dot{x} = Ax$$

The eqillibrium of the above system is given by:

$$Ax = 0 \Rightarrow x \in \mathcal{N}(A)$$

where, $\mathcal{N}(A)$ is the null space of the matrix A.

No such existence of the eqillibrium is guaranteed in the case of the nonlinear systems. Thus, we can have a nonlinear system with no eqillibrium, or multiple eqillibria.

Example (Nonlinear Systems)**.** Here are some examples of nonlinear systems with no, multiple or infiite equillibria.

1. $$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   The above system has infiite equillibrium points, given by:

   $$x^\star = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \forall \alpha \in \mathbb{R}$$

2. $$\dot{x} = x^2 + a \quad a > 0$$

   This system has no eqillibrium point.

3. $$\dot{x} = x^2 - a \quad a > 0$$

   This system has two eqillibrium points, given by $x^\star = \pm\sqrt{a}$.

1.1.2 Finite Escape Time

In general, the solution of the nonlinear system is not guaranteed to be bounded for all time, and can escape to infinity in a finite time. This is not possible in the case of the linear systems.

Example (Linear Systems)**.** Consider the case of the linear system,

$$\dot{x} = x \Rightarrow x(t) = e^t x_0$$

The solution approaches infinity as $t \to \infty$, in an asymptotic manner, but never reaches infinity in a finite time.

But, in the case of nonlinear systems, the solution can escape to infinity in a finite time.

Example (Nonlinear Systems). Consider the case of the nonlinear system,

$$\dot{x} = 1 + x^2 \quad x \in \mathbb{R}$$

Integrating the system, we get:

$$\frac{dx}{1+x^2} = dt$$
$$\Rightarrow \tan^{-1} x\big|_{x_0}^{x(t)} = t\big|_0^t$$
$$\Rightarrow x(t) = \tan(t + \tan^{-1}(x_0))$$

Let $x_0 = 0$, then we have:

$$x(t) = \tan(t)$$

The solution of the above system, approaches ∞ as $t \to \frac{\pi}{2}$. Since the solution goes unbounded in a finite time, we say that the system has a finite escape time.

1.2 Uniqueness of Solution

In general, the solution of the nonlinear system is not unique.

Example (Uniqueness of Solution). Consider the following system:

$$\dot{x} = \sqrt{2}, x(0) = 0, x \in \mathbb{R}$$

For this system,

$$x \equiv 0 \text{ is a solution}$$

But, we can also have the following solution:

$$x_\alpha = \begin{cases} \dfrac{(t-\alpha)^2}{4}, & t \geq \alpha, \alpha > 0; \\ 0, & t < \alpha; \end{cases}$$

for each $\alpha \in \mathbb{R}$, with the same initial conditions, we have infiite solutions

Thus, the solution of the nonlinear system is not unique. This is a big problem, as we cannot predict the behaviour of the system, which can lead to break down of the system.

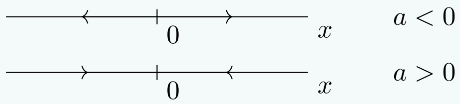
9 Jan 2024

2 Lecture 2 — Phase Portraits

Consider the system $\dot{x} = f(x)$ which has an equillibrium point $x^\star \in \mathbb{R}$ Thus,

$$\dot{x}\big|_{x=x^\star} \Leftrightarrow f(x^\star) = 0$$

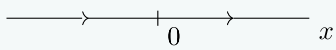
Example (Linear System). Consider the case of a linear system given by $\dot{x} = ax$. Then, the equillibrium point is $x^\star = 0$. The solution to the system is given by $x(t) = e^{at}x_0$. The qualitative behaviour of the system depending upon the value of a is given by the following "1-D" phase portrait diagram.



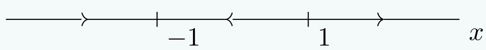
Where the arrows shows the evolution of the trajectory of the system. At $a = 0$, the the entire real line is the equillibria of the system.

Example (Nonlinear System). We will consider a bunch of non linear systems with varying qualitative behaviour.

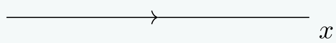
- $\dot{x} = x^2 \quad x \in \mathbb{R}$



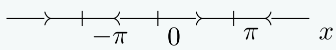
- $\dot{x} = x^2 - 1 \quad x \in \mathbb{R}$



- $\dot{x} = 1 \quad x \in \mathbb{R}$



- $\dot{x} = \sin(x) \quad x \in \mathbb{R}$



2.1 2D Phase Portraits

Let the two dimensional system be defined by

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2) \Rightarrow \dot{x} = f(x)$$

The phase portrait in some rough sense can be thought of as the "trajectory" of the system, and where we care about the slope of the vector field of $f(x)$ is given by:

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x)}{f_1(x)}$$

The phase portrait of the linear system can be easily understood via the jordan normal form of the system.

Consider the system given by, $\dot{x} = Ax$, and the linear transformation given by $y = Px$, Thus we get,

$$\dot{y} = P\dot{x} = PAx = PAP^{-1}y = Jy$$

where J is the jordan normal form of A.

We can have one of the 3 cases of the jordan form. The forms with their characteristic polynomials are given by:

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow p(s) = (s - \lambda_1)(s - \lambda_2)$$
$$J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \Rightarrow p(s) = (s - \lambda_1)^2$$
$$J_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \Rightarrow p(s) = (s - \alpha)^2 + \beta^2$$

The solution of the equation in the transformed coordinates is given by:

$$y(t) = e^{Jt}y(0) \Rightarrow x(t) = P^{-1}e^{Jt}Px(0)$$

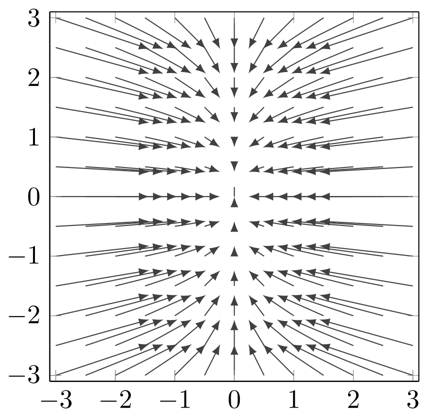
Thus, the solution can be split into three cases similar to the one done above

Case 1:

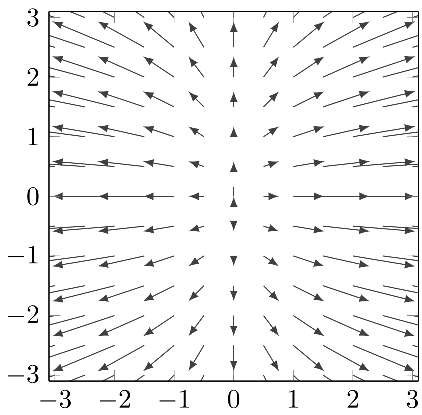
$$y_1(t) = e^{\lambda_1 t}y_1(0) \quad y_2(t) = e^{\lambda_2 t}y_2(0)$$

Thus, depending on the value of λ_1 and λ_2, we can have the following phase portraits:

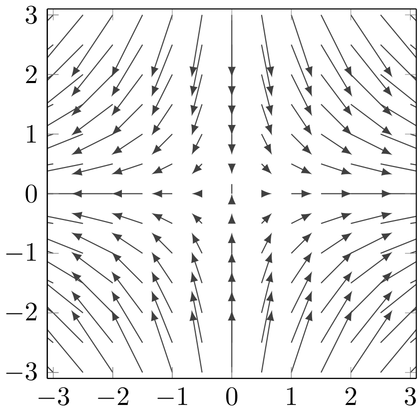
$\lambda_1 < \lambda_2 < 0$: Stable Node



$\lambda_1 > \lambda_2 > 0$: Unstable Node



$\lambda_1 > 0; \lambda_2 < 0$: Saddle

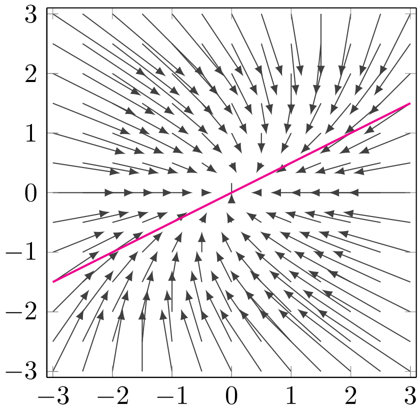


Case 2:

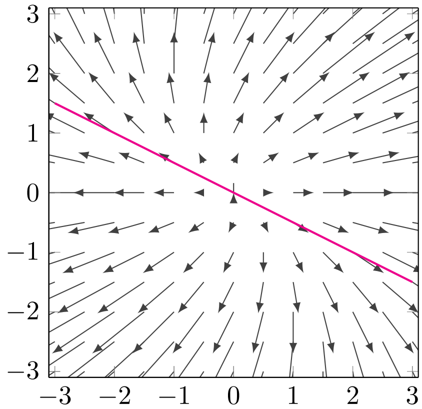
$$y_1(t) = e^{\lambda t}\left(y_1(0) + t y_2(0)\right)$$
$$y_2(t) = e^{\lambda t} y_2(0)$$

Simillary, we can have the following phase portraits depending on the value of λ:

$\lambda < 0$: Improper Stable Node



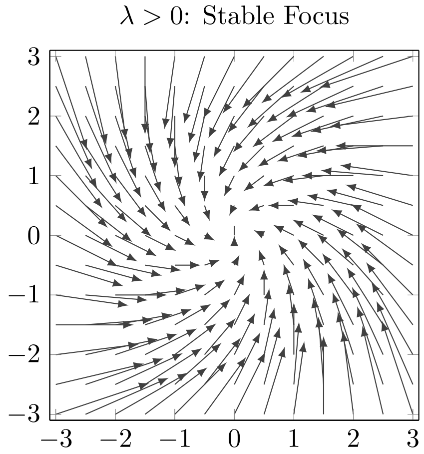
$\lambda > 0$: Improper Unstable Node

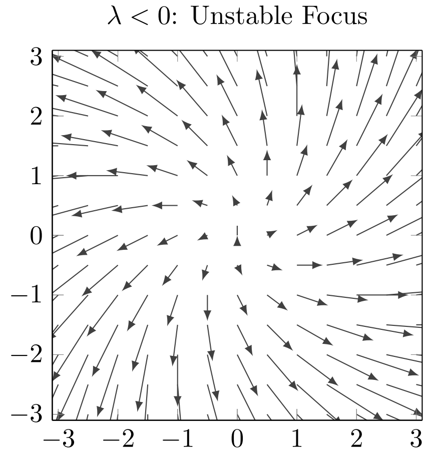


Case 3: Let $r \coloneqq \sqrt{y_1^2 + y_{2^2}}$ and $\theta \coloneqq \tan^{-1}\left(\frac{y_2}{y_1}\right)$ Substituing the values of y_1 and y_2 in the equation, and simplifying, we get:

$$\dot{r} = \alpha r \quad \dot{\theta} = \beta$$

Thus, switching in the polar form, the transformed system are evolving independently of each other. The phase portrait is given by:


λ > 0: Stable Focus
3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3


λ < 0: Unstable Focus
3
2
1
0
−1
−2
−3
−3 −2 −1 0 1 2 3

11 Jan 2024

3 Lecture 3 — Linearization

3.1 Linearization

Consider a non linear system $\dot{x} = f(x)$ with an equillibrium point $x^\star = 0$. Then using first order Taylor series expansion, we have:

$$\dot{x} = f(x) = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + \mathcal{O}(x^2)$$

Thus, the linearised system can be written as:

$$\dot{x} = Ax \quad \text{where} \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

The matrix A is called the *Jacobian* of the system $\dot{x} = f(x)$ evaluated at the equillibrium point $x^\star = 0$. The system can also be calculated for an equillibrium point not at origin, by shifting the origin to the equillibrium point.

3.1.1 Pertubations of the eigenvalues

The study of the behaviour of linear systems about the equillibrium point $x = 0$ is important beacuse in many cases the behaviour of a nonlinear systems, near an equillibrium point can be deduced by linearsing the system about the equillibrium point.

The validity and the accuracy of the linearization of the nonlinear system, and the resultant qualitative behaviour depends on the placement of the eigenvalues of the linearised system. This can be understood by considering the linear pertubations. Let the linearized system be given by $\dot{x} = Ax, x \in \mathbb{R}$, with distinct eigenvalues, and let $\Delta A \in \mathbb{R}^{2\times 2}$ be a small pertubation matrix, whose elements have arbitrary small magnitudes. We know, that the eigenvalues of the matrix $A + \Delta A$ depend continuously on the parameters of the matrix.

Thus, when the matrix A is pertubed by ΔA, any eigenvalues of A that lies in the left or the right half plane, wil remain in its respective half plane. But, the eigenvalues on the imaginary axis, when perturbed might go into either the left or the right half plane.

Consquently, we can say that if the equillibrium point $x = 0$ of $\dot{x} = Ax$ is a node, focus, or saddle, then the equillibrium point $x = 0$ of $\dot{x} = (A + \Delta A)x$ will be of the same type, provided that the pertubations are small enough. This situation is quite different in the case of center. Since the qualitative behaviour of the stable focus and unstable focus are different from that of a center, the center equillibrium point will not persist under pertubations.

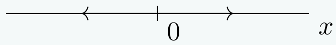
The node, focus, and saddle equillibrium point are said to be *structurally stable* because they maintain their qualitative behaviour under infinitesimally small pertubations.

Definition 3.1 (Hyperbolic Equillibrium Point). An equillibrium point of the system $\dot{x} = f(x)$ with $f(x^\star) = 0$ is said to be hyperbolic if:

$$\Re\left(A = \left.\frac{\partial f}{\partial x}\right|_{x=x^\star}\right) \neq 0$$

i.e. the jacobian of the system evaluated at the equillibrium point has no eigenvalues with zero real part.

Example. Consider the system $\dot{x} = x^3, x \in \mathbb{R}$ The system has its only equillibrium point at $x^\star = 0$. The phase portrait of the system is given by:



linearsing the system we have $\dot{x} = 3x^2 \Rightarrow \dot{x} = 0$.

Thus we can say that for any arbitrary point close to the equillibrium point, the behaviour described by the linear system is different from the non linear system

Example. Consider the system given by:

$$\begin{aligned}\dot{x}_1 &= -x_2 - \mu x_1(x_1^2 + x_2^2)\\ \dot{x}_2 &= x_1 - \mu x_2(x_1^2 + x_2^2)\end{aligned}$$

The system has an equillibrium point at $x^\star = 0$. Linearsing the system around the origin we have:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda = \pm i$$

Note that the linearised system corresponds to a simple harmoic oscillator

Shifting, into polar coordinates to analyse the system, we have:

$$r := \sqrt{x_1^2 + x_2^2} \quad \theta := \arctan\left(\frac{x_2}{x_1}\right)$$

After some simplification, we have:

$$\dot{r} = -\mu r^3 \quad \dot{\theta} = 1$$

Depending upon the value of μ, we have the following:

$$\begin{aligned}&\mu > 0 \Rightarrow \text{Stable Focus}\\ &\mu = 0 \Rightarrow \text{Center}\\ &\mu < 0 \Rightarrow \text{unstable Focus}\end{aligned}$$

Thus, the qualitative behaviour is same for the linearised system and the nonlinear system, only at $\mu = 0$, which can also be trivally observed from the state equations.

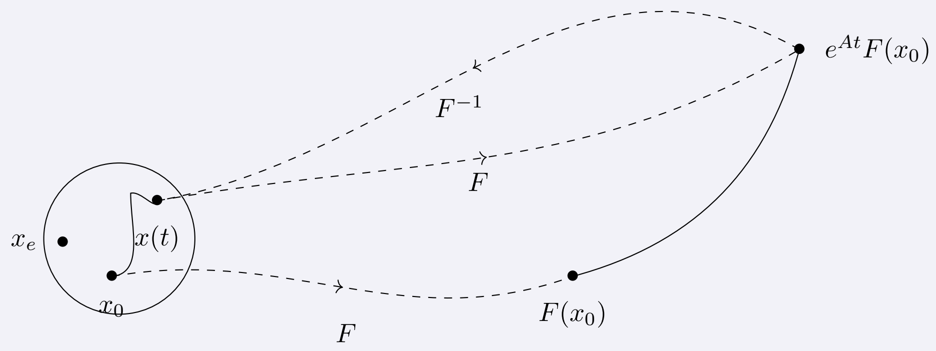
Thus, we have a sufficient condition for the qualitative behaviour of the nonlinear system to be same as the linearised system, which is that the equillibrium point of the nonlinear system should be hyperbolic.

Theorem 3.1 (Hartman-Grobman Theorem)**.** Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ be a nonlinear system with an hyperbolic equillibrium point x_e. Let ϕ_t^f be the flow map for $\dot{x} = f(x)$ i.e. $\dot{x} = f(x), x(0) = x_0$ has the solution $x(t) = \phi_t^f(x_0)$. Then for a $\delta > 0$ and a ball $B_\delta(x \in \mathbb{R}^n | \|x - x^\star\| < \delta)$ there exits as map $F : B_\delta \subset \mathbb{R}^n \to \mathbb{R}^n$, $\exists T > 0$ such that,

$$F(x_e) = 0, \text{ F is one to one on } B_\delta \text{ onto } F(B_\delta)$$

and both F and F^{-1} exits and are continuous, such that $F(\phi_t^f) = e^{At}F(x)\forall |t| < T$. Where

$A = \frac{\partial f}{\partial x}\Big|_{x=x^\star}$ and $x \in B_\delta$



The map F in Theorem 3.1 maps from the nonlinear world to the linear world.

3.1.2 Pendulumn

One example of periodicity in phase portraits is given by the pendulumn. This type of periodic behaviour is not found in linear systems.

Example (Pendulumn). Consider a pendulumn, with normalised dynamics given by:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\sin(x_1) - x_2$$

The equillibrium points of this system are given by:

$$x_1 = k\pi \quad x_2 = 0 \quad k \in \mathbb{Z}$$

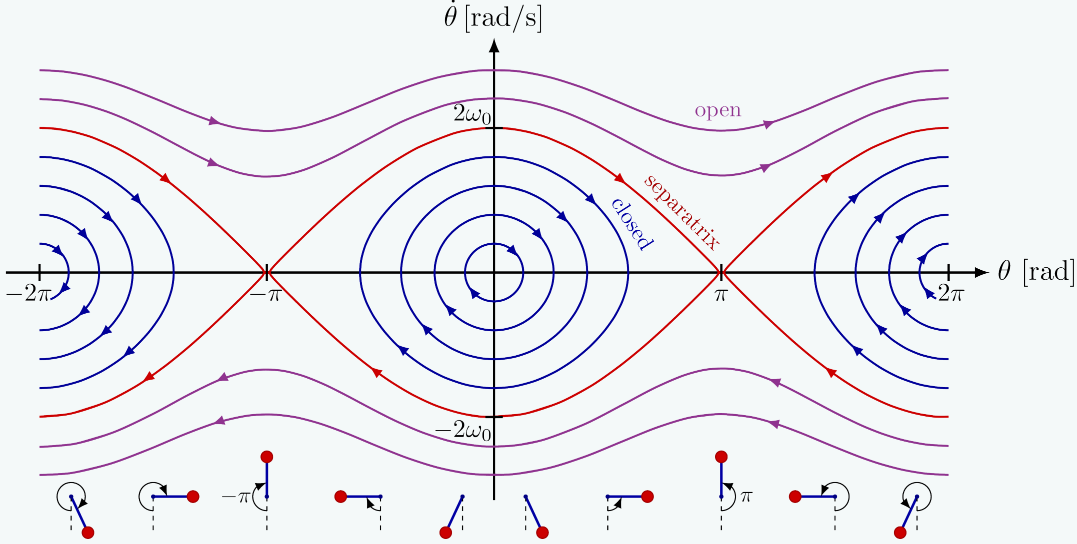
Calculating the jacobian of the system, we have:

$$A = \frac{\partial f}{\partial x}\Big|_{x=x^\star} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -1 \end{bmatrix}$$

Evalating the jacobian at $(0,0)$ and $(\pi, 0)$ we have :

$$\frac{\partial f}{\partial x}\Big|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm \sqrt{3}}{2}$$
$$\frac{\partial f}{\partial x}\Big|_{x=(\pi,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm \sqrt{5}}{2}$$

The phase portrait of the system is given by:



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16 Jan 2024

4 Lecture 4 — Limit Cycles and Periodic Orbits

Periodic orbits in the phase plane, which have a closed trajectory is usually called a perioidc orbit or closed orbit. An isolated perioidc orbit is called as a limit cycle.

For examples a harmonic oscillator has there is a continuum of cloed orbits as shown in the Figure 1 , while in the case of Van der Pol oscillator, there is only one closed orbit i.e. a limit cycle as shown in Figure 2.

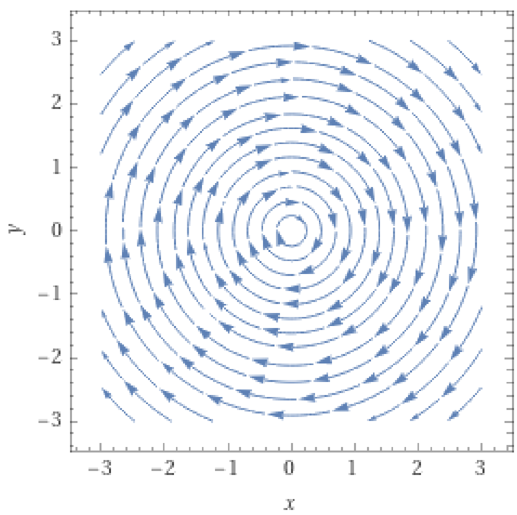

Figure 1: Harmonic Oscillator

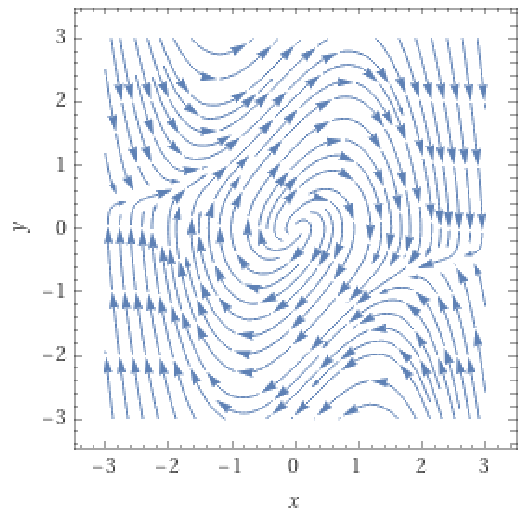

Figure 2: Limit Cycle in Vander Pol Oscillator

4.1 Poincaré-Bendixson Theorem and Bendixson's Criterion

Periodic orbits divide the plane into a region inside the orbit and a region outside it. This allows us to obtains criteria for the existence of periodic orbits in a region of the plane. The analysis of such a kind doesn;t extend to higher dimensions, as there is no concept of inside and outside in higher dimensions. One such criteria is given by the Poincaré-Bendixson theorem.

Theorem 4.1 (Poincaré-Bendixson Theorem). Let $\dot{x} = f(x), x \in \mathbb{R}^2$ and there exists a closed and bounded set $M \subset \mathbb{R}^2$ such that:

- M contains no equillibrium points of $\dot{x} = f(x)$
- Every trajectory that starts in M remains in M for all $t \geq 0$

Then M contains a periodic orbit of $\dot{x} = f(x)$.

Math Review:

$$\operatorname{div} f = \text{ divergence of vector field } f$$

$$\operatorname{div} f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \quad \text{where } f = (f_1, f_2, \ldots, f_n)$$

For a 2D vector field F we have:

$$\operatorname{div} F = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \quad x \in \mathbb{R}^2$$

If there are no sources or sinks in a region (which are equillibrium points in context of dynamical systems), then the divergence of the vector field is zero.

Theorem 4.2 (Divergence and Green's Theorem)**.** THe divergence theorem states that the volume integral of the divergence of a vector field is equal to the surface integral of the vector field over the boundary of the volume. i.e.

$$\iiint V \operatorname{div} F dV = \iint_S F \cdot n dS$$

where n is the outward normal to the surface S.

Green's theorem states that the line integral of a vector field over a closed curve is equal to the surface integral of the curl of the vector field over the region bounded by the curve. i.e.

$$\oint_C F \cdot dr = \iint_S \operatorname{curl} F \cdot n dS$$

$$\Rightarrow \iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dS = \oint f_2(x_1, x_2) dx_1 - f_1(x_1, x_2) dx_2$$

Thus, we can now state teh Bendixson's criterion as follows:

Theorem 4.3 (Bendixson's Criterion)**.** Let D be a simply connected region in $\mathbb{R}^2$. Suppose f is such that $\operatorname{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero in D, and doesn't change sign in D. Then there are no periodic orbits of $\dot{x} = f(x)$ entirely in D.

Proof. Proof by contradiction: Suppose there exists a perioidc orbit γ that in entirly in D, for any trajectory of $\dot{x} = f(x) \quad x \in \mathbb{R}^2$

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x)}{f_1(x)} \Rightarrow f_2(x)\mathrm{d}x_2 - f_1(x)\mathrm{d}x_1 = 0$$

$$\Rightarrow \oint_\gamma f_2(x)\mathrm{d}x_2 - f_1(x)\mathrm{d}x_1 = 0$$

By Green's theorem, we have:

$$\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = 0$$

This contradictcs the fact the divergence of f is not identically zero in D, thus volating our assumption. Hence Proved. ☺

4.2 Index of a curve

Definition 4.1 (Index of a curve)**.** For a sufficiently smooth vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a closed curve γ not passing through any equillibrium points of $\dot{x} = f(x)$, the index of γ w.r.t.

f is:

$$I_\gamma^f = \frac{1}{2\pi}\oint_\gamma d\theta_f \qquad \theta = \tan^{-1}\left(\frac{f_2}{f_1}\right)$$

There are certain properties of the index of a curve listed below:

Property (1)**.** If we have two closed curves γ_1 and γ_2 such that γ_1 can be continuously deformed into γ_2 without passing through any equillibrium points then,

$$I_{\gamma_1}^f = I_{\gamma_2}^f$$

Property (2)**.** If γ doesn not enclose any equillibrium points, then

$$I_\gamma^f = 0$$

Property (3)**.**

$$I_\gamma^f = I_\gamma^{-f}$$

This can be intituitively understood as follows:

$$\theta_f = \pi + \theta_{-f} \Rightarrow d\theta_f = -d\theta_{-f}$$

Thus equating the line integral

Property (4)**.** If γ is a closed orbit of $\dot{x} = f(x)$ then

$$I_\gamma^f = 1$$

Property (5)**.** If γ encloses one equillibrium point x^*, and $x^\star$ is a hyperbolic equillibrium point, then:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^\star} \Rightarrow I_\gamma^f = I_\gamma^{Ax}$$

Using the above properties, we can now define the index of the equillibrium point.

Definition 4.2 (Index of an equillibrium point)**.** $I(x_i^\star)$ is the index of the equillibrium point $x_i^\star$ of $\dot{x} = f(x)$ where $I(x_i^\star)$ is the index of any closed curve γ enclosing only $x_i^\star$ and no other equillibrium points.

Lemma 4.1. Index of a node, focus or a center is 1, while that of a saddle is -1.

Lemma 4.2. Index of a closed curve is the sum of the indices of the equillibrium points enclosed by the curve γ

Thus, we have the following corollary:

Corollary 4.1. Inside any perioidc orbit γ there exists at least one equillibrium point. Suppose all the equillibrium points inside γ are hyperbolic. Let N be the number of nodes, foci and centers inside γ and S be the number of saddles inside γ. then, the index of γ is:

$$I_\gamma^f = N - S$$