Nonlinear Systems and Control

Varad Vaidya

February 17, 2024

Lecture notes from the 2024 graduate course Nonlinear Systems and Control, given by professor Pavankumar Tallapragada at IISc Banglore in the academic year 2023-23. Credit for the material in these notes is due to professor Pavankumar The credit for the typesetting is my own, along with any contributions from the PRs, of the notes hosted on GitHub.

Disclaimer: This document will inevitably contain some mistakes— both simple typos and legitimate errors. Keep in mind that these are the notes of a graduate student in the process of learning the material, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email at vaidyavarad2001@gmail.com.

For more notes like this, visit varadVaidya.

Varad Vaidya,
Spring Term: 2023 – 24,
Last Update: February 17, 2024,

Contents

4 Jan 2024

1 Introduction to Nonlinear Systems

In general, a nonlinear system can be defined with the following:

$$\begin{aligned} f &: \mathbb{R}^n \to \mathbb{R}^n \\ \frac{dx}{dt} &= \dot{x} = f(t, x, u, d) \\ y &= g(t, x, u, d) \end{aligned}$$

with the initial condition,

$$x(t_0) = x_0$$

where, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, and $d \in \mathbb{R}^q$ is the disturbance.

We can also write the control signal u as a function of the state x and some parameter vector θ. Thus, using this "state feedback" we have:

$$\begin{aligned} u &= h(t, x, \theta) \\ \Rightarrow \dot{x} &= \bar{f}(t, x, \theta) \end{aligned}$$

This free choice of θ allows us to design the system to meet certain requirements, or to optimize some performance metric etc.

One of the special case of nonlinear systems is called as autonomous systems, where the system does not depend on time explicitly. Thus, we have:

$$\dot{x} = f(x)$$

It is possible to convert a non-autonomous system to an autonomous system by adding a new state variable z, such that:

$$\dot{z} = 1$$

and incorporating the new variable z, in the state vector, increasing the order of the system.

Note:-

Note that, the state z and hence the state x, is unbounded with time in this conversion.

In the case of the linear systems, we have the following:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

with the solution of the above system being:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

with the state transition matrix $\Phi(t, \tau)$. The state transition matrix simplifies even further under the case of LTI systems as:

$$\Phi(t, \tau) = e^{A(t-\tau)}$$

In genreal the solution of the nonlinear system cannot be wriiten in a closed form. However, there are certain observations that are present in the nonlinear systems, that are not present in the linear systems.

1.1 Observations in Nonlinear Systems (vs Linear Systems)

1.1.1 Equillibrium

In general the eqillibrium of any system $\dot{x} = f(x)$ is defined as the point $x^\star$ where:

$$f(x)|_{x^\star} = 0 \Leftrightarrow \dot{x}|_{x^\star} = 0$$

Example (Linear Systems)**.** Consider the case of the linear system,

$$\dot{x} = Ax$$

The eqillibrium of the above system is given by:

$$Ax = 0 \Rightarrow x \in \mathcal{N}(A)$$

where, $\mathcal{N}(A)$ is the null space of the matrix A.

No such existence of the eqillibrium is guaranteed in the case of the nonlinear systems. Thus, we can have a nonlinear system with no eqillibrium, or multiple eqillibria.

Example (Nonlinear Systems)**.** Here are some examples of nonlinear systems with no, multiple or infiite equillibria.

1. $$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

   The above system has infiite equillibrium points, given by:

   $$x^\star = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \forall \alpha \in \mathbb{R}$$

2. $$\dot{x} = x^2 + a \quad a > 0$$

   This system has no eqillibrium point.

3. $$\dot{x} = x^2 - a \quad a > 0$$

   This system has two eqillibrium points, given by $x^\star = \pm\sqrt{a}$.

1.1.2 Finite Escape Time

In general, the solution of the nonlinear system is not guaranteed to be bounded for all time, and can escape to infinity in a finite time. This is not possible in the case of the linear systems.

Example (Linear Systems)**.** Consider the case of the linear system,

$$\dot{x} = x \Rightarrow x(t) = e^t x_0$$

The solution approaches infinity as $t \to \infty$, in an asymptotic manner, but never reaches infinity in a finite time.

But, in the case of nonlinear systems, the solution can escape to infinity in a finite time.

Example (Nonlinear Systems). Consider the case of the nonlinear system,

$$\dot{x} = 1 + x^2 \quad x \in \mathbb{R}$$

Integrating the system, we get:

$$\begin{aligned} \frac{dx}{1+x^2} &= dt \\ \Rightarrow \tan^{-1} x\Big|_{x_0}^{x(t)} &= t\Big|_0^t \\ \Rightarrow x(t) &= \tan(t + \tan^{-1}(x_0)) \end{aligned}$$

Let $x_0 = 0$, then we have:

$$x(t) = \tan(t)$$

The solution of the above system, approaches ∞ as $t \to \frac{\pi}{2}$. Since the solution goes unbounded in a finite time, we say that the system has a finite escape time.

1.2 Uniqueness of Solution

In general, the solution of the nonlinear system is not unique.

Example (Uniqueness of Solution). Consider the following system:

$$\dot{x} = \sqrt{2}, x(0) = 0, x \in \mathbb{R}$$

For this system,

$$x \equiv 0 \text{ is a solution}$$

But, we can also have the following solution:

$$x_\alpha = \begin{cases} \dfrac{(t-\alpha)^2}{4}, & t \geq \alpha, \alpha > 0; \\ 0, & t < \alpha; \end{cases}$$

for each $\alpha \in \mathbb{R}$, with the same initial conditions, we have infiite solutions

Thus, the solution of the nonlinear system is not unique. This is a big problem, as we cannot predict the behaviour of the system, which can lead to break down of the system.

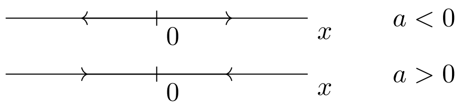
9 Jan 2024

2 Phase Portraits and Limit Cycles

Consider the system $\dot{x} = f(x)$ which has an equillibrium point $x^\star \in \mathbb{R}$ Thus,

$$\dot{x}\big|_{x=x^\star} \Leftrightarrow f(x^\star) = 0$$

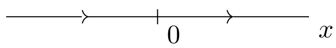
Example (Linear System)**.** Consider the case of a linear system given by $\dot{x} = ax$. Then, the equillibrium point is $x^\star = 0$. The solution to the system is given by $x(t) = e^{at}x_0$. The qualitative behaviour of the system depending upon the value of a is given by the following "1-D" phase portrait diagram.



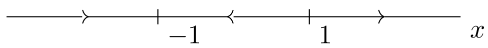
Where the arrows shows the evolution of the trajectory of the system. At $a = 0$, the the entire real line is the equillibria of the system.

Example (Nonlinear System)**.** We will consider a bunch of non linear systems with varying qualitative behaviour.

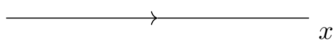
- $\dot{x} = x^2 \quad x \in \mathbb{R}$



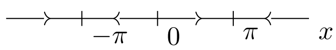
- $\dot{x} = x^2 - 1 \quad x \in \mathbb{R}$



- $\dot{x} = 1 \quad x \in \mathbb{R}$



- $\dot{x} = \sin(x) \quad x \in \mathbb{R}$



2.1 2D Phase Portraits

Let the two dimensional system be defined by

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2) \Rightarrow \dot{x} = f(x)$$

The phase portrait in some rough sense can be thought of as the "trajectory" of the system, and where we care about the slope of the vector field of $f(x)$ is given by:

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x)}{f_1(x)}$$

The phase portrait of the linear system can be easily understood via the jordan normal form of the system.

Consider the system given by, $\dot{x} = Ax$, and the linear transformation given by $y = Px$, Thus we get,

$$\dot{y} = P\dot{x} = PAx = PAP^{-1}y = Jy$$

where J is the jordan normal form of A.

We can have one of the 3 cases of the jordan form. The forms with their characteristic polynomials are given by:

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow p(s) = (s - \lambda_1)(s - \lambda_2)$$

$$J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \Rightarrow p(s) = (s - \lambda_1)^2$$

$$J_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \Rightarrow p(s) = (s - \alpha)^2 + \beta^2$$

The solution of the equation in the transformed coordinates is given by:

$$y(t) = e^{Jt}y(0) \Rightarrow x(t) = P^{-1}e^{Jt}Px(0)$$

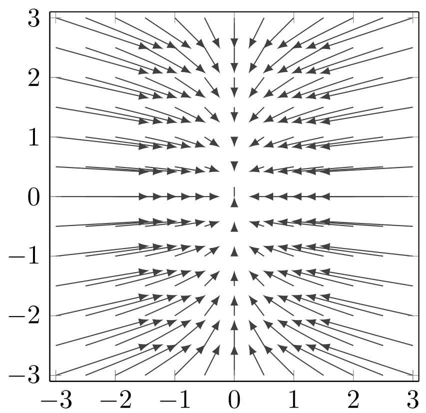
Thus, the solution can be split into three cases similar to the one done above

Case 1:

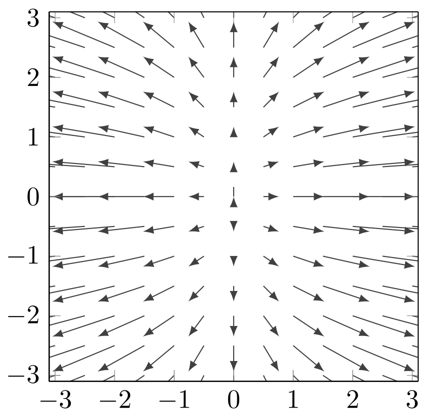
$$y_1(t) = e^{\lambda_1 t}y_1(0) \quad y_2(t) = e^{\lambda_2 t}y_2(0)$$

Thus, depending on the value of λ_1 and λ_2, we can have the following phase portraits:

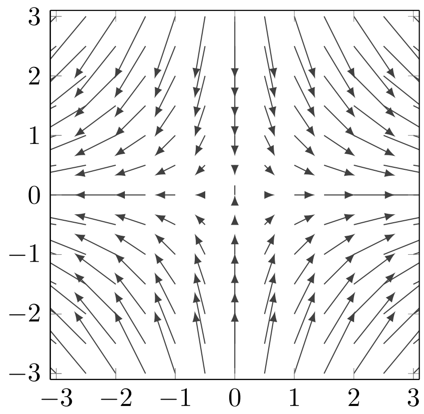
$\lambda_1 < \lambda_2 < 0$: Stable Node



$\lambda_1 > \lambda_2 > 0$: Unstable Node



$\lambda_1 > 0; \lambda_2 < 0$: Saddle

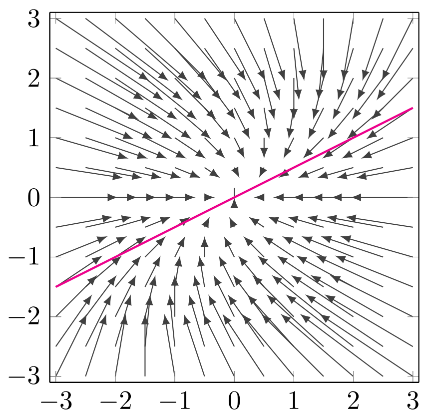


Case 2:

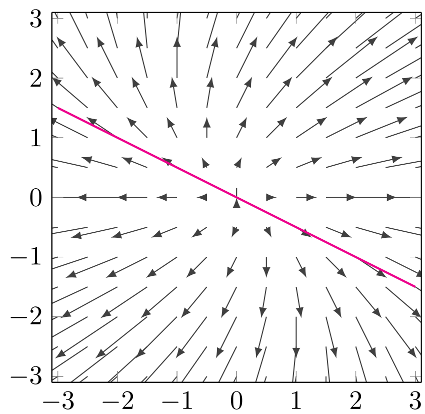
$$y_1(t) = e^{\lambda t}\left(y_1(0) + t y_2(0)\right)$$
$$y_2(t) = e^{\lambda t} y_2(0)$$

Simillary, we can have the following phase portraits depending on the value of λ:

$\lambda < 0$: Improper Stable Node



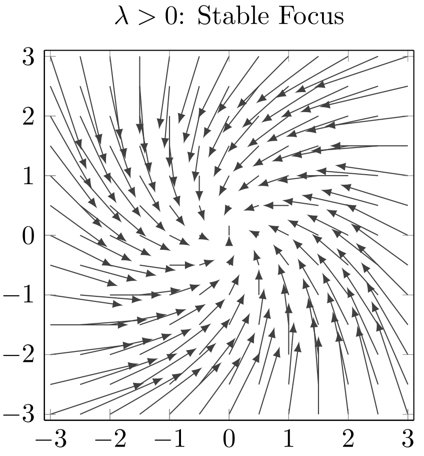
$\lambda > 0$: Improper Unstable Node

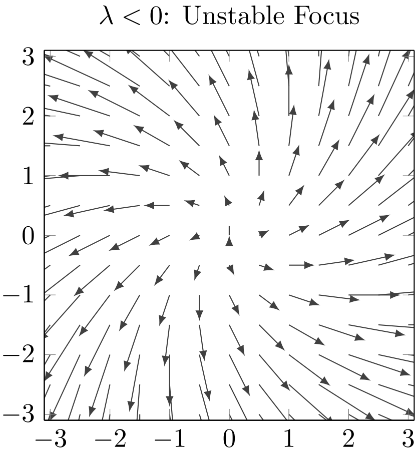


Case 3: Let $r \coloneqq \sqrt{y_1^2 + y_{2^2}}$ and $\theta \coloneqq \tan^{-1}\left(\frac{y_2}{y_1}\right)$ Substituing the values of y_1 and y_2 in the equation, and simplifying, we get:

$$\dot{r} = \alpha r \quad \dot{\theta} = \beta$$

Thus, switching in the polar form, the transformed system are evolving independently of each other. The phase portrait is given by:





2.2 Linearization

Consider a non linear system $\dot{x} = f(x)$ with an equillibrium point $x^\star = 0$. Then using first order Taylor series expansion, we have:

$$\dot{x} = f(x) = f(0) + \left.\frac{\partial f}{\partial x}\right|_{x=0} x + \mathcal{O}(x^2)$$

Thus, the linearised system can be written as:

$$\dot{x} = Ax \quad \text{where} \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=0}$$

The matrix A is called the *Jacobian* of the system $\dot{x} = f(x)$ evaluated at the equillibrium point $x^\star = 0$. The system can also be calculated for an equillibrium point not at origin, by shifting the origin to the equillibrium point.

2.2.1 Pertubations of the eigenvalues

The study of the behaviour of linear systems about the equillibrium point $x = 0$ is important beacuse in many cases the behaviour of a nonlinear systems, near an equillibrium point can be deduced by linearsing the system about the equillibrium point.

The validity and the accuracy of the linearization of the nonlinear system, and the resultant qualitative behaviour depends on the placement of the eigenvalues of the linearised system. This can be understood by considering the linear pertubations. Let the linearized system be given by $\dot{x} = Ax, x \in \mathbb{R}$, with distinct eigenvalues, and let $\Delta A \in \mathbb{R}^{2\times 2}$ be a small pertubation matrix, whose elements have arbitrary small magnitudes. We know, that the eigenvalues of the matrix $A + \Delta A$ depend continuously on the parameters of the matrix.

Thus, when the matrix A is pertubed by ΔA, any eigenvalues of A that lies in the left or the right half plane, wil remain in its respective half plane. But, the eigenvalues on the imaginary axis, when perturbed might go into either the left or the right half plane.

Consquently, we can say that if the equillibrium point $x = 0$ of $\dot{x} = Ax$ is a node, focus, or saddle, then the equillibrium point $x = 0$ of $\dot{x} = (A + \Delta A)x$ will be of the same type, provided that the pertubations are small enough. This situation is quite different in the case of center. Since the qualitative behaviour of the stable focus and unstable focus are different from that of a center, the center equillibrium point will not persist under pertubations.

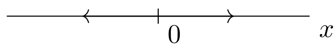
The node, focus, and saddle equillibrium point are said to be *structurally stable* because they maintain their qualitative behaviour under infinitesimally small pertubations.

Definition 2.1 (Hyperbolic Equillibrium Point)**.** An equillibrium point of the system $\dot{x} = f(x)$ with $f(x^\star) = 0$ is said to be hyperbolic if:

$$\Re\left(A = \frac{\partial f}{\partial x}\bigg|_{x=x^\star}\right) \neq 0$$

i.e. the jacobian of the system evaluated at the equillibrium point has no eigenvalues with zero real part.

Example. Consider the system $\dot{x} = x^3, x \in \mathbb{R}$ The system has its only equillibrium point at $x^\star = 0$. The phase portrait of the system is given by:



linearsing the system we have $\dot{x} = 3x^2 \Rightarrow \dot{x} = 0$.

Thus we can say that for any arbitrary point close to the equillibrium point, the behaviour described by the linear system is different from the non linear system

Example. Consider the system given by:

$$\dot{x}_1 = -x_2 - \mu x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 - \mu x_2(x_1^2 + x_2^2)$$

The system has an equillibrium point at $x^\star = 0$. Linearsing the system around the origin we have:

$$A = \frac{\partial f}{\partial x}\bigg|_{x=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda = \pm i$$

Note that the linearised system corresponds to a simple harmoic oscillator

Shifting, into polar coordinates to analyse the system, we have:

$$r := \sqrt{x_1^2 + x_2^2} \quad \theta := \arctan\left(\frac{x_2}{x_1}\right)$$

After some simplification, we have:

$$\dot{r} = -\mu r^3 \quad \dot{\theta} = 1$$

Depending upon the value of μ, we have the following:

$$\mu > 0 \Rightarrow \text{Stable Focus}$$
$$\mu = 0 \Rightarrow \text{Center}$$
$$\mu < 0 \Rightarrow \text{unstable Focus}$$

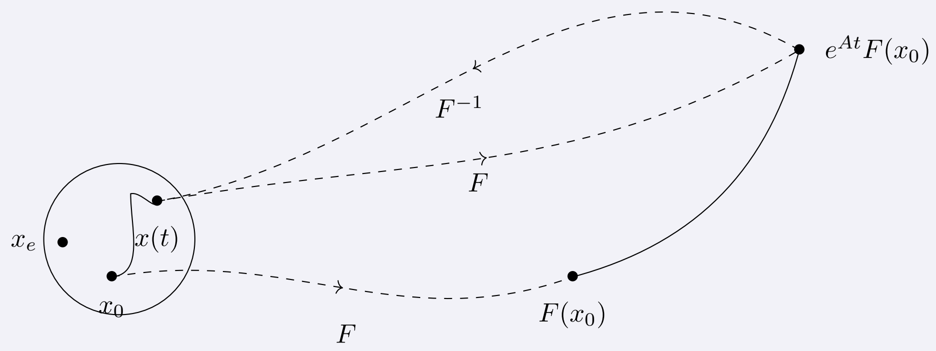
Thus, the qualitative behaviour is same for the linearised system and the nonlinear system, only at $\mu = 0$, which can also be trivally observed from the state equations.

Thus, we have a sufficient condition for the qualitative behaviour of the nonlinear system to be same as the linearised system, which is that the equillibrium point of the nonlinear system should be hyperbolic.

Theorem 2.1 (Hartman-Grobman Theorem)**.** Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ be a nonlinear system with an hyperbolic equillibrium point x_e. Let ϕ_t^f be the flow map for $\dot{x} = f(x)$ i.e. $\dot{x} = f(x)$, $x(0) = x_0$ has the solution $x(t) = \phi_t^f(x_0)$. Then for a $\delta > 0$ and a ball $B_\delta(x \in \mathbb{R}^n | \|x - x^\star\| < \delta)$ there exits as map $F : B_\delta \subset \mathbb{R}^n \to \mathbb{R}^n$, $\exists T > 0$ such that,

$$F(x_e) = 0, \text{ F is one to one on } B_\delta \text{ onto } F(B_\delta)$$

and both F and F^{-1} exits and are continuous, such that $F(\phi_t^f) = e^{At}F(x) \quad \forall |t| < T$. Where $A = \frac{\partial f}{\partial x}\Big|_{x=x^\star}$ and $x \in B_\delta$



The map F in Theorem 2.1 maps from the nonlinear world to the linear world.

2.2.2 Pendulumn

One example of periodicity in phase portraits is given by the pendulumn. This type of periodic behaviour is not found in linear systems.

Example (Pendulumn)**.** Consider a pendulumn, with normalised dynamics given by:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) - x_2\end{aligned}$$

The equillibrium points of this system are given by:

$$x_1 = k\pi \quad x_2 = 0 \quad k \in \mathbb{Z}$$

Calculating the jacobian of the system, we have:

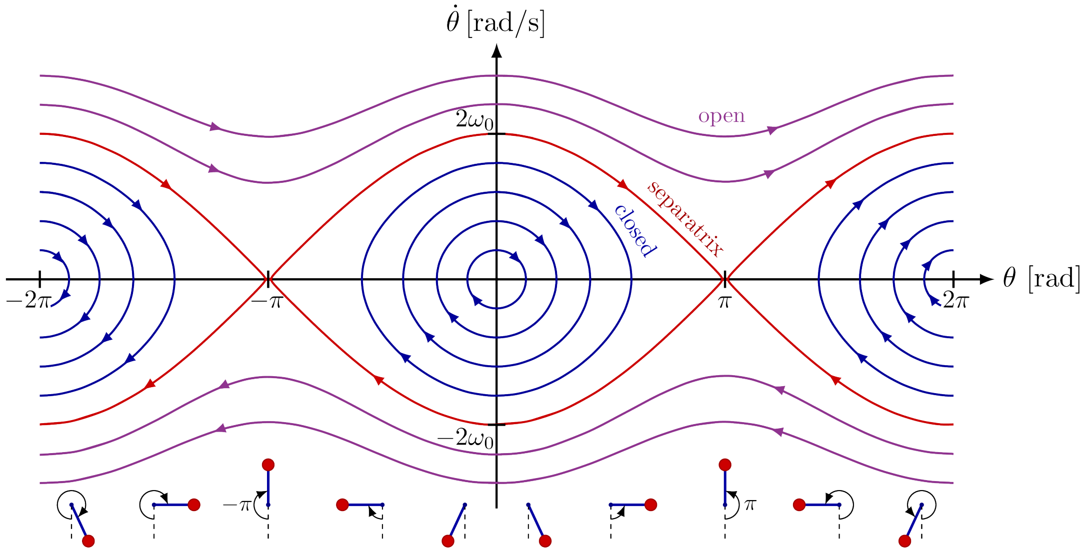
$$A = \frac{\partial f}{\partial x}\Big|_{x=x^\star} = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -1 \end{bmatrix}$$

Evalating the jacobian at $(0, 0)$ and $(\pi, 0)$ we have :

$$\left.\frac{\partial f}{\partial x}\right|_{x=(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm \sqrt{3}}{2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=(\pi,0)} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow \lambda = \frac{-1 \pm \sqrt{5}}{2}$$

The phase portrait of the system is given by:



The image source is linked here

2.3 Limit Cycles and Periodic Orbits

Periodic orbits in the phase plane, which have a closed trajectory is usually called a perioidc orbit or closed orbit. An isolated perioidc orbit is called as a limit cycle.

For examples a harmonic oscillator has there is a continuum of cloed orbits as shown in the Figure 1 , while in the case of Van der Pol oscillator, there is only one closed orbit i.e. a limit cycle as shown in Figure 2.

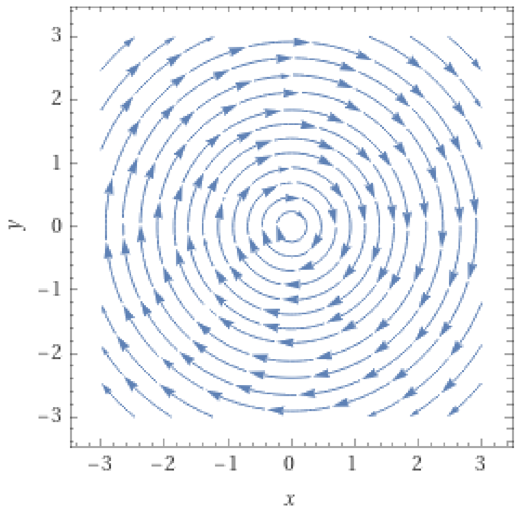


Figure 1: Harmonic Oscillator

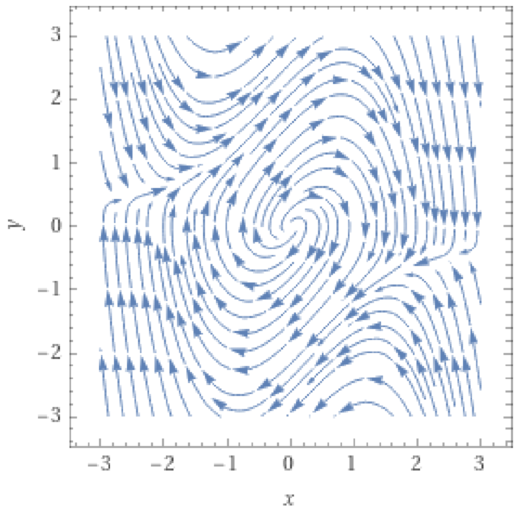


Figure 2: Limit Cycle in Vander Pol Oscillator

2.4 Poincaré-Bendixson Theorem and Bendixson's Criterion

Periodic orbits divide the plane into a region inside the orbit and a region outside it. This allows us to obtains criteria for the existence of periodic orbits in a region of the plane. The analysis of such a kind doesn;t extend to higher dimensions, as there is no concept of inside and outside in higher dimensions. One such criteria is given by the Poincaré-Bendixson theorem.

Theorem 2.2 (Poincaré-Bendixson Theorem)**.** Let $\dot{x} = f(x), x \in \mathbb{R}^2$ and there exists a closed and bounded set $M \subset \mathbb{R}^2$ such that:

- M contains no equillibrium points of $\dot{x} = f(x)$
- Every trajectory that starts in M remains in M for all $t \geq 0$

Then M contains a periodic orbit of $\dot{x} = f(x)$.

Math Review:

$$\operatorname{div} f = \text{ divergence of vector field } f$$

$$\operatorname{div} f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \quad \text{where } f = (f_1, f_2, \ldots, f_n)$$

For a 2D vector field F we have:

$$\operatorname{div} F = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \quad x \in \mathbb{R}^2$$

If there are no sources or sinks in a region (which are equillibrium points in context of dynamical systems), then the divergence of the vector field is zero.

Theorem 2.3 (Divergence and Green's Theorem)**.** THe divergence theorem states that the volume integral of the divergence of a vector field is equal to the surface integral of the vector field over the boundary of the volume. i.e.

$$\iiint V \operatorname{div} F dV = \iint_S F \cdot n dS$$

where n is the outward normal to the surface S.

Green's theorem states that the line integral of a vector field over a closed curve is equal to the surface integral of the curl of the vector field over the region bounded by the curve. i.e.

$$\oint_C F \cdot dr = \iint_S \operatorname{curl} F \cdot n dS$$

$$\Rightarrow \iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dS = \oint f_2(x_1, x_2) dx_1 - f_1(x_1, x_2) dx_2$$

Thus, we can now state teh Bendixson's criterion as follows:

Theorem 2.4 (Bendixson's Criterion)**.** Let D be a simply connected region in $\mathbb{R}^2$. Suppose f is such that $\operatorname{div} f = \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_2}{\partial x_2}$ is not identically zero in D, and doesn't change sign in D. Then there are no periodic orbits of $\dot{x} = f(x)$ entirely in D.

Proof. Proof by contradiction: Suppose there exists a perioidc orbit γ that in entirly in D, for any trajectory of $\dot{x} = f(x) \quad x \in \mathbb{R}^2$

$$\frac{\mathrm{d}x_2}{\mathrm{d}x_1} = \frac{f_2(x)}{f_1(x)} \Rightarrow f_2(x)\mathrm{d}x_2 - f_1(x)\mathrm{d}x_1 = 0$$

$$\Rightarrow \oint_\gamma f_2(x)\mathrm{d}x_2 - f_1(x)\mathrm{d}x_1 = 0$$

By Green's theorem, we have:

$$\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) = 0$$

This contradictcs the fact the divergence of f is not identically zero in D, thus volating our assumption. Hence Proved. ☺

2.5 Index of a curve

Definition 2.2 (Index of a curve)**.** For a sufficiently smooth vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$ and a closed curve γ not passing through any equillibrium points of $\dot{x} = f(x)$, the index of γ w.r.t. f is:

$$I_\gamma^f = \frac{1}{2\pi} \oint_\gamma d\theta_f \qquad \theta = \tan^{-1} \left(\frac{f_2}{f_1} \right)$$

There are certain properties of the index of a curve listed below:

Property (1)**.** If we have two closed curves γ_1 and γ_2 such that γ_1 can be continuously deformed into γ_2 without passing through any equillibrium points then,

$$I_{\gamma_1}^f = I_{\gamma_2}^f$$

Property (2)**.** If γ doesn not enclose any equillibrium points, then

$$I_\gamma^f = 0$$

Property (3).

$$I_\gamma^f = I_\gamma^{-f}$$

This can be intituitively understood as follows:

$$\theta_f = \pi + \theta_{-f} \Rightarrow d\theta_f = -d\theta_{-f}$$

Thus equating the line integral

Property (4). If γ is a closed orbit of $\dot{x} = f(x)$ then

$$I_\gamma^f = 1$$

Property (5). If γ encloses one equillibrium point x^*, and $x^\star$ is a hyperbolic equillibrium point, then:

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^\star} \Rightarrow I_\gamma^f = I_\gamma^{Ax}$$

Using the above properties, we can now define the index of the equillibrium point.

Definition 2.3 (Index of an equillibrium point). $I(x_i^\star)$ is the index of the equillibrium point $x_i^\star$ of $\dot{x} = f(x)$ where $I(x_i^\star)$ is the index of any closed curve γ enclosing only $x_i^\star$ and no other equillibrium points.

Lemma 2.1. Index of a node, focus or a center is 1, while that of a saddle is -1.

Lemma 2.2. Index of a closed curve is the sum of the indices of the equillibrium points enclosed by the curve γ

Thus, we have the following corollary:

Corollary 2.1. Inside any perioidc orbit γ there exists at least one equillibrium point. Suppose all the equillibrium points inside γ are hyperbolic. Let N be the number of nodes, foci and centers inside γ and S be the number of saddles inside γ. then, the index of γ is:

$$I_\gamma^f = N - S$$

4 Jan 2024

3 Contraction Mapping and Comparison Lemma

3.1 Math Review

3.1.1 Vector Spaces

The set of all n dimensional vector $x = [x_1, x_2, ..., x_n]^\top$ with each element being a real number is denoted as $\mathbb{R}^n$ and defines the n-dimensional Euclidean space. The inner product of two vectors x and y is defined as:

$$x^\top y = \sum_{i=1}^{n} x_i y_i$$

The norm of a vector x is a real valued function with the properties:

- $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$ with $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha| \|x\| \quad \forall \alpha \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$

The class of p-norm is defined as:

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty$$

with the special case of $p = \infty$ being:

$$\|x\|_\infty = \max_i |x_i|$$

The most commonly used norms are the 1-norm, 2-norm and ∞-norm are defined as:

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

$$\|x\|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

$$\|x\|_\infty = \max_i |x_i|$$

All p-norms are equivalent in the sense that if $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are two p-norms, then there exists two positive constants c_1 and c_2 such that:

$$c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha \quad \forall x \in \mathbb{R}^n$$

An important property of the p-norm is the Holder's inequality:

$$|x^\top y| \leq \|x\|_p \|y\|_q \quad , \frac{1}{p} + \frac{1}{q} = 1, \quad \forall x, y \in \mathbb{R}^n$$

Many times when the properties deduced from the basic properties satisfied by any norm, in such cases the norm is denoted by $\|\cdot\|$ without any subscript, indicating that the norm can be any p-norm.

The matrix A defines a linear mapping $y = Ax$ from $\mathbb{R}^n$ to $\mathbb{R}^m$. The induced norm of the matrix A is defined as:

$$\|A\|_p = \sup_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \max_{\|x\|_p=1} \|Ax\|_p \quad 1 \leq p \leq \infty$$

This norm for p = 1, 2 and ∞ is defined as:

$$\|A\|_1 = \max_{1\leq j\leq n} \sum_{i=1}^{m} |a_{ij}|$$
$$\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)}$$
$$\|A\|_\infty = \max_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}|$$

where $\lambda_{\max}(A^\top A)$ is the largest eigenvalue of the matrix $A^\top A$.

Some useful properties of the induced matrix norm are for real matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times l}$ are:

$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty$$
$$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$$
$$\|A\|_2 \leq \sqrt{\|A\|_1\|A\|_\infty}$$
$$\|AB\|_p \leq \|A\|_p\|B\|_p$$

3.1.2 Sequence, Series and Sets

Convergence of Sequence: A sequence of vectors $x_0, x_1, \ldots, x_k, \ldots$ in $\mathbb{R}^n$ denoted by $\{x_k\}$, is said to converge to a a limit vector x if :

$$\|x_k - x\| \to 0 \quad \text{as} \quad k \to \infty$$

which is equivalent to:

$$\|x_k - x\| < \varepsilon \quad \forall k \geq N \text{ for some } N \in \mathbb{N} \text{ and } \varepsilon > 0$$

A vector x is an accumulation point of the sequence $\{x_k\}$ if there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ that converges to x.

A bounded sequence $\{x_k\}$ in $\mathbb{R}^n$ has atleast one accumulation point in $\mathbb{R}^n$. A sequence r_k is said to be increasing if $r_k \leq r_{k+1}$ for all $k \in \mathbb{N}$. It is said to be strictly increasing if $r_k < r_{k+1}$ for all $k \in \mathbb{N}$. And similarly for decreasing and strictly decreasing sequences.

An increasing sequence bounded from above converges to a real number. Similarly, a decreasing sequence bounded from below converges to a real number.

Sets: A subset $S \subset \mathbb{R}^n$ is said to be open if for every $x \in S$, there exists a neighborhood of x:

$$\mathcal{N}_\varepsilon(x) = \{z \in \mathbb{R}^n \mid \|z - x\| < \varepsilon\}$$

such that $\mathcal{N}_\varepsilon(x) \subset S$. A set is said to be closed if its complement is open. Equivalently, a set is closed if and only if every convergent sequence in the set S has its limit in S.

A set is said to be bounded if there is $r > 0$ such that

$$\|x\| < r \quad \forall x \in S$$

Thus, a compact set is a set that is closed and bounded.

A point p is a boundary point of a set S if every neighborhood of p contains at least one point in S and at least one point not in S. The set of all boundary points of S is called the boundary of

S and is denoted by ∂S. A closed set contains all its boundary points, with open sets containing none of its boundary points. The interior of a set S is the set of all points in S that are not boundary points of S i.e. $S \setminus \partial S$. The closure of a set S is the union of S and its boundary i.e. $\overline{S} := S \cup \partial S$.

A open set S is connected if every pair of points in S can be joined by a curve in S. The set S is called convex if for every pair of points $x, y \in S$, and for every $\lambda \in [0,1]$, $\lambda x + (1-\lambda)y \in S$.

3.1.3 Normed Linear Spaces

A linear space $\mathcal{X}$ is a normed linear space if, to each vector $x \in \mathcal{X}$, there is a real valued norm $\|x\|$ that satifies the following properties:

- $\|x\| \geq 0 \quad \forall x \in \mathcal{X}$ with $\|x\| = 0 \Leftrightarrow x = 0$
- $\|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathcal{X}$

To denote that a norm $\|\cdot\|$ is a norm on the linear space $\mathcal{X}$, we write the norm as $\|\cdot\|_{\mathcal{X}}$.

Convergence A sequence $\{x_k\} \in \mathcal{X}$ is said to converge to $x \in \mathcal{X}$ if:

$$\|x_k - x\| \to 0 \quad \text{as} \quad k \to \infty$$

Closed Set: A set $S \subset \mathcal{X}$ is said to be closed if every convergent sequence in S has its limit in S.

Cauchy Sequence: A sequence $\{x_k\} \in \mathcal{X}$ is said to be a Cauchy sequence if:

$$\|x_k - x_l\| \to 0 \quad \text{as} \quad k, l \to \infty$$

Every convergent sequence is a Cauchy sequence, the converse is true if the space $\mathcal{X}$ is complete.

Banch Space: A normed linear space $\mathcal{X}$ is said to be a complete space or a Banch space if every Cauchy sequence in $\mathcal{X}$ converges to a vector in $\mathcal{X}$.

3.2 Contraction Mapping

The motivation for using contraction mapping in the non linear systems is that the solution of the non linear system can be said as a *fixed point* of the non linear map.

Consider the system $\dot{x} = f(t, x(t))$ $x(t_0) = x_0$ $t \in [t_0, t_f]$. The solution of the the system can be written as:

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(\tau))d\tau$$

Envisioning this as solving the system in one go we can the above as:

$$x(\cdot) = x_0 + \int_{t_0}^{(\cdot)} f(s, x(\tau))d\tau$$

In other words, the solution of the system is a fixed point of the map F:

$$x(\cdot) = F(x(\cdot)) \quad \text{where } F \text{ is some map}$$

This can be explained using a linear system. Consider the system $\dot{x} = Ax,\ x(t_0) = x_0$. Then the map F is defined as:

$$F(x(\cdot))(t) = x_0 + \int_{t_0}^{t} Ax(\tau)d\tau$$

A common method of finding the solution of finding such a fixed point is using the successive approximation method. Thus, the iterations are defined as:

$$x_{k+1}(t) = x_0 + \int_{t_0}^{t} Ax_k(\tau)d\tau$$

Thus, we have the following

$$\begin{aligned} x_0(s) &= x_0 \quad \forall s \in [t_0, t_1] \\ x_1(t) &= x_0 + \int_{t_0}^{t} Ax_0(\tau)d\tau = [I + A(t-t_0)]\, x_0 \\ x_2(t) &= \cdots = \left[I + A(t-t_0) + \frac{A^2(t-t_0)^2}{2!}\right] x_0 \\ \Rightarrow x_k(t) &= e^{A(t-t_0)}x_0 \end{aligned}$$

Thus, we can now define the contraction theorem as follows:

Theorem 3.1 (Contraction Theorem)**.** Let $\mathcal{X}$ be a complete normed linear space and let $\mathcal{S}$ be a closed subset of $\mathcal{X}$. Let $T : \mathcal{S} \to \mathcal{S}$ such that:

$$\|T(x) - T(y)\| \leq \rho\|x - y\| \quad \forall x, y \in \mathcal{S},\ \rho \in [0, 1)$$

Then,

- there exits unique vector $x^\star \in \mathcal{S}$ such that $T(x^\star) = x^\star$
- $x^\star$ can be obtained by fixed point iteration i.e.

$$x^\star = \lim_{k\to\infty} x_k \quad \text{where, } x_{k+1} = T(x_k),\ x_0 \in \mathcal{S}$$

Proof. Choose an arbitrary $x_0 \in \mathcal{S}$ and define,

$$x_{k+1} = T(x_k) \quad T : \mathcal{S} \to \mathcal{S} \Rightarrow x_k \in \mathcal{S} \quad \forall k \geq 0$$

The proof follows the following series of claims:

Claim. $\{x_k\}$ is a Cauchy sequence.

$$\begin{aligned} \|x_{k+1} - x_k\| = \|T(x_k) - T(x_{k-1})\| &\leq \rho\|x_k - x_{k-1}\| \\ &\leq \rho^2\|x_{k-1} - x_{k-2}\| \\ &\leq \rho^k\|x_1 - x_0\| \end{aligned}$$

Now,

$$\begin{aligned} \|x_{k+r} - x_k\| &\leq \|x_{k+r} - x_{k+r-1}\| + \|x_{k+r-1} - x_{k+r-2}\| + \cdots + \|x_{k+1} - x_k\| \\ &\leq \left(\rho^{k+r-1} + \rho^{k+r-2} + \cdots + \rho^k\right)\|x_1 - x_0\| \\ &\leq \frac{\rho^k}{1-\rho}\|x_1 - x_0\| \Rightarrow \{x_k\} \text{ is a Cauchy sequence} \end{aligned}$$

Claim. $\{x_k\}$ converges to some $x^\star \in \mathcal{S}$ as $k \to \infty$.

Since $\mathcal{S}$ is complete and closed, $\{x_k\}$ converges to some $x^\star \in \mathcal{S}$.

Claim. $x^\star$ is a fixed point of T. i.e. $T(x^\star) = x^\star$

We have:

$$\begin{aligned}\|x^\star - T(x^\star)\| &\leq \|x^\star - x_k\| + \|x_k - T(x^\star)\| \\ &= \|x^\star - x_k\| + \|T(x_k) - T(x^\star)\| \\ &\leq \underbrace{\|x^\star - x_k\| + \rho\|x_k - x^\star\|}_{\to 0 \text{ as } k\to\infty} \\ &\Rightarrow \|x^\star - T(x^\star)\| = 0 \Rightarrow x^\star = T(x^\star)\end{aligned}$$

Claim. $x^\star$ is unique fixed point of $T \in \mathcal{S}$.

By Contradiction: Let $x^\star$ and $y^\star$ be two fixed points of T such that $y^\star = T(y^\star)$, $x^\star = T(x^\star)$ and $x^\star \neq y^\star$. Then,

$$\begin{aligned}\|x^\star - y^\star\| &= \|T(x^\star) - T(y^\star)\| \\ &\leq \rho\|x^\star - y^\star\| \Rightarrow \|x^\star - y^\star\| = 0 \\ &\Rightarrow x^\star = y^\star \quad \text{which is a contradiction}\end{aligned}$$

Thus, $x^\star$ is the unique fixed point of T. ☺

We can now use this result to prove the local existence and uniqueness of the solution of the the nonlinear system $\dot{x} = f(t, x(t)),\ x(t_0) = x_0,\ t \in [t_0, t_f]$, where f is piecewise continuous.

Theorem 3.2 (Local Existence and Uniqueness). Let $\dot{x} = f(t, x(t)),\ x(t_0) = x_0,\ t \in [t_0, t_f]$ be a nonlinear system where f is piecewise continuous, and satifies the lipschitz condition,

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|,\ t \in [t_0, t_f],\ L > 0 \quad \forall x, y \in B_r(x_0) := \{x \mid \|x - x_0\| \leq r\}$$

Then, $\exists\, \delta > 0$ such that the system has a unique solution on the interval $[t_0, t_0 + \delta]$.

Proof. Define the map P as:

$$P(\cdot)(t) := x_0 + \int_{t_0}^{t} f(s, x(\tau))d\tau \Rightarrow x(t) = P(x(\cdot))(t)$$

Let,

$$\mathcal{X} := C[t_0, t_0 + \delta] \to \text{ set of continuous functions on } [t_0, t_0 + \delta]$$

Note on abuse of notation.

$$x \in \mathcal{X} \quad \text{but} \quad x(t) \in \mathbb{R}^n$$

Defining the norm on $\mathcal{X}$ as:

$$\|x\|_C := \max_{t\in[t_0, t_0+\delta]} \|x(t)\|$$

The above norm is genrelising ∞-norm to function spaces. Thus, we have:

$$\mathcal{S} := \{x \in \mathcal{X} \mid \|x - x_0\|_C \leq r\}$$

Thus, we have $\mathcal{S} : \mathcal{X} \to \mathcal{X}$. But we need to show that $P : \mathcal{S} \to \mathcal{S}$. To achieve this we note the following two observations:

Observe (1). f is piecewise continuous on $[t_0, t_0+\delta]$ and thus $\|f(t,x)\|$ is piecewise continuous too. Let,

$$h := \max_{t \in [t_0, t_0+\delta]} \|f(t,x)\|$$

Observe (2).

$$\begin{aligned}
\|P(x(\cdot))(t) - x_0\| &= \left\| \int_{t_0}^{t} f(\tau, x(\tau)) d\tau \right\| \\
&\leq \int_{t_0}^{t} \|f(\tau, x(\tau)) - f(\tau, x_0) + f(\tau, x_0)\| \, d\tau \\
&\leq \int_{t_0}^{t} \|f(\tau, x(\tau)) - f(\tau, x_0)\| + \|f(\tau, x_0)\| d\tau \\
&\leq \int_{t_0}^{t} L\|x(\tau) - x_0\| + h d\tau \leq \int_{t_0}^{t} Lr + h d\tau = (t - t_0)(Lr + h) \\
&= \delta(Lr + h)
\end{aligned}$$

Since this is true for all $t \in [t_0, t_0 + \delta]$, we have:

$$\|P(x(\cdot)) - x_0\| \leq \delta(Lr + h) \leq \underbrace{r}_{\text{enforce this}}$$

The condition on δ is:

$$\delta(Lr + h) \leq r \Rightarrow \delta \leq \frac{r}{Lr + h}$$

Now, choose δ such that $P : \mathcal{S} \to \mathcal{S}$ is a contraction mapping. Thus,

$$\begin{aligned}
\|P(x(\cdot))(t) - P(y(\cdot))(t)\| &= \left\| \int_{t_0}^{t} f(\tau, x(\tau)) - f(\tau, y(\tau)) d\tau \right\| \\
&\leq \int_{t_0}^{t} \|f(\tau, x(\tau)) - f(\tau, y(\tau))\| d\tau \\
&\leq \int_{t_0}^{t} L\|x(\tau) - y(\tau)\| d\tau \\
&\leq L\delta \|x - y\|_C \quad \forall\, t \in [t_0, t_0 + \delta] \\
&\leq \rho \|x(\cdot) - y(\cdot)\|_C \quad \rho \in [0, 1) \\
&\Rightarrow \delta \leq \frac{\rho}{L}
\end{aligned}$$

Picking $\delta \leq \min\left\{\frac{r}{Lr+h}, \frac{\rho}{L}, t_1 - t_0\right\}$ we have the contraction on the P. Thus, $P(x(\cdot))$ has a unique fixed point $x^\star \in \mathcal{S}$. Now, we need to extend the result from map $\mathcal{S}$ to map $\mathcal{X}$.

Even if there are multiple solutions in $\mathcal{X}$, it must be that $x(t) \in B_r(x_0) \forall t \in [t_0, t_0 + \delta]$ for some $\mu > 0$. Let, $t_0 + \mu$ be the first time when $\|x(t) - x_0\| < r$. Then, we have:

$$\begin{aligned} r &= \left\| \int_{t_0}^{t_0+\mu} f(\tau, x(\tau)) d\tau \right\| \\ &\leq \int_{t_0}^{t_0+\mu} Lr + h d\tau = \mu(Lr + h) \\ &\Rightarrow \mu \geq \frac{r}{Lr + h} \geq \delta \end{aligned}$$

Thus, the same result holds for $\mathcal{X}$ too, completing the proof. 😊

3.3 Lipschitz Condition

A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to satisfy the Lipschitz condition in x at x_0 if $\exists L > 0, r > 0$ such that:

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad \forall x, y \in B_r(x_0) \; \forall t \in [t_0, t_f]$$

Definition 3.1 (Locally Lipschitz). A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be *locally lipschitz* on a set $D \subset \mathbb{R}^n$ if $\forall x_0 \in D$, the function f satisfies the lipschitz condition for lipschitz constant $L(x_0)$.
The function is said to be lipschitz on D if the function satifies the lipschitz condition for all $x \in D$ with a uniform lipschitz constant L.

The function $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be globally lipschitz if the function is lipschitz on the entire domain $\mathbb{R}^n$.

Example (Lipschitz Condition). Let $f : I \subset \mathbb{R} \to \mathbb{R} \ni |f'(x)| \leq k, \forall x \in I$. Then, f satisfies the Lipschitz condition on I with $L = k$:

$$|f(x) - f(y)| = \left| \int_x^y f'(s) ds \right| \leq \int_x^y |f'(s)| ds \leq k|x - y|$$

Lemma 3.1 (Lipschitz Condition for a Convex Set). Let $f : [a, b] \times D \to \mathbb{R}^n$ be a continuous on some domain $D \subset \mathbb{R}^n$. Suppose $\frac{\partial f}{\partial x}$ exists and is continuous on $[a, b] \times D$. If for a convex subset $W \subset D$, $\exists L > 0$ such that:

$$\left\| \frac{\partial f}{\partial x}(t, x) \right\| \leq L \quad \forall (t, x) \in [a, b] \times W$$

Then,

$$\|f(t, x) - f(t, y)\| \leq L\|x - y\| \quad \forall \; (t, x), (t, y) \in [a, b] \times W$$

Proof. For a convex set, we have:

$$\gamma(s) = x_1(1-s) + x_2 s \quad s \in [0,1]$$
$$g(s) := f(t, \gamma(s))$$

$$\begin{aligned}
\Rightarrow \|f(t,x_1) - f(t,x_2)\| = \|g(0) - g(1)\| &= \left\| \int_0^1 \frac{\partial g}{\partial s} \mathrm{d}s \right\| \\
&\leq \int_0^1 \left\| \frac{\partial g}{\partial \gamma} \frac{\partial \gamma}{\partial s} \right\| \mathrm{d}s \\
&\leq \int_0^1 L \|x_2 - x_1\| \mathrm{d}s = L\|x_2 - x_1\|
\end{aligned}$$

Thus, f satisfies the Lipschitz condition on W. ☺

Lemma 3.2. If $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous on $[a,b] \times D$, or some domain $D \subset \mathbb{R}^n$, then f is locally lipschitz in x on $[a,b] \times D$.

Lemma 3.3. Supoose $f(t,x)$ and $\frac{\partial f}{\partial x}(t,x)$ are continuous on $[a,b] \times \mathbb{R}^n$. Then, f is globally lipschitz in x on $[a,b] \times \mathbb{R}^n$ iff $\frac{\partial f}{\partial x}(t,x)$ is uniformly bounded on $[a,b] \times \mathbb{R}^n$.

Example.

$$f(x) := \begin{bmatrix} -ax_1 + x_1 x_2 \\ ax_2 - x_1 x_2 \end{bmatrix} \quad a > 0$$
$$\frac{\partial f}{\partial x}(x) := \begin{bmatrix} -a + x_2 & x_1 \\ -x_2 & a - x_1 \end{bmatrix}$$

Since $f(x)$ is differentiable and continuous over $\mathbb{R}^2$, $f(x)$ is locally lipschitz. But since $\frac{\partial f}{\partial x}(x)$ is not uniformly bounded, $f(x)$ is not globally lipschitz.

But, it can be lipschitz on some compact set. Let,

$$W := \{x \in \mathbb{R}^2 \mid |x_1| < a_1, |x_2| < a_2\}$$

Here, we can explot the equivalence of p-norms to chose and find the lipschitz constant. Choose induced ∞ norm:

$$\left\| \frac{\partial f}{\partial x}(x) \right\|_\infty = \max\{|-a + x_2|, |x_1|, |-x_2|, |a - x_1|\} \leq a + a_1 + a_2 =: L$$

3.4 Existence and Uniqueness of Solutions

Theorem 3.3. Let $f(t,x)$ be a piecewise continuous function in t, and locally lipschitz in x, $\forall t \geq t_0$, and $\forall x \in D \subset \mathbb{R}^n$. Let, W is a compact subset of D and $x_0 \in W$. Suppose that every solution of $\dot{x} = f(t,x)$, $x(t_0) = x_0$ lies entirly in W, $\forall t \geq t_0$. Then, there exists a unique solution with $x(t_0) = x_0$, $\forall t \geq t_0$

Proof. TODO: misunderstood and wrote the wrong proof. ☺

Lemma 3.4 (Comparison Lemma)**.** Consider the scalar ODE: $\dot{u} = f(t, u), u(t_0) = u_0$. Supoose $ft(t, u)$ is continuous in t and locally lipschitz in u, $\forall t \geq t_0$, and, $u(t) \in J \subset \mathbb{R}$, $\forall t \geq t_0$. Let $[t_0, T]$ be the maximal interval of existence of the solution $u(t) \in J$, $\forall t \in [T_o, T)$. Let $v(t)$ be continuous such that:

$$D^+ v(t) \leq f(t, v(t)) \quad v(t_0) \leq u_0, with\ v(t) \in J,\ \forall t \in [t_0, T)$$

Then, $v(t) \leq u(t)$, $\forall t \in [t_0, T)$.

Proof. Assume that $V(\cdot)$ is differentiable. The proof holds otherwise too, but assumption greatly simplifies the proof. Proof by contraction: Suppose $\exists a, b \in [t_0, t_1]$ such that:

$$v(a) = u(a) \quad v(t) < u(t) \quad \forall t \in (a, b]$$

$$\Rightarrow v(t) - v(a) > u(t) - u(a) \quad \forall t \in (a, b]$$

$$\lim_{h \to 0^+} \frac{v(a+h) - v(a)}{h} \geq \lim_{h \to 0^+} \frac{u(a+h) - u(a)}{h}$$

$$\Rightarrow \dot{v}(a) \geq \dot{u}(a) \quad \text{which is a contradiction}$$

☺

Theorem 3.4. Let f(t,x) be piecewise continuous in t, and locally lipschitz in x on $[t_0, t_1] \times W$, with Lipschitz constant L, and $W \subset \mathbb{R}^n$ is a open connected set. Let, $y(t), z(t)$ be the solutions of $\dot{y} = f(t, y)$, $y(t_0) = y_0$, $\dot{z} = f(t, z) + g(t, z)$, $z(t_0) = z_0$, such that $y(t), z(t) \in W$, $\forall t \in [t_0, t_1]$. Suppose $\|g(t, x)\| \leq \mu\ \forall (t, x) \in [t_0, t_1] \times W$, $\mu > 0$, $\|y_0 - z_0\| \leq \gamma$, Then,

$$\|y(t) - z(t)\| \leq \gamma e^{L(t-t_0)} + \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right) \quad \forall t \in [t_0, t_1]$$

Proof. Note that, if we know more about, the system, then the upper bound can be made tighter and hence, the above upper bound can be useless at times.

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) ds$$

$$z(t) = z_0 + \int_{t_0}^{t} f(s, z(s)) + g(s, z(s)) ds$$

$$\Rightarrow \|y(t) - z(t)\| \leq \|y_0 - z_0\| + \int_{t_0}^{t} \|f(s, y(s)) - f(s, z(s))\| ds + \int_{t_0}^{t} \|g(s, z(s))\| ds$$

$$\leq \gamma + \mu(t - t_0) + \int_{t_0}^{t} L\|y(s) - z(s)\| ds$$

Using, the resutls of gronwell-bellman inequality, we have:

$$\|y(t) - z(t)\| \leq \gamma e^{L(t-t_0)} + \frac{\mu}{L}\left(e^{L(t-t_0)} - 1\right) \quad \forall t \in [t_0, t_1]$$

Thus, completing the proof. 😄

4 Jan 2024

4 Lyaupunov Stability