Nonlinear Systems and Control

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Lecture notes from the 2024 graduate course Nonlinear Systems and Control, given by professor Pavankumar Tallapragada at IISc Banglore in the academic year 2023-23. Credit for the material in these notes is due to professor Pavankumar The credit for the typesetting is my own, along with any contributions from the PRs, of the notes hosted on GitHub.

Disclaimer: This document will inevitably contain some mistakes— both simple typos and legitimate errors. Keep in mind that these are the notes of a graduate student in the process of learning the material, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email at vaidyavarad2001@gmail.com.

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1 Lecture 1 — Introduction to Nonlinear Systems

In general, a nonlinear system can be defined with the following:

$$f : \mathbb{R}^n \to \mathbb{R}^n$$
$$\frac{dx}{dt} = \dot{x} = f(t, x, u, d)$$
$$y = g(t, x, u, d)$$

with the initial condition,

$$x(t_0) = x_0$$

where, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, and $d \in \mathbb{R}^q$ is the disturbance.

We can also write the control signal u as a function of the state x and some parameter vector θ. Thus, using this "state feedback" we have:

$$u = h(t, x, \theta)$$
$$\Rightarrow \dot{x} = \bar{f}(t, x, \theta)$$

This free choice of θ allows us to design the system to meet certain requirements, or to optimize some performance metric etc.

One of the special case of nonlinear systems is called as autonomous systems, where the system does not depend on time explicitly. Thus, we have:

$$\dot{x} = f(x)$$

It is possible to convert a non-autonomous system to an autonomous system by adding a new state variable z, such that:

$$\dot{z} = 1$$

and incorporating the new variable z, in the state vector, increasing the order of the system.

> **Note:-**
>
> Note that, the state z and hence the state x, is unbounded with time in this conversion.

In the case of the linear systems, we have the following:

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x + D(t)u$$

with the solution of the above system being:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

with the state transition matrix $\Phi(t, \tau)$. The state transition matrix simplifies even further under the case of LTI systems as:

$$\Phi(t, \tau) = e^{A(t-\tau)}$$

In genreal the solution of the nonlinear system cannot be wriiten in a closed form. However, there are certain observations that are present in the nonlinear systems, that are not present in the linear systems.

1.1 Observations in Nonlinear Systems (vs Linear Systems)

1.1.1 Equillibrium

In general the eqillibrium of any system $\dot{x} = f(x)$ is defined as the point $x^{\star}$ where:

$$f(x)|_{x^{\star}} = 0 \Leftrightarrow \dot{x}|_{x^{\star}} = 0$$

Example (Linear Systems). Consider the case of the linear system,

$$\dot{x} = Ax$$

The eqillibrium of the above system is given by:

$$Ax = 0 \Rightarrow x \in \mathcal{N}(A)$$

where, $\mathcal{N}(A)$ is the null space of the matrix A.

No such existence of the eqillibrium is guaranteed in the case of the nonlinear systems. Thus, we can have a nonlinear system with no eqillibrium, or multiple eqillibria.

Example (Nonlinear Systems). Here are some examples of nonlinear systems with no, multiple or infiite equillibria.

1.
$$\dot{x} = Ax \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The above system has infiite equillibrium points, given by:

$$x^{\star} = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \forall \alpha \in \mathbb{R}$$

2.
$$\dot{x} = x^2 + a \quad a > 0$$

This system has no eqillibrium point.

3.
$$\dot{x} = x^2 - a \quad a > 0$$

This system has two eqillibrium points, given by $x^{\star} = \pm\sqrt{a}$.

1.1.2 Finite Escape Time

In general, the solution of the nonlinear system is not guaranteed to be bounded for all time, and can escape to infinity in a finite time. This is not possible in the case of the linear systems.

Example (Linear Systems). Consider the case of the linear system,

$$\dot{x} = x \Rightarrow x(t) = e^t x_0$$

The solution approaches infinity as $t \to \infty$, in an asymptotic manner, but never reaches infinity in a finite time.

But, in the case of nonlinear systems, the solution can escape to infinity in a finite time.

Example (Nonlinear Systems). Consider the case of the nonlinear system,

$$\dot{x} = 1 + x^2 \quad x \in \mathbb{R}$$

Integrating the system, we get:

$$\frac{dx}{1 + x^2} = dt$$

$$\Rightarrow \tan^{-1} x \Big|_{x_0}^{x(t)} = t \Big|_0^t$$

$$\Rightarrow x(t) = \tan(t + \tan^{-1}(x_0))$$

Let $x_0 = 0$, then we have:

$$x(t) = \tan(t)$$

The solution of the above system, approaches ∞ as $t \to \frac{\pi}{2}$. Since the solution goes unbounded in a finite time, we say that the system has a finite escape time.

1.2 Uniqueness of Solution

In general, the solution of the nonlinear system is not unique.

Example (Uniqueness of Solution). Consider the following system:

$$\dot{x} = \sqrt{2}, x(0) = 0, x \in \mathbb{R}$$

For this system,

$$x \equiv 0 \text{ is a solution}$$

But, we can also have the following solution:

$$x_\alpha = \begin{cases} \dfrac{(t - \alpha)^2}{4}, & t \geq \alpha, \alpha > 0; \\ 0, & t < \alpha; \end{cases}$$

for each $\alpha \in \mathbb{R}$, with the same initial conditions, we have infiite solutions

Thus, the solution of the nonlinear system is not unique. This is a big problem, as we cannot predict the behaviour of the system, which can lead to break down of the system.

2   Lecture 2 — Phase Portraits

Consider the system $\dot{x} = f(x)$ which has an equillibrium point $x^\star \in \mathbb{R}$ Thus,

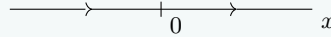$$\dot{x}|_{x=x^\star} \Leftrightarrow f(x^\star) = 0$$

Example (Linear System). Consider the case of a linear system given by $\dot{x} = ax$. Then, the equillibrium point is $x^\star = 0$. The solution to the system is given by $x(t) = e^{at}x_0$. The qualitative behaviour of the system depending upon the value of a is given by the following "1-D" phase portrait diagram.

$$x \qquad a < 0$$

$$x \qquad a > 0$$
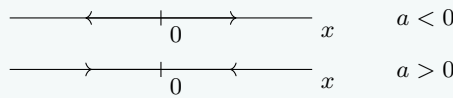
Where the arrows shows the evolution of the trajectory of the system. At $a = 0$, the the entire real line is the equillibria of the system.

Example (Nonlinear System). We will consider a bunch of non linear systems with varying qualitative behaviour.
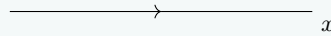
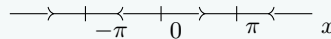- $\dot{x} = x^2 \quad x \in \mathbb{R}$

$$x$$

- $\dot{x} = x^2 - 1 \quad x \in \mathbb{R}$

$$x$$

- $\dot{x} = 1 \quad x \in \mathbb{R}$

$$x$$

- $\dot{x} = \sin(x) \quad x \in \mathbb{R}$

$$x$$
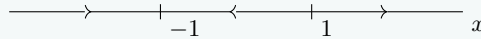
2.1   2D Phase Portraits

Let the two dimensional system be defined by

$$\dot{x}_1 = f_1(x_1, x_2) \quad \dot{x}_2 = f_2(x_1, x_2) \Rightarrow \dot{x} = f(x)$$

The phase portrait in some rough sense can be thought of as the "trajectory" of the system, and where we care about the slope of the vector field of $f(x)$ is given by:

$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{f_2(x)}{f_1(x)}$$

The phase portrait of the linear system can be easily understood via the jordan normal form of the system.

Consider the system given by, $\dot{x} = Ax$, and the linear transformation given by $y = Px$, Thus we get,

$$\dot{y} = P\dot{x} = PAx = PAP^{-1}y = Jy$$

where J is the jordan normal form of A.

We can have one of the 3 cases of the jordan form. The forms with their characteristic polynomials are given by:

$$J_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow p(s) = (s - \lambda_1)(s - \lambda_2)$$

$$J_2 = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \Rightarrow p(s) = (s - \lambda_1)^2$$

$$J_3 = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \Rightarrow p(s) = (s - \alpha)^2 + \beta^2$$

The solution of the equation in the transformed coordinates is given by:

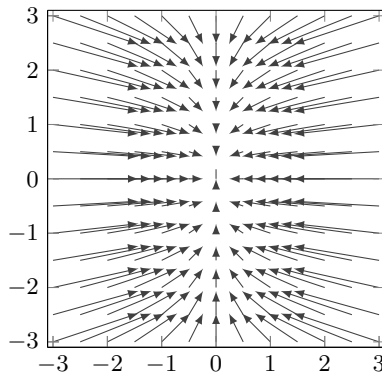$$y(t) = e^{Jt}y(0) \Rightarrow x(t) = P^{-1}e^{Jt}Px(0)$$

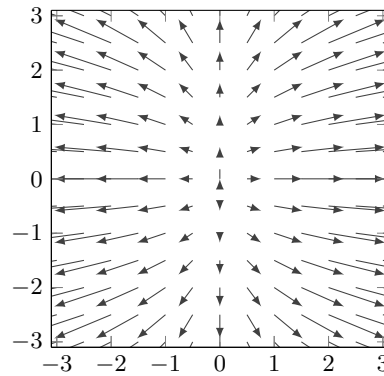Thus, the solution can be split into three cases similar to the one done above

Case 1:

$$y_1(t) = e^{\lambda_1 t}y_1(0) \quad y_2(t) = e^{\lambda_2 t}y_2(0)$$

Thus, depending on the value of λ_1 and λ_2, we can have the following phase portraits:

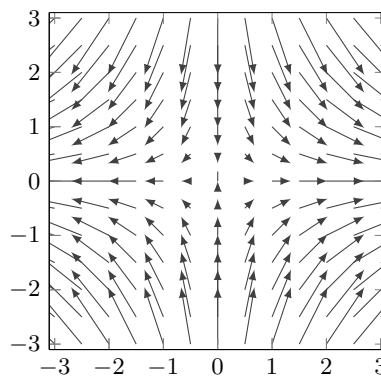$\lambda_1 < \lambda_2 < 0$: Stable Node $\quad\quad\quad\quad$ $\lambda_1 > \lambda_2 > 0$: Unstable Node
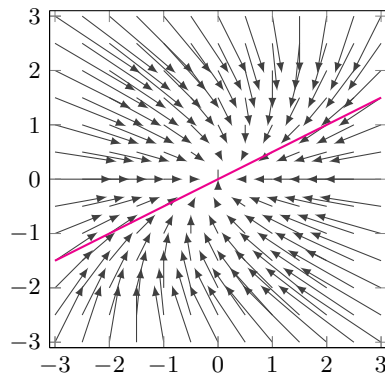

$\lambda_1 > 0; \lambda_2 < 0$: Saddle

Case 2:

$$y_1(t) = e^{\lambda t}\left(y_1(0) + ty_2(0)\right)$$
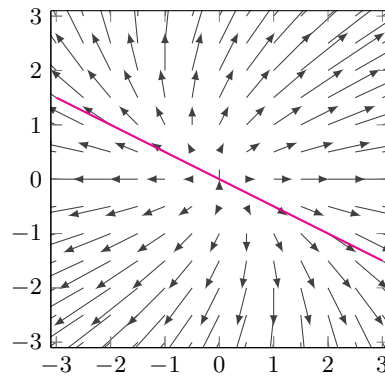$$y_2(t) = e^{\lambda t}y_2(0)$$

Simillary, we can have the following phase portraits depending on the value of λ:

$\lambda < 0$: Improper Stable Node

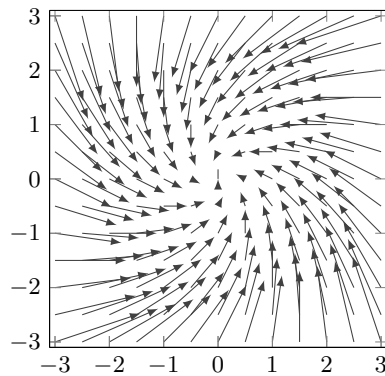$\lambda > 0$: Improper Unstable Node

Case 3: Let $r := \sqrt{y_1^2 + y_{2^2}}$ and $\theta := \tan^{-1}\left(\frac{y_2}{y_1}\right)$ Substituing the values of y_1 and y_2 in the equation, and simplifying, we get:

$$\dot{r} = \alpha r \quad \dot{\theta} = \beta$$

Thus, switching in the polar form, the transformed system are evolving independently of each other. The phase portrait is given by:

$\lambda > 0$: Stable Focus

$\lambda < 0$: Unstable Focus

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