

Nonlinear Systems and Control

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Disclaimer: This document will inevitably contain some mistakes— both simple typos and legitimate errors. Keep in mind that these are the notes of a graduate student in the process of learning the material, so take what you read with a grain of salt. If you find mistakes and feel like telling me, I will be grateful and happy to hear from you, even for the most trivial of errors. You can reach me by email at vaidyavarad2001@gmail.com.

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1 Lecture 1 — Introduction to Nonlinear Systems

In general, a nonlinear system can be defined with the following:

$$\begin{aligned} f : \mathbb{R}^n &\rightarrow \mathbb{R}^n \\ \frac{dx}{dt} &= \dot{x} = f(t, x, u, d) \\ y &= g(t, x, u, d) \end{aligned}$$

with the initial condition,

$$x(t_0) = x_0$$

where, $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the output, and $d \in \mathbb{R}^q$ is the disturbance.

We can also write the control signal u as a function of the state x and some parameter vector θ . Thus, using this “state feedback” we have:

$$\begin{aligned} u &= h(t, x, \theta) \\ \Rightarrow \dot{x} &= \bar{f}(t, x, \theta) \end{aligned}$$

This free choice of θ allows us to design the system to meet certain requirements, or to optimize some performance metric etc.

One of the special case of nonlinear systems is called as autonomous systems, where the system does not depend on time explicitly. Thus, we have:

$$\dot{x} = f(x)$$

It is possible to convert a non-autonomous system to an autonomous system by adding a new state variable z , such that:

$$\dot{z} = 1$$

and incorporating the new variable z , in the state vector, increasing the order of the system.

Note:-

Note that, the state z and hence the state x , is unbounded with time in this conversion.

In the case of the linear systems, we have the following:

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x + D(t)u \end{aligned}$$

with the solution of the above system being:

$$x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

with the state transition matrix $\Phi(t, \tau)$. The state transition matrix simplifies even further under the case of LTI systems as:

$$\Phi(t, \tau) = e^{A(t-\tau)}$$

In general the solution of the nonlinear system cannot be written in a closed form. However, there are certain observations that are present in the nonlinear systems, that are not present in the linear systems.

1.1 Observations in Nonlinear Systems (vs Linear Systems)

1.1.1 Equilibrium

In general the equilibrium of any system $\dot{x} = f(x)$ is defined as the point x^* where:

$$f(x)|_{x^*} = 0 \Leftrightarrow \dot{x}|_{x^*} = 0$$

Example (Linear Systems). Consider the case of the linear system,

$$\dot{x} = Ax$$

The equilibrium of the above system is given by:

$$Ax = 0 \Rightarrow x \in \mathcal{N}(A)$$

where, $\mathcal{N}(A)$ is the null space of the matrix A .

No such existence of the equilibrium is guaranteed in the case of the nonlinear systems. Thus, we can have a nonlinear system with no equilibrium, or multiple equilibria.

Example (Nonlinear Systems). Here are some examples of nonlinear systems with no, multiple or infinite equilibria.

1.

$$\dot{x} = Ax \quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The above system has infinite equilibrium points, given by:

$$x^* = \alpha \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \forall \alpha \in \mathbb{R}$$

2.

$$\dot{x} = x^2 + a \quad a > 0$$

This system has no equilibrium point.

3.

$$\dot{x} = x^2 - a \quad a > 0$$

This system has two equilibrium points, given by $x^* = \pm\sqrt{a}$.

1.1.2 Finite Escape Time

In general, the solution of the nonlinear system is not guaranteed to be bounded for all time, and can escape to infinity in a finite time. This is not possible in the case of the linear systems.

Example (Linear Systems). Consider the case of the linear system,

$$\dot{x} = x \Rightarrow x(t) = e^t x_0$$

The solution approaches infinity as $t \rightarrow \infty$, in an asymptotic manner, but never reaches infinity in a finite time.

But, in the case of nonlinear systems, the solution can escape to infinity in a finite time.

Example (Nonlinear Systems). Consider the case of the nonlinear system,

$$\dot{x} = 1 + x^2 \quad x \in \mathbb{R}$$

Integrating the system, we get:

$$\begin{aligned}\frac{dx}{1+x^2} &= dt \\ \Rightarrow \tan^{-1} x \Big|_{x_0}^{x(t)} &= t \Big|_0^t \\ \Rightarrow x(t) &= \tan(t + \tan^{-1}(x_0))\end{aligned}$$

Let $x_0 = 0$, then we have:

$$x(t) = \tan(t)$$

The solution of the above system, approaches ∞ as $t \rightarrow \frac{\pi}{2}$. Since the solution goes unbounded in a finite time, we say that the system has a finite escape time.

1.2 Uniqueness of Solution

In general, the solution of the nonlinear system is not unique.

Example (Uniqueness of Solution). Consider the following system:

$$\dot{x} = \sqrt{2}, x(0) = 0, x \in \mathbb{R}$$

For this system,

$$x \equiv 0 \text{ is a solution}$$

But, we can also have the following solution:

$$x_\alpha = \begin{cases} \frac{(t-\alpha)^2}{4}, & t \geq \alpha, \alpha > 0; \\ 0, & t < \alpha; \end{cases}$$

for each $\alpha \in \mathbb{R}$, with the same initial conditions, we have infinite solutions

Thus, the solution of the nonlinear system is not unique. This is a big problem, as we cannot predict the behaviour of the system, which can lead to break down of the system.