॥ श्री गणेशाय नमः ॥

Optimal Control

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Abstract

Contents

Lecture 20: Rocket Soft Landing and Convexification

> As previously seen. In previous lectures, we covered Stochastic Optimal Control (LQG) and State Estimation (Kalman Filter). Today, we shift gears to a specific, high-impact application of optimal control: **Rocket Soft Landing**. This problem motivated a major breakthrough in trajectory optimization known as "Lossless Convexification," which allows us to solve non-convex powered descent problems using convex solvers (SOCP) with global optimality guarantees.

1.1 The Rocket Soft-Landing Problem

The goal is to navigate a rocket from an initial state to a target landing site (usually at $z = 0$) with zero terminal velocity, while minimizing fuel consumption and respecting physical constraints.

- **Objective:** Minimize fuel consumption (maximize remaining mass) or landing error.
- **Constraints:** Thrust limits (min/max), glide slope (pointing angle), safety zones.
- **Real-world Examples:**
  - NASA Curiosity "Sky Crane" (Mars, 2012).
  - SpaceX Falcon 9 Boostback and Landing.
  - NASA Perseverance with Terrain Relative Navigation (TRN) (Mars, 2021).
  - SpaceX Starship.

1.2 The "Full Stack" Control Architecture

Landing a rocket requires a hierarchical control approach to handle different timescales and physics.

1. **State Estimation:**
   - **Earth (SpaceX):** GPS + IMU + Altimeter. Provides accurate position ($\sim$ meters), velocity, and attitude.
   - **Mars (NASA):** No GPS. Relies on IMU integration, Radar Altimeter, and Vision (Terrain Relative Navigation) to avoid boulders and achieve $\sim$ 30m accuracy.
2. **High-Level Position Controller (Guidance):**
   - Solves the trajectory optimization problem.
   - Reasons about safety, fuel, and thrust limits.
   - Treats the rocket as a **point mass**.
   - Runs at low frequency ($\sim$ 1-10 Hz).
   - Outputs: Desired acceleration vector (which implies thrust magnitude and pointing direction).
3. **Low-Level Attitude Controller:**
   - Tracks the desired pointing direction commanded by the guidance layer.
   - Deals with rigid body dynamics, aerodynamics, and disturbances.
   - Handles **fluid slosh** (often modeled as a pendulum) and **flexible modes** (bending of the rocket body, handled via notch filters).
   - Runs at high frequency ($\sim$ 50-100 Hz).

1.3 Rocket Dynamics

For the guidance problem, we typically use a 3-DOF point mass model with variable mass.

$$\dot{\mathbf{r}}(t) = \mathbf{v}(t) \tag{1.1}$$
$$\dot{\mathbf{v}}(t) = \mathbf{g} + \frac{\mathbf{T}(t)}{m(t)} \tag{1.2}$$
$$\dot{m}(t) = -\alpha\|\mathbf{T}(t)\| \tag{1.3}$$

where:

- $\mathbf{r}, \mathbf{v} \in \mathbb{R}^3$ are position and velocity.
- $\mathbf{g} = [0, 0, -g]^\top$ is gravity.
- $\mathbf{T}$ is the thrust vector in the world frame.
- α is the mass depletion rate coefficient (related to specific impulse I_{sp}).

> **Note.** Fuel mass is significant (can be 80% of initial mass), so $\dot{m}$ cannot be ignored. Aerodynamic forces are often ignored in the final landing burn phase (Mars atmosphere is thin, or speeds are low).

1.4 Convex Relaxation

The dynamics above are non-linear due to the $\frac{\mathbf{T}}{m}$ term, and the thrust constraints can be non-convex. To solve this efficiently (online), we need to cast it as a convex problem.

> **Intuition (Convex Relaxation).** Consider an optimization problem constrained to the boundary of a set, $S_1 = \{\mathbf{x} \mid \|\mathbf{x}\| = 1\}$. This is non-convex (a spherical shell). We can **relax** the constraint to the convex set $S_2 = \{\mathbf{x} \mid \|\mathbf{x}\| \leq 1\}$ (the solid ball).
>
> $$\min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \quad \text{s.t.} \quad \|\mathbf{x}\| = 1 \Rightarrow \min_{\mathbf{x}} \mathbf{c}^\top \mathbf{x} \quad \text{s.t.} \quad \|\mathbf{x}\| \leq 1$$
>
> If the cost function "pushes" the solution to the boundary (e.g., minimizing $\mathbf{c}^\top\mathbf{x}$ pushes $\mathbf{x}$ as far as possible in the direction $-\mathbf{c}$), the optimal solution to the relaxed problem will satisfy $\|\mathbf{x}^*\| = 1$. When this happens, we call the relaxation **tight** or **lossless**. We solved the easier convex problem but got the solution to the hard non-convex one.

1.5 Convexification of Thrust Constraints

We need to constrain the thrust vector $\mathbf{T}$.

1.5.1 1. Maximum Thrust (Convex)

We have an upper limit on engine power:

$$\|\mathbf{T}\| \leq T_{\max}$$

This describes a solid sphere (or ball), which is a **convex** set (specifically, a Second-Order Cone constraint).

1.5.2 2. Glide Slope / Pointing Angle (Convex)

The rocket engine cannot point too far away from the vertical (to avoid tipping or sensor occlusion). Let $\hat{\mathbf{n}}$ be the vertical axis. We require the angle θ between $\mathbf{T}$ and $\hat{\mathbf{n}}$ to be small:

$$\theta \leq \theta_{\max} \Rightarrow \cos\theta \geq \cos\theta_{\max}$$

Using the dot product $\mathbf{T} \cdot \hat{\mathbf{n}} = \|\mathbf{T}\| \cos\theta$:

$$\hat{\mathbf{n}}^\top \mathbf{T} \geq \|\mathbf{T}\| \cos\theta_{\max}$$

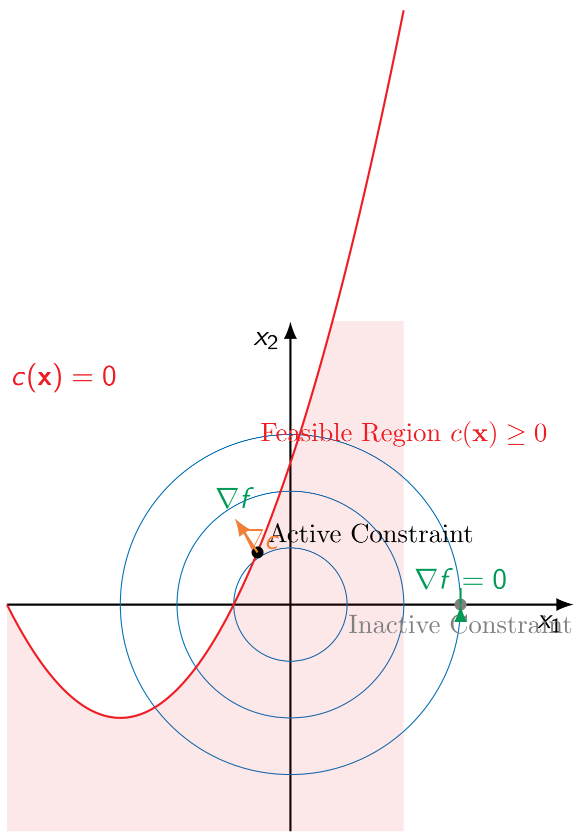
This is a **Second-Order Cone (SOC)** constraint, which is convex.

1.5.3 3. Minimum Thrust (Non-Convex)

Rocket engines cannot throttle down to zero continuously; they have a minimum stable thrust level (or they flame out).

$$\|\mathbf{T}\| \geq T_{\min}$$

This removes a small ball from the center of the feasible thrust set. The resulting set (a hollow shell between $T_{\min}$ and $T_{\max}$) is **non-convex**.



This non-convexity prevents the direct use of convex solvers like SOCP.

1.6 Lossless Convexification

Acikmese et al. (2007) introduced a "slack variable" trick to convexify the minimum thrust constraint.

Step 1: Change of Variables. Introduce a new scalar decision variable $\Gamma(t)$ representing the thrust magnitude. We impose constraints on $\mathbf{T}$ and Γ:

1. $\|\mathbf{T}\| = \Gamma$ (Non-convex equality constraint: $\mathbf{T}$ must lie *on* the cone defined by Γ).
2. $T_{\min} \leq \Gamma \leq T_{\max}$ (Linear/Convex box constraint).
3. $\hat{\mathbf{n}}^\top \mathbf{T} \geq \Gamma \cos(\theta_{\max})$ (Linear/Convex cone constraint in $\mathbf{T}, \Gamma$).

Step 2: Relaxation. We relax the non-convex equality (1) to an inequality:

$$\|\mathbf{T}\| \leq \Gamma$$

This is a convex Second-Order Cone constraint.

Step 3: The Result. The final optimization problem is:

$$\begin{aligned}
\min \quad & \int_0^{t_f} \Gamma(t)\,dt \quad \text{(Minimize fuel/thrust magnitude)} \\
\text{s.t.} \quad & \text{Dynamics (linearized or discretized)} \\
& \|\mathbf{T}(t)\| \leq \Gamma(t) \\
& T_{\min} \leq \Gamma(t) \leq T_{\max} \\
& \hat{\mathbf{n}}^\top \mathbf{T}(t) \geq \Gamma(t) \cos(\theta_{\max})
\end{aligned}$$

Theorem (Lossless Convexification): For the minimum fuel problem, the optimal solution to the relaxed problem **always** satisfies $\|\mathbf{T}^*(t)\| = \Gamma^*(t)$. Therefore, the relaxation is tight, and we can solve the non-convex rocket landing problem using efficient convex solvers (like ECOS or OSQP) in real-time.

> Note. This works because minimizing $\int \Gamma\,dt$ encourages Γ to be as small as possible. Since Γ is bounded from below by $\|\mathbf{T}\|$, the optimizer squeezes Γ down until it hits the boundary $\|\mathbf{T}\|$.