Title

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Lecture notes from the Coursera specialization *Spacecraft Dynamics and Control*

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Part I

Dynamics and Control

1 Kinematics: Describing the Motions of Spacecraft

1.1 Introduction to Kinematics

1.1.1 Particle Kinematics and Vector Frames

Something with direction and magnitude.

$$\begin{aligned}\boldsymbol{r} &= x\hat{\boldsymbol{e}}_{\boldsymbol{1}} + y\hat{\boldsymbol{e}}_{\boldsymbol{2}} + z\hat{\boldsymbol{e}}_{\boldsymbol{3}} \\ &= r\hat{\boldsymbol{e}}_{\boldsymbol{r}} \\ &= {}^{\mathcal{E}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}\end{aligned}$$

Example. Let the frame $\mathcal{E}$ be defined as: $\mathcal{E}\colon \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$, with the vector, $\boldsymbol{q} = 3\hat{\boldsymbol{e}}_{\boldsymbol{1}} + 2\hat{\boldsymbol{e}}_{\boldsymbol{3}}$.
Thus, the vector $\boldsymbol{q}$ can be written in matrix form as:

$$\boldsymbol{q} = {}^{\mathcal{E}}\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

Coordinate Frames

Let a coordinate frame $\mathcal{B}$ be defined through 3 unit orthogonal vectors, $\{\hat{\boldsymbol{b}}_{\boldsymbol{1}}, \hat{\boldsymbol{b}}_{\boldsymbol{2}}, \hat{\boldsymbol{b}}_{\boldsymbol{3}}\}$. Let the origin of the frame be given by $\mathcal{O}_{\mathcal{B}}$. Then the frame is defined as:

$$\mathcal{B}\colon \{\mathcal{O}_{\mathcal{B}}, \hat{\boldsymbol{b}}_{\boldsymbol{1}}, \hat{\boldsymbol{b}}_{\boldsymbol{2}}, \hat{\boldsymbol{b}}_{\boldsymbol{3}}\}$$

Often the frame origin is ignored, and the shorthand notation is used to define the frame as:

$$\mathcal{B}\colon \{\hat{\boldsymbol{b}}_{\boldsymbol{1}}, \hat{\boldsymbol{b}}_{\boldsymbol{2}}, \hat{\boldsymbol{b}}_{\boldsymbol{3}}\}$$

Angular Velocity

The angular velocity vector of a rigid body is often defined as:

$$\boldsymbol{\omega} = \omega_1\hat{\boldsymbol{b}}_{\boldsymbol{1}} + \omega_2\hat{\boldsymbol{b}}_{\boldsymbol{2}} + \omega_3\hat{\boldsymbol{b}}_{\boldsymbol{3}}$$

$${}^{\mathcal{B}}\omega = {}^{\mathcal{B}}\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

where, $\omega_1, \omega_2, \omega_3$ are the instantaneous angular velocities about the $\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3$ axes respectively. The angular velocity frame components is shown in Figure 1.

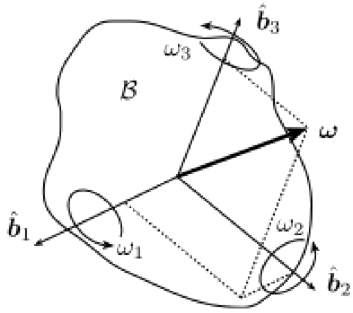

Figure 1: Angular velocity body frame components

Rotation about a Fixed Axis

Let a body $\mathcal{B}$ rotate about the axis AB as shown in Figure 2. Let the origin $\mathcal{O}$ of the coordinate system $\mathcal{B}$ located on the axis of rotation. Let P be a body fixed point located relative to O by the vector $\boldsymbol{r}$. Thus, if the vector $\boldsymbol{r}$ is defined in the $\mathcal{B}$ frame, then the following can be said, from the geometery of the problem:

$$
\begin{aligned}
|\dot{\boldsymbol{r}}| &= (r \sin \theta)\omega \\
\dot{\boldsymbol{r}} &= \boldsymbol{\omega} \times \boldsymbol{r}
\end{aligned}
$$

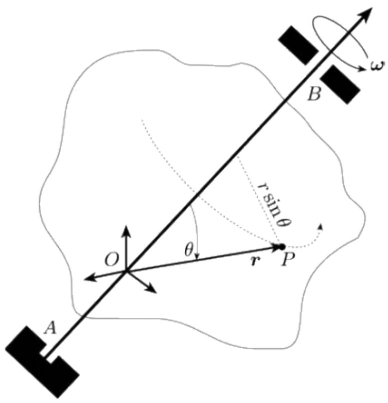

Figure 2: Rotation about a fixed axis

1.1.2 Vector Differentiation and Transport Theorem

Let $\mathcal{N}$ be an inertially fixed frame denoted by the traid, $\{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}$. Let $\mathcal{B}$ be a another frame denoted by the traid, $\{\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{b}}_3\}$. For simplicity, the let the origin of the two frames coincide.

Let $\boldsymbol{r}$ be a vector in $\mathcal{B}$ frame,

$$\boldsymbol{r} = r_1\boldsymbol{\hat{b}_1} + r_2\boldsymbol{\hat{b}_2} + r_3\boldsymbol{\hat{b}_3}$$

Let the angular velocity vector $\omega_{\mathcal{B}/\mathcal{N}}$ define the angular velocity of the $\mathcal{B}$ frame with respect to the $\mathcal{N}$ frame. The angular velocity vector is defined usually in the $\mathcal{B}$ frame as,

$$\omega_{\mathcal{B}/\mathcal{N}} = \omega_1\boldsymbol{\hat{b}_1} + \omega_2\boldsymbol{\hat{b}_2} + \omega_3\boldsymbol{\hat{b}_3}$$

To denote that a derivative of a vector $\boldsymbol{r}$ as seen by the $\mathcal{B}$ frame, we use the notation,

$$\frac{{}^{\mathcal{B}}d\boldsymbol{x}}{dt}$$

The derivative of the vector $\boldsymbol{r}$ as seen by the $\mathcal{B}$ frame is given by,

$$\begin{aligned}\frac{{}^{\mathcal{B}}d\boldsymbol{r}}{dt}dt &= \frac{{}^{\mathcal{B}}d}{dt}\left(r_1\boldsymbol{\hat{b}_1} + r_2\boldsymbol{\hat{b}_2} + r_3\boldsymbol{\hat{b}_3}\right)\\ &= \dot{r}_1\boldsymbol{\hat{b}_1} + \dot{r}_2\boldsymbol{\hat{b}_2} + \dot{r}_3\boldsymbol{\hat{b}_3}\end{aligned}$$

Since the derivative of the unit vectors $\boldsymbol{\hat{b}_i}$ wrt the frame $\mathcal{B}$ is 0.

Using chain rule of differentiation, we can write,

$$\frac{{}^{\mathcal{N}}d}{dt}$$

And,

$$\frac{{}^{\mathcal{N}}d}{dt}\boldsymbol{\hat{b}_i} = \boldsymbol{\omega_{\mathcal{B}/\mathcal{N}}} \times \boldsymbol{\hat{b}_i}$$

Thus, combining the above two equations, we get,

$$\frac{{}^{\mathcal{N}}d}{dt}\boldsymbol{r} = \frac{{}^{\mathcal{B}}d\boldsymbol{r}}{dt} + \boldsymbol{\omega_{\mathcal{B}/\mathcal{N}}} \times \boldsymbol{r}$$

Theorem 1.1 (Transport Theorem)**.** Let $\mathcal{N}$ and $\mathcal{B}$ be two frames with a relative angular velocity $\boldsymbol{\omega_{\mathcal{B}/\mathcal{N}}}$. Let $\boldsymbol{r}$ be a generic vector. Then the derivative of the vector $\boldsymbol{r}$ as seen by the $\mathcal{N}$ frame can be related to the derivative of the vector $\boldsymbol{r}$ as seen by the $\mathcal{B}$ frame as,

$$\frac{{}^{\mathcal{N}}d}{dt}\boldsymbol{r} = \frac{{}^{\mathcal{B}}d\boldsymbol{r}}{dt} + \boldsymbol{\omega_{\mathcal{B}/\mathcal{N}}} \times \boldsymbol{r}$$

Note:-

When writing compact notations, the following will hold,

$$\frac{{}^{\mathcal{N}}d}{dt}\boldsymbol{x} \equiv \dot{\boldsymbol{x}}$$

Example. Find the inertial velocity of the point S shown in Figure 3

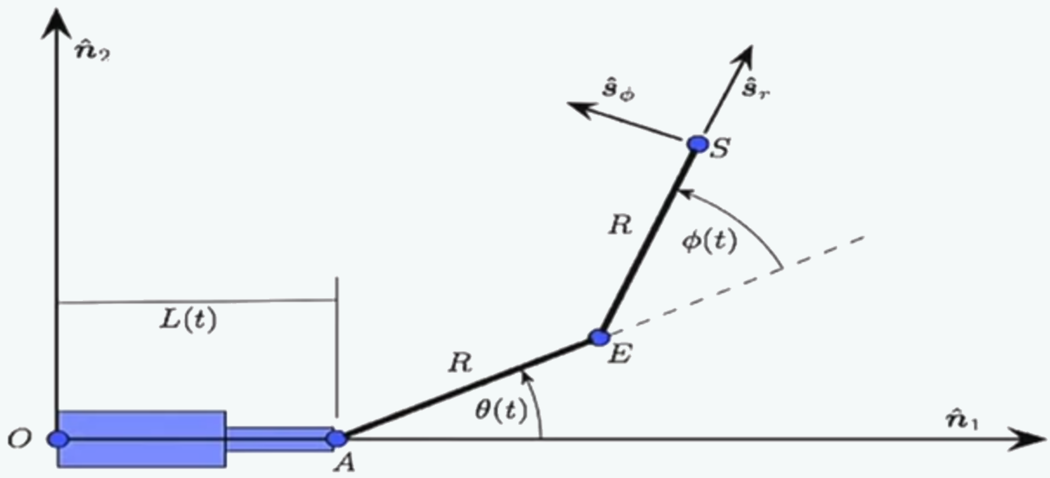

Figure 3: Figure for

Assigning the frames we have,

$$\begin{aligned}\mathcal{N}&: \{\hat{\boldsymbol{n}}_1, \hat{\boldsymbol{n}}_2, \hat{\boldsymbol{n}}_3\}\\ \mathcal{E}&: \{\hat{\boldsymbol{e}}_r, \hat{\boldsymbol{e}}_\theta, \hat{\boldsymbol{n}}_3\}\\ \mathcal{S}&: \{\hat{\boldsymbol{s}}_r, \hat{\boldsymbol{s}}_\phi, \hat{\boldsymbol{n}}_3\}\end{aligned}$$

And the angular velocities of the frames relative to each other can be expressed as:

$$\begin{aligned}\boldsymbol{\omega}_{\mathcal{E}/\mathcal{N}} &= \dot{\theta}\hat{\boldsymbol{n}}_3\\ \boldsymbol{\omega}_{\mathcal{S}/\mathcal{E}} &= \dot{\phi}\hat{\boldsymbol{n}}_3\\ \boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}} &= \boldsymbol{\omega}_{\mathcal{E}/\mathcal{N}} + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{E}}\\ &= (\dot{\theta} + \dot{\phi})\hat{\boldsymbol{n}}_3\end{aligned}$$

Thus,

$$\begin{aligned}\dot{\boldsymbol{r}} &= L\hat{\boldsymbol{n}}_1 + R\hat{\boldsymbol{e}}_r + R\hat{\boldsymbol{s}}_r\\ &= {}^{\mathcal{N}}\frac{d}{dt}\boldsymbol{r} = {}^{\mathcal{N}}\frac{d}{dt}L\hat{\boldsymbol{n}}_1 + {}^{\mathcal{E}}\frac{d}{dt}R\hat{\boldsymbol{e}}_r + \boldsymbol{\omega}_{\mathcal{E}/\mathcal{N}}(R\hat{\boldsymbol{e}}_r) + {}^{\mathcal{S}}\frac{d}{dt}R\hat{\boldsymbol{s}}_r + \boldsymbol{\omega}_{\mathcal{S}/\mathcal{N}}(R\hat{\boldsymbol{s}}_r)\\ &= \dot{L}\hat{\boldsymbol{n}}_1 + (\dot{\theta}\hat{\boldsymbol{n}}_3) \times R\hat{\boldsymbol{e}}_r + \left(\dot{\theta} + \dot{\phi}\right)\hat{\boldsymbol{n}}_3 \times R\hat{\boldsymbol{e}}_r\\ &= \dot{L}\hat{\boldsymbol{n}}_1 + R\dot{\theta}\hat{\boldsymbol{e}}_\theta + R\left(\dot{\theta} + \dot{\phi}\right)\hat{\boldsymbol{s}}_\phi\end{aligned}$$

Example. Find the inertial velocity of the point P shown in Figure 4

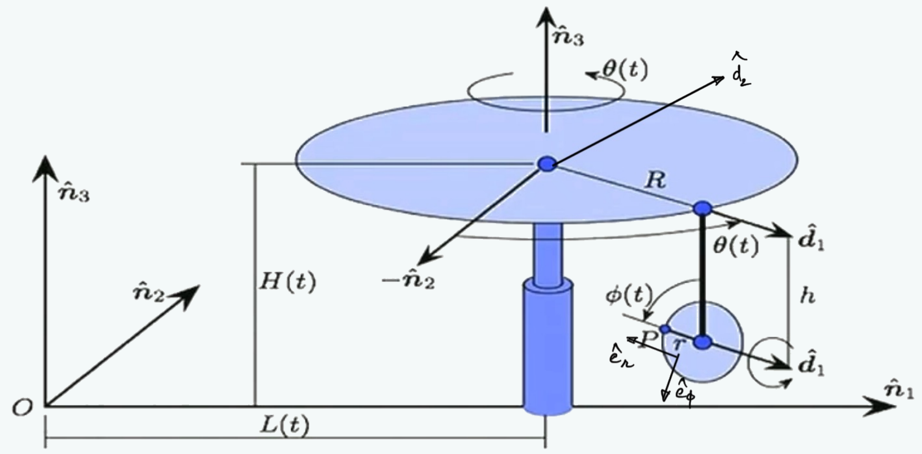

Figure 4: Figure for

Assigning the frames we have,

$$\mathcal{N}\colon \{\boldsymbol{\hat{n}_1}, \boldsymbol{\hat{n}_2}, \boldsymbol{\hat{n}_3}\}$$
$$\mathcal{D}\colon \{\boldsymbol{\hat{d}_1}, \boldsymbol{\hat{d}_2}, \boldsymbol{\hat{n}_3}\}$$
$$\mathcal{E}\colon \{\boldsymbol{\hat{e}_r}, \boldsymbol{\hat{e}_\phi}, \boldsymbol{\hat{d}_1}\}$$

And the angular velocities of the frames relative to each other can be expressed as:

$$\begin{aligned}
\boldsymbol{\omega_{\mathcal{D}/\mathcal{N}}} &= \dot{\theta}\boldsymbol{\hat{n}_3} \\
\boldsymbol{\omega_{\mathcal{E}/\mathcal{D}}} &= \dot{\phi}\boldsymbol{\hat{d}_1} \\
\boldsymbol{\omega_{\mathcal{E}/\mathcal{N}}} &= \boldsymbol{\omega_{\mathcal{D}/\mathcal{N}}} + \boldsymbol{\omega_{\mathcal{E}/\mathcal{D}}} \\
&= \dot{\theta}\boldsymbol{\hat{n}_3} + \dot{\phi}\boldsymbol{\hat{d}_1}
\end{aligned}$$

Thus,

$$\begin{aligned}
\boldsymbol{r} &= L\boldsymbol{\hat{n}_1} + H\boldsymbol{\hat{n}_3} + R\boldsymbol{\hat{d}_1} - h\boldsymbol{\hat{n}_3} + r\boldsymbol{\hat{e}_r} \\
\dot{r} &= \frac{{}^{\mathcal{N}}d}{dt}\boldsymbol{r} \\
&= \dot{L}\boldsymbol{\hat{n}_1} + \dot{H}\boldsymbol{\hat{n}_3} + \left[\cancelto{0}{\frac{{}^{\mathcal{D}}d}{dt}R\boldsymbol{\hat{d}_1}} + \omega_{\mathcal{D}/\mathcal{N}} \times R\boldsymbol{\hat{d}_1}\right]
\end{aligned}$$

1.2 Rigid Body Kineamtics

Kinematics means to study the description of motion of the body.

Attitude coordinates are set of coordinates hat describe attitude of body and/or a reference frame.

- ∞ number of attitude coordinates choice exists.
- good choice $\rightarrow$ simplifies the maths of the problem.
- bad choice $\rightarrow$ introduces singularities and adds non-linear maths to the problem

A minimum of 3 coordinates are required to describe the orientation of a rigid body in space, or describe the relative angular displacement between two frames. But, any set of three coordinates will contain atleast one geometrical orientation where the coordinates are singular i.e. atleast two coordinates are undefined or not unique. At or near such a geometric singularity, the corresponding kinematic differential equations are also singular. Such geometric singularities and associated numerical difficulties can be avoided altogether through regularization. Redundant sets of four ot more coordinates exist that are universally valid.

1.2.1 Direction Cosine Matrix

Notes