Complex Variables

Analytic Function

I (1) If $f(z) = u + iv$ is analytic with constant modulus, show that $f(z)$ is constant.

(2) If $f(z)$ is analytic with constant argument, show that $f(z)$ is constant.

(3) If $u+iv$ and $u-iv$ are analytic, show that both are constants.

(4) If $u+iv$ and $v+iu$ are analytic, show that both are constants.

II If $f(z)$ is analytic, then show that

(1) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^n = n^2 |f(z)|^{n-2} |f'(z)|^2$

Hence deduce the result for $n=2$

(2) $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^n) = n(n-1)\, u^{n-2} |f'(z)|^2$

Hence deduce the result for $n=3$

(3) $\nabla^2 \left[\log |f'(z)|\right] = 0 \qquad \left[\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$

(4) $\left[\frac{\partial}{\partial x} |f(z)|\right]^2 + \left[\frac{\partial}{\partial y} |f(z)|\right]^2 = |f'(z)|^2 \qquad$ Hint: $\left[\text{Use: } |f(z)| = \sqrt{u^2+v^2}\right]$

III (1) Find the values of a, b, c, d so that the funⁿ.

$$f(z) = (x^2 + axy + by^2) + i\,(cx^2 + dxy + y^2) \text{ is analytic}$$

(2) Find 'p' such that

$$f(z) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(\frac{px}{y}\right) \text{ is analytic}$$

(3) Find a, b so that

$$f(z) = \cos x\,[\cosh(y) + a\sinh(y)] + i\sin x\,[\cosh(y) + b\sinh(y)]$$ is analytic

(4) Find 'b' such that $u = e^{bx}\cos(5y)$ is harmonic

Ans. (1) $a=2=d$; $b=-1=c$ (2) $p=-1$ (3) $a=b=-1$, (4) $b=\pm 5$ P.T.O.

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⑤ Find a, b, c, d, e if

$$f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$$

Ans : {2, −6, −3, 6, 6}

is analytic

⑥ Find a, b, c, d, e if

$$f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$$

Ans: {1, −6, 1, 2, 4}

is analytic

⑦ Find a & b such that

$$f(z) = (-r^2\sin(a\theta) + r\sin\theta) + i(r^2\cos(2\theta) - r\cos(b\theta) + 2)$$

{a = 2, b = 1}

is analytic

IV ⓐ S.T. the following funⁿˢ $u(x,y)$ / $v(x,y)$ are harmonic [$u(x,y)$ can be a real part of an ana. funⁿ.]

ⓑ Construct the corresponding analytic funⁿ

ⓒ Determine the harmonic conjugate & hence the analytic funⁿ. $f(z)$

ⓓ Obtain the orthogonal trajectories of the families of curves $u(x,y) = \text{const}$.

ⓔ Test whether it can be the R.P or I.P of an analytic funⁿ

Type II Method Type III

① $u = x^3y - xy^3$

② $u = x^2 - y^2 - y$

③ $v = x^2 - y^2 + \dfrac{x}{x^2+y^2}$ → (Please omit)

④ $u = 3xy^2 - x^3$

⑤ $u = e^{-x}(x\sin y - y\cos y)$

⑥ $u = \dfrac{x^2-y^2}{(x^2+y^2)^2}$

⑦ $u = \frac{1}{2}\log(x^2+y^2)$

Ans:-

① $v = \dfrac{3x^2y^2}{2} - \dfrac{y^4}{4} - \dfrac{x^4}{4} + c'$
$f(z) = -i\dfrac{z^4}{4} + c$

② $v = 2xy + x + c'$
$f(z) = z^2 + iz + c$

③ $u = -2xy + \dfrac{y}{x^2+y^2} + c'$
$f(z) = i\left[z^2 + \frac{1}{z}\right] + c$

④ $v = y^3 - 3x^2y + c'$
$f(z) = -z^3 + c$

⑤ $v = e^{-x}(y\sin y + x\cos y) + c'$
$f(z) = ize^{-z} + c$

⑥ $v = \dfrac{-2xy}{(x^2+y^2)^2} + c'$
$f(z) = \dfrac{1}{z^2} + c$

⑦ $v = \tan^{-1}(y/x) + c'$
$f(z) = \log z + c$

(8) $u = e^{-2xy}\sin(x^2-y^2)$

(9) $u = \dfrac{\sin(2x)}{\cosh(2y)-\cos(2x)}$

(1)

(10) If $u = \dfrac{x}{2}\log(x^2+y^2) - y\tan^{-1}\left(\dfrac{y}{x}\right) + \sin x\cosh(y)$

find the ana. funⁿ. $f(z)$

(11) (a) $v = e^{-x}(x\sin y + y\cos y)$

(b) $v = e^{-x}(y\sin y + x\cos y)$ → (2)

(12) $v = \sinh(x)\cos y$

(13) $v = \cos x\cosh(y)$

(14) (a) If $v = \dfrac{x}{x^2+y^2} + \cosh(x)\cos y$, (1)

find the ana. funⁿ $f(z)$

(b) $v = (x^4-6x^2y^2+y^4)+(x^2-y^2)+2xy$

(15) If $v = e^{-x}\left[2xy\cos y + (y^2-x^2)\sin y\right]$

find the ana. funⁿ. $f(z)$

Ans :

(03)

(8) $v = -e^{-2xy}\cos(x^2-y^2)+c$

$f(z) = -i\,e^{iz^2}+c$

(9) $v = \dfrac{-\sinh(2y)}{\cosh(2y)-\cos(2x)}$

$f(z) = \cot z + c$

(10) $f(z) = z\log z + \sin z + c$

$-(z+2)e^{-z}+c$ Verify

(11) (a) $f(z) = ze^{-z}+c$

(b) $f(z) = ize^{-z}+c$

(12) $f(z) = i\sinh(z)+c$

(13) $f(z) = i\cos z + c$

(14) (a) $f(z) = i\left[\dfrac{1}{z} + \cosh(z)\right]+c$

(b) $f(z) = z^2 + i(z^4+z^2)+c$

(15) $f(z) = e^{-z}z^2+c$

(V) Verify whether the following funⁿˢ. can be the real part / imag. part of an analytic funⁿ. If so, find the corresponding ana. funⁿ.

(1) $u = e^{x}\cos y + x^3 - 3xy^2$

(2) $u = x + e^{xy} + y + e^{-xy}$

(3) $u = e^{x}\cos y$

(4) $u = 3x^2y + 2x^2 - y^3 - 2y^2$

(5) $u = x^2 - y^2 - 2xy - 2x + 3y$

(3)

(6) $u = (x-1)^3 - 3xy^2 + 3y^2$, find v & hence $f(z)$

(7) $u = 3x^3y + 2x^2 - y^3 - 2y^2$

(1) Yes

(2) No

(3) Yes ; $f(z) = e^{z}+c$

(4) Yes

$f(z) = 2z^2 - iz^3 + c$

(5) Yes

$f(z) = (z^2 - 2z) + i(z^2 - 3z) + c$

(6) Yes; $f(z) = (z-1)^3 + c$

$v = 3x^2y - 6xy + 3y - y^3 + c$

(7) No

(P.T.O)

(8) $u(r,\theta) = r^2\cos(2\theta) - r\sin\theta$

(9) $u(r,\theta) = r^n\cos(n\theta)$

(10) $u(x,y) = e^{x^2-y^2}\cos(2xy)$

(11) $v(x,y) = \log\left[(x-1)^2+(y-2)^2\right]$
[except at $(1,2)$]

(12) $v = e^{2x}\left[x\cos(2y) - y\sin(2y)\right]$

(13) $u = 3x^2 + \sin x + y^2 + 5y + 4$; (14) $u = 3x^2 - 2x^2y + y^2$

(VI) Find the ana. funⁿ $f(z) = u + iv$ given

(1) $u - v = (x-y)(x^2+4xy+y^2)$

(2) $u - v = x^3 + x^2 - 3xy^2 - y^2 - 3x^2y + y^3 - 2xy$

(3) (a) $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - e^{y} - e^{-y}}$ subject to $f(\frac{\pi}{2}) = 0$

(b) $u - v = \dfrac{e^{y} - \cos x + \sin x}{\cosh(y) - \cos x}$; $f(\frac{\pi}{2}) = \frac{3-i}{2}$

(4) $u + v = \dfrac{x}{x^2+y^2}$

(5) $u + v = e^x(\cos y + \sin y) + \dfrac{x-y}{x^2+y^2}$

(6) (a) $u + v = \dfrac{2x}{x^2+y^2}$; $f(1) = i$ (b) $u+v = \dfrac{2\sin(2x)}{e^{2y}+e^{-2y}-2\cos(2x)}$

(7) $u - v = \dfrac{\sin(2x)}{-\cos(2x)+\cosh(2y)}$ u+v

(8) $v = \dfrac{\sin x \sinh(y)}{\cos(2x)+\cosh(2y)}$; $f(0) = 1$

(9) $v = e^{-2y}\left[y\cos(2x) + x\sin(2x)\right]$ find $f(z)$ first & hence find 'u'

(10) $3u + 2v = y^2 - x^2 + 16xy$
[Hint: Diff. p.w.r.t. x ; p.w.r.t. y , solve for u_x, v_x]

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(8) Yes, $f(z) = z^2 + z + c$

(9) Yes; $f(z) = z^n + c$

(10) Yes, $f(z) = e^{z^2} + c$

(11) Yes, $f(z) = 2i\log(z-1-2i) + c$

(12) Yes; $f(z) = ize^{2z} + c$

(13) No (14) No

(1) $f(z) = -iz^3 + c$

(2) $f(z) = z^3 + z^2 + c$

(3) (a) $f(z) = -\frac{1}{2}\cot(\frac{z}{2}) + c$
$c = 1/2$

(b) $f(z) = \cot(\frac{z}{2})$; $c = \frac{1-i}{2}$

(4) $f(z) = \left(\dfrac{i}{1+i}\right)\left(\dfrac{1}{z}\right) + c$

(5) $f(z) = e^z + \frac{1}{z} + c$

(6) (a) $f(z) = \dfrac{1+i}{z} - 1$

(b) $f(z) = \dfrac{i}{1+i}\cot z + c$

(7) $f(z) = \dfrac{i\cot z}{1+i} + c$
verify

(8) $f(z) = \frac{1}{2}(1+\sec z) + c$
$c = 0$
or
$\frac{1}{2}\sec z + c'$
$c' = \frac{1}{2}$

(9) $f(z) = ze^{i2z} + c$
$u = e^{-2y}\left[x\cos(2x) - y\sin(2x)\right]$

(10) $f'(z) = 2z - 4iz$
$\Rightarrow f(z) = (1-2i)z^2 + c$

P.T.O.

VII Verify whether the foll.. funⁿ.. are analytic. If so find $f'(z)$ in two ways. [Hint: Verify C.R. equⁿ.. & use: $f'(z) = u_x + i v_x$] & verify using direct differentiation

① $f(z) = \cosh(z)$ [$= \cos(iz) = \cos[ix - y] =$]	Yes; $f'(z) = \sinh(z)$
② $\cos(z)$	Yes, $f'(z) = -\sin z$
③ $e^x [\cos y - i \sin y]$	No
④ $e^{-x} [\cos y - i \sin y]$	Yes; $f'(z) = -e^{-\bar{z}}$
⑤ $x^2 - y^2 + 2ixy$	Yes; $f'(z) = 2z$
⑥ $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$	Yes, $f'(z) = 3z^2 + 3$
⑦ ze^{2z}	Yes; $f'(z) = 2ze^{2z} + e^{2z} = e^{2z}(2z+1)$
⑧ e^{2z}	Yes; $f'(z) = 2e^{2z}$

VIII. Find the orthogonal trajectories of

① $3xy^2 + 2x^2 - x^3 - 2y^2 = \text{const}$	① $4xy - 3x^2y + y^3 = c'$
② $e^x \cos y - xy = c$	② $e^x \sin y + \frac{1}{2}(x^2 - y^2) = c'$
③ $x^2 - y^2 - 2xy + 2x - 3y = c$	$3x + 2xy + 2y + x^2 - y^2 = c$

IX Find the values of z for which the foll. funⁿ ceases to be analytic [Hint: Find 'z' for which $f'(z) \to \infty$]

① $f(z) = \dfrac{z}{z^2 - 1}$	$z = \pm 1$
② $\dfrac{z^2 - 4}{z^2 + 1}$	$z = \pm i$
③ $\dfrac{z + i}{(z - i)^2}$	$z = i$
④ $z^3 - 4z - 1$	analytic everywhere

Bi-linear Transformation (B.T.)

I Find the fixed points of the foll. B.T: - [Solve: $w = f(z) = z$]

1. $w = \dfrac{2z+6}{z+7}$
2. $w = \dfrac{3z-5i}{iz-1}$
3. $w = \dfrac{z-1-i}{z+2}$ [Hint: $z = \dfrac{-1 \pm \sqrt{-3-4i}}{2} = \dfrac{-1 \pm i\sqrt{3+4i}}{2}$
   $3+4i = 4-1+4i$
   $2^2 + 2(2i) + i^2 = (2+i)^2$]
4. $w = -\dfrac{2z+4i}{iz+1}$; Also show that the 2 fixed points together with any point z and its image w form a set of four points having constant cross-ratio
   [i.e. S.T. $(z_1, z_2, z_3, z_4) = 5/2$ a const]
5. $w = \dfrac{2z-2+iz}{i+z}$ 6. $w = \dfrac{2z-5}{z+4}$

Tut 06

Ans

1. $\{1, -6\}$
2. $\{i, -5i\}$
3. $\{-i ; -1+i\}$
4. $\{4i, -i\}$

$z_1 = z$;
$z_2 = w = -\left[\dfrac{2z+4i}{iz+1}\right]$
$z_3 = 4i$
$z_4 = -i$

5. $1 \pm i$ 6. $-1 \pm 2i$

II Find the B.T. that maps the points

1.	$z = \{-1, 1, \infty\}$ onto $w = \{-i, -1, i\}$	$w = \dfrac{iz+i-2}{z+1-2i}$
2.	$z = \{2, 1, 0\}$ onto $w = \{1, 0, i\}$	$w = \dfrac{2i(z-1)}{z(1+i)-2}$
✓ 3.	$z = \{-1, 0, 1\}$ onto $w = \{-1, -i, 1\}$	$w = \dfrac{z-i}{1-iz}$
Tut. 4.	$z = \{-i, 0, i\}$ onto $w = \{-1, i, 1\}$	$w = \dfrac{i(1-z)}{1+z}$
5.	$z = \{i, -1, 1\}$ onto $w = \{0, 1, \infty\}$	$w = \dfrac{2(z-i)}{(1+i)(z-1)}$
6.	$z = \{0, 1, \infty\}$ onto $w = \{-5, -1, 3\}$	$w = \dfrac{3z-5}{z+1}$
7.	$z = \{\infty, i, 0\}$ onto $w = \{0, i, \infty\}$	$w = -\dfrac{1}{z}$
8.	$z = \{1+i, -i, 2-i\}$ onto $w = \{0, 1, i\}$	$w = \dfrac{2z-2-2i}{(i-1)z-3-5i}$

⑨ $z = \{0, -i, -1\}$ onto $w = \{i, 1, 0\}$ — Ans: $w = -i\left[\frac{z+1}{z-1}\right]$ (02)

⑩ $z = \{0, -i, 2i\}$ onto $w = \{5i, \infty, -i/3\}$ — $w = \frac{3z-5i}{iz-1}$

⑪ $z = \{0, -1, \infty\}$ onto $w = \{-1, -2-i, i\}$ — $w = \frac{iz-2}{z+2}$

(III) Find the images under B.T :-

① Find the image of the real axis of z-plane under $w = \frac{1}{z+i}$

Ans: $u^2+v^2+v=0$; Centre: $(0, -\frac{1}{2})$, ra $= \frac{1}{2}$

② S.T. $w = \frac{i-z}{i+z}$ maps the unit circle $|z|=1$ in z-plane onto the imaginary axis of the w-plane. → v-axis i.e. $u=0$

③ S.T. $w = \frac{2z+3}{z-4}$ maps the circle $x^2+y^2-4x=0$ in z-Plane into the st. line $4u+3=0$ in w-plane.

④ S.T. $w = \frac{iz+2}{4z+i}$ maps the real axis in z-plane into a circle $u^2+v^2+\frac{7}{4}v-\frac{1}{2}=0$; Find the pre-image of the centre of this circle

Ans: Centre: $(0, -\frac{7}{8})$, ra $= \frac{9}{8}$

$z = \frac{i}{4} = (0, \frac{1}{4})$

⑤ S.T. $w = \frac{1+iz}{1-iz}$ maps the interior of the unit circle $|z|<1$ in z-Plane into the ~~upper~~ left half of the w-plane $u<0$

Tut (06)

⑥ S.T. $w = \frac{z}{1-z}$ maps the upper half of z-plane $(y>0)$ onto the upper half of w-plane $v>0$. Find the image of the circle $|z|=1$ under this trans^n.

Ans: $u = -\frac{1}{2}$

⑦ S.T. $w = \frac{z-i}{iz-1}$ maps $\text{Im}(z) \geq 0$ $y \geq 0$ onto $|w| \leq 1$

⑧ Find the image of the segment of the real axis between $z=+1$ & $z=-1$ under $w = \frac{1+iz}{z+i}$

Ans: lower half of $|w|=1$

$u_y = (e^{-x})\left[x\cos y - \{\cos y - y\sin y\}\right]$

$(e^{-x})\left[x\cos y - \cos y + y\sin y\right]$

$u_{yy} = (e^{-x})\left[-x\sin y + \sin y + y\cos y + \sin y\right]$

$= e^{-x}\left[-x\sin y + 2\sin y + y\cos y\right]$

$= -x e^{-x}\sin y + 2\sin y\,(e^{-x}) + y\cos y\,(e^{-x})$

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