ENGS203P: MATHEMATICAL MODELLING AND ANALYSIS II

TOPIC 2: TRANSFORMS

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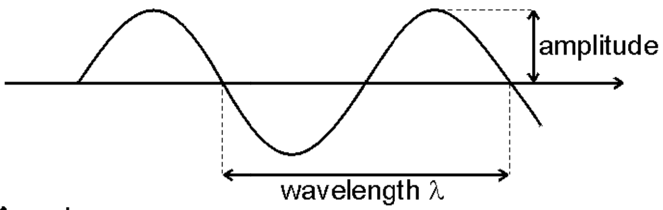
Introduction

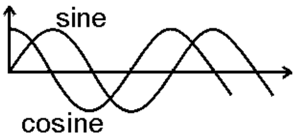
A *transform* is a mathematical tool for changing the way something is represented. A transform can always be reversed. In other words, an original something and its transformed version are always equivalent; they contain the same information, without anything added or taken away.

2.1 Fourier transform

2.1.1 A non-mathematical description

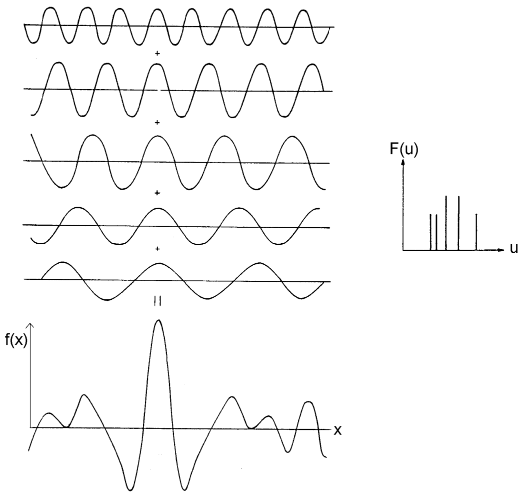
A sinusoidal waveform is one whose shape resembles a sine or cosine wave, which has an *amplitude* (height) and a *wavelength* (distance between consecutive peaks). The number of peaks per unit distance along the length of the waveform is called the *frequency*. Wavelength is inversely proportional to frequency. The *phase* of the waveform tells us the position of the peaks relative to some fixed point on the horizontal axis (such as the zero point). A sine wave is said to have a 90 degree (or $\pi/2$ radian) *phase shift* with respect to a cosine wave.





The *Fourier transform* is a way of representing something as a *sum of waveforms*. Suppose we have a stream of numbers which represent the values of something (e.g. intensity of light, voltage, or whatever) at different points in space, or at different times. We will represent these set of values by a function $\mathrm{f}(x)$, where x is the variable (space or time). The **Fourier Transform** of $\mathrm{f}(x)$ provides us with a knowledge of the *amplitudes*, *frequencies*, and *phases* of all the waveforms which, when added together, would generate $\mathrm{f}(x)$. To avoid confusion, frequencies are labelled with the letter u, and the Fourier Transforms of functions are denoted using upper-case letters. Thus the Fourier Transform of $\mathrm{f}(x) = \mathrm{F}(u)$.

The amplitude (or modulus) of $\mathrm{F}(u)$ is a distribution representing waveform amplitude as a function of frequency u. The figure below shows five sinusoidal waveforms added together. A fairly strong peak appears in the sum, where peaks in all the individual waveforms happened to coincide.



The Fourier Transform of this sum is represented by a graph which shows the amplitude of the five waveforms plotted against their frequency. This is also shown. Incidentally, anything plotted against frequency is generally known as a *spectrum*. The Fourier Transform of the sum (which only consists of five points!) contains exactly the same information as the sum itself.

2.1.2 Fourier integrals

In section 1.7 on Fourier Series it was shown how a (periodic) mathematical function *f(x)* can be re-expressed as a sum of cosine and sine functions. In the complex form (section 1.7.2), the complex coefficients c_n describing the sines and cosines are found by solving a set of integrals as follows:

$$f(x) = \sum_{n=-\infty}^{n=+\infty} c_n e^{jnx}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x).\, e^{-jnx} dx$.

Although sometimes useful, this is a time-consuming process when many different frequency waves (i.e. values of n) in the series are required. While the Fourier series is used to represent a *periodic* function as a set of *discrete* frequencies, the **Fourier transform** enables us to represent a function which is *not* periodic, and as a *continuous* range of frequencies, which is far more useful. This is achieved by replacing the series summation with an integral (i.e. replacing integer n with a frequency variable u). The continuous set of complex coefficients c_n thus becomes the summation of waveforms $\mathrm{F}(u)$ described above, and we obtain the following mathematical definitions of the Fourier transform and its inverse:

$$F(u) = \int_{-\infty}^{\infty} f(x).\, e^{-2\pi jux} dx$$

$$f(x) = \int_{-\infty}^{\infty} F(u).\, e^{2\pi jux} du \quad .$$

The inverse transform enables a set of waveforms to be transformed back into the original function. These beautifully simple expressions are of immense use in engineering. Note that the Fourier and inverse Fourier transforms are often represented as follows:

$$F(u) = FT[f(x)] \qquad \text{and} \qquad f(x) = FT^{-1}[F(u)] \quad .$$

Example: Calculate the Fourier transform of a rectangular function expressed mathematically as follows:

$f(x) = 1$	for	$-a/2 < x < a/2$
$f(x) = 0$	for	$-a/2 \geq x \geq a/2$

The Fourier transform of $f(x)$ is obtained by inserting 1 into the Fourier integral and reducing the limits of integration to $\pm a/2$ since the function $f(x)$ is zero outside this range:

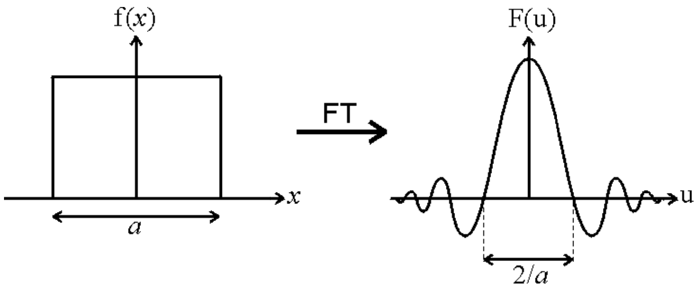
$$\Rightarrow \quad F(u) = \frac{-1}{2\pi ju}\left[e^{-2\pi jux}\right]_{-a/2}^{a/2}$$

$$\Rightarrow \quad F(u) = \frac{1}{2\pi ju}\left(e^{j\pi ua} - e^{-j\pi ua}\right)$$

From section 1.7.2 we already know that $\sin x = (e^{jx} - e^{-jx})/2j$.

$$\Rightarrow \quad F(u) = \frac{\sin(\pi au)}{\pi u}$$

A function of the form $\sin(x)/x$ is known as a "sinc" function. It has values of zero when $\pi au = \pm\pi$ (i.e. $\pm$ 180 degrees), which occurs when $u = \pm 1/a$. And therefore $F(u)$ has a "width" equal to $2/a$ as illustrated along with the rectangular function *f(x)* below.



Note that the function $F(u)$ is complex. Thus for each frequency u, the value of $F(u)$ has both real (a) and imaginary (jb) components. The *amplitude* of each waveform a frequency u is described by the modulus $R = (a^2 + b^2)^{1/2}$, and the sketch of $F(u)$ above actually represents the modulus of $F(u)$. Meanwhile the *phase* of each waveform (see section 6.2.1 of ENGS103P notes on complex numbers) is described by $\theta = \tan^{-1}(b/a)$. Since $a + jb = R(\cos\theta + j\sin\theta)$, the components a and b represent the cosine and sine components of the waveform. If $a >> b$ then the cosine component dominates, and if $a << b$ then the sine component dominates. Consequently, if *f(x)* is strongly symmetric about the line $x = 0$, then at most frequencies $F(u)$ will have $a >> b$, and vice versa.

2.1.3 Conceptual interpretation

While engineers are not frequently required to calculate the Fourier transforms of specific functions, it is essential that they have a very good grasp of the concept of representing functions as a spectrum of frequencies. This concept is explored below for two different physical definitions of the variable x.

a) x is a measure of time (units of seconds).

In this case, we would replace x with the more familiar time variable t, and *f(t)* represents a time-varying quantity. The Fourier transform of *f(t)* is then a function of frequency u expressed in units of Hertz (s^{-1}). For example, suppose *f(t)* represents the intensity of audible sound produced by a symphony orchestra, recorded over a short period of time. The Fourier transform $F(u)$ would represent the spectrum of frequencies contained within that sound. The modulus of $F(u)$ at lower values of u would depend on the loudness of the low-pitched instruments (such as cellos and tubas), whereas the modulus at higher values of u would depend on the loudness of high-pitched instruments (such as violins and flutes). Sound recording systems often display the spectral content of the sound during recording and playback (typically averaged over discrete bands of frequencies).

b) x is a measure of distance (units of metres).

When the variable x represents distance, the Fourier transform becomes a function of *spatial frequency* u expressed in units of inverse metres (m^{-1}). For example, suppose $f(x)$ represents the variation in brightness across a single line across an image. If the image is smooth, the intensity will vary slowly across the image, and the modulus of $F(u)$ will only be non-zero at lower values of u. If the image contains a lot of sharp detail, then the modulus of $F(u)$ will be non-zero at much higher values of u. Thus sharp detail in $f(x)$ implies that it contains waveforms at high spatial frequencies. This concept will be explored further below in section 2.3 on Convolution.

In general, it is observed that:

"width" of $f(x)$ x "width" of $F(\underline{u}) \approx$ a constant, roughly equal to 1

This tells us that narrower functions have broader Fourier Transforms and vice versa. A broader Fourier Transform implies that more higher frequencies are included within it (higher frequencies provide more sharp detail). The rectangular function above had a "width" of a, whereas the "width" of its Fourier transform was around $2/a$. Thus the product is a constant (2), independent of the value of a.

2.2 Delta functions

The *Dirac delta function* $\delta(x - x_0)$ is a function which is equal to 1 at a position $x = x_0$, and is zero everywhere else. This unusual function is very useful for representing physical phenomena which have negligible (≈ 0) duration or spatial extent. Examples include the force of a hammer blow and the image of a distant star.

The Fourier and inverse Fourier transforms of the delta function are given by:

$$FT\left[\delta(x - x_0)\right] = \int_{-\infty}^{\infty} \delta(x - x_0)\, e^{-2\pi i u x}\, dx = e^{-2\pi i u x_0} = \cos 2\pi u x_0 - i \sin 2\pi u x_0$$

$$FT^{-1}\left[\delta(u - u_0)\right] = \int_{-\infty}^{\infty} \delta(u - u_0)\, e^{2\pi i u x}\, du = e^{2\pi i x u_0} = \cos 2\pi x u_0 + i \sin 2\pi x u_0$$

Note that the result in each case is a *single-frequency waveform* which has both cosine (real) and sine (imaginary) components. Note that the frequency of these waveforms depends on the position of the delta function (i.e. x_0 or u_0). Also note that for $x_0 = 0$ and $u_0 = 0$:

$$FT\left[\delta(x)\right] = \int_{-\infty}^{\infty} \delta(x)\, e^{-2\pi i u x}\, dx = e^0 = 1$$

$$FT^{-1}\left[\delta(u)\right] = \int_{-\infty}^{\infty} \delta(u)\, e^{2\pi i u x}\, du = e^0 = 1$$

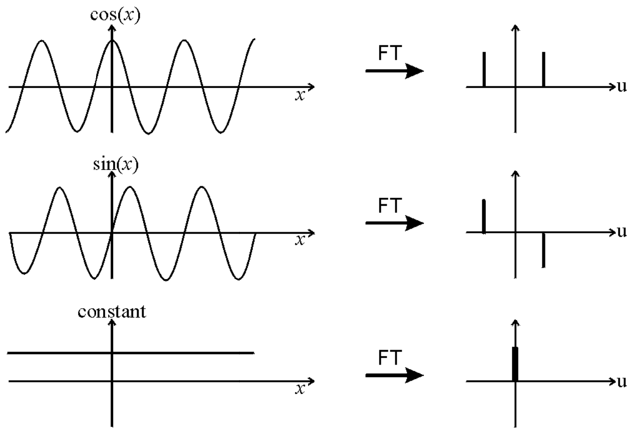
Thus the Fourier Transform of a delta function centred at $x_0 = 0$ is equal to 1 at all frequencies (i.e. at all values of *u*). This tells us that a delta function requires an infinite range of frequencies to represent it. Likewise, the inverse Fourier Transform of a delta function is a constant if the frequency is zero. In other words, a constant (at all values of *x*) is equivalent to a single waveform with a zero frequency (i.e. with an infinite wavelength).

It can also be shown (although it should already be obvious) that the Fourier transform of a sinusoidal waveform is a delta function, occurring at the point on the frequency (*u*) axis corresponding to the frequency of the waveform (i.e. the function contains just a single frequency component). In practice, the Fourier transform yields *two* delta functions at positive and negative frequencies. If the waveform is a pure cosine wave, the delta functions are real. However, if the waveform is a pure sine wave, the delta functions are imaginary.

$$\mathrm{FT}[\cos 2\pi u_0 x] = \frac{1}{2}\left(\delta(u-u_0) + \delta(u+u_0)\right)$$

$$FT[\sin 2\pi u_0 x] = \frac{1}{2i}\left(\delta(u-u_0) - \delta(u+u_0)\right) = -\frac{i}{2}\left(\delta(u-u_0) - \delta(u+u_0)\right)$$

These transforms are represented graphically below:

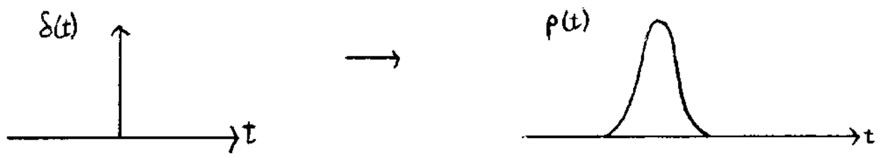


2.3 Convolution

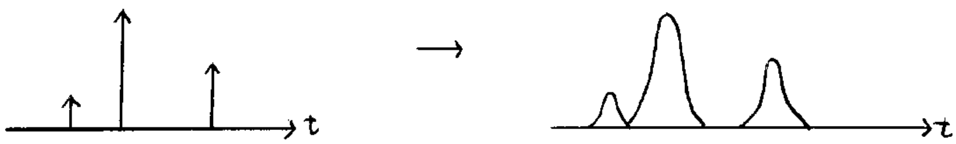
Now that we know a little about the Fourier transform, we can examine one of its most fundamental applications in physics and engineering, known as convolution.

2.3.1 Point response function

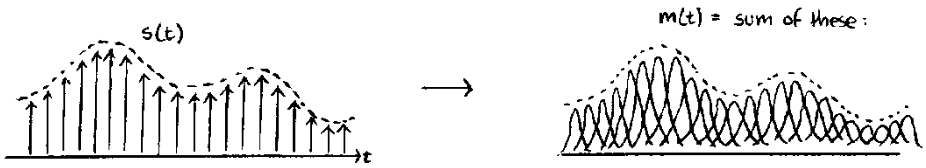
Consider a situation where we attempt to make a measurement of an instantaneous signal represented by the Dirac delta function $\delta(t)$. Remember that $\delta(t)$ has a value of one at $t = 0$, and is zero everywhere else. However, suppose the device we use to make the measurement is unable to respond infinitely fast, and instead of recording a delta function, it records a smooth distribution of finite width. We call this distribution the *point-response function* of the device.



Let us denote this function by $p(t)$. If we attempted to record a whole series of delta functions of arbitrary amplitude, our actual measurement would naturally consist of a series of point responses functions with each amplitude dependent on that of the corresponding delta function:



If the delta functions are spaced very close together, obviously our device will generate a series of overlapping point response functions added together.



Now suppose that, instead of a delta function, the signal we wish to measure is a continuous signal represented by $s(t)$. We can consider that $s(t)$ is composed of an infinite number of closely-spaced delta functions with a varying amplitude dependent on the shape of $s(t)$. Thus the recorded measurement of $s(t)$ will consist of an infinite sum of displaced point response functions with amplitude varying according to $s(t)$.

The observed measurement $m(t)$ is known as the *convolution* of $s(t)$ with $p(t)$, and can be represented by the following mathematical expression known as the *convolution integral*:

$$m(t') = \int_{-\infty}^{\infty} s(t).p(t'-t).\mathrm{d}t$$

The integral denotes the sum, and the $(t' - t)$ denotes the shift of $p(t)$ along the time axis.

2.3.2 Fourier representation of convolution

To avoid having to write out the integral, the convolution of two functions is often represented using a star symbol as follows:

$$m(t) = s(t) * p(t)$$

However, when we need to calculate a convolution in practice, we almost never bother to solve the integral because we have a much more convenient method. It can be shown that <u>the Fourier Transform of the convolution of two functions is equivalent to the product of the Fourier Transforms of the two functions</u>! Thus:

$$\mathrm{FT}[\ m(t)\ ] = \mathrm{FT}[\ s(t)\ ]\ .\ \mathrm{FT}[\ p(t)\ ] \qquad \text{or} \qquad \mathrm{M}(u) = \mathrm{S}(u)\ .\ \mathrm{P}(u)$$

$$\Rightarrow \qquad m(t) = \mathrm{FT}^{-1}[\ \mathrm{FT}[\ s(t)\ ]\ .\ \mathrm{FT}[\ p(t)\ ]\ ]$$

Here is the proof. It is slightly easier if we work backwards, starting with the product of two Fourier transforms:

$$\mathrm{M}(u) = \mathrm{S}(u)\ .\ \mathrm{P}(u)$$

$$\Rightarrow \qquad m(t) = \int_{-\infty}^{\infty} \mathrm{S}(u).\mathrm{P}(u).e^{2\pi \mathrm{i}ut}\mathrm{d}u$$

The function $\mathrm{S}(u)$ is the Fourier transform of $s(t)$. Therefore:

$$\Rightarrow \qquad m(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} s(t).e^{-2\pi \mathrm{i}ut'}\mathrm{d}t' \right] \mathrm{P}(u).e^{2\pi \mathrm{i}ut}\mathrm{d}u$$

where t' is a dummy variable.

Changing the order of integration we get:

$$\Rightarrow \qquad m(t) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \mathrm{P}(u).e^{2\pi \mathrm{i}u(\mathrm{t}-\mathrm{t}')}\mathrm{d}u \right] s(t)\mathrm{d}t'$$

The inverse Fourier transform of $\mathrm{P}(u)$ is given by :

$$p(t) = \int_{-\infty}^{\infty} \mathrm{P}(u).e^{2\pi \mathrm{i}ut}\mathrm{d}u$$

and therefore:

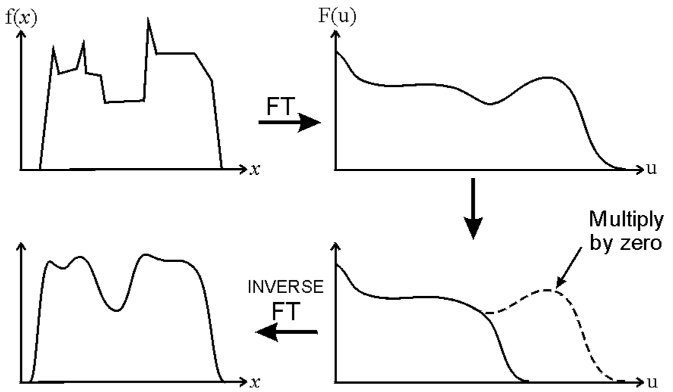
$$p(t-t') = \int_{-\infty}^{\infty} \mathrm{P}(u).e^{2\pi \mathrm{i}u(\mathrm{t}-\mathrm{t}')}\mathrm{d}u$$

$$\Rightarrow \qquad m(t) = \int_{-\infty}^{\infty} p(t' - t).s(t)\mathrm{d}t' = s(t) * p(t)$$

This is the convolution integral. Thus we have proven that the Fourier transform of the convolution of $s(t)$ with $p(t)$ is equal to the product of the Fourier transforms of $s(t)$ and $p(t)$.

2.3.3 An example of convolution: smoothing

Often engineering need to remove noise from data by process known as smoothing. This can be achieved, for example, by averaging adjacent values in the data. However, the smoothing is performed, the process is equivalent to the removal of higher frequencies from the Fourier transform of the data. Consider the function $\mathrm{f}(x)$ exhibited graphically at the top left of the diagram below, and let us suppose that it has the Fourier Transform $\mathrm{F}(u)$ as shown at the top right. Remember that each value of $\Phi(\upsilon)$ represent the amplitude of the waveform with a frequency u which, when added to all the other waveforms gives us $\mathrm{f}(x)$.



Now suppose that, having generated $\mathrm{F}(u)$ using a computer, we multiply the values of $\mathrm{F}(u)$ by zero for the higher values of u. The result of this is illustrated in the bottom right graph. An Inverse Fourier

Transform of this modified version of $F(u)$ will produce a smoothed version of $f(x)$, as shown at the bottom left. The sharp detail is lost when the higher frequency information is removed.

Note that, strictly speaking, the above process is irreversible. A multiplication by zero cannot be undone by a division. Nevertheless, engineers commonly employ so-called “deconvolution” techniques which attempt to recover the true signal *s(t)* from a measurement *m(t)* using knowledge of the point spread function *p(t)*. However, all such technique are necessarily approximate.

2.4 Laplace transform

2.4.1 The Laplace transform of functions

The Laplace Transform is a very useful tool for solving *linear differential equations*. As we know, solving such an equation means finding an expression for the function $y(x)$ which can be inserted into the differential equation to make it true. Like any other transform (such as the Fourier Transform), the Laplace Transform converts a function to an equivalent form. The Laplace transform of a function $y(t)$ is defined as follows:

$$\mathrm{Y(s)} = \int_0^\infty \mathrm{e}^{-st}\, \mathrm{y(t)}\, \mathrm{dt}$$

Note that the transform is a function of a new variable s. Instead of writing the integral each time, we shall use the following shorthand notation:

$$\mathrm{Y(s)} = \Gamma\left[y(\mathrm{t})\right]$$

And similarly we can define an *inverse* Laplace Transform, which converts $\mathrm{Y}(s)$ back into $y(t)$:

$$y(\mathrm{t}) = \Gamma^{-1}\left[\mathrm{Y(s)}\right]$$

Whereas the Fourier transform expresses a function as a sum of sinusoidal waveforms, the Laplace transform expresses a function as a sum of moments (e.g. mean, variance, skew, etc.). The Laplace transform is commonly employed when needing to simplify the way that a system is described mathematically. For example, Laplace transformation from the time domain to the frequency domain transforms differential equations into algebraic equations and (like the Fourier transform) convolution into multiplication.

We will now illustrate how Laplace transforms are calculated for a pair of simple functions.

a) The Laplace transform of a constant: ***y(t) = k***

$$\Gamma[\mathrm{k}] = \int_0^\infty \mathrm{k}.\mathrm{e}^{-st}\, \mathrm{dt} = \mathrm{k}.-\frac{1}{\mathrm{s}}\left|\mathrm{e}^{-st}\right|_0^\infty = \frac{\mathrm{k}}{s}$$

b) The Laplace transform of an exponential: ***y(t) = e^at^***

$$\Gamma\left[\mathrm{e}^{at}\right] = \int_0^\infty \mathrm{e}^{at}.\mathrm{e}^{-st}\, \mathrm{dt} = \int_0^\infty \mathrm{e}^{(a-s)t}\, \mathrm{dt} = \frac{-1}{(a-s)}\left|\mathrm{e}^{-(a-s)t}\right|_0^\infty = \frac{1}{(s-a)}$$

Tables of Laplace transforms of common functions are available in many mathematics textbooks, such as page 1131 of *Engineering Mathematics* by Stroud & Booth (6^{th} Edition, 2007), and online:

https://en.wikipedia.org/wiki/Laplace_transform

2.4.2 The Laplace transform of derivatives

The principal reason why Laplace Transforms are so useful is because the transforms of derivatives of a function are related in a simple way to the transform of the function itself. Consider the Laplace transform of $\mathrm{d}y/\mathrm{d}t$:

$$\Gamma\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]=\int_0^\infty\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right).\mathrm{e}^{-\mathrm{st}}\,\mathrm{d}t$$

Integrating by parts:

$$\Rightarrow \qquad \Gamma\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]=\left|y.\mathrm{e}^{-st}\right|_0^\infty-\int_0^\infty \mathrm{y}.\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{-\mathrm{st}}\right)\mathrm{d}t$$

$$\Rightarrow \qquad \Gamma\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]=-\,\mathrm{y}(0)-(-\mathrm{s})\int_0^\infty \mathrm{y}.\mathrm{e}^{-\mathrm{st}}\,\mathrm{d}t$$

$$\Rightarrow \qquad \Gamma\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]=s\mathrm{Y}(s)-y(0)$$

This result says that the Laplace Transform of the *derivative of y(t)* is equal to *s* multiplied by the Laplace Transform of $y(t)$ plus something called the *initial value*. The "initial value" is just the value of $y(t)$ at the point $t=0$. It can be shown that the transform of the *second derivative of y(t)* is given by:

$$\Gamma\left[\frac{\mathrm{d}^2 y}{\mathrm{d}t^2}\right]=s^2\,\mathrm{Y}(s)-\frac{\mathrm{d}y}{\mathrm{d}t}(0)-sy(0)$$

Note that now the solution contains *two* initial values: the value of the first derivative of y at $t=0$, and the value of y at $t=0$. Similar expressions are available for the Laplace Transforms of higher-order derivatives. In fact there is a general formula which allows the transform of any order of derivative to be calculated very easily:

$$\Gamma\left[\frac{\mathrm{d}^{\mathrm{n}} y}{\mathrm{d}t^{\mathrm{n}}}\right]=s^{\mathrm{n}}\,\mathrm{Y}(s)-\frac{\mathrm{d}^{\mathrm{n}-1}y}{\mathrm{d}t^{n-1}}(0)-s\frac{\mathrm{d}^{\mathrm{n}-2}y}{\mathrm{d}t^{n-2}}(0)-s^2\frac{\mathrm{d}^{\mathrm{n}-3}y}{\mathrm{d}t^{n-3}}(0)-\ldots.-s^{\mathrm{n}-1}y(0)$$

2.4.3 Solving differential equations given initial conditions

Solving a linear differential equation using Laplace Transforms will now be illustrated using an example.

Example: Suppose we have the following equation which we are told describes the relationship between voltage and time for an electrical circuit:

$$\frac{\mathrm{d}y}{\mathrm{d}t}+3y=\sin(t) \quad .$$

Suppose we also know that at a time $t=0$ the voltage y was 2 volts. We wish to solve this equation, which means that we want to find an expression for $y(t)$ that describes how the voltage changes with time. The first step is to replace each term by its Laplace Transform. The transform of the first two terms (y and the first derivative of y) were given above. The transform of $\sin(t)$, found in a table of transforms, is given by:

$$\Gamma[\sin at] = \frac{a}{s^2+a^2} \quad .$$

Thus we get:

$$sY(s) - y(0) + 3Y(s) = \left(\frac{1}{s^2+1}\right) \quad .$$

By rearranging this expression and inserting $y(0) = 2$, we get:

$$Y(s) = \frac{2}{s+3} + \frac{1}{(s+3)(s^2+1)} \quad .$$

Before we can continue, the term on the right needs to be simplified a little further. To accomplish this, we need to use the method of *partial fractions*. This simplification gives us:

$$\mathrm{Y(s)} = \frac{2}{s+3} + \frac{1}{10(s+3)} - \frac{s}{10(s^2+1)} + \frac{3}{10(s^2+1)} \quad .$$

The function $y(t)$ is simply the *inverse* Laplace Transform of the above expression:

$$y(t) = 2\,\Gamma^{-1}\left[\frac{1}{s+3}\right] + \frac{1}{10}\Gamma^{-1}\left[\frac{1}{s+3}\right] - \frac{1}{10}\Gamma^{-1}\left[\frac{s}{s^2+1}\right] + \frac{3}{10}\Gamma^{-1}\left[\frac{1}{s^2+1}\right] \quad .$$

By inserting each of the four inverse transform terms which can be found in a table, we obtain our solution:

$$y(t) = \frac{1}{10}\left[21\mathrm{e}^{-3t} - \cos(t) + 3\sin(t)\right] \quad .$$

This is clearly quite a complicated expression. However, obtaining the result required nothing more complicated than some simple algebra and looking up some transforms in a table!

2.4.4 The Shift Theorem

It can be shown that the Laplace Transform of $y(t)$ multiplied by a factor e^{at} is given by:

$$\Gamma\left[y(t).\mathrm{e}^{at}\right] = \mathrm{Y}(s-a)$$

Note that this is simply $\mathrm{Y}(s)$ *shifted along the s axis* by an amount a. This theorem can be very useful when finding inverse transforms. For example, suppose we need to find the inverse transform:

$$\Gamma^{-1}\left[\frac{1}{(s-1)^2}\right] \quad .$$

Since our table of Laplace transforms tells us that $\Gamma[t] = 1/s^2$, by the shift theorem with $a = 1$ we get:

$$\Rightarrow \qquad \Gamma[\mathrm{e}^t\, t] = 1/(s-1)^2.$$

$$\Rightarrow \qquad \Gamma^{-1}\left[\frac{1}{(s-1)^2}\right] = t.\mathrm{e}^t$$

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Summary

This section has addressed the following key concepts:

- Mathematical transformation to change the way functions or signals are represented.
- Representing functions as a sum of sinusoidal waveforms.
- Use of the Fourier integral to find the spectral content of a function.
- Delta functions.
- Convolution.
- Smoothing as a convolution process.
- Place transforms of functions and derivatives.
- Use of the Laplace transform to solve differential equations.