Modelling & Analysis II
Partial Differential Equations II

Prof. M. Zangeneh and Peng Wang

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Solution of Heat Equation in a Bar

Example (Bar with Constant Temperature Ends)

Consider the problem of determining the temperature distribution in a thin homogeneous bar of length L, given the initial temperature throughout the bar and the temperature at both ends at all times. The governing equation is;

$$\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

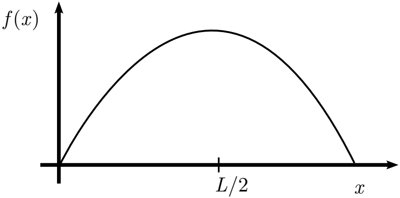
where $\beta = \sqrt{K/c\rho}$.

Boundary conditions:

i $u(0, t) = 0$
ii $u(L, t) = 0$

Initial condition:

i $u(x, 0) = f(x)$



Let

$$f(x) = \frac{4x}{L}\left(-\frac{x}{L} + 1\right).$$

Look for solutions of the form

$$u(x,t) = T(t)X(x)$$
$$\therefore \frac{\partial u}{\partial t} = X(x)\,T'(t) \qquad \& \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)\,T(t).$$

Substituting into (1)

$$X''(x)\,T(t) = \frac{1}{\beta^2}\,T'(t)X(x)$$
$$\therefore \frac{1}{\beta^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \alpha \text{ (a constant).}$$

Again, the only way the above equality can hold true is for the LHS and RHS ratios to be a constant.

Case 1 - $\alpha > 0$

Let $\alpha = \lambda^2$

$$X'' = \lambda^2 X \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

Applying boundary condition (i) and (ii) we can show, as in the case of the Wave equation, that a trivial solution is obtained.

Case 2 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$X'' = -\lambda^2 X \Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

From boundary condition (i);

$$u(0,t) = 0 \Rightarrow X(0) = 0$$
$$\therefore A = 0.$$

From boundary condition (ii);

$$u(L,t) = 0 \Rightarrow B\sin(\lambda L) = 0$$
$$\therefore \lambda L = n\pi$$
$$\lambda = \frac{n\pi}{L}$$
$$X_n = B_n \sin\left(\frac{n\pi x}{L}\right)$$

Also;

$$
\begin{aligned}
T_n(t) &= D_n \exp\left[-\frac{n^2\pi^2}{L^2}\beta^2 t\right] \\
u_n(x,t) &= c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left[-\frac{n^2\pi^2}{L^2}\beta^2 t\right],
\end{aligned}
$$

To satisfy the initial condition we use the superposition principle, *i.e.* ;

$$
u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left[-\frac{n^2\pi^2}{L^2}\beta^2 t\right].
$$

Thus at $t = 0$, we have;

$$
u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) = \frac{4x}{L}\left(-\frac{x}{L} + 1\right).
$$

This is a half-range Fourier series, so;

$$c_n = \frac{2}{L}\int_0^L \left[-\frac{4x^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{4x}{L}\sin\left(\frac{n\pi x}{L}\right)\right]\mathrm{d}x$$

$$= -\frac{16}{n^3\pi^3}\left[\cos\left(n\pi\right) - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{32}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases}$$

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{32}{(2n-1)^3\pi^3}\sin\left[\frac{(2n-1)\pi x}{L}\right]\exp\left[-\frac{(2n-1)^2\pi^2}{L}\beta^2 t\right].$$

Example (Bar with Insulated Ends)

Consider a bar with insulated ends *i.e.*
Boundary conditions:

i $\dot{Q} = K\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x}(0,t) = 0.$

ii $\frac{\partial u}{\partial x}(L,t) = 0$

Initial condition:

i $u(x,0) = x(L-x), \qquad 0 \le x \le L.$

Assume solutions of the form;

$$u(x,t) = T(t)X(x).$$

So as before we get;

$$\frac{1}{\beta^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \alpha \text{ a constant.}$$

Case 1 - $\alpha = 0$

$$X(x) = Ax + B$$

From boundary condition (i):

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow X'(0) = A = 0$$
$$\therefore A = 0,$$

similarly, this satisfies boundary condition (ii),

$$\frac{\partial u}{\partial x}(L,t) = 0.$$
$$X(x) = B.$$

So $\alpha = 0$ is an eigenvalue, which gives eigenfunction

$$X(x) = \text{ a constant.}$$

Case 2 - $\boldsymbol{\alpha} > 0$

Let $\alpha = \lambda^2$

$$X'' - \lambda^2 X = 0 \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

or

$$X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$$

Now from boundary condition (i);

$$\begin{aligned} &\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow X'(0) = 0 \Rightarrow A\lambda\sinh(\lambda x) + B\lambda\cosh(\lambda x). \\ &\therefore B = 0 \end{aligned}$$

Now from boundary condition (ii);

$$\begin{aligned} &\frac{\partial u}{\partial x}(L,t) = 0 \\ &A\lambda\sinh(\lambda L) = 0 \Rightarrow A = 0 \end{aligned}$$

which gives a trivial solution. So this problem has no eigenvalues $\alpha > 0$.

(2)

Case 3 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$\therefore X'' + \lambda^2 X = 0 \Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

From boundary condition (i):

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow -A\sin(\lambda x) + B\cos(\lambda x).$$
$$\therefore B = 0$$

From boundary condition (ii):

$$\frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow -AL\sin(\lambda L) = 0$$
$$\therefore \sin(\lambda L) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

So the eigenfunction for this eigenvalue are given by

$$X_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right)$$

Also

$$
\begin{aligned}
&T'(t) + \frac{n^2\pi^2}{L^2}\beta^2 T = 0 \\
&\therefore\ T_n(t) = D_n \exp\left[\frac{-n^2\pi^2}{L^2}\beta^2 t\right] \\
&\therefore\ u_n(x,t) = c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left[\frac{-n^2\pi^2{}^2}{\beta} t\right]
\end{aligned}
$$

where $c_n = a_n d_n$. To satisfy the initial condition we need to use the superposition principle, *i.e.*

$$
\begin{aligned}
&u(x,t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left[\frac{n^2\pi^2}{L^2}\beta^2 t\right] \\
&\therefore\ u(x,0) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = f(x)
\end{aligned}
$$

This is a cosine series expansion, so

$$
\begin{aligned}
c_0 &= \frac{2}{L}\int_0^L x(L-x)\mathrm{d}x = \frac{L^2}{3} \\
c_n &= \frac{2}{L}\int_0^L x(L-x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \begin{cases} -\frac{4L^2}{n^2\pi^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

$$
\therefore u(x,t) = \frac{L^2}{6} - \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{4n^2}\cos\left(\frac{2n\pi x}{L}\right)\exp\left[\frac{-4n^2\pi^2}{L^2}\beta^2 t\right]
$$

Steady State Heat Conduction in a Plate

Consider the heat equation

$$\nabla^2 T = \frac{c\rho}{K}\frac{\partial T}{\partial t}$$

If the temperature T is steady (*i.e.* does not vary with time) the equation reduces to the Laplace's equation

$$\nabla^2 T = 0$$

Example (Temperature Distribution in a Plate)

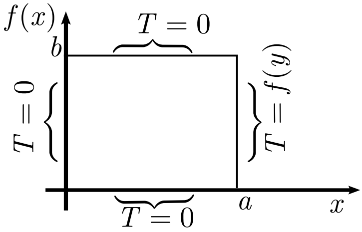
Consider a rectangular Plate: $0 \leq x \leq a$, $0 \leq y \leq b$, which is subjected to the following temperatures at its edges:

i $T(x,0) = 0$ for $0 \leq x \leq a$

ii $T(x,b) = 0$ for $0 \leq x \leq a$

iii $T(0,y) = 0$ for $0 \leq y \leq b$

iv $T(a,y) = f(y)$ for $0 \leq y \leq b$



$f(y) = \frac{4y}{b^2}(b - y)$

The governing equation is;

$$\nabla^2 T = 0$$

$$\Rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Assume a solution of the form

$$T(x, y) = X(x)\,Y(y)$$

$$\therefore X''Y + Y''X = 0$$

$$\therefore \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \alpha \text{ a constant.}$$

Case 1 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$\begin{aligned}
\therefore X'' &= -\lambda^2 X \\
\Rightarrow X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\
Y'' &= +\lambda^2 Y \\
\Rightarrow Y(y) &= Ce^{\lambda y} + De^{-\lambda y} \text{ or } = C\cosh(\lambda y) + D\sinh(\lambda y)
\end{aligned}$$

Consider boundary conditions on $y = 0$ and $y = b$

$$\begin{aligned}
&T(x,0) = 0 \Rightarrow X(x)\,Y(0) = 0 \\
&\therefore Y(0) = 0 \Rightarrow C = 0 \\
&T(x,b) = 0 \Rightarrow X(x)\,Y(b) = 0 \\
&\therefore Y(b) = 0 \Rightarrow D\sinh(\lambda b) = 0 \Rightarrow D = 0
\end{aligned}$$

So $\alpha < 0$ results in a trivial solution.

(3)

Case 2 - $\alpha > 0$

Let $\alpha = \lambda^2$

$$\therefore X'' = \lambda^2 X \Rightarrow X(x) = A\sinh(\lambda x) + B\cosh(\lambda x)$$
$$Y'' = -\lambda^2 Y \Rightarrow Y(y) = C\cos(\lambda y) + D\sin(\lambda y)$$

Consider boundary conditions on $y = 0$ and $y = b$

$$T(x,0) = 0 \Rightarrow X(x)\,Y(0) = 0 \Rightarrow Y(0) = 0$$
$$\therefore C = 0$$
$$T(x,b) = 0 \Rightarrow X(x)\,Y(b) = 0 \Rightarrow Y(b) = 0$$
$$\therefore D\sinh(\lambda b) = 0 \Rightarrow \lambda b = n\pi$$
$$\therefore \lambda = \frac{n\pi}{b}$$
$$\therefore Y_n(y) = D\sin\left(\frac{n\pi y}{b}\right)$$
$$X_n(x) = A\sinh\left(\frac{n\pi x}{b}\right) + B\cosh\left(\frac{n\pi x}{b}\right)$$

Applying boundary condition on $x = 0$ and $x = a$

$$T(0, y) = 0 \Rightarrow X(0)\,Y(y) = 0 \Rightarrow X(0) = 0$$

$$\therefore B = 0$$

$$T_n(x, y) = C_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right)$$

$$T(a, y) = f(y) = \frac{4y}{b^2}(b - y)$$

Now using superposition;

$$T(x, y) = \sum_{n=1}^{\infty} T_n(x, y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{b}\right) \sinh\left(\frac{n\pi x}{b}\right)$$

Now from the boundary condition;

$$\therefore\; T(a,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi a}{b}\right)$$

$$C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)}\int_0^b \frac{4y(b-y)}{b^2}\sin\left(\frac{n\pi y}{b}\right)\mathrm{d}y$$

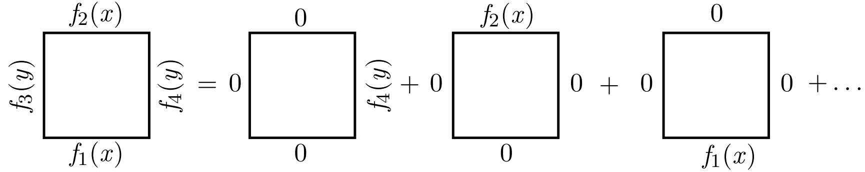
$$= \frac{-16}{n^3\pi^3}\left[\cos(n\pi)-1\right]\frac{1}{\sinh\left(\frac{n\pi a}{b}\right)}$$

$$\therefore\; T(x,y) = \sum_{n=1}^{\infty}\frac{32}{(2n-1)^3\pi^3}\frac{1}{\sinh\left(\frac{n\pi a}{b}\right)}\sin\left[\frac{(2n-1)\pi y}{b}\right]\sinh\left[\frac{(2n-1)\pi x}{b}\right]$$

In practice boundary conditions can be more complicated. For example T can be specified on all edges, *e.g.*

$$\begin{aligned} T(x,0) &= f_1(x), \qquad & T(0,y) &= f_3(y) \\ T(x,b) &= f_2(x), \qquad & T(0,y) &= f_4(y). \end{aligned}$$

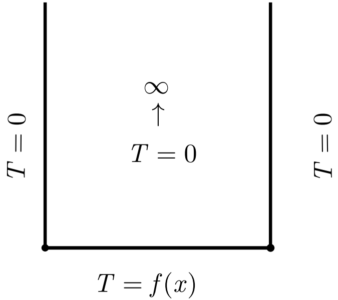
This problem can be solved by superposition, *i.e.* ;



Example (Temperature Distribution in a Semi-infinite Strip)

Consider the equation of Laplace's equation in the strip $y > 0$, $0 \le x \le a$;

1. $T(0, y) = 0$ for all $y > 0$
2. $T(a, y) = 0$ for all $y > 0$
3. $T(x, 0) = f(x)$ for $0 \le x \le a$
4. $T(x, y) \to 0$ as $y \to \infty$.



The governing equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Assume solutions of the form:

$$T(x, y) = X(x)\,Y(y).$$

$$\therefore X'' = \alpha X, \qquad Y'' = -\alpha\,Y.$$

Case 1 - $\alpha > 0$

Let $\alpha = \lambda^2$

$$X'' = \lambda^2 X \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$
$$Y'' = -\lambda^2 Y \Rightarrow Y(y) = C\cos(\lambda y) + D\sin(\lambda y)$$

From boundary condition (iv):

$$T(x, y) \to 0 \text{ as } y \to \infty$$
$$\therefore Y(y) \to 0 \text{ as } y \to \infty$$

Hence $C = D = 0$ which results in a trivial solution.

Case 2 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$\therefore X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$
$$Y(y) = Ce^{\lambda y} + De^{-\lambda y}.$$

Applying boundary condition (iv):

$$T(c, y) \to 0 \text{ as } y \to \infty$$
$$\therefore C = 0$$

Applying boundary condition (i):

$$T(0, y) = 0 \Rightarrow X(0) = 0$$
$$A = 0$$

Applying boundary condition (ii):

$$\begin{aligned} T(a, y) &= 0 \Rightarrow X(a) = 0 \\ \therefore B\sin(\lambda a) &= 0 \Rightarrow \lambda a = n\pi \\ \therefore \lambda &= \frac{n\pi}{a} \\ \therefore T_n(x, y) &= K_n \sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{n\pi y}{a}\right) \end{aligned}$$

Using principle of superposition:

$$T(x, y) = \sum_{n=1}^{\infty} T_n(x, y) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{n\pi y}{a}\right)$$

Applying (iii) we get;

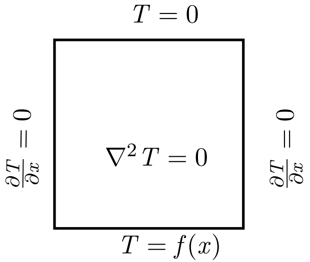
$$\begin{aligned} T(x, 0) &= \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right) = f(x) \\ \therefore K_n &= \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)\mathrm{d}x \end{aligned}$$

Insulated Boundaries for a Rectangular Plate

Consider a rectangular plate
with T specified on $y = 0$ and $y = b$ and
with insulated edges on $x = 0$ and $x = a$.

By applying the method of separation
of variables it is possible to show that

$$T = C + Dy$$



and that

$$T = \cos\left(\frac{n\pi x}{a}\right)\left[A_n \exp\left(\frac{n\pi y}{a}\right) + B_n \exp\left(-\frac{n\pi y}{a}\right)\right]$$

which satisfies conditions on edges $x = 0$ and $x = a$.