Modelling & Analysis in Engineering II

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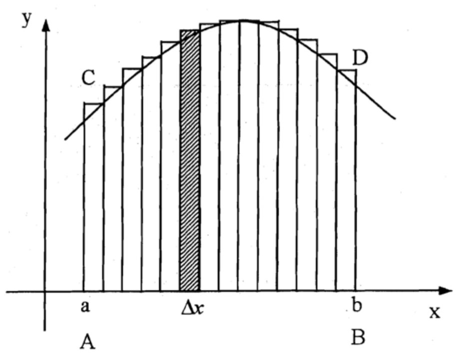
1 Multiple Integrals

1.1 One Dimensional Integral

Consider a function, $y = f(x)$, which defines a curve on the plane $y - x$ as shown:



If we divide the area underneath curve $y = f(x)$ between $x = a$ and $x = b$ into equal increments of length Δx, then δA = area of region $= f(x)\,\Delta x$.

So as $\Delta x \to 0$, we can see that

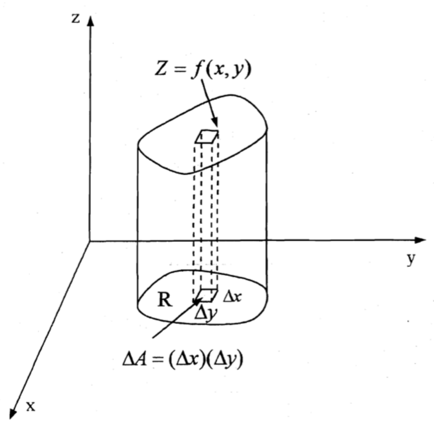
$$\text{Area } ACDB \to \sum_{n=1}^{\infty} f(x)\,\Delta x = \int f(x)\,\mathrm{d}x.$$

$\int_a^b f(x)\,\mathrm{d}x$ is the area underneath the curve $y = f(x)$ between $x = A$ and $x = B$.

1.2 Two Dimensional Integral

Similarly, if we consider a function of two variables, *i.e.* $Z = f(x, y)$ which defines a surface in space, we can show that the volume below this curve is related to the double integral of the function.

Consider the volume enclosed by the cylinder erected on the plane region R in the (x, y) plane and below surface $Z = f(x, y)$. The region R can be divided into a large number of elementary area, δA. Note that $\delta A = \delta x \delta y$. The volume of an elementary cylinder is then $\delta V = f(x, y)\,\delta A$.



So as $\delta A \to 0$, we can see that

$$V = \lim_{\delta A \to 0} \left\{ \sum_{n=1}^{\infty} f(x,y)\,\delta A \right\} = \iint_R f(x,y)\,\mathrm{d}A.$$

$\iint f(x,y)\,\mathrm{d}A$ is the double integral of $f(x,y)$ over the region of integration R.

1.3 Evaluation of Double Integrals

1.3.1 Rectangular region

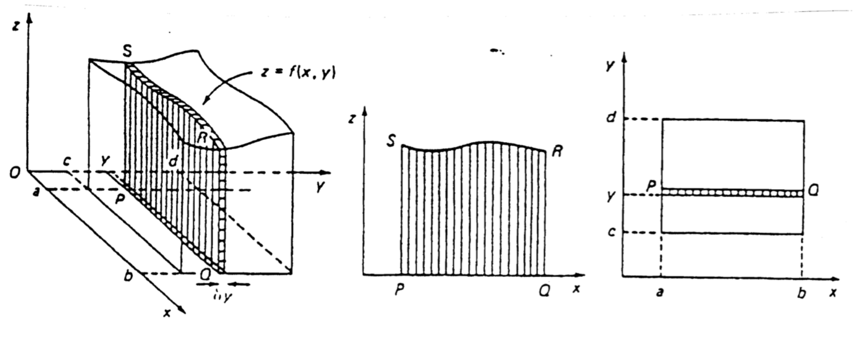
Consider the volume represented by $\iint_R f(x,y)\,\mathrm{d}A$, where R is the rectangular region;

$$R = \{(x,y) : a \le x \le b \text{ and } c \le y \le d\}.$$

Area $PQRS = \int_a^b f(x,y)\,\mathrm{d}x$ at a given value of y, so δV, the volume of strip PQRS with a thickness δy, is given by $\delta V = \left[\int_a^b f(x,y)\,\mathrm{d}x\right]\mathrm{d}y$. Therefore, the volume is given by

$$V = \int_c^d \left\{ \int_a^b f(x,y)\,\mathrm{d}x \right\} \mathrm{d}y.$$

One performs two ordinary integrations.



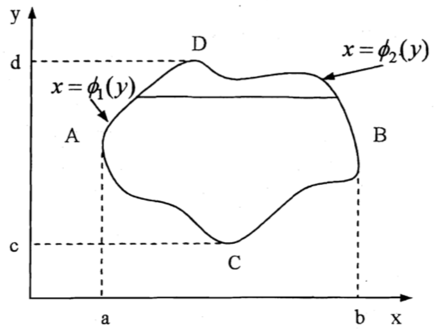
(a) (b) (c)

1.3.2 Non-rectangular region

Consider a region of integration

$$R = \{(x, y) : \phi_1(y) \leq x \leq \phi_2(y) \text{ and } c \leq y \leq d\},$$

where $\phi_1(y)$ is the equation of CAD and $\phi_2(y)$ the equation of CBD.



Alternatively, we can describe R as

$$R = \{(x, y) : \phi_1(x) \leq y \leq \phi_2(x) \text{ and } a \leq x \leq b\},$$

where $\phi_1(x)$ is the equation of ACB and $\phi_2(x)$ is the equation of ADB.

The volume of elementary strips PQ parallel to the x-axis below $Z = f(x, y)$ is

$$\delta V = \left\{ \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x, y)\,\mathrm{d}x \right\} \mathrm{d}y,$$

such that the volume is then given by

$$V = \int_c^d \int_{\phi_1}^{\phi_2} f(x, y)\,\mathrm{d}x\mathrm{d}y = \int_a^b \int_{\phi_1}^{\phi_2} f(x, y)\,\mathrm{d}y\mathrm{d}x.$$

1.4 General Transformation

Let the relationship between the original independent variables x, y and the new variables u and v be described by:

$$x = X(u, v) \text{ and } y = Y(u, v).$$

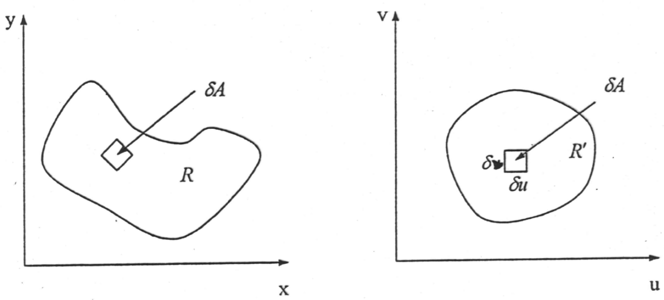
This pair of equations defines the transformation or mapping of the u-v plane and gives the coordinates of a point in the x-y plane corresponding to a point in the u-v plane. Alternatively:

$$u = U(x, y) \text{ and } v = V(x, y),$$

which is inverse transformation or mapping.

Now we want to express $\iint_R f(x, y)\,\mathrm{d}A$ in terms of u and v in the new region R'. The difficulty is to express $\mathrm{d}A = \mathrm{d}x\mathrm{d}y$ in terms of u, v, $\mathrm{d}u$ and $\mathrm{d}v$.

Let us consider:



A typical elemental rectangle $\delta A' = \delta u \delta v$ in region R' corresponds to a parallelogram δA in region R where δu and δv are small. It can be shown that

$$\delta A = \frac{\partial(x, y)}{\partial(u, v)} \delta u \delta v,$$

where $\partial(x, y)/\partial(u, v)$ is the Jacobian of the coordinate transformation and is defined by

$$J := \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

The Jacobian gives the local magnification of the transformation. Hence,

$$\iint_R f(x, y)\,\mathrm{d}A = \iint_{R'} f(X(u, v), Y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \mathrm{d}A'.$$

1.4.1 Properties of Jacobians

Chain rule:
$$\frac{\partial(x,y)}{\partial(u,v)}\frac{\partial(u,v)}{\partial(r,s)}=\frac{\partial(x,y)}{\partial(r,s)}.$$
By choosing $r=x$ and $s=y$, we can deduce that
$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{1}{\frac{\partial(u,v)}{\partial(x,y)}}.$$

1.5 Triple Integrals

The idea involved in the integration of a function of two variable x and y over a region R of the x-y plane can be extended to the integration of a function $f(x,y,z)$ of the three variables x, y, z over a volume V. Thus, if V is the volume described by

$$Z_1(x,y)\leq Z\leq Z_2(x,y),\ Y_1(x)\leq Y\leq Y_2(x) \text{ and } a\leq x\leq b,$$

then
$$\iiint\limits_V f(x,y,z)\mathrm{d}V=\int_a^b\int_{y_1}^{y_2}\int_{z_1}^{z_2}f(x,y,z)\mathrm{d}z\mathrm{d}y\mathrm{d}x,$$

where
$$J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} x_u & y_u & z_u\\ x_v & y_v & z_v\\ x_w & y_w & z_w\end{vmatrix}.$$

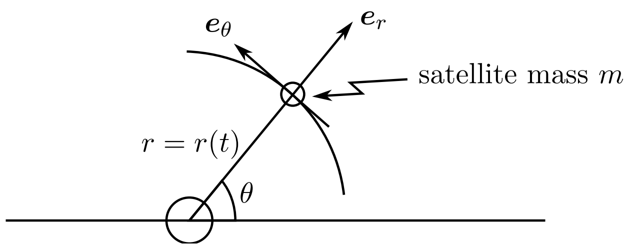
2 Vector Calculus I

2.1 Overviews

Vector calculus deals with the calculus of vector quantities. It has application in Mechanics: Equation of motion of a body or particle in 3D space, *e.g.* it is planned to launch a satellite into a geostationary circular orbit above the equator of the earth. We need to know:

i The satellite speed if it is going to remain stationary relative to earth

ii The height of the orbit above the earth



If we have the position of the satellite at a point, then

$$r(t) = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}.$$

Applying Newton's inverse square law:

$$Force = \gamma \frac{mM}{r^2} \hat{\boldsymbol{e}}_{\boldsymbol{r}},$$

where γ is the universal gravitational constant, m is the mass of the satellite and M is the mass of the earth. Using Newton's 2^{nd} Law:

$$Force = mass \times acceleration$$

so we need to know acceleration and how it varies with time.

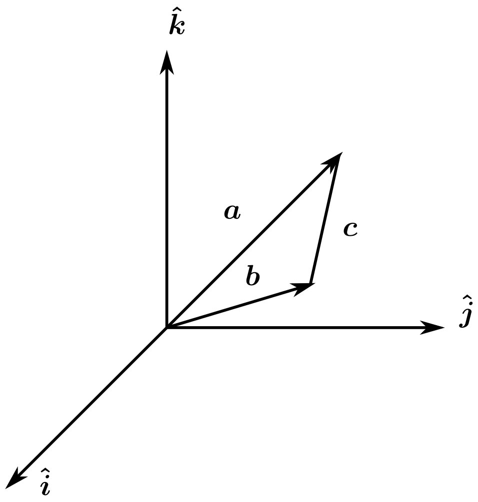
2.2 Revision of Vectors

$$\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}$$

where,

$$\boldsymbol{a} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}}$$

$$\text{Length} = |\boldsymbol{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$



2.2.1 Dot Product (scalar product)

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta,$$

Where $\hat{\boldsymbol{n}}$ is a unit vector in the direction perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$ (right hand rule).

Properties of the Scalar Product:

$$\begin{aligned} \boldsymbol{a} \cdot \boldsymbol{b} &= \boldsymbol{b} \cdot \boldsymbol{a} \\ \boldsymbol{a} \cdot \boldsymbol{a} &= |a|^2 \\ \boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) &= \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c} \\ \boldsymbol{a} \cdot \boldsymbol{b} = 0 &\Rightarrow \boldsymbol{a} = 0, \end{aligned}$$

or

$$\boldsymbol{a} \perp \boldsymbol{b}$$

$$\begin{array}{ccc} \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{i}} = 1 & \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{j}} = 1 & \hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}} = 1 \\ \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{j}} = 0 & \hat{\boldsymbol{i}} \cdot \hat{\boldsymbol{k}} = 0 & \hat{\boldsymbol{j}} \cdot \hat{\boldsymbol{k}} = 0. \end{array}$$

If

$$\hat{\boldsymbol{a}} = a_1\hat{\boldsymbol{i}} + a_2\hat{\boldsymbol{j}} + a_3\hat{\boldsymbol{k}},$$

and

$$\boldsymbol{b} = b_1\hat{\boldsymbol{i}} + b_2\hat{\boldsymbol{j}} + b_3\hat{\boldsymbol{k}},$$

then it follows that

$$\hat{\boldsymbol{a}} \cdot \hat{\boldsymbol{b}} = (a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}}) \cdot (b_1\hat{\boldsymbol{\imath}} + b_2\hat{\boldsymbol{\jmath}} + b_3\hat{\boldsymbol{k}}).$$

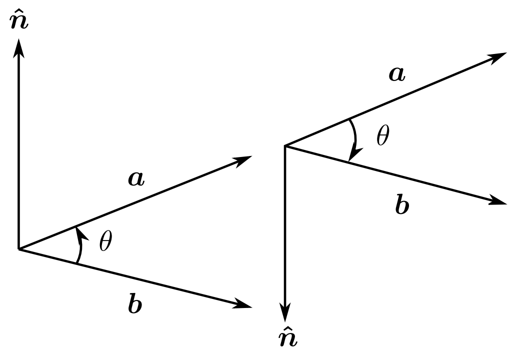
Hence, the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{ab}}{|\boldsymbol{a}|\,|\boldsymbol{b}|}\right).$$

2.2.2 Cross Product (vector product)

$$\boldsymbol{a} \times \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\,\hat{\boldsymbol{n}}\sin\theta,$$

where $\hat{\boldsymbol{n}}$ is a unit vector in the direction perpendicular to the plane containing $\boldsymbol{a}$ and $\boldsymbol{b}$ (right hand rule).



Properties of the Vector Product:

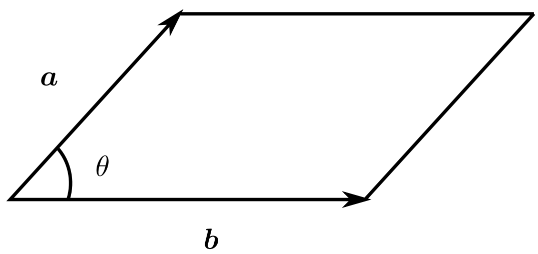
$$\begin{aligned}
&\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{a} \times \boldsymbol{b}\\
&\boldsymbol{a} \times \boldsymbol{a} = 0\\
&\boldsymbol{a} \times \boldsymbol{b} = 0 \Rightarrow \boldsymbol{a} = 0,
\end{aligned}$$

or

$$\boldsymbol{b} = 0,$$

or

$$\begin{aligned}
\boldsymbol{a} &\parallel \boldsymbol{b}\\
|\boldsymbol{a} \times \boldsymbol{b}| &= \text{ area of parallelogram with } \boldsymbol{a} \text{ and } \boldsymbol{b}
\end{aligned}$$



$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}} = \hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}} = 0$$

$$\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}} = \hat{\boldsymbol{\imath}} \qquad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{\imath}} \qquad \hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}} = \hat{\boldsymbol{\jmath}}$$
$$\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{k}} = -\hat{\boldsymbol{\jmath}} \qquad \hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}} = -\hat{\boldsymbol{k}}$$

$$\boldsymbol{a} \times \boldsymbol{b} = (a_1\hat{\boldsymbol{\imath}} + a_2\hat{\boldsymbol{\jmath}} + a_3\hat{\boldsymbol{k}})$$

$$\begin{aligned} &= a_1b_1(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\imath}}) + a_1b_2(\hat{\boldsymbol{\imath}} \times \hat{\boldsymbol{\jmath}}) + a_2b_2(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{\jmath}}) \\ &+ a_2b_3(\hat{\boldsymbol{\jmath}} \times \hat{\boldsymbol{k}}) + a_3b_1(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\imath}}) + a_3b_2(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\jmath}}) + a_3b_3(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{k}}) \\ &= a_1b_2\hat{\boldsymbol{k}} - a_1b_3\hat{\boldsymbol{\jmath}} - a_2b_1\hat{\boldsymbol{k}} + a_2b_3\hat{\boldsymbol{\imath}} + a_3b_1\hat{\boldsymbol{\jmath}} - a_3b_2\hat{\boldsymbol{\imath}} \end{aligned}$$

$$\therefore \boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

Example 1.

$$\boldsymbol{a} = (2, -1, 6), \qquad \boldsymbol{b} = (-3, 5, 1)$$

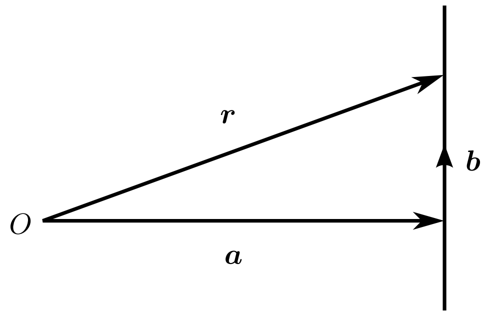
$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix}.$$

2.2.3 Line in 3 D

Line through point $\boldsymbol{a}$ in direction $\boldsymbol{b}$

$\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b}$ for all λ, unit vector is $\frac{\boldsymbol{b}}{|\boldsymbol{b}|}$. In components:

$$x = a_1 + \lambda b_1, \qquad y = a_2 + \lambda b_2, \qquad z = a_1 + \lambda b_3$$



2.2.4 Plane in 3 D

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c},$$

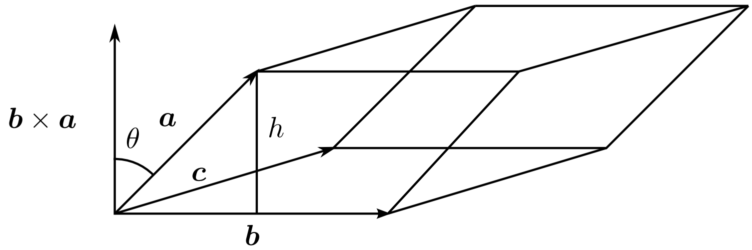
where $\boldsymbol{b}$ and $\boldsymbol{c}$ are vectors in the plane and are not co-linear. The plane contains lines $\boldsymbol{a} + \lambda \boldsymbol{b}$ and $\boldsymbol{a} + \mu \boldsymbol{c}$. The plane has normal vector $\boldsymbol{b} \times \boldsymbol{c}$. Unit normal vector is:

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{b} \times \boldsymbol{c}}{|\boldsymbol{b} \times \boldsymbol{c}|}.$$

2.2.5 Triple Scalar Product

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (a_1 \hat{\boldsymbol{i}} + a_2 \hat{\boldsymbol{j}} + a_3 \hat{\boldsymbol{k}}) \cdot \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},$$

$$= a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$



$$\begin{aligned}
\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) &= |\boldsymbol{b}\times\boldsymbol{c}|\,(a\cdot\hat{\boldsymbol{n}}) \qquad (|\boldsymbol{a}|\cos\theta = \text{height})\\
&= \text{area of base}\times\text{height}\\
&= \text{area of parallelepiped}
\end{aligned}$$

Properties of the Triple Scalar Product:

$$\begin{aligned}
&\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\\
\therefore\ &\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}\\
&\boldsymbol{a}\cdot(\boldsymbol{c}\times\boldsymbol{b}) = \boldsymbol{c}\cdot(\boldsymbol{b}\times\boldsymbol{a}) = \boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c}) = -\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})\\
\text{If }&\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = 0 \Rightarrow \boldsymbol{b}\parallel\boldsymbol{c} \qquad \text{or} \qquad a=0, \qquad b=0, \qquad c=0
\end{aligned}$$

or a, b, c are co-planar, *i.e.* in parallel planes.

Properties of Triple Vector Product[1]

$$\begin{aligned}
\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) &= (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}\\
(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c} &= (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b} - (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a}.
\end{aligned}$$

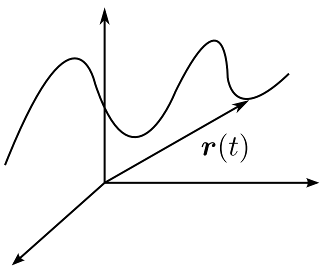
In general,

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}.$$

2.3 Parametric Representation of Curves

Consider the path of a moving particle which follows a curve C in 3 D given by a vector function:

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}.$$



This is known as parametric representation, particularly useful in motion of particles in 3 D, *e.g.* a satellite in orbit.

[1] The detailed derivation of triple vector product can be found in K. A. Stroud, 'Advanced Engineering Mathematics,' Fourth Edition, Appendix 3, pp. 992-993

2.3.1 Tangent to Curve

By definition, the tangent to a curve at P is the limiting position of the line L through P and a neighbouring point Q as Q approaches P. But suppose the curve C is given by function $\boldsymbol{r}(t)$, the tangent at:

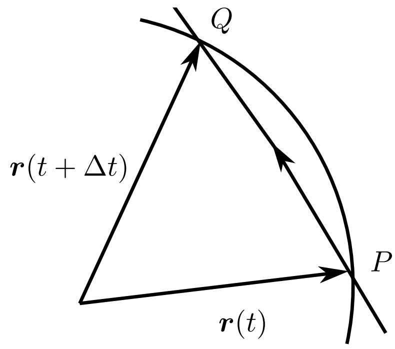
$$P = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \lim_{\substack{\Delta t \to 0 \\ Q \to P}} \left(\frac{\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)}{\Delta t} \right) = \dot{\boldsymbol{r}}.$$

2.3.2 Arc Length

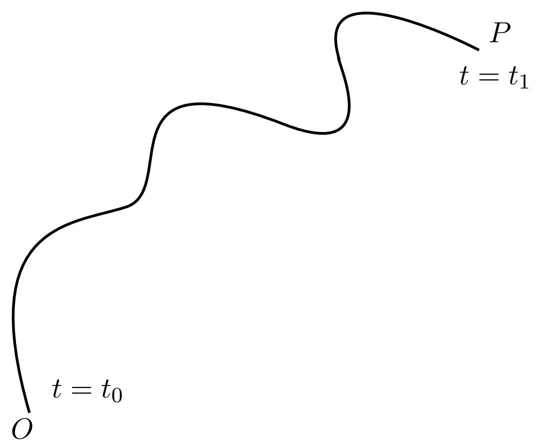
$$\overrightarrow{PQ} = \boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)$$
$$\left|\overrightarrow{PQ}\right| = |\boldsymbol{r}(t+\Delta t) - \boldsymbol{r}(t)| \approx |\dot{\boldsymbol{r}}\Delta t|\,,$$

So a small length of arc $\mathrm{d}s$ is given by

$$\mathrm{d}s = \dot{\boldsymbol{r}}(t)\mathrm{d}t$$
$$i.e.\ \frac{\mathrm{d}s}{\mathrm{d}t} = |\dot{\boldsymbol{r}}(t)|\,.$$



Hence, if we chose an origin O from which to measure arch length and t at O, then the distance following the curve to P where $t = t$ is:



$$\delta = \int_{t=t}^{t=t} \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_{t_0}^{t_1} |\dot{\boldsymbol{r}}(t)|\,\mathrm{d}t.$$

2.4 Velocity and Acceleration

Imagine a particle moving in 3 D space with position vector

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}.$$

In any time interval $t_1 \leq t \leq t_2$, the particle moves a distance

$$\int_{t_1}^{t_2} |\dot{\boldsymbol{r}}(t)| \, \mathrm{d}t.$$

The velocity of the particle at any time t is defined by

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \frac{\mathrm{d}x}{\mathrm{d}t}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}y}{\mathrm{d}t}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}z}{\mathrm{d}t}\hat{\boldsymbol{k}}.$$

Note: This is the direction of tangent to the trajectory, where speed $v = |\boldsymbol{v}|$. The acceleration is the rate of change of velocity, so

$$\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \frac{\mathrm{d}^2x}{\mathrm{d}t^2}\hat{\boldsymbol{\imath}} + \frac{\mathrm{d}^2y}{\mathrm{d}t^2}\hat{\boldsymbol{\jmath}} + \frac{\mathrm{d}^2z}{\mathrm{d}t^2}\hat{\boldsymbol{k}}.$$

2.5 Decomposition of Acceleration into Tangential and Centrifugal Components

This is quite useful in mechanics but we first need to define a scalar quantity called the radius of curvature, R, which is defined in terms of a concept curvature.

Consider a point P on the curve C. The curvature of C at point P is a measure of the amount the tangent is changing direction at P.

Curvature K of C at point P is the magnitude of the rate of change with respect to arc length of unit tangent to C, *i.e.* if $\boldsymbol{T} = \boldsymbol{T}(s)$ is the unit tangent, then

$$K = \left|\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}\right|$$

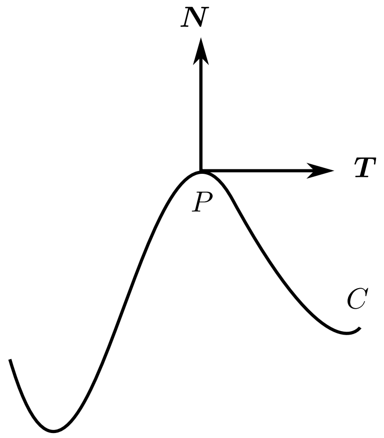
$$\because \boldsymbol{T} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|}$$

$$\boldsymbol{T}' = \frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s} = \frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\dot{\boldsymbol{T}}}{\left|\dot{\boldsymbol{T}}\right|}.$$

Theorem 1. *At any point P on curve C the acceleration of a particle moving along C can be written as*

$$\boldsymbol{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{T} + \frac{v^2}{\rho}\boldsymbol{N},$$

where $\boldsymbol{T}$ and $\boldsymbol{N}$ are the unit tangent and unit normal to the curve C at point P and ρ is the radius of curvature, $1/K$.



Proof.

$$\boldsymbol{T} = \frac{\dot{\boldsymbol{r}}}{|\dot{\boldsymbol{r}}|} = \frac{\boldsymbol{v}}{v}$$
$$\therefore \boldsymbol{v} = v\boldsymbol{T}.$$

Then

$$\begin{aligned}\boldsymbol{a} &= \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v\boldsymbol{T}) \\ &= \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{T} + v\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{T} + v\left(\frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}\right).\end{aligned}$$

Since $\frac{\mathrm{d}s}{\mathrm{d}t} = 0$,

$$\boldsymbol{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{T} + v^2\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s},$$

so if we define $\boldsymbol{N} = \rho\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}$ then we have

$$\boldsymbol{a} = \frac{\mathrm{d}v}{\mathrm{d}t}\boldsymbol{T} + \frac{v^2}{\rho}\boldsymbol{N}.$$

Now we have to show that $\boldsymbol{N}$ is orthogonal to $\boldsymbol{T}$. We know that $\boldsymbol{T}\cdot\boldsymbol{T}=1$, differentiating with respect to s we get:

$$\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}\boldsymbol{T}+\boldsymbol{T}\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}=0.$$

Hence $\boldsymbol{T}$ is orthogonal to $\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}s}$ which is $\boldsymbol{N}$. Finally, $|\boldsymbol{N}|=\rho|\mathrm{d}\boldsymbol{T}/\mathrm{d}s|=\rho K=1$ (unit vector).

$\frac{\mathrm{d}\boldsymbol{T}}{\mathrm{d}t}$ is the tangential component of acceleration

$\frac{v^2}{\rho}$ is the centrifugal component of acceleration.

Alternatively,

$$\begin{aligned}\boldsymbol{a}&=a_t\boldsymbol{T}+a_n\boldsymbol{N}\\ |\boldsymbol{a}|^2&=a_t^2+a_n^2|\boldsymbol{a}|^2=|\ddot{\boldsymbol{r}}(t)|\\ a_t&=\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}s}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}|\dot{\boldsymbol{r}}(t)|\\ \therefore a_n^2&=|\boldsymbol{a}|^2-a_t^2.\end{aligned}$$

2.6 Directional Derivatives; Gradients

Suppose we know the temperature $T(x,y,a)$ at every point of an object. Obviously temperature increases in some directions and decreases in others. Thus the rate of change of temperature with distance depends on the direction we consider, *i.e.* it's a *directional derivative.*

We are interested in the rate of change of T with position P (a point in space) in the direction $\hat{\boldsymbol{b}}\cdot\hat{\boldsymbol{b}}=1$. Let Q be a point on the line through P in the direction of $\hat{\boldsymbol{b}}$. Let $\left|\overrightarrow{PQ}\right|=\Delta s$, then the directional derivative is

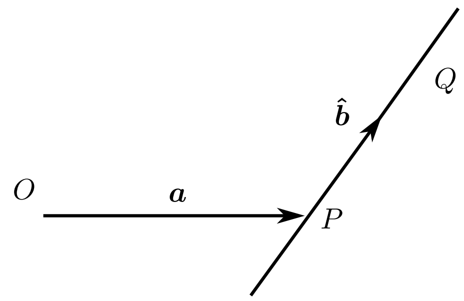
$$\lim_{\Delta t\to 0}\frac{T(Q)-T(P)}{\Delta s}=\frac{\partial T}{\partial s},$$

i.e. the derivative of T at P in the direction of $\hat{\boldsymbol{b}}$. Let the position vector of P be $\boldsymbol{a}$, the line $\overrightarrow{PR}$ is

$$\boldsymbol{r} = \boldsymbol{a} + s\hat{\boldsymbol{b}} = x(s)\hat{\boldsymbol{\imath}} + y(s)\hat{\boldsymbol{\jmath}} + z(s)\hat{\boldsymbol{k}}.$$

Now for a point on $\overrightarrow{PQ}$;

$$T(x, y, z) = T[x(s), y(s), z(s)]$$



Hence by chain rule;

$$\begin{aligned}\frac{\partial T}{\partial s} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s} \\ = \left(\frac{\partial T}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial T}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{k}}\right) \cdot \left(\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}}\right).\end{aligned}$$

But

$$\dot{r} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} = \hat{\boldsymbol{b}}.$$

Hence

$$\frac{\partial T}{\partial s} = (\nabla T) \cdot \hat{\boldsymbol{b}},$$

where

$$\nabla T = \frac{\partial T}{\partial x}\hat{\boldsymbol{\imath}} + \frac{\partial T}{\partial y}\hat{\boldsymbol{\jmath}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{k}}$$

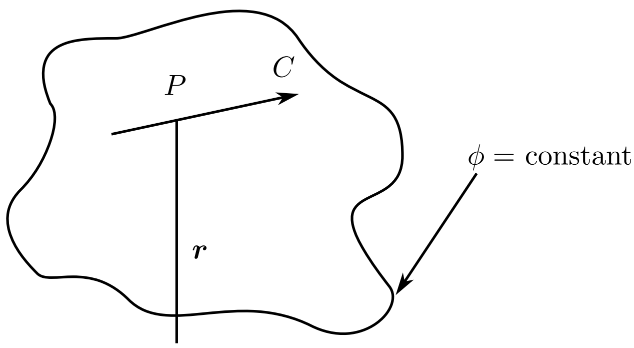
Now $\frac{\partial T}{\partial s} = |\nabla T| \cos\gamma$, where γ is the angle between $\hat{\boldsymbol{b}}$ and ∇T and since $\cos\gamma \leq 1$, $\frac{\partial T}{\partial s}$ has maximum value when $\gamma = 0$, *i.e.* $\hat{\boldsymbol{b}}$ is the same direction as ∇T.

N.B. Direction of ∇T gives the direction of the maximum rate of change of T and $|\nabla T|$ is the magnitude of this change.

2.7 Geometric Interpretation of Gradients

Consider a surface $\phi(x, y, z) =$ constant. The equation of a curve C on this surface is given by

$$\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}.$$



Then at point P, $\phi(x(t), y(t), z(t)) =$ constant. Hence,

$$\frac{\partial \phi}{\partial x}\dot{x} + \frac{\partial \phi}{\partial y}\dot{y} + \frac{\partial \phi}{\partial z}\dot{z} = 0.$$

Therefore $\nabla\phi \cdot \dot{\boldsymbol{r}} = 0$, which implies that $\nabla\phi$ is perpendicular to $\dot{\boldsymbol{r}}$ but $\dot{\boldsymbol{r}}$ is the tangent to the surface ϕ at P. If we consider the tangents to all the curves on the surface at P, they form a plane, *i.e.* the tangent plane to the surface at P (obviously $\dot{\boldsymbol{r}}$ lines on this plane) but $\nabla\phi$ is perpendicular to $\dot{\boldsymbol{r}}$ and hence $\nabla\phi$ is perpendicular to the tangent plane.

N.B. The direction of the largest rate of change of a given function ϕ with distance is perpendicular to the surface ϕ =constant or equi-potentials.

2.8 Line Integral

2.8.1 Work done by a force

Consider a particle moving along a curve C given by $\boldsymbol{r}(t)$. Suppose at any point on that curve, the particle is acted on by the force $\boldsymbol{F}$. Then the work done by the force $\boldsymbol{F}$ when the particle moves from $\boldsymbol{r}$ to $\boldsymbol{r} + \delta\boldsymbol{r}$ is;

$$\mathrm{d}W = \boldsymbol{F} \cdot \delta\boldsymbol{r}.$$

Therefore the total work done by $\boldsymbol{F}$ as the particle moves from P to Q is:

$$W = \int_Q^P \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}.$$

This is the line integral.

Note: The amount of work done in moving from P to Q depends on the path taken. Suppose the particle moves from P at $t = t_0$ and arrives at Q at $t = t_1$, then;

$$W = \int_{t_0}^{t_1} \left(\boldsymbol{F} \cdot \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \right) \mathrm{d}t = \int_{t_0}^{t_1} \boldsymbol{F} \cdot \dot{\boldsymbol{r}} \mathrm{d}t = \int_{t_0}^{t_1} (\boldsymbol{F} \cdot \boldsymbol{v}) \, \mathrm{d}t,$$

but $\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}$ by Newton, hence;

$$W = m \int_{t_0}^{t_1} \boldsymbol{v} \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} \mathrm{d}t = \left[\frac{1}{2} m \left(\boldsymbol{v} \cdot \boldsymbol{v} \right) \right]_{t_0}^{t_1} = \left[\frac{1}{2} m |\boldsymbol{v}|^2 \right].$$

Hence work done equals the change in Kinetic Energy.

N.B. If

$$\boldsymbol{F} = A_1(x, y, z)\hat{\boldsymbol{\imath}} + A_2(x, y, z)\hat{\boldsymbol{\jmath}} + A_3(x, y, z)\hat{\boldsymbol{k}},$$

then

$$\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = \int (A_1 \mathrm{d}x + A_2 \mathrm{d}y + A_3 \mathrm{d}z),$$

where

$$\boldsymbol{r} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}},$$

and

$$\dot{\boldsymbol{r}} \mathrm{d}t = \mathrm{d}x\hat{\boldsymbol{\imath}} + \mathrm{d}y\hat{\boldsymbol{\jmath}} + \mathrm{d}z\hat{\boldsymbol{k}}.$$

2.8.2 Conservative vector fields

In general, the work done depends on the path taken between two points. If the work done is the same for every path taken then we say that $\boldsymbol{F}$ is conservative (*i.e.* no friction).

Theorem 2. *$\boldsymbol{F}$ is conservative if, and only if, $\boldsymbol{F} = \nabla\phi$ for some scalar function ϕ*

Proof. Let $\boldsymbol{F} = \nabla\phi$. Then

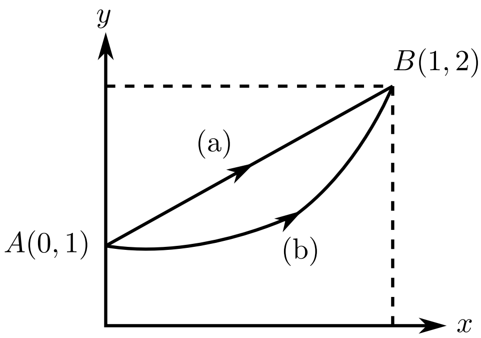
$$\begin{aligned}
W &= \int_A^B \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{r} = \int_{t_0}^{t_1} \boldsymbol{F}\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\mathrm{d}t = \int_{t_0}^{t_1}\left(\nabla\phi\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right)\mathrm{d}t \\
&= \int_{t_0}^{t_1}\left(\frac{\partial\phi}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial\phi}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial\phi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}t \\
&= \int_{t_0}^{t_1}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left[\phi(x(t),y(t),z(t))\right]\right\}\mathrm{d}t \\
&= \phi(x(t_1),y(t_1),z(t_1)) - \phi(x(t_0),y(t_0),z(t_0)) \\
&= \phi_B - \phi_A.
\end{aligned}$$

Independent of path and hence conservative.

N.B. If C is a closed curve and if $\boldsymbol{F}$ is a conservative force field then the work done by the force in displacing a particle around the closed curve, *i.e.* $\oint_C \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{r} = 0$ is zero.

Example 2. Evaluate $\displaystyle\int_{A(0,1)}^{B(0,2)} (x^2-y)\mathrm{d}x + (y^2+x)\mathrm{d}y$ along

a) a straight line $x = t$, $y = t+1$
b) parabola $x = t$, $y = t^2+1$



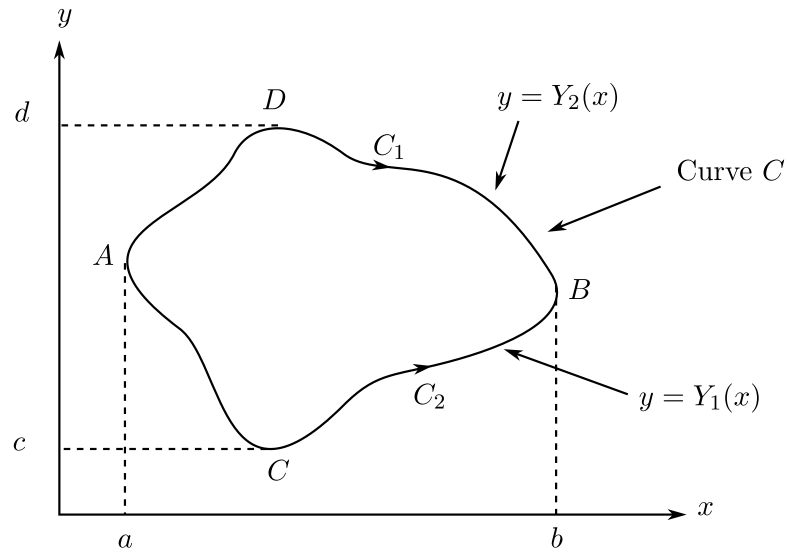
Answer:

a) $\frac{5}{3}$

b) 2

2.8.3 Green's theorem on a plane

Let P, Q be continuous and differentiable in region R bounded by a closed curve C.



Then

$$\oint (P\mathrm{d}x + Q\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y.$$

Proof. Let R be described by $R = \{(x, y) : a \leq x \leq b,\ Y_1(x) \leq y \leq Y_2(x)\}$
Then

$$\begin{aligned}
\iint_R \frac{\partial P}{\partial y}\mathrm{d}x\mathrm{d}y &= \int_a^b \left\{\int_{Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\mathrm{d}y\right\} \mathrm{d}x \\
&= \int_a^b \left[P(x, y)\right]_{Y_1(x)}^{Y_2(x)} \mathrm{d}x \\
&= \int_a^b \left[P(x, Y_2(x)) - P(x, Y_1(x))\right] \mathrm{d}x \\
&= \int_a^b P(x, Y_2(x))\mathrm{d}x - \int_a^b P(x, Y_1(x))\mathrm{d}x \\
&= \int_A^B (P\mathrm{d}x)_{C_1} - \int_A^B (P\mathrm{d}x)_{C_2},
\end{aligned}$$

where C_1 and C_2 are parts of the curve C

$$\begin{aligned}
&= -\int_A^B (P\mathrm{d}x)_{C_1} - \int_A^B (P\mathrm{d}x)_{C_1} \\
&= -\oint P\mathrm{d}x.
\end{aligned}$$

By the same argument:

$$\iint_R \frac{\partial Q}{\partial x}\mathrm{d}x\mathrm{d}y = \oint Q\mathrm{d}y.$$

Hence

$$\oint (P\mathrm{d}x + Q\mathrm{d}y) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y.$$

Conservative Fields:

If P and Q were the components of force

$$\boldsymbol{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}},$$

then

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r},$$

is the work done in moving a particle around the closed path C.

If $\boldsymbol{F}$ is conservative, then

$$\int_A^B (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r})_{C_1} = \int_A^B (\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r})_{C_2}.$$

Hence $\oint \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$. If $\boldsymbol{F}$ is conservative, then $\oint_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r} = 0$ for any closed curve C. Hence

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x\mathrm{d}y = 0,$$

for any region R, so $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$.

Therefore:

$$\boldsymbol{F} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}},$$

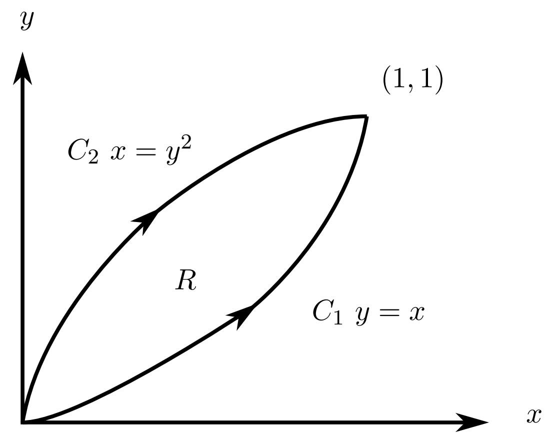
is conservative if, and only if;

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Hence, $\boldsymbol{F} = \nabla\phi$.

Example 3. Determine that $\boldsymbol{F} = 2xy\hat{\boldsymbol{\imath}} + (x^2 + y^2)\hat{\boldsymbol{\jmath}}$ is conservative.

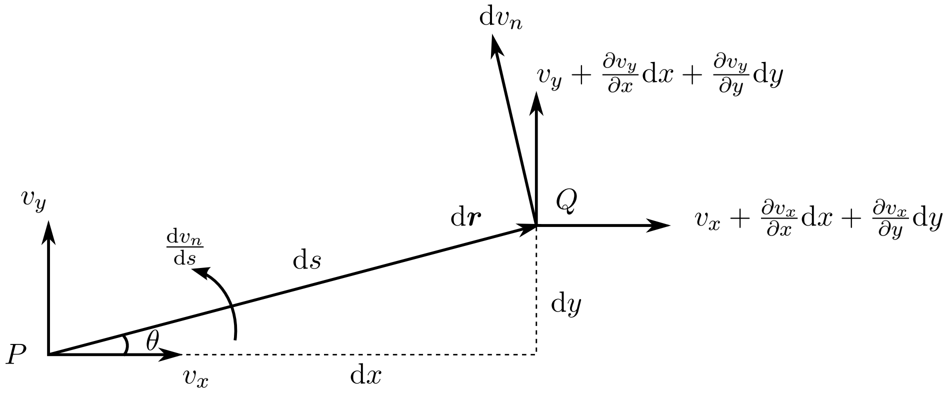
Example 4. Verify Green's Theorem in a plane by the direct evaluation of integration involved for $\oint_C (2xy - x^2)\mathrm{d}x + (x^2 + y^2)\mathrm{d}y$ where C is given by the following:



3 Vector Calculus II

3.1 Vorticity, Fluid Rotation and the Curl Operator

Consider the flow of a fluid in the (x, y) plane. Two adjacent points P and Q in this plane are separated by a small displacement $\delta\boldsymbol{r}$ of length δs.



v_x and v_y are the velocity components at P. The velocity components at Q can be found by using Taylor series, so:

$$\mathrm{d}\boldsymbol{v} = \left(\mathrm{d}x\frac{\partial}{\partial x} + \mathrm{d}y\frac{\partial}{\partial y}\right) \cdot \left(v_x\hat{\boldsymbol{\imath}} + v_y\hat{\boldsymbol{\jmath}}\right).$$

If $\mathrm{d}v_n$ is the component of $\mathrm{d}\boldsymbol{v}$ normal to $\overrightarrow{PQ}$, then the anti-clockwise velocity of $\overrightarrow{PQ}$ is $\frac{\mathrm{d}v_n}{\mathrm{d}s}$. By resolving we get:

$$\mathrm{d}v_n = \left(\frac{\partial v_y}{\partial x}\mathrm{d}x + \frac{\partial v_y}{\partial y}\mathrm{d}y\right)\cos\theta - \left(\frac{\partial v_x}{\partial x}\mathrm{d}x + \frac{\partial v_x}{\partial y}\mathrm{d}y\right)\sin\theta.$$

But

$$\mathrm{d}x = \mathrm{d}s\cos\theta,$$

and

$$\mathrm{d}y = \mathrm{d}s\sin\theta,$$

so

$$\mathrm{d}v_n = \left(\frac{\partial v_y}{\partial x}\mathrm{d}s\cos\theta + \frac{\partial v_y}{\partial y}\mathrm{d}s\sin\theta\right)\cos\theta - \left(\frac{\partial v_x}{\partial x}\mathrm{d}s\cos\theta + \frac{\partial v_x}{\partial y}\mathrm{d}s\sin\theta\right)\sin\theta$$

$$\therefore \frac{\mathrm{d}v_n}{\mathrm{d}s} = \frac{\partial v_y}{\partial x}\cos^2\theta + \frac{\partial v_x}{\partial y}\sin^2\theta + \frac{1}{2}\left(\frac{\partial v_y}{\partial y} - \frac{\partial v_x}{\partial x}\right)\sin 2\theta.$$

This result shows that the fluids angular velocity at P is not unique and depends on θ. Now we can try and evaluate the *average*, taken over all values of θ from 0 to 2π. Now mean value of $\cos^2\theta = \sin^2\theta = 1/2$ and mean value of $\sin 2\theta = 0$. Therefore, the mean angular velocity at $P = \frac{1}{2}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$.

In fluid mechanics, one usually defines a new vector $\boldsymbol{\omega}$, called the *vorticity*, defined as twice the mean angular velocity, or;

$$\boldsymbol{\omega} = \hat{\boldsymbol{k}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

It can be shown that in 3 D the vorticity vector is given by:

$$\boldsymbol{\omega} = \hat{\boldsymbol{\imath}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\boldsymbol{\jmath}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\boldsymbol{k}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right).$$

The vector $\boldsymbol{\omega}$ is a measure of the non-uniformity of the vector field $\boldsymbol{v}$. The operation on the vector $\boldsymbol{v}$ to get $\boldsymbol{\omega}$ is known as *curl*, *i.e.*

$$\textbf{curl}\ \boldsymbol{v} = \boldsymbol{\omega}.$$

By using the del operator

$$\nabla = \hat{\boldsymbol{\imath}}\frac{\partial}{\partial x} + \hat{\boldsymbol{\jmath}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z},$$

it is clear that:

$$\textbf{curl}\ \boldsymbol{v} = \nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}.$$

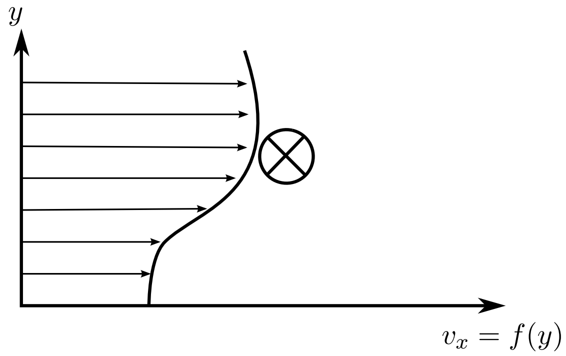
Example 5. Calculate the vorticity vector $\boldsymbol{\omega}$ if $\boldsymbol{v} = 2xy\hat{\boldsymbol{\imath}} + e^y\hat{\boldsymbol{\jmath}} + 2z\hat{\boldsymbol{k}}$.

Solution

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & e^y & 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(e^y)\right]\hat{\boldsymbol{\imath}} - \left[\frac{\partial}{\partial x}(2z) - \frac{\partial}{\partial z}(2xy)\right]\hat{\boldsymbol{\jmath}} + \left[\frac{\partial}{\partial x}(e^y) - \frac{\partial}{\partial y}(2xy)\right]\hat{\boldsymbol{k}}$$

$\therefore \boldsymbol{\omega} = -2x\hat{\boldsymbol{k}}$



Example 6 (Rectilinear shear flow)**.** *i.e.* $v_x = f(y)$, $v_y = 0$, $v_z = 0$
The vorticity in this case is given by:

$$\boldsymbol{\omega} = \nabla \times \boldsymbol{v} = -\hat{\boldsymbol{k}}\frac{\mathrm{d}v_x}{\mathrm{d}y}.$$

The value of the mean rotation of the fluid elements can in this case be measured by using a small diameter paddle wheel consisting of a crossed pair of vanes, which will rotate as a result of the different velocities acting on its vanes.

The rotation of a fluid particle can be caused only by a torque applied by shear forces on the sides of the particle.

3.1.1 Incompressible Potential (Irrotational) Flow

If the velocity field is conservative, then we can say that:

$$\boldsymbol{v} = \nabla\phi.$$

Now the continuity equation for incompressible flow is given by:

$$\nabla \cdot \boldsymbol{v} = 0,$$

or

$$\nabla \cdot \nabla\phi = \nabla^2\phi = \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) = 0.$$

This is known as the Laplace's Equation and is the governing equation of Incompressible Potential Flow.

3.1.2 Vector Identity $\nabla \times \nabla\phi = 0$

Consider a potential function ϕ then

$$\nabla \times \nabla\phi = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right)\hat{\boldsymbol{i}} - \left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial z}\partial x\right)\hat{\boldsymbol{j}} + \left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right)\hat{\boldsymbol{k}}$$

$$= 0.$$

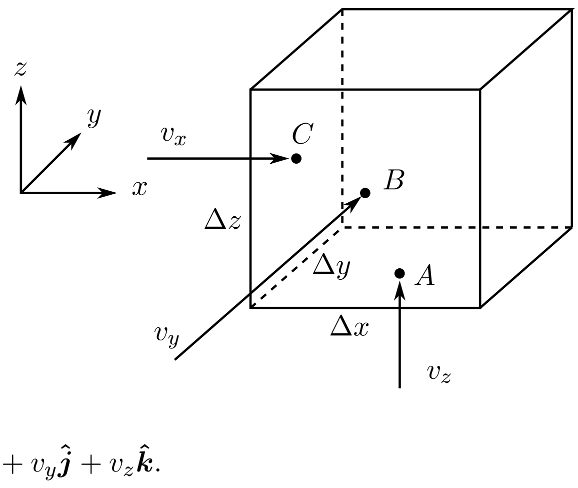
A conservative vector field $\boldsymbol{v}$ with a potential ϕ such that $\boldsymbol{v} = \nabla\phi$ is always irrotational

3.1.3 Circulation and Vorticity

A vector field is conservative, *i.e.* $\boldsymbol{v} = \nabla\phi$ if

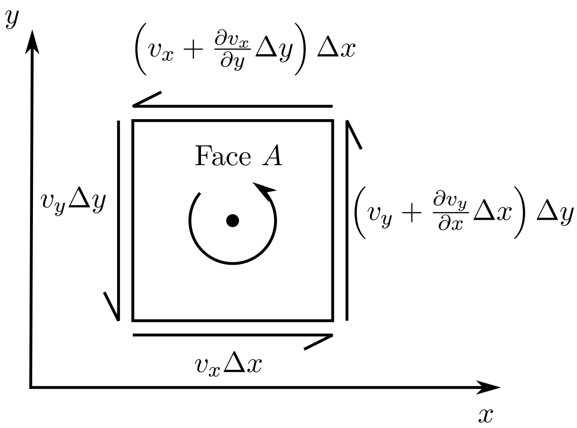
$$\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} \Longleftrightarrow \nabla \times \boldsymbol{v} = 0.$$

This implies a relationship between $\oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r}$ (the circulation integral) and vorticity. To see whether there is a relationship, let us consider a small rectangular element; $\Delta x\Delta y\Delta z$



where $\boldsymbol{v} = v_x\hat{\boldsymbol{i}} + v_y\hat{\boldsymbol{j}} + v_z\hat{\boldsymbol{k}}$.

Consider face A where $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$ and so $\mathrm{d}A_z$ = elemental area of the surface with $\hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$ is $\Delta x\Delta y$.

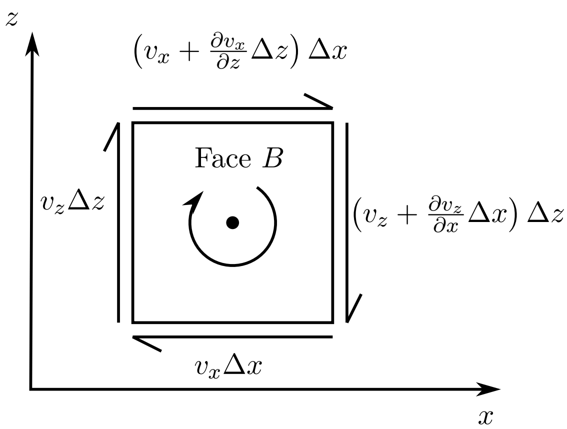


$$
\begin{aligned}
\mathrm{d}P_A &= \oint_A \boldsymbol{v}\mathrm{d}\boldsymbol{r} \\
&= v_x\Delta x + \left(v_y + \frac{\partial v_y}{\partial x}\Delta x\right)\Delta y - \left(v_x + \frac{\partial v_x}{\partial y}\Delta y\right)\Delta x - v_y\Delta y \\
&= \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\Delta x\Delta y
\end{aligned}
$$

$$
\mathrm{d}P_a = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\mathrm{d}A_z = \omega_z\mathrm{d}A_z\,,
$$

where $\boldsymbol{\omega} = \nabla \times \boldsymbol{N} = \omega_x\hat{\boldsymbol{i}} + \omega_y\hat{\boldsymbol{j}} + \omega_z\hat{\boldsymbol{k}}$.

Considering face B:



$$\hat{\boldsymbol{n}} = \hat{\boldsymbol{j}}, \qquad \mathrm{d}A_y = \Delta x \Delta z$$

$$\begin{aligned}
\mathrm{d}P_B &= \oint_B \boldsymbol{v}\mathrm{d}\boldsymbol{r} \\
&= v_z \Delta z + \left(v_x + \frac{\partial v_x}{\partial z}\Delta z \right)\Delta x - \left(v_z + \frac{\partial v_z}{\partial x}\Delta x \right)\Delta z - v_x \Delta x \\
&= \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} \right)\Delta z \Delta x
\end{aligned}$$

$$\mathrm{d}P_B = \omega_y \mathrm{d}A_y \,.$$

Similarly, for face C:

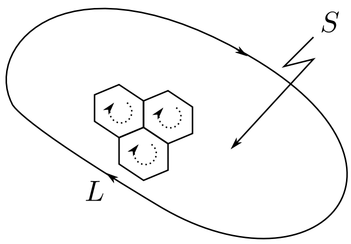
$$\mathrm{d}P_C = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} \right)\mathrm{d}A_x = \omega_x \mathrm{d}A_x$$

$$\mathrm{d}P = (\nabla \times \boldsymbol{v}) \cdot \hat{\boldsymbol{n}}\mathrm{d}S \,.$$

The circulation of $\boldsymbol{v}$ around a small loop equals the flux of curl of $\boldsymbol{v}$ through it. So *curl* is some kind of circulation density per unit area and can be contrasted with *divergence*, which is flux density per unit volume. It measures the rigour with which the field departs from the state of being conservative.

3.2 Stoke's Theorem

The small element $\mathrm{d}\boldsymbol{A}$ can be part of any finite surface S spanning a given loop L. If the surface is divided into small elements and the circulation $\mathrm{d}\Gamma$ around the elements are added up together, the circulation from boundary lines between pairs of adjoining elements cancel out and only contribution from those parts of the element boundaries which form part of L survive, such that:

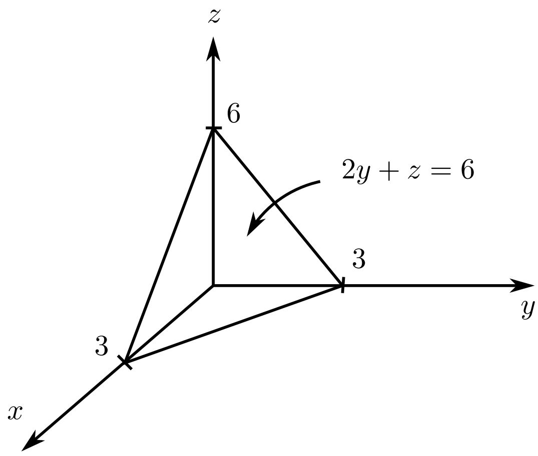


$$\Gamma = \oint_L \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{v}) \cdot \mathrm{d}\boldsymbol{A} \,.$$

This is Stoke's Theorem for any surface S spanning a loop L.

N.B. This only applies to open surfaces such as Butterfly net.

Example 7. Verify by direct evaluation of the integrals involved the divergence theorem for $\boldsymbol{A} = (2xy + z)\hat{\boldsymbol{i}} + y^2\hat{\boldsymbol{j}} - (x + 3y)\hat{\boldsymbol{k}}$ taken over the region bounded by the four surfaces: $2x+2y+z = 6, \quad x = 0, \quad y = 0 \quad , z{-}0$



Divergence Theorem:

$$\iint\limits_S \boldsymbol{A}\cdot\hat{\boldsymbol{n}}\mathrm{d}S = \iiint\limits_V \nabla\cdot\boldsymbol{A}\mathrm{d}V.$$

Fluxes - Surface Integrals

i Surface $x = 0$
On $x - 0$, $\boldsymbol{A} = z\hat{\boldsymbol{\imath}} + y^2\hat{\boldsymbol{\jmath}} - 3y\hat{\boldsymbol{k}}$ and outward pointing normal is $-\hat{\boldsymbol{\imath}}$, so:

$$\iint\limits_{\substack{x=0\\ \text{plane}}} \boldsymbol{A}\cdot\hat{\boldsymbol{n}}\mathrm{d}S = \iint(-z)\mathrm{d}S$$

$$= \int_{z=0}^{6}\left\{\int_{y=0}^{\frac{6-z}{2}}(-z)\mathrm{d}y\right\}\mathrm{d}z$$

$$= -\int_0^6 (3 - v_z z)\,\mathrm{d}z$$

$$= +\int_0^6\left(\frac{1}{2}z^2 - 3z\right)\mathrm{d}z$$

$$= +18$$

ii Surface $y = 0$

$$\boldsymbol{A} = z\hat{\boldsymbol{\imath}} - x\hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{n}} = -\hat{\boldsymbol{\jmath}},$$

$$\therefore \boldsymbol{A}\cdot\hat{\boldsymbol{n}} = 0$$

so zero flux across this surface.

iii Surface $z = 0$

so

$$
\begin{aligned}
\boldsymbol{A} &= 2xy\hat{\boldsymbol{i}} + y^2\hat{\boldsymbol{j}} - (x+3y)\hat{\boldsymbol{k}}, \qquad \hat{\boldsymbol{n}} = -\hat{\boldsymbol{k}} \\
\therefore \boldsymbol{A}\cdot\hat{\boldsymbol{n}} &= x + 3y \\
\text{Flux } &= \iint\limits_{\substack{z=0\\ \text{plane}}} (x+3y)\mathrm{d}S \\
&= \int_{y=0}^{3} \left\{ \int_{x=0}^{3-y} (x+3y)\mathrm{d}x \right\} \mathrm{d}y \\
&= \left[-\frac{5}{6}y^3 + \frac{12}{4}y^2 + \frac{9}{2}y \right]_0^3 \\
&= 18
\end{aligned}
$$

iv On sloping face $z = 6 - 2x - 2y$:

$$
\begin{aligned}
\boldsymbol{A} &= (2xy + z)\hat{\boldsymbol{i}} + y^2\hat{\boldsymbol{j}} - (x+3y)\hat{\boldsymbol{k}} \\
\hat{\boldsymbol{n}} &= \frac{\nabla\phi}{|\nabla\phi|} = \frac{2}{3}\hat{\boldsymbol{i}} + \frac{2}{3}\hat{\boldsymbol{j}}\frac{1}{3}\hat{\boldsymbol{k}}
\end{aligned}
$$

where

$$
\begin{aligned}
\phi &= z + 2x + 2y \\
\therefore \boldsymbol{A}\cdot\hat{\boldsymbol{n}} &= \frac{4}{3}xy + \frac{2}{3}z + \frac{2}{3}y^2 - \left(\frac{x+3y}{3}\right)
\end{aligned}
$$

But on this surface

$$
z = 6 - 2x - 2y
$$

so

$$
\begin{aligned}
\hat{\boldsymbol{A}}\cdot\hat{\boldsymbol{n}} &= \frac{4}{5}xy + 4 - \frac{4}{3}x - \frac{4}{3}y + \frac{2}{3}y^2 - \frac{1}{3}x - y \\
&= \frac{4}{3}xy + \frac{2}{3}y^2 - \frac{5}{3}x - \frac{7}{3}y + 4 \\
\text{Flux } &= \iint\limits_{S} \boldsymbol{A}\cdot\hat{\boldsymbol{n}}\mathrm{d}S = \iint\limits_{R} (\boldsymbol{A}\cdot\hat{\boldsymbol{n}})\sec\gamma\mathrm{d}x\mathrm{d}y
\end{aligned}
$$

where

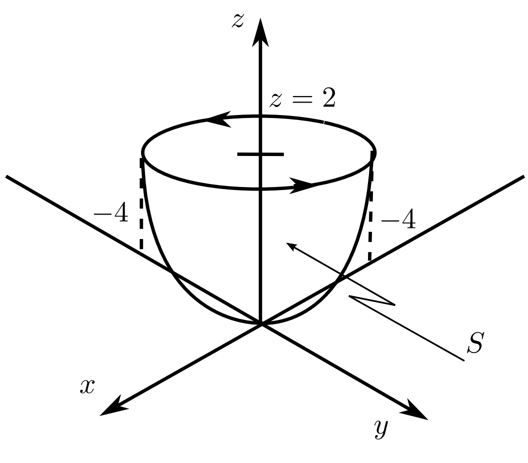
$$
\begin{aligned}
\cos\gamma &= \hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}} = \frac{1}{3} \\
\therefore \text{Flux } &= \int_{x=0}^{3} \left\{ \int_{y=0}^{3-x} \left[\frac{4}{3}xy - \frac{5}{3}x - \frac{7}{3}y + \frac{2}{3}y^2 + 4 \right] 3\mathrm{d}y \right\} \mathrm{d}x \\
&= -18
\end{aligned}
$$

So the net flux out of the region is +18, *i.e.* it contains a source.

Evaluating $\iiint_V \nabla \cdot \boldsymbol{A} \mathrm{d}V$:

$$\begin{aligned}
\nabla \cdot \boldsymbol{A} &= \frac{\partial}{\partial x}(2xy+z) + \frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial z}(x+3y) \\
&= 2y + 2y = 4y. \\
\therefore \iiint \nabla \cdot \boldsymbol{A} \mathrm{d}V &= 4\int_{y=0}^{3}\int_{x=0}^{3-y}\int_{z=0}^{6-2x-2y} y \mathrm{d}z\mathrm{d}x\mathrm{d}y. \\
&= 4\int_{y=0}^{3} y\left\{\int_{x=0}^{3-y}(6-2x-2y)\mathrm{d}x\right\}\mathrm{d}y \\
&= 4\int_{0}^{3}\left[6(3-y)-(3-y)^2-2y(3-y)\right] y\mathrm{d}y \\
&= 18
\end{aligned}$$

Example 8. Verify Stokes's theorem for $\boldsymbol{A} = 3y\hat{\boldsymbol{\imath}} - xz\hat{\boldsymbol{\jmath}} + yz^2\hat{\boldsymbol{k}}$ where S is the surface of the parabloid $2z = x^2 + y^2$ and C is the curve ($x^2 + y^2 = 1$ and $z = 2$).



To verify: $\oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} = \iint_S (\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} \mathrm{d}S$

i Line Integral

$$\text{Curve } C: \qquad \boldsymbol{r} = 2\cos t\hat{\boldsymbol{\imath}} + 2\sin t\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}$$

$$\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = -2\sin t\hat{\boldsymbol{\imath}} + 2\cos t\hat{\boldsymbol{\jmath}}, \qquad \boldsymbol{A} = 3y\hat{\boldsymbol{\imath}} - xz\hat{\boldsymbol{\jmath}} + yz^2\hat{\boldsymbol{k}}$$

$$\therefore \int_C \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{r} = \int_{L=0}^{2\pi} [6\sin t\,(-2\sin t) - 4\cos t\,(2\cos t)]\,\mathrm{d}t$$

$$= 4\int_0^{2\pi} \left(-3\sin^2 t - 2\cos^2 t\right)\mathrm{d}t = -20\pi$$

Consider $\iint_S (\nabla\times\boldsymbol{A})\cdot\hat{\boldsymbol{n}}\mathrm{d}S$:

Now:

$$\nabla\times\boldsymbol{A} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix} = (z^2+x)\hat{\boldsymbol{\imath}} - (3+z)\hat{\boldsymbol{k}}.$$

but:

$$\hat{\boldsymbol{n}} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2x\hat{\boldsymbol{\imath}} - 2y\hat{\boldsymbol{\jmath}} + 2\hat{\boldsymbol{k}}}{2\sqrt{1+x^2+y^2}}$$

where $\phi = 2z - x^2 - y^2$

$$\therefore (\nabla\times\boldsymbol{A})\cdot\hat{\boldsymbol{n}} = \frac{-x(z^2+x) - (z+3)}{\sqrt{1+x^2y^2}}$$

Now

$$\iint\limits_S (\nabla\times\boldsymbol{A})\cdot\mathrm{d}S = \iint\limits_R (\nabla\times\boldsymbol{A})\cdot\hat{\boldsymbol{n}}\left(\frac{1}{\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}}}\right)\mathrm{d}x\mathrm{d}y$$

But

$$\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{k}} = \frac{1}{\sqrt{1+x^2+y^2}}$$

$$\therefore \iint\limits_S (\nabla\times\boldsymbol{A})\cdot\hat{\boldsymbol{n}}\mathrm{d}S = -\iint\limits_R (xz^2+x^2+z+3)\mathrm{d}x\mathrm{d}y$$

using $z = \frac{1}{2}(x^2+y^2)$

$$= -\iint\limits_R \left[\frac{1}{4}x(x^2+y^2)^2 + \frac{3}{2}x^2 + \frac{1}{2}y^2 + 3\right]\mathrm{d}x\mathrm{d}y$$

using polar coordinates:

$$R' = \{(r, \theta) : \qquad 0 \le r \le 2, \qquad -\pi \le \theta \le \pi\}$$

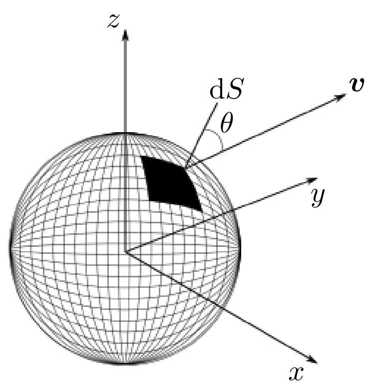
$$\therefore \iint\limits_S (\nabla \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} \mathrm{d}S = -\int_{-\pi}^{\pi} \int_0^2 \left[\frac{1}{4}r^5 \cos\theta + \frac{3}{2}r^2 \cos^2\theta + \frac{1}{2}r^2 \sin^2\theta + 3\right] r \mathrm{d}r \mathrm{d}\theta$$

$$= -\int_0^2 \left[\frac{3}{2}\pi r^3 + \frac{1}{2}\pi r^3 + 6\pi r\right] \mathrm{d}r$$

$$= -20\pi$$

3.3 Fluid Flow

Consider a control volume V through which a fluid is flowing. Mass flow rate of fluid going out of small elemental surface area $\mathrm{d}S = \rho|\boldsymbol{v}|\mathrm{d}S\cos\theta$ or if we define $\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\mathrm{d}S$ where $\hat{\boldsymbol{n}}$ is the unit normal to $\mathrm{d}S$ then the volume flow rate $= \rho\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$.



If we divide the surface into a large number of small elemental areas, then the total mass flux through the surface is

$$\text{Flux} = \sum_{\substack{\text{over the}\\ \text{surface}}} \rho\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$$

Now the rate of decrease of matter inside the control volume is

$$= \frac{\partial}{\partial t} \iiint\limits_V \rho \mathrm{d}V$$

$$= \iiint\limits_V \frac{\partial \rho}{\partial t} \mathrm{d}V$$

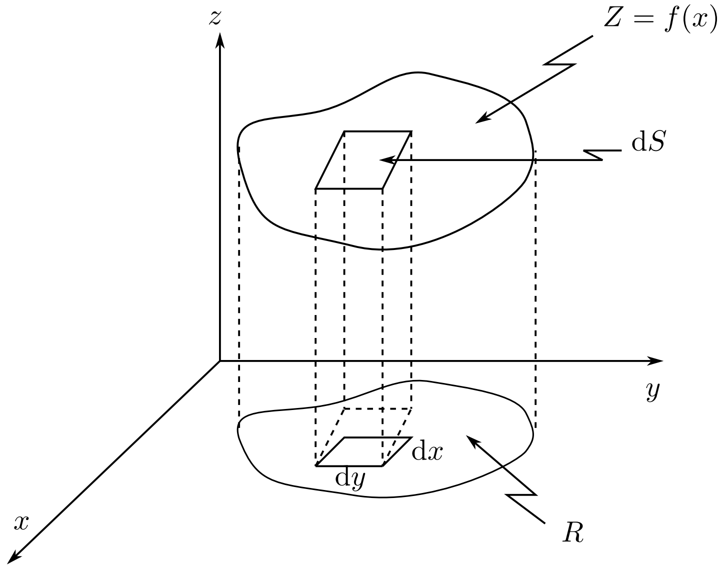
For conservation of mass;

$$\sum_{\substack{\text{over the}\\ \text{surface}}} \rho\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = -\iiint\limits_V \frac{\partial \rho}{\partial t} \mathrm{d}V$$

We want to obtain a PDE for $\boldsymbol{v}$, so that we can express mass conservation as a governing equation for $\boldsymbol{v}$. To do so we need to be able to evaluate surface integrals.

3.4 Surface Integrals

Let us consider a small area $\mathrm{d}S$ on a surface $Z = f(x, y)$ whose projection on the (x, y) plane is given by $\mathrm{d}x\mathrm{d}y$.

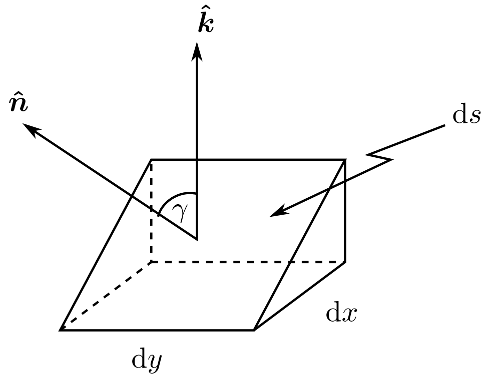


Now let γ be the acute angle between $\mathrm{d}S$ and the (x, y) plane.

Hence $\mathrm{d}x\mathrm{d}y = \mathrm{d}S\cos\gamma$. The surface is then;

$$\iint \mathrm{d}S = \iint_R \sec\gamma \mathrm{d}x\mathrm{d}y$$

where R is the projected area of surface Z on to the (x, y) plane.



Now the acute angle between the two planes is the same as the acute angle between the normal to the planes. If $\hat{\boldsymbol{n}}$ is the normal to the surface at $\mathrm{d}S$, then γ is the acute angle between the $\hat{\boldsymbol{n}}$ and the z-axis, *i.e.* between vectors $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{k}}$.

$$\therefore |\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}| = \cos\gamma$$

The equation of the surface can be written as;

$$\begin{aligned} \phi(x, y, z) &= \text{ a constant} \\ \therefore \nabla\phi &= \frac{\partial\phi}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial\phi}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial\phi}{\partial z}\hat{\boldsymbol{k}} \end{aligned}$$

and

$$\begin{aligned} \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} &= \frac{\nabla\phi}{|\nabla\phi|} = \text{ unit normal} \\ \therefore \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}} &= \frac{\hat{\boldsymbol{k}} \cdot \nabla\phi}{|\nabla\phi|} = \frac{\partial\phi/\partial z}{|\nabla\phi|} \end{aligned}$$

But

$$\begin{aligned} \sec\gamma &= \frac{1}{\cos\gamma} = \frac{1}{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{k}}} \\ \therefore \sec\gamma &= \frac{|\nabla\phi|}{\partial\phi/\partial z} = \frac{\sqrt{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2}}{\partial\phi/\partial z} \end{aligned}$$

or if we say

$$\phi(x, y, z) = Z - f(x, y)$$

then

$$\begin{aligned} \frac{\partial\phi}{\partial z} &= 1 \\ \therefore \sec\gamma &= \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \end{aligned}$$

$$\iint_S \mathrm{d}S = \iint_R \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}\mathrm{d}x\mathrm{d}y$$

Example 9. Find the surface area of the paraboloid $z = 2 - (x^2 + y^2)$. Projection onto (x, y) plane is a circle of radius $\sqrt{2}$.

$$\begin{aligned} \text{Area } &= \iint_S \mathrm{d}S = \iint_R \sqrt{1 + fx^2 + fy^2}\mathrm{d}\mathrm{d}y \\ &= \iint_R \sqrt{1 + 4x^2 + 4y^2}\mathrm{d}x\mathrm{d}y \\ &= \int_{r=0}^{\sqrt{2}} \int_{\theta=-\pi}^{\pi} \sqrt{1 + 4r^2} r\mathrm{d}r \end{aligned}$$

let $u = r^2$ and $\mathrm{d}u = 2r\mathrm{d}r$

$$= \pi \int_{u=0}^{2} \sqrt{1+4u}\mathrm{d}u$$
$$= \frac{1}{4}\left[\frac{2\pi}{3}(1+4u)^{3/2}\right]_0^2 = \frac{13}{3}\pi$$

Example 10. Find the surface area of the hemisphere $x^2 + y^2 + z^2 \geq a^2$, $z \geq 0$. Projection onto (x, y) plane is a circle of radius a.

$$\text{Surface Area } = \iint\limits_S \mathrm{d}S = \iint\limits_R \frac{\sqrt{\phi x^2 + \phi y^2 + \phi z^2}}{\phi z}\mathrm{d}x\mathrm{d}y$$

where $\phi = x^2 + y^2 + z^2$

$$\text{Surface Area} = \iint\limits_R \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}\mathrm{d}x\mathrm{d}y$$
$$= a\iint\limits_R \frac{\mathrm{d}x\mathrm{d}y}{\sqrt{a^2 - x^2 - y^2}}$$
$$= a\iint\limits_R \frac{r\mathrm{d}r\mathrm{d}\theta}{\sqrt{a^2 - r^2}}$$
$$= a\int_{r=0}^{a}\int_{\theta=-\pi}^{\pi} \frac{r}{\sqrt{a^2 - r^2}}\mathrm{d}r\mathrm{d}\theta$$
$$= 2\pi a\int_{r=0}^{a} \frac{r}{\sqrt{a^2 - r^2}}\mathrm{d}r$$

let $r = a\sin\theta$

$$= 2\pi a^2$$

3.5 Flux of a Vector

We showed earlier that the rate at which mass is transported across an arbitrary surface element at point P where density is ρ and velocity is $\boldsymbol{v}$ is given by

$$\rho\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\mathrm{d}S,$$

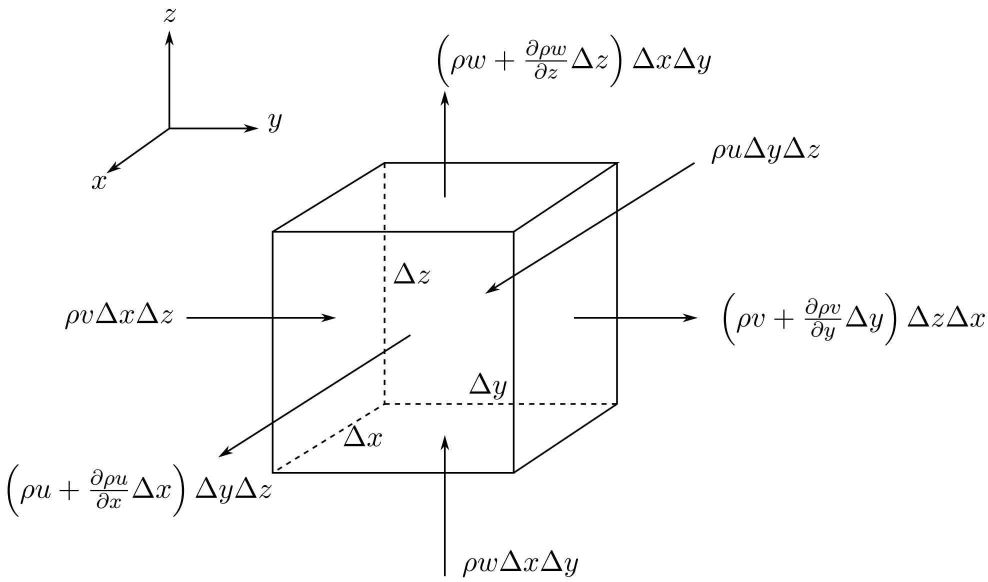
where $\mathrm{d}S$ is the area of the surface element and $\hat{\boldsymbol{n}}$ is the unit normal to the surface. The vector field $\rho\boldsymbol{v}$ is called the *mass flux vector*.

In general, the flux of any vector field $\boldsymbol{F} = \boldsymbol{F}(x, y, z)$ through a surface element $\mathrm{d}\boldsymbol{S} = \hat{\boldsymbol{n}}\cdot\mathrm{d}\boldsymbol{S}$ is given by

$$\boldsymbol{F}\cdot\mathrm{d}\boldsymbol{S} = \boldsymbol{F}\cdot\hat{\boldsymbol{n}}\mathrm{d}S.$$

3.6 Divergence

Consider the flow of fluid through a small rectangular elemental volume of dimensions $\Delta x \Delta y \Delta z$ with velocity $\boldsymbol{v} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$ and density ρ.



Now the net flux of mass;

$$= \frac{\partial \rho u}{\partial x}\Delta x\Delta y\Delta z + \frac{\partial \rho v}{\partial y}\Delta x\Delta y\Delta z + \frac{\partial \rho w}{\partial z}\Delta x\Delta y\Delta z$$

$$= \left[\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}\right]\Delta V$$

$$\therefore \nabla \cdot (\rho \boldsymbol{v}) = \text{ Net flux of mass per unit volume}$$

$$= \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho v}{\partial z}.$$

where $\nabla(\rho\boldsymbol{v})$ is the *divergence* of the mass flux vector.

3.7 Gauss's Divergence Theorem

Consider a finite volume V bounded by surface S. The volume V can be divided into small elemental volumes with dimensions $\Delta x \Delta y \Delta z$. The fluxes through the neighbouring faces cancel out and what remains is the flux through the surface.

$$\sum_{\substack{\text{over all}\\ \text{elements}}} \rho \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \mathrm{d}S = \iiint_V \nabla \cdot (\rho \boldsymbol{v}) \mathrm{d}V$$

$$\therefore \boxed{\iint_S \rho \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \mathrm{d}S = \iiint_V \nabla \cdot (\rho \boldsymbol{v}) \mathrm{d}V.}$$

This is known as the Divergence theorem.

3.8 Application of Divergence Theorem

3.8.1 Continuity Equation of Fluid Dynamics

Consider a control volume V with control surface S. The net flux of fluid out of this surface S is given by

$$\iint_S \rho \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \mathrm{d}S.$$

As a result the quantity fluid inside V is changed by

$$-\iiint_V \frac{\partial \rho}{\partial t} \mathrm{d}V.$$

Hence, by conservation of mass:

$$\iint_S \rho \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \mathrm{d}S = -\iiint_V \frac{\partial \rho}{\partial t} \mathrm{d}V.$$

Applying the Divergence theorem;

$$\iiint_V \nabla(\rho \boldsymbol{v}) \mathrm{d}V + \iiint_V \frac{\partial \rho}{\partial t} \mathrm{d}V = 0,$$

or in differential form

$$\nabla \cdot (\rho \boldsymbol{v}) + \frac{\partial \rho}{\partial t} = 0.$$

This differential equation is known as the Continuity Equation of Fluid Dynamics.

3.9 Substantial Derivatives

Assume we know the density ρ as a function of x, y, z and t, *i.e.*

$$\rho = \rho(x, y, z, t).$$

Now if a fluid particle moves along a path $\boldsymbol{r}(t) = x(t)\hat{\boldsymbol{\imath}} + y(t)\hat{\boldsymbol{\jmath}} + z(t)\hat{\boldsymbol{k}}$, then we see that along this path,

$$\begin{aligned}
\rho &= \rho\,[x(t), y(t), z(t), t] \\
\frac{\mathrm{d}\rho}{\mathrm{d}t} &= \left[\frac{\partial \rho}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial \rho}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial \rho}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} + \frac{\partial \rho}{\partial t}\right] \\
&= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \\
\therefore \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho,
\end{aligned}$$

When ρ is steady, *i.e.* is not a function of time, we find that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \boldsymbol{v} \cdot \nabla \rho,$$

i.e. is not equal to zero.

$\boldsymbol{v} \cdot \nabla \rho = \nabla \rho \cdot \boldsymbol{v}$ is a directional derivative and is the rate of change of density along a streamline, *i.e.* $\boldsymbol{v} \cdot \nabla \rho$ measures the contribution due to spatial non-uniformity.

Now using the substantial derivative, the continuity equation becomes;

$$\nabla \cdot (\rho \boldsymbol{v}) + \frac{\mathrm{d}\rho}{\mathrm{d}t} - \boldsymbol{v} \cdot \nabla \rho = 0,$$

or

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \boldsymbol{v} \cdot \nabla \rho - \nabla \cdot (\rho \boldsymbol{v}).$$

But if we let $\boldsymbol{v} = u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}$, then;

$$
\begin{aligned}
\nabla \cdot (\rho \boldsymbol{v}) &= \nabla \left[\rho (u\hat{\boldsymbol{\imath}} + v\hat{\boldsymbol{\jmath}} + w\hat{\boldsymbol{k}}) \right] \\
&= \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \\
&= \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} \\
&= \rho \nabla \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \rho \\
\therefore \frac{\mathrm{d}\rho}{\mathrm{d}t} &= \cancel{\boldsymbol{v} \cdot \nabla \rho} - \cancel{\boldsymbol{v} \cdot \nabla \rho} - \rho \nabla \cdot \boldsymbol{v} = -\rho \nabla \cdot \boldsymbol{v} \\
\therefore \nabla \cdot \boldsymbol{v} &= -\frac{1}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}.
\end{aligned}
$$

For water, $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$ *i.e.* incompressible. Hence,

$$\nabla \cdot \boldsymbol{v} = 0,$$

i.e. the velocity field of incompressible flow is *solenoidal.*

When $\nabla \cdot \boldsymbol{v} = 0$, this implies that

$$\iint_S \boldsymbol{v} \cdot \hat{\boldsymbol{n}} \mathrm{d}S = \iiint_V (\nabla \cdot \boldsymbol{v}) \mathrm{d}V = 0,$$

i.e. the net flux of $\boldsymbol{v}$ out of any surface S is zero.

Example 11. Does the velocity $\boldsymbol{v} = (x^2 - yz)\hat{\boldsymbol{\imath}} - 2yz\hat{\boldsymbol{\jmath}} + (z^2 - 2zx)\hat{\boldsymbol{k}}$ represent an incompressible flow?

Solution For any incompressible flow, $\nabla \cdot \boldsymbol{v} = 0$. Now

$$
\begin{aligned}
\nabla \cdot \boldsymbol{v} &= \frac{\partial}{\partial x}(x^2 - yz) - \frac{\partial}{\partial y}(-2yz) + \frac{\partial}{\partial z}(z^2 - 2zx) \\
&= 2x - 2z + 2z - 2x \\
&= 0.
\end{aligned}
$$

So $\boldsymbol{v}$ is solenoidal and represents velocity of incompressible flow.

3.10 Equation of Heat Conduction

Consider a conducting medium, volume V with density $\rho(x, y, z)$, thermal conductivity $\kappa(x, y, z)$ and heat capacity c. Let $T(x, y, z, t)$ be the temperature. Now the rate of storage or loss of heat from V is

$$\iiint_V c\rho \frac{\partial T}{\partial t} \mathrm{d}V.$$

If $\boldsymbol{q}$ is the conduction vector (or heat transfer unit area) is given by;

$$\iint\limits_S \boldsymbol{q} \cdot \hat{\boldsymbol{n}} \mathrm{d}S.$$

In the absence of heat source and radiation;

$$\iint\limits_S \boldsymbol{q} \cdot \hat{\boldsymbol{n}} \mathrm{d}S = -\iiint\limits_V \left(c\rho \frac{\partial T}{\partial t} \right) \mathrm{d}V.$$

Applying Divergence theorem;

$$\iint\limits_S \boldsymbol{q} \cdot \hat{\boldsymbol{n}} \mathrm{d}S = \iiint\limits_V \nabla \cdot \hat{\boldsymbol{q}} \mathrm{d}V$$

$$\therefore \iiint\limits_V \left(\nabla \cdot \boldsymbol{q} + c\rho \frac{\partial T}{\partial t} \right) \mathrm{d}V$$

$$\therefore \nabla \cdot \boldsymbol{q} + c\rho \frac{\partial T}{\partial t} = 0. \tag{1}$$

But from Fourier's law of heat conduction;

> *"Heat transfer per unit area is proportional to the normal temperature gradient."*

i.e. for a slab,

$$q_x \propto \frac{\partial T}{\partial x},$$

or

$$q_x = -\kappa \frac{\partial T}{\partial x},$$

where κ is the coefficient of thermal conductivity and the sign is negative because heat goes along the temperature gradient from high to low temperature. Similarly,

$$q_y = -\kappa \frac{\partial T}{\partial t},$$

and in general

$$\boldsymbol{q} = -\kappa \nabla T. \tag{2}$$

Substituting (2) into (1) we get;

$$-\nabla \cdot \kappa \nabla T + c\rho \frac{\partial \rho}{\partial t} = 0.$$

But

$$\nabla(\kappa\nabla T) = \kappa\nabla\cdot\nabla T + \nabla T\cdot\nabla K,$$

and so

$$\begin{aligned}\nabla\cdot\nabla T &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial T}{\partial z}\right)\\ &= \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\\ &= \nabla^2 T,\end{aligned}$$

where ∇^2 is known as the Laplacian operator. So the equation of heat conduction is;

$$\kappa\nabla^2 T + \nabla\kappa\cdot\nabla T = c\rho\frac{\partial T}{\partial t},$$

and when κ is constant;

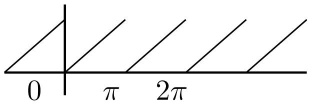
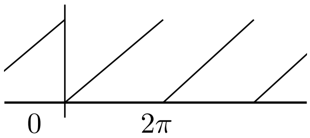
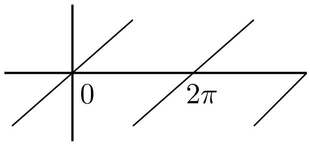
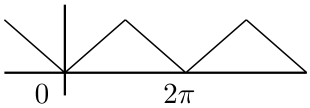
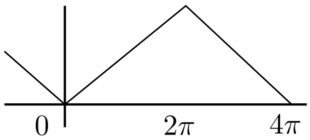
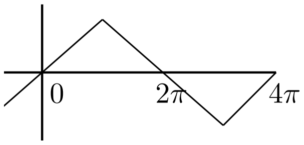
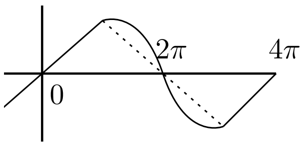
$$\frac{\partial T}{\partial t} = \frac{\kappa}{c\rho}\nabla^2 T.$$

For steady state case, $\frac{\partial T}{\partial t} = 0$, so we get

$$\nabla^2 T = 0,$$

i.e. Laplace's Equation.

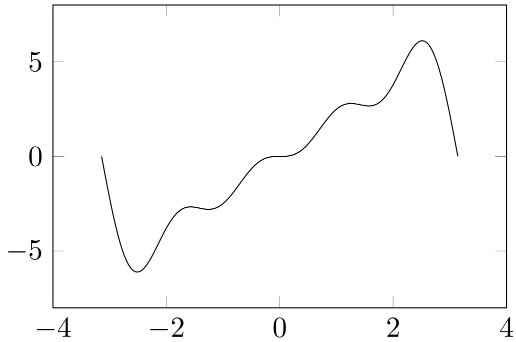
4 Fourier Series

	Example	Period	Periodic function
1		π	$\frac{1}{2} - \frac{1}{\pi}\left(\sin 2x + \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x \ldots\right)$
2		2π	$1 - \frac{2}{\pi}\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right)$
3		2π	$\frac{2}{\pi}\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right)$
4		2π	$\frac{1}{2} - \frac{1}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$
5		4π	$1 - \frac{8}{\pi^2}\left(\cos \frac{1}{2}x + \frac{1}{3^2}\cos \frac{3}{2}x - \frac{1}{5^2}\cos \frac{5}{2}x + \ldots\right)$
6		4π	$\frac{8}{\pi^2}\left(\sin \frac{1}{2}x - \frac{1}{3^2}\sin \frac{3}{2}x + \frac{1}{5^2}\sin \frac{5}{2}x - \ldots\right)$
7		4π	$\frac{8}{\pi^2}\left(\frac{4}{3}\sin \frac{1}{2}x + \frac{1}{45}\sin \frac{3}{2}x - \frac{4}{525}\sin \frac{5}{2}x + \ldots\right)$

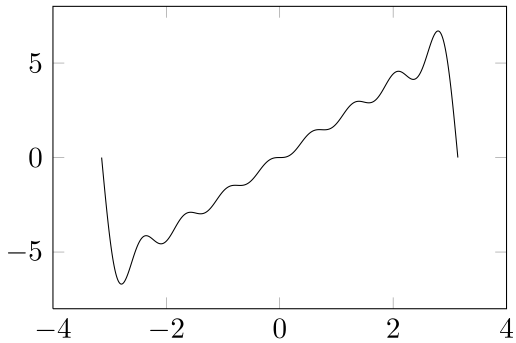
4.1 Partial Summation

$$f(x) = 2x, \qquad -\pi \le x \le \pi$$

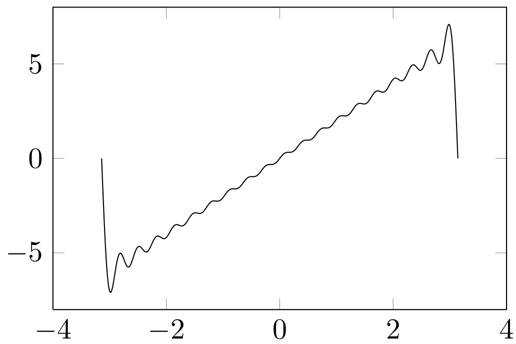
$$S_4(x) = \sum_{n=1}^{4} \frac{4}{n}(-1)^{n+1} \sin nx$$



$$S_8(x) = \sum_{n=1}^{8} \frac{4}{n}(-1)^{n+1} \sin nx$$

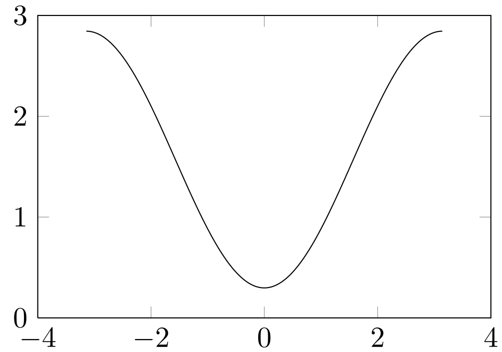


$$S_{19}(x) = \sum_{n=1}^{19} \frac{4}{n}(-1)^{n+1} \sin nx$$

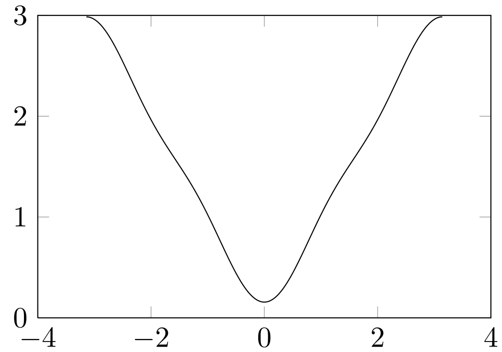


$$f(x) = |x|, \qquad -\pi \le x \le \pi$$

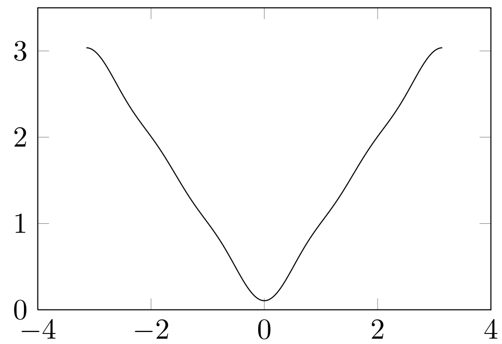
$S_1(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x$



$S_2(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$



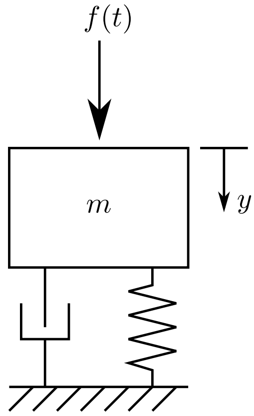
$S_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$
$- \frac{4}{25\pi}\cos 5x$



Example 12. A body of mass 1 kg is attached to a rigid support by a spring of modulus 36 kg/s^2 and a dashpot, with damping constant 0.05 kg/s. The system is kept in motion by an external force $f(t)$ Newtons which is periodic with period $2T$ and is given by:

$$f(t) = f(t+2T) = \begin{cases} C(t+T) & \text{in } -T \le t < 0; \\ Ct & 0 \le t < T. \end{cases}$$

Find the steady state solution.



Solution. The governing equation is

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 0.05\frac{\mathrm{d}y}{\mathrm{d}t} + 36y = f(t), \tag{3}$$

where $y(t)$ is the displacement in metres. Now

$$\begin{aligned}
y(t) &= \underbrace{u(t)}_{\text{C.F. (or transient)}} + \underbrace{v(t)}_{\text{P.I. (or steady state solution)}} \\
u(t) &= \text{C.F.} = \text{solution of homogeneous equation} \\
&= e^{-\alpha t}\left[Ae^{\sqrt{\alpha^2\omega^2}t} + Be^{-\sqrt{\alpha^2\omega^2}t}\right],
\end{aligned}$$

where $\alpha = 0.05$ and $\omega^2 = 36$. This solution tends to zero after sufficiently long time. To calculate the steady state solution $v(t)$ we first need to express the forcing function $f(t)$ by a Fourier Series:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi t}{T}\right) + b_n\sin\left(\frac{n\pi t}{T}\right)\right],$$

where:

$$\begin{aligned}
a_0 &= \frac{1}{\int}{}^{T}_{-T} f(t)\mathrm{d}t = \frac{1}{T}\int_{-T}^{0} C(t+T)\mathrm{d}t + \frac{1}{T}\int_{0}^{T} Ct\mathrm{d}t \\
&= \frac{C}{T}\left[\frac{t^2}{2} + Tt\right]_{-T}^{0} + \frac{C}{T}\left[\frac{t^2}{2}\right]_{0}^{T} \\
&= \frac{CT}{2} + \frac{CT}{2} = CT \\
a_n &= \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t \\
&= \frac{C}{T}\int_{-T}^{0}\underbrace{(t+T)\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t}_{\mathrm{I}} + \frac{C}{T}\int_{0}^{T}\underbrace{t\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t}_{\mathrm{II}}.
\end{aligned}$$

Using integration by parts:

$$\mathrm{I} = \left[\frac{(t+T)}{p}\cancel{\sin}(pt)\right]_{-T}^{0} - \int_{-T}^{0} \frac{1}{p}\sin(pt)\mathrm{d}t$$

where $p = \frac{n\pi}{T}$.

$$\therefore \mathrm{I} = \left[\frac{1}{p^2}\cos(pt)\right]_{-T}^{0}.$$

Similarly:

$$\begin{aligned}
\mathrm{II} &= \left[\frac{t}{p}\cancel{\sin}(pt)\right]_{0}^{T} - \int_{0}^{T} \frac{1}{p}\sin(pt)\mathrm{d}t \\
&= \left[\frac{1}{p^2}\cos(pt)\right]_{0}^{T} \\
\mathrm{I} + \mathrm{II} &= \frac{1}{p^2}[1 - \cos(pt) + \cos(pt) - 1] = 0 \\
\therefore a_n &= 0 \\
b_n &= \frac{C}{T}\left[\int_{-T}^{0} (t+T)\sin\left(\frac{n\pi t}{T}\right)\mathrm{d}t + \int_{0}^{T} t\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t\right] \\
\therefore b_n &= \begin{cases} -\frac{2CT}{n\pi} & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd.} \end{cases} \\
\therefore f(t) &= \frac{CT}{2} - \frac{2CT}{\pi}\sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi t}{T}\right). \qquad (4)
\end{aligned}$$

Now, by superposition, let

$$V(t) = V_0(t) + \sum_{n=1}^{\infty} V_n(t), \qquad (5)$$

where

$$V_n = A_n\cos\left(\frac{n\pi t}{T}\right) + B_n\sin\left(\frac{n\pi t}{T}\right). \qquad (6)$$

Differentiating (6) and substituting into (3) we get:

$$\cos\left(\frac{n\pi t}{T}\right) : -\left(\frac{n\pi}{T}\right)^2 A_n + 0.05\left(\frac{n\pi}{T}\right) B_n + 36A_n = 0$$
$$\sin\left(\frac{n\pi t}{T}\right) : -\left(\frac{n\pi}{T}\right)^2 B_n - 0.05\left(\frac{n\pi}{T}\right) A_n + 36B_n = b_n$$
$$\text{constant} : V_0 = \frac{CT}{72}$$
$$\therefore A_n = \frac{-0.05\frac{n\pi}{T}b_n}{\left(\frac{n^2\pi^2}{T^2} - 36\right)^2 + \left(0.05\frac{n\pi}{T}\right)^2}$$
$$B_n = \frac{-\left(\frac{n^2\pi^2}{T^2} - 36\right)b_n}{\left(\frac{n^2\pi^2}{T^2} - 36\right)^2 + \left(0.05\frac{n\pi}{T}\right)^2}$$
$$V(t) = \frac{CT}{36} + \sum_{n=2,4,6\ldots}^{\infty} \left(A_n^2 + B_n^2\right)^{1/2}\left[\frac{A_n\cos\left(\frac{n\pi t}{T}\right) + B_n\sin\left(\frac{n\pi t}{T}\right)}{(A_n^2 + B_n^2)^{1/2}}\right].$$

But if $\tan(\theta_n) = u$ and $\cos(\theta_n) = 1/\sqrt{1+u^2}$ then

$$V(t) = \frac{CT}{72} + \sum_{n=1}^{\infty} y_n \cos\left(\frac{n\pi t}{T} - \theta_n\right),$$

where $y_n = (A_n^2 + B_n^2)^{1/2}$ and $\theta_n = \arctan(B_n/A_n)$. The amplitude of vibration is y_n, which is given by:

$$y_n = \frac{b_n}{\sqrt{\left(\frac{n^2\pi^2}{T^2} - 36\right)^2 + \left(0.05\frac{n\pi}{T}\right)^2}}, \qquad \text{for } n = 2, 4, 6, \ldots$$

when $\frac{n\pi}{T} \approx 6$ then the denominator becomes very small, so the ratio y_n/b_n becomes very large. These harmonics are known as the resonance frequencies of the system.

5 Partial Differential Equations I

Most physical phenomena (in a continuous media) can be described in terms of fields *i.e.* as a function of position and time *e.g.* temperature, velocity, magnetism, *etc.*

So $T = T(x, y, z, t)$, $\boldsymbol{v} = \boldsymbol{v}(x, y, z, t)$, *etc.*

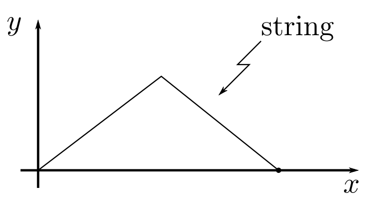
The way a physical quantity represented by a field is related from point to point or time to time is by a partial differential equation (PDE). The PDE is the mathematical statement of the way in which laws of nature govern the phenomenon.

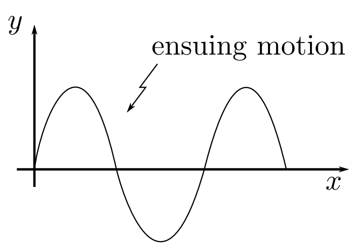
Example 13. By applying the principle of conservation of energy (a law of nature) to a conducting medium we derived the Heat Equation

$$\frac{\partial T}{\partial t} = \frac{K}{c\rho}\nabla^2 T,$$

where K is conductivity, c is specific heat capacity, and ρ is density. The Heat Equation governs the temperature distribution in a conducting medium. We also derived the Laplace's equation $\nabla^2\phi = 0$ which governs steady state heat conduction and inviscid fluid flow.

5.1 The Wave Equation





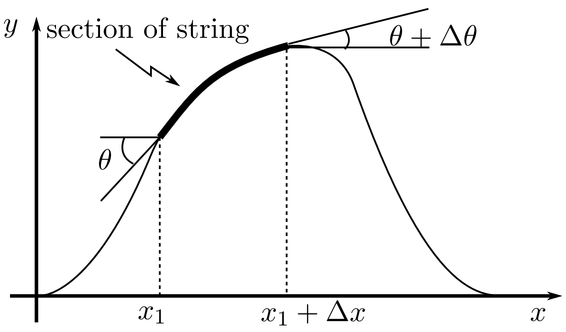
Another important equation is the wave equation, which governs the motion of a vibrating string or membrane. Consider a flexible elastic string fixed at two end points being plucked and we are interested in the ensuing motion of the string. In order to obtain the governing equation for the displacement, let us assume;

(i) The string has uniform density ρ (mass/unit length).

(ii) The string is elastic but offers no resistance to bending, so the tension in the string always acts tangentially.

(iii) Gravitational effects & air friction can be ignored.

i.e. we are considering small transverse vibrations of a flexible string under large tension.



Now consider a segment of string between x_1 and $x_1 + \Delta x$. Applying Newton's second law to this segment and considering vertical forces only;

$$T(x + \Delta x)\sin(\theta + \Delta\theta) - T(x,t)\sin(\theta) = \rho\Delta x\frac{\partial^2 y}{\partial t^2}(\bar{x}, t), \tag{7}$$

where $\bar{x}$ is the centre of mass of the segment. Then;

$$\frac{T(x + \Delta x)\sin(\theta + \Delta\theta) - T(x,t)\sin(\theta)}{\Delta x} = \rho\frac{\partial^2 y}{t^2}(\bar{x}, t).$$

Let $T_v(x,t)$ represent the vertical component of $T(x,t)$, so;

$$\begin{aligned}\frac{T_v(x + \Delta x) - T_v(x,t)}{\Delta x} &= \rho\frac{\partial^2 y}{\partial t^2}(\bar{x}, t)\\ \therefore \frac{\partial T_v}{\partial x} &= \rho\frac{\partial^2 y}{\partial t^2},\end{aligned} \tag{8}$$

as $\Delta x \to$ and $\Delta\bar{x} \to x$. Considering the horizontal component only;

$$\begin{aligned}T(x + \Delta x, t)\cos(\theta + \Delta\theta) - T(x,t)\cos(\theta) &= 0\\ \therefore T_h(x + \Delta x, t) - T_h(x,t) &= 0,\end{aligned}$$

i.e.

$$\frac{\partial T_h}{\partial x} = 0.$$

But

$$T_v = T_h\tan(\theta) \Rightarrow T_h\frac{\partial y}{\partial x}. \tag{9}$$

Substituting (9) into (8) we get;

$$\frac{\partial}{\partial x}\left(T_h\frac{\partial y}{\partial x}\right) = \rho\frac{\partial^2 y}{\partial t^2}$$

$$\therefore T_h\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\therefore \frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2},} \tag{10}$$

where a^2 is T_h/ρ. This is the one dimensional Wave Equation.

Summary			**Others**		
Heat Eq.	:	$\nabla^2 u = \frac{1}{K}\frac{\partial u}{\partial t}$	Poisson's Eq.	:	$\nabla^2 u = f(x, y, z)$
Laplace's Eq.	:	$\nabla^2 = 0$	Helmholtz Eq.	:	$\nabla^2 u + \lambda u = 0$
Wave Eq.	:	$\nabla^2 u = \frac{1}{a}\frac{\partial^2 u}{\partial t^2}$	Biharmonic Wave Eq.	:	$\nabla^4 u = -\frac{1}{\rho^2}\frac{\partial^2 u}{\partial t^2}$

6 Method of Separation of Variables

6.1 The Solution of the Wave Equation

The Wave equation is;

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{11}$$

Let us look for solutions of the form;

$$u(x,t) = X(x)T(t).$$

$$\begin{aligned} \therefore \frac{\partial^2 u}{\partial x^2} &= \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}T(t) \qquad \& \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\mathrm{d}^2 T}{\mathrm{d}t^2}X(x), \\ &= X''(x)T(t) \qquad \& \qquad = T''(t)X(x). \end{aligned}$$

Hence, (11) becomes;

$$\begin{aligned} X''(x)T(t) &= \frac{1}{c^2}T''(b)X(x) \\ \therefore \frac{X''(x)}{X(x)} &= \frac{1}{c^2}\frac{T''(t)}{T(t)} \end{aligned}$$

i.e. the right hand side is a function of t and the left hand side only a function of x;

$$\therefore \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \text{ a constant } = \alpha$$

Now we need to consider different cases.

Case 1 - $\alpha < 0$

let $\alpha = \lambda^2$

$$\begin{aligned} X'' &= \lambda^2 X & \& \qquad T &= \lambda^2 T c^2 \\ \therefore X(x) &= Ae^{\lambda x} + Be^{-\lambda x} & \& \qquad T(t) &= De^{\lambda t} + Ee^{-\lambda t} \end{aligned}$$

where A, B, D & E are arbitrary constants, which are determined from boundary conditions.

Case 2 - $\alpha < 0$

let $\alpha = -\lambda^2$

$$\begin{aligned} X'' &= -\lambda^2 X & \& \qquad T'' &= -c^2\lambda^2 T \\ \therefore X(x) &= A\cos(\lambda x) + B\sin(\lambda x) & \& \qquad T(t) &= D\cos(\lambda ct) + E\sin(\lambda ct) \end{aligned}$$

Case 3 - $\alpha = 0$

let $\alpha = 0$

$$X(x) = Ax + B \qquad \& \qquad T(t) = Dt + E$$

6.2 Boundary Conditions

Consider a stretched string fixed at both ends, with zero initial velocity *i.e.*

(i) $u(0,t) = 0$

(ii) $u(L,t) = 0$

(iii) $\frac{\partial}{\partial t}u(x,0) = 0$

(iv) $u(x,0) = f(x)$ where $f(x)$ is the initial displacement

Let us see which solution is suitable for the problem.

Case $\alpha = \lambda^2$

From boundary condition (i);

$$\begin{aligned} u(0,t) = 0 &\Rightarrow X(0)T(t) = 0, \\ \therefore X(0) = 0 &\Rightarrow A + B = 0. \end{aligned}$$

Now from boundary condition (ii);

$$\begin{aligned} u(L,t) = 0 &\Rightarrow X(L)T(t) = 0, \\ \therefore X(L) = 0 &\Rightarrow Ae^{\lambda L} + Be^{\lambda L} = 0. \end{aligned}$$

Since $e^{\lambda L} \neq 0$, then to satisfy the condition we need to have

$$A = B = 0.$$

i.e. this leads to a trivial solution.

Case $\alpha = 0$

From boundary condition (i);

$$\begin{aligned} u(0,t) = 0 &\Rightarrow X(0) = 0, \\ \therefore B &= 0. \end{aligned}$$

From boundary condition (ii);

$$\begin{aligned} u(L,t) = 0 &\Rightarrow X(L) = 0, \\ \therefore AL + B = 0 &\Rightarrow A = 0, \end{aligned} \tag{12}$$

i.e. this case also leads to a trivial solution.

Case $\boldsymbol{\alpha < 0}$

$$\therefore X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

From boundary condition (i);

$$\begin{aligned} u(0.t) = 0 &\Rightarrow X(0)T(0) = 0 \\ \therefore X(0) = 0 &\Rightarrow A = 0 \end{aligned}$$

From boundary condition (ii);

$$\begin{aligned} u(L,t) = 0 &\Rightarrow X(L)T(t) = 0 \\ \therefore X(L) = 0 &\Rightarrow B\sin(\lambda L) = 0 \end{aligned}$$

Now either $B = 0$, which results in a trivial solution again, or $\sin(\lambda L) = 0$. Hence,

$$\sin(\lambda L) = 0 \Rightarrow \lambda L = n\pi,$$

where n is an integer.

$$\therefore \lambda = \frac{n\pi}{L}, \tag{13}$$

where λ is known as the *eigenvalue* of the problem. So finally,

$$\begin{aligned} X(x) &= B\sin\left(\frac{n\pi x}{L}\right) \\ T(t) &= D\cos\left(\frac{n\pi ct}{L}\right) + E\sin\left(\frac{n\pi ct}{L}\right) \end{aligned}$$

alternatively;

$$\begin{aligned} u_n(x,t) &= \sin\left(\frac{n\pi x}{L}\right)\left[BD\cos\left(\frac{n\pi ct}{L}\right) + BE\sin\left(\frac{n\pi ct}{L}\right)\right] \\ &= \sin\left(\frac{n\pi x}{L}\right)\left[D_n\cos\left(\frac{n\pi ct}{L}\right) + E_n\sin\left(\frac{n\pi ct}{L}\right)\right], \end{aligned}$$

where $D_n = BD$ & $E_n = BE$ and $n = 1, 2, 3,$ *etc.*

The displacement $u_n(x,t)$ is referred to as the n^{th} eigenfunction or the n^{th} normal mode of the vibrating string. Now from initial condition (iii):

$$\begin{aligned} &\frac{\partial u}{\partial t}(x,0) = 0 \\ \therefore\ &X(x)T'(t) = 0 \Rightarrow T'(0) = 0. \end{aligned}$$

Now

$$T'(t) = -\frac{n\pi c}{L} D_n \sin\left(\frac{n\pi ct}{L}\right) + \frac{n\pi c}{L} E_n \cos\left(\frac{n\pi ct}{L}\right)$$
$$\therefore T'(0) = 0 \Rightarrow E_n = 0$$
$$\boxed{u_n(xt) = D_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right).}$$

Compare this to simple harmonic motion where;

$$y = A \sin \omega t \qquad \& \qquad x = A \cos \omega t,$$
$$i.e.\ D_n \sin\left(\frac{n\pi x}{L}\right) = \text{ amplitude of oscillation,}$$
$$\text{Period of oscillation } = \frac{2\pi}{\omega} = \frac{2\pi}{(n\pi c)/t} = \frac{2L}{nc}.$$

So the n^{th} normal mode vibrates with a period $2L/nc$ seconds which corresponds to a frequency of $nc/2L$ cycles per seconds. Since $c^2 = T_h/\rho$ then the frequency is equal to

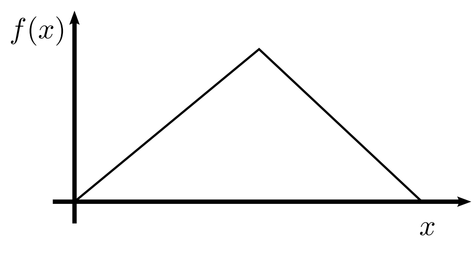
$$\frac{n}{2L}\left(\frac{T_h}{\rho}\right)^{1/2}.$$

So if the string is vibrating in a normal mode, it's pitch may be sharpened (frequency increased) by either decreasing L or increasing T_h (the tension).

Finally we need to satisfy the final initial condition;

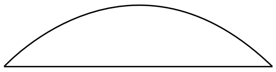
$$u(x, 0) = f(x).$$

If the string is plucked in the middle, then $f(x)$ could look like the sketch opposite.

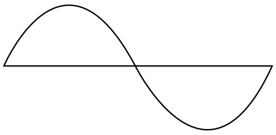


Clearly, a single normal mode of oscillation can not satisfy the initial condition. So we have to use the superposition principle

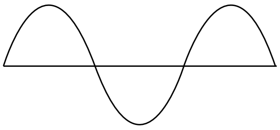
Normal modes



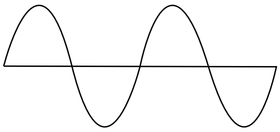
$n = 1:$ frequency $= \frac{c}{2L}$ = fundamental



$n = 2:$ frequency $= \frac{c}{L}$



$n = 3:$ frequency $= \frac{3c}{2L}$



$n = 4:$ frequency $= \frac{2c}{L}$

7 Principle of Superposition

Theorem 3. *If u_1 and u_2 are any solutions of a linear homogeneous Partial Differential Equation in some region then*

$$u = c_1 u_1 + c_2 u_2,$$

where c_1 and c_2 are constants and are also a solution of that equation in that region.

Definition 1. A partial differential equation is said to be linear if it is of the first degree in the field variables and its partial derivatives, *e.g.*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \quad \text{is linear but}$$
$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial t}\right)^2 \quad \text{is non-linear.}$$

Definition 2. A linear partial differential equation is said to be homogenous if every term in the equation contains either the field variable or one of its derivatives, *e.g.*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{is homogenous, but}$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + 3 \quad \text{is inhomogeneous.}$$

Proof. Consider $\partial^2 u/\partial x^2 = K^{-1}\partial u/\partial t$. Suppose u_1 and u_2 are solutions, then;

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{K}\frac{\partial u_1}{\partial t} \qquad \& \qquad \frac{\partial^2 u_2}{\partial x^2} = \frac{1}{K}\frac{\partial u_2}{\partial t}.$$

Now

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(u_1 c_1 + u_2 c_2\right) &= \frac{\partial^2}{\partial x^2}\left(c_1 u_1\right) + \frac{\partial^2}{\partial x^2}\left(c_2 u_2\right) \\
&= c_1 \frac{\partial u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} \\
&= c_1 \left(\frac{1}{K}\frac{\partial u_1}{\partial t}\right) + c_2 \left(\frac{1}{K}\frac{\partial u_2}{\partial t}\right) \\
&= \frac{1}{K}\frac{\partial}{\partial t}\left(c_1 u_1 + c_2 u_2\right).
\end{aligned}$$

Now since the wave equation is a linear homogenous partial differential equation, we can use the superposition principle, *i.e.* we can say that

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\ &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right), \end{aligned}$$

i.e. $u(x,t)$, which is the sum of all harmonics, is also a solution of the P.D.E.

Now applying the initial condition $u(x,0) = f(x)$ we get;

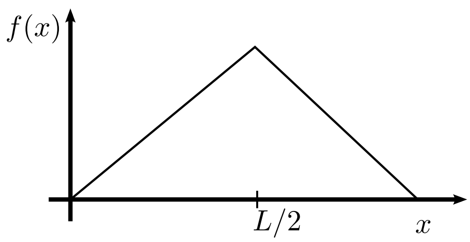
$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right),$$

i.e. this is a half-range Fourier series, which can be solved to find c_n, *i.e.*

$$\boxed{c_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.}$$

Example 14. The string is plucked in the middle according to:

$$f(x) = \begin{cases} x & \text{for } 0 \le x \le L/2 \\ L - x & \text{for } L/2 \le x \le L. \end{cases}$$



$$\begin{aligned} \therefore c_n &= \frac{2}{L}\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x + \frac{2}{L}\int_{L/2}^{L}(L-x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\ &= \frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

$$\boxed{\therefore u(x,t) = \sum_{n=1}^{\infty}\frac{4L}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right).}$$

This gives the displacement of the string.

8 Partial Differential Equations II

8.1 Solution of Heat Equation in a Bar

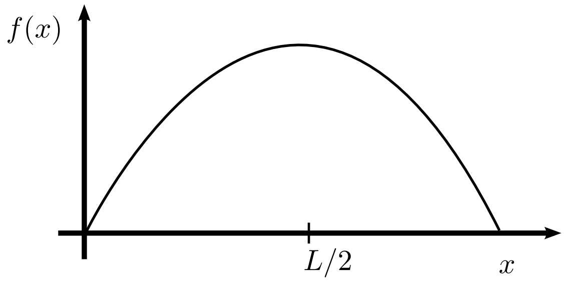
Example 15 (Bar with Constant Temperature Ends)**.** Consider the problem of determining the temperature distribution in a thin homogeneous bar of length L, given the initial temperature throughout the bar and the temperature at both ends at all times. The governing equation is;

$$\frac{\partial u}{\partial t} = \beta^2 \frac{\partial^2 u}{\partial x^2}, \tag{14}$$

where $\beta = \sqrt{K/c\rho}$.

Boundary conditions:
i $u(0,t) = 0$
ii $u(L,t) = 0$

Initial condition:
i $u(x,0) = f(x)$



Let

$$f(x) = \frac{4x}{L}\left(-\frac{x}{L} + 1\right).$$

Look for solutions of the form

$$u(x,t) = T(t)X(x)$$
$$\therefore \frac{\partial u}{\partial x} = X(x)T'(t) \qquad \& \qquad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t).$$

Substituting into (14)

$$X''(x)T(t) = \frac{1}{\beta^2}T'(t)X(x)$$
$$\therefore \frac{1}{\beta^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \alpha \text{ (a constant).}$$

Again, the only way the above equality can hold true is for the LHS and RHS ratios to be a constant.

Case 1 - $\alpha > 0$

Let $\alpha = \lambda^2$

$$X'' = \lambda^2 X \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

Applying boundary condition (i) and (ii) we can show, as in the case of the Wave equation, that a trivial solution is obtained.

Case 2 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$X'' = -\lambda^2 X \Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x)\,.$$

From boundary condition (i);

$$u(0,t) = 0 \Rightarrow X(0) = 0$$

$$\therefore A = 0\,.$$

From boundary condition (ii);

$$u(L,t) = 0 \Rightarrow B\sin(\lambda L) = 0$$

$$\therefore \lambda L = n\pi$$

$$\lambda = \frac{n\pi}{L}$$

$$X_n = B_n \sin\left(\frac{n\pi x}{L}\right)$$

Also;

$$T_n(t) = D_n \exp\left[-\frac{n^2\pi^2}{L^2}\beta^2 t\right]$$

$$u_n(x,t) = c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left[-\frac{n^2\pi^2}{L^2}\beta^2 t\right],$$

which satisfies the heat conduction equation and boundary conditions.

To satisfy the initial condition we use the superposition principle, *i.e.* ;

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)\exp\left[-\frac{n^2\pi^2}{L^2}\beta^2 t\right].$$

Thus at $t = 0$, we have;

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = f(x) = \frac{4x}{L}\left(-\frac{x}{L} + 1\right).$$

This is a half-range Fourier series, so;

$$
\begin{aligned}
c_n &= \frac{2}{L}\int_0^L \left[-\frac{4x^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{4x}{L}\sin\left(\frac{n\pi x}{L}\right)\right] \mathrm{d}x \\
&= -\frac{16}{n^3\pi^3}\left[\cos\left(n\pi\right) - 1\right] = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{32}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}
$$

$$
\therefore u(x,t) = \sum_{n=1}^{\infty} \frac{32}{(2n-1)^3\pi^3} \sin\left[\frac{(2n-1)\pi x}{L}\right] \exp\left[-\frac{(2n-1)^2\pi^2}{L}\beta^2 t\right].
$$

Example 16 (Bar with Insulated Ends)**.** Consider a bar with insulated ends *i.e.*

Boundary conditions:

i $\dot{Q} = K\frac{\partial u}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x}(0,t) = 0.$

ii $\frac{\partial u}{\partial x}(L,t) = 0$

Initial condition:

i $u(x,0) = x(L-x), \qquad 0 \leq x \leq L.$

Assume solutions of the form;

$$
u(x,t) = T(t)X(x).
$$

So as before we get;

$$
\frac{1}{\beta^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \alpha \text{ a constant.}
$$

<u>**Case 1 - $\alpha = 0$**</u>

$X(x) = Ax + B$

From boundary condition (i):

$$
\begin{aligned}
&\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow X'(0) = 0 \\
&\therefore A = 0,
\end{aligned}
$$

similarly, this satisfies boundary condition (ii),

$$
\begin{aligned}
&\frac{\partial u}{\partial x}(L,t) = 0. \\
&X(x) = B.
\end{aligned}
$$

So $\alpha = 0$ is an eigenvalue, which gives eigenfunction

$$
X(x) = \text{ a constant }.
$$

Case 2 - α

Let $\alpha = \lambda^2$

$$X'' - \lambda^2 X = 0 \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

or

$$X(x) = A\cosh(\lambda x) + B\sinh(\lambda x)$$

Now from boundary condition (i);

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow X'(0) = 0 \Rightarrow A\lambda\sinh(\lambda x) + B\lambda\cosh(\lambda x).$$
$$\therefore B = 0$$

Now from boundary condition (ii);

$$\frac{\partial u}{\partial x}(L,t) = 0$$
$$A\lambda\sinh(\lambda L) = 0 \Rightarrow A = 0$$

which gives a trivial solution. So this problem has no eigenvalues $\alpha > 0$.

Case 3 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$\therefore X'' + \lambda^2 X = 0 \Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

From boundary condition (i):

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow -A\cos(\lambda x) + B\sin(\lambda x).$$
$$\therefore B = 0$$

From boundary condition (ii):

$$\frac{\partial u}{\partial x}(L,t) = 0 \Rightarrow -AL\sin(\lambda L) = 0$$
$$\therefore \sin(\lambda L) = 0 \Rightarrow \lambda = \frac{n\pi}{L}$$

So the eigenfunction for this eigenvalue are given by

$$X_n(x) = A_n\cos\left(\frac{n\pi x}{L}\right)$$

Also

$$
\begin{aligned}
&T'(t) + \frac{n^2\pi^2}{L^2}\beta^2 T = 0 \\
&\therefore T_n(t) = D_n \exp\left[\frac{-n^2\pi^2}{L^2}\beta^2 t\right] \\
&\therefore u_n(x,t) = c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left[\frac{-n^2\pi^2{}^2}{\beta} t\right]
\end{aligned}
$$

where $c_n = a_n d_n$. To satisfy the initial condition we need to use the superposition principle, *i.e.*

$$
\begin{aligned}
&u(x,t) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) \exp\left[\frac{n^2\pi^2}{L^2}\beta^2 t\right] \\
&\therefore u(x,0) = \sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) = f(x)
\end{aligned}
$$

This is a cosine series expansion, so

$$
\begin{aligned}
c_0 &= \frac{1}{L}\int_0^L x(L-x)\mathrm{d}x = \frac{L^2}{6} \\
c_n &= \frac{2}{L}\int_0^L x(L-x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x \\
&= \begin{cases} -\frac{4L^2}{n^2\pi^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

$$
\therefore u(x,t) = \frac{L^2}{6} - \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{4n^2}\cos\left(\frac{2n\pi x}{L}\right)\exp\left[\frac{-4n^2\pi^2}{L^2}\beta^2 t\right]
$$

8.2 Steady State Heat Conduction in a Plate

Consider the heat equation

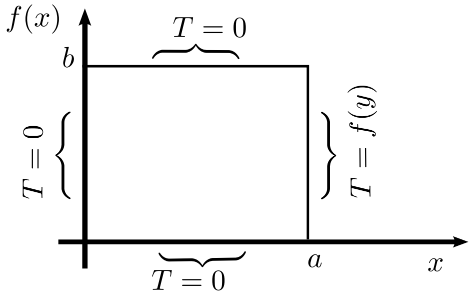
$$
\nabla^2 T = \frac{c\rho}{K}\frac{\partial T}{\partial t}
$$

If the temperature T is steady (*i.e.* does not vary with time) the equation reduces to the Laplace's equation

$$
\nabla^2 T = 0
$$

Example 17 (Temperature Distribution in a Plate)**.** Consider a rectangular Plate: $0 \le x \le a$, $0 \le y \le b$, which is subjected to the following temperatures at its edges:

i $T(x,0) = 0$ for $0 \le x \le a$
ii $T(x,b) = 0$ for $0 \le x \le a$
iii $T(0,y) = 0$ for $0 \le y \le b$
iv $T(a,y) = f(y)$ for $0 \le y \le b$



The governing equation is;

$$\nabla^2 T = 0$$
$$\Rightarrow \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Assume a solution of the form

$$T(x,y) = X(x)Y(y)$$
$$\therefore X''Y + Y''X = 0$$
$$\therefore \frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} = \alpha \text{ a constant.}$$

Case 1 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$\therefore X'' = -\lambda^2 X$$
$$\Rightarrow X(x) = A\cos(\lambda x) + B\sin(\lambda x)$$
$$Y'' = +\lambda^2 Y$$
$$\Rightarrow Y(y) = Ce^{\lambda y} + De^{-\lambda y} \text{ or } = C\cosh(\lambda y) + D\sinh(\lambda y)$$

Consider boundary conditions on $y = 0$ and $y = b$

$$T(x,0) = 0 \Rightarrow X(x)Y(0) = 0$$
$$\therefore Y(0) = 0 \Rightarrow C = 0$$
$$T(x,b) = 0 \Rightarrow X(x)Y(b) = 0$$
$$\therefore Y(b) = 0 \Rightarrow D\sinh(\lambda b) = 0 \Rightarrow D = 0$$

So $\alpha < 0$ results in a trivial solution.

Case 2 - $\alpha < 0$

Let $\alpha = \lambda^2$

$$\therefore X'' = \lambda^2 X \Rightarrow X(x) = A\sinh(\lambda x) + B\cosh(\lambda x)$$
$$Y'' = -\lambda^2 Y \Rightarrow Y(y) = C\cos(\lambda y) + D\sin(\lambda y)$$

Consider boundary conditions on $y = 0$ and $y = b$

$$\begin{aligned}
T(x,0) = 0 \Rightarrow X(x)Y(0) = 0 &\Rightarrow Y(0) = 0 \\
\therefore C &= 0 \\
T(x,b) = 0 \Rightarrow X(x)Y(b) = 0 &\Rightarrow Y(b) = 0 \\
\therefore D\sinh(\lambda b) = 0 &\Rightarrow \lambda b = n\pi \\
\therefore \lambda &= \frac{n\pi}{b} \\
\therefore Y_n(y) &= D\sin\left(\frac{n\pi y}{b}\right) \\
X_n(x) &= A\sinh\left(\frac{n\pi x}{b}\right) + B\cosh\left(\frac{n\pi x}{b}\right)
\end{aligned}$$

Applying boundary condition on $x = 0$ and $x = a$

$$\begin{aligned}
&T(0,y) = 0 \Rightarrow X(0)Y(y) = 0 \Rightarrow X(0) = 0 \\
&\therefore B = 0 \\
&T_n(x,y) = C_n \sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi x}{b}\right) \\
&T(a,y) = f(y) = \frac{4y}{b^2}(b-y)
\end{aligned}$$

Now using superposition;

$$T(x,y) = \sum_{n=1}^{\infty} T_n(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi x}{a}\right)$$

Now from the boundary condition;

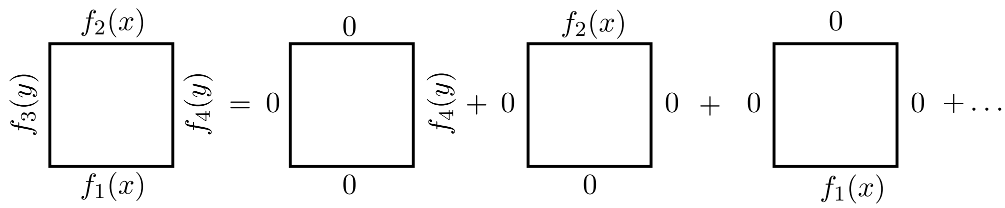
$$\begin{aligned}
&\therefore T(a,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{b}\right)\sinh\left(\frac{n\pi a}{b}\right) \\
&C_n = \frac{2}{b\sinh\left(\frac{n\pi a}{b}\right)} \int_0^b \frac{4y(b-y)}{b^2} \sin\left(\frac{n\pi y}{b}\right) \mathrm{d}y \\
&= \frac{-16}{n^3\pi^3}\left[\cos(n\pi) - 1\right] \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)}
\end{aligned}$$

$$\therefore T(x,y) = \sum_{n=1}^{\infty} \frac{32}{(2n-1)^3\pi^3} \frac{1}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left[\frac{(2n-1)\pi y}{b}\right] \sinh\left[\frac{(2n-1)\pi x}{b}\right]$$

In practice boundary conditions can be more complicated. For example T can be specified on all edges, *e.g.*

$$\begin{aligned}
T(x,0) = f_1(x), \qquad T(0,y) = f_3(y) \\
T(x,b) = f_2(x), \qquad T(0,y) = f_4(y).
\end{aligned}$$

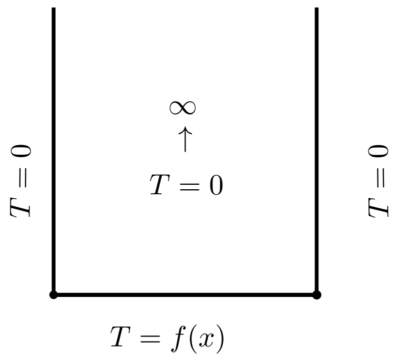
This problem can be solved by superposition, *i.e.* ;



Example 18 (Temperature Distribution in Semi-infinite Strip)**.** Consider the equation of Laplace's equation in the strip $y > 0$, $0 \leq x \leq a$;

Where;

1. $T(0, y) = 0$ for all $y > 0$
2. $T(a, y) = 0$ for all $y > 0$
3. $T(x, 0) = f(x)$ for $0 \leq x \leq a$
4. $T(x, y) \to 0$ as $y \to \infty$.



The governing equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

Assume solutions of the form:

$$T(x, y) = X(x)Y(y).$$
$$\therefore X'' = \alpha X, \qquad Y'' = -\alpha Y.$$

Case 1 - $\alpha > 0$

Let $\alpha = \lambda^2$

$$X'' = \lambda^2 X \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$
$$Y'' = -\lambda^2 Y \Rightarrow Y(y) = C\cos(\lambda y) + D\sin(\lambda y)$$

From boundary condition (iv):

$$T(x, y) \to 0 \text{ as } y \to \infty$$
$$\therefore Y(y) \to 0 \text{ as } y \to \infty$$

Hence $C = D = 0$ which results in a trivial solution.

Case 2 - $\alpha < 0$

Let $\alpha = -\lambda^2$

$$\begin{aligned} \therefore X(x) &= A\cos(\lambda x) + B\sin(\lambda x) \\ Y(y) &= Ce^{\lambda y} + De^{-\lambda y}. \end{aligned}$$

Applying boundary condition (iv):

$$\begin{aligned} T(c, y) &\to 0 \text{ as } y \to \infty \\ \therefore C &= 0 \end{aligned}$$

Applying boundary condition (i):

$$\begin{aligned} T(0, y) = 0 &\Rightarrow X(0) = 0 \\ A &= 0 \end{aligned}$$

Applying boundary conditions (ii):

$$\begin{aligned} T(a, y) = 0 &\Rightarrow X(0) = 0 \\ A &= 0 \end{aligned}$$

Applying boundary condition (ii):

$$\begin{aligned} T(a, y) = 0 &\Rightarrow X(a) = 0 \\ \therefore B\sin(\lambda a) = 0 &\Rightarrow \lambda a = n\pi \\ \therefore \lambda &= \frac{n\pi}{a} \\ \therefore T - n(x, y) &= K_n \sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{n\pi y}{a}\right) \end{aligned}$$

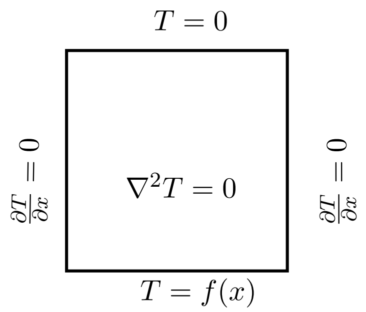
Using principle of superposition:

$$T(x, y) = \sum_{n=1}^{\infty} T_n(x, y) = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right)\exp\left(-\frac{n\pi y}{a}\right)$$

Applying (iii) we get;

$$\begin{aligned} T(x, 0) &= \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right) = f(x) \\ \therefore K_n &= \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) \mathrm{d}x \end{aligned}$$

8.3 Insulated Boundaries for a Rectangular Plate



Consider a rectangular plate with T specified on $y = 0$ and $y = b$ and with insulated edges on $x = 0$ and $x = a$. By applying the method of separation of variables it is possible to show that

$$T = C + Dy$$

and that

$$T = \cos\left(\frac{n\pi x}{a}\right)\left[A_n \exp\left(\frac{n\pi y}{a}\right) + B_n \exp\left(-\frac{n\pi y}{a}\right)\right]$$

which satisfies conditions on edges $x = 0$ and $x = a$.

