Modelling & Analysis II
Fourier Series

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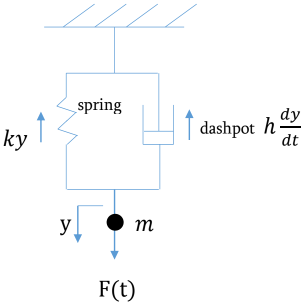
Forced Damped Vibrations



Consider the motion of a mass oscillating on a spring in a resisting medium.

- y is the downward displacement of mass m from its equilibrium;
- spring force $= ky$;
- Dashpot force $= h\frac{dy}{dt}$.

Forced Damped Vibrations

The variable force $F(t)$ is applied to the mass by some external agency. The equation of motions is:

$$m\frac{d^2y}{dt^2} = -ky - h\frac{dy}{dt} + F(t)$$

$$\therefore m\frac{d^2y}{dt^2} + h\frac{dy}{dt} + ky = F(t) \tag{1}$$

Let $\omega^2 = k/m$ and $h/m = 2\alpha$

$$\therefore \frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega^2 y = \frac{F(t)}{m} \tag{2}$$

Many other physical systems lead to equation of the above form, e.g., small oscillations of simple pendulum in a resisting medium, etc.

Forced Damped Vibrations

Of particular importance are the cases in which $F(t)$ is a periodic function whose values repeat itself at constant interval.
The simplest and most common periodic functions are $\sin$ and $\cos$.
So consider $F(t) = F_0 \cos pt$

$$\therefore \frac{d^2y}{dt^2} + 2\alpha\frac{dy}{dt} + \omega^2 y = f_0 \cos pt \tag{3}$$

where $f_0 = \frac{F_0}{m}$

$$y(t) = \underbrace{u(t)}_{\text{C.F. (or transient)}} + \underbrace{v(t)}_{\text{P.I. (or steady state solution)}}$$

$u(t)$ is solution of the homogeneous equation;

$$\frac{d^2u}{dt^2} + 2\alpha\frac{du}{dt} + \omega^2 u = 0$$

Forced Damped Vibrations

Let $u = e^{\lambda}$

$$\therefore \lambda = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega^2}}{2}$$

$\therefore \lambda_1 = -\alpha + \sqrt{\alpha^2 - \omega^2}$ & $\lambda_2 = -\alpha - \sqrt{\alpha^2 - \omega^2}$
$\therefore u(t) = e^{-\alpha t}\left[Ae^{\sqrt{\alpha^2-\omega^2}t} + Be^{-\sqrt{\alpha^2-\omega^2}t}\right]$, where $A \& B$ are constants.
P.I. is solution of the form:

$$v(t) = C\cos pt + D\sin pt,$$

where $C \& D$ are constants.

$$\begin{aligned} \frac{dv}{dt} &= -Cp\sin pt + Dp\cos pt \\ \frac{d^2v}{dt^2} &= -Cp^2\cos pt - Dp^2\sin pt \end{aligned}$$

Forced Damped Vibrations

This must satisfy:

$$\frac{d^2v}{dt^2} + 2\alpha\frac{dv}{dt} + \omega^2 v = f_0 \cos pt$$

$\therefore -Cp^2 \cos pt - Dp^2 \sin pt - 2\alpha Cp \sin pt + 2\alpha Dp \cos pt +$
$\omega^2 C \cos pt + \omega^2 D \sin pt = f_0 \cos pt$

$$\begin{aligned}
\cos pt : (-Cp^2 + 2\alpha Dp + \omega^2 C) &= f_0 \\
\sin pt : (-Dp^2 - 2\alpha Cp + \omega^2 D) &= 0
\end{aligned}$$

$$\begin{aligned}
C &= \frac{(\omega^2 - p)f_0}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2} \\
D &= \frac{2\alpha p f_0}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2}
\end{aligned}$$

Forced Damped Vibrations

$$
\begin{aligned}
y(t) &= u(t) + v(t) \\
&= \underbrace{e^{-\alpha t}\left[Ae^{(\sqrt{\alpha^2-\omega^2})t} + Be^{-(\sqrt{\alpha^2-\omega^2})t}\right]}_{\text{transient}} \\
&\quad + \underbrace{\frac{(\omega^2 - p)f_0}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2}\cos pt + \frac{2\alpha p f_0}{(\omega^2 - p^2)^2 + 4\alpha^2 p^2}\sin pt}_{\text{steady state solution}}
\end{aligned}
$$

Unfortunately in practice the function $F(t)$ is rarely a sine or cosine ! We need to use Fourier Series which can *represent* any arbitrary periodic function as an infinite series of sines or cosines.

Trigonomegtric Preliminaries

$$\begin{aligned}\cos(A+B) &= \cos A\cos B - \sin A\sin B\\ \cos(A-B) &= \cos A\cos B + \sin A\sin B\end{aligned}$$

Add up: $\cos A\cos B = \frac{1}{2}\Big[\cos(A-B)+\cos(A+B)\Big]$

Subtract: $\sin A\sin B = \frac{1}{2}\Big[\cos(A-B)-\cos(A+B)\Big]$

Consider $\int_{-\pi}^{\pi}\cos px\cos qx\ dx$ and $\int_{-\pi}^{\pi}\sin px\sin qx\ dx$, where p and q are integers.

If $p \neq q$,

$$\begin{aligned}\int_{-\pi}^{\pi}\cos px\cos qx\ dx &= \frac{1}{2}\int_{-\pi}^{\pi}\Big[\cos(p-q)x+\cos(p+q)x\Big]dx\\ &= \frac{1}{2}\Big[\frac{\sin(p-q)x}{(p-q)}+\frac{\sin(p+q)x}{(p+q)}\Big]_{-\pi}^{\pi}\\ &= 0\end{aligned}$$

Trigonomegtric Preliminaries

If $p = q \neq 0$,

$$\begin{aligned}\int_{-\pi}^{\pi} \cos px \cos qx \ dx &= \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 2px)\,dx \\ &= \frac{1}{2}\Big[x + \frac{\sin 2px}{2p}\Big]_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

If $p = q = 0$,

$$\int_{-\pi}^{\pi} \cos px \cos qx \ dx = \frac{1}{2}\int_{-\pi}^{\pi}(1+1)\,dx = 2\pi$$

Trigonomegtric Preliminaries

Similarly,

$$\begin{aligned}\int_{-\pi}^{\pi} \sin px \sin qx dx &= \frac{1}{2}\int_{-\pi}^{\pi} \Big[\cos(p-q)x - \cos(p+q)x\Big] dx \\ &= 0 \quad \text{if } p \neq q \\ &= \pi \quad \text{if } p = q \neq 0 \\ &= 0 \quad \text{if } p = q = 0\end{aligned}$$

Now

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$
$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

Add up: $\sin A \cos B = \frac{1}{2}\Big[\sin(A+B) + \sin(A-B)\Big]$

Trigonomegtric Preliminaries

$$
\begin{aligned}
\int_{-\pi}^{\pi} \sin px \cos qx dx &= \frac{1}{2}\int_{-\pi}^{\pi} \Big[\sin(p+q)x + \sin(p-q)x\Big] dx \\
&= -\frac{1}{2}\Big[\frac{\cos(p+q)x}{p+q} + \frac{\cos(p-q)x}{p-q}\Big]_{-\pi}^{\pi} \\
&= 0 \quad \text{if } p \neq q \\
&= 0 \quad \text{if } p = q \neq 0 \\
&= 0 \quad \text{if } p = q = 0
\end{aligned}
$$

Fourier Coefficients

Consider a function $f(\theta)$ which is periodic with fundamental period 2π and suppose it can be represented in $-\pi \leq \theta \leq \pi$ by the Fourier Series,

$$f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) \tag{4}$$

a_0, a_n, & b_n are the Coefficientss of Fourier Series.

- Coefficients a_0

Integrate both sides of Eq. (4);

$$\int_{-\pi}^{\pi} f(\theta)\,d\theta = \int_{-\pi}^{\pi} \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)\right] d\theta$$

Assume R.H.S can be integrated term by term,

Fourier Coefficients

$$
\begin{aligned}
\int_{-\pi}^{\pi} f(\theta)\, d\theta &= \frac{1}{2} a_0 \int_{-\pi}^{\pi} d\theta \\
&+ \sum_{n=1}^{\infty} \left(a_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta\, d\theta}_{0} + b_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta\, d\theta}_{0} \right)
\end{aligned}
$$

Therefore,

$$
\int_{-\pi}^{\pi} f(\theta)\, d\theta = \pi a_0 \quad \text{or} \quad a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta
$$

Fourier Coefficients

- Coefficients a_n

Multiply both sides by $\cos m\theta$ when m is a fixed *positive* integer & then integrate,

$$\int_{-\pi}^{\pi} f(\theta)\cos m\theta\, d\theta = \int_{-\pi}^{\pi}\Big[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta)\Big]\cos m\theta\, d\theta$$

Term by term integration,

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta)\cos m\theta\, d\theta &= \frac{1}{2}a_0 \underbrace{\int_{-\pi}^{\pi} \cancelto{0}{\cos m\theta\, d\theta}}_{\text{I}} \\
&+ \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \cos m\theta\, d\theta}_{\text{II}} \\
&+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi} \cancelto{0}{\sin n\theta \cos m\theta\, d\theta}}_{\text{III}}
\end{aligned}$$

Fourier Coefficients

$$\begin{aligned} \text{II} &= 0 \;\; \text{for } m \neq n \\ &= 2\pi \;\; \text{for } n = m = 0\text{, but } m \text{ and } n \text{ are both positive integers !} \\ &= \pi \;\; \text{for } n = m \neq 0 \end{aligned}$$

Since m and n are both *positive* integers,

$$\int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta = \pi a_n$$

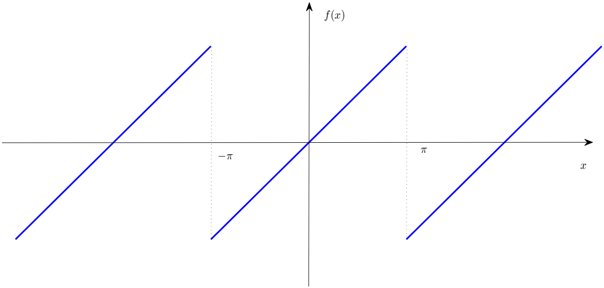
$$\therefore \boxed{a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta}$$

Similarly b_n can be calculated by multiplying both sides of Eq. (4) by $\sin m\theta$ and integrating from $-\pi$ to π.

$$\therefore \boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta}$$

Example 1

Find the Fourier Coefficientss of the periodic function defined by $f(x) = x$ for $-\pi \leq x \leq \pi$ and $f(x + 2\pi) = f(x)$.



Solution of Example 1

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_{-\pi}^{\pi} = 0$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx \\ &= \frac{1}{\pi}\left[\frac{x\sin nx}{n}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin nx}{n}dx \\ &= \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_{-\pi}^{\pi} = 0 \end{aligned}$$

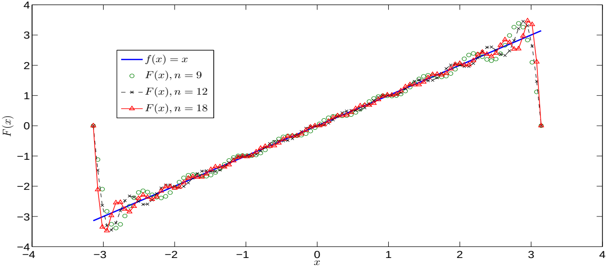
Also,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx \\ &= \frac{1}{\pi}\left[\frac{-x\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos nx}{n}dx \\ &= \frac{2(-1)^{n+1}}{n} \end{aligned}$$

Solution of Example 1

Therefore,

$$F(x) = 2\Big(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\Big) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n}$$



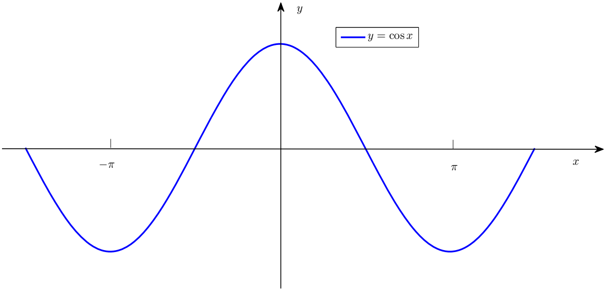
Note:

$F(\pi) = 0$! **In general the Fourier Series at the point of discontinuity converges to average values off on either side.**

Odd & Even functions

- Even function

A function $y = g(x)$ is *even* if $g(-x) = g(x)$ for all values of x, e.g.,



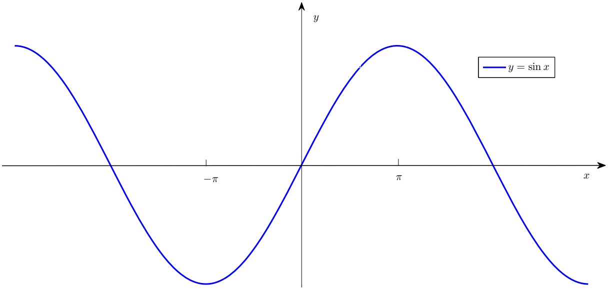
Therefore,

$$\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx$$

Odd & Even functions

- Odd function

A function $y = f(x)$ is *odd* if $f(-x) = -f(x)$ for all values of x, e.g.,



Therefore,

$$\int_{-a}^{a} f(x)dx = 0$$

Odd & Even functions

Properties

If $h(x) = g(x)f(x)$, then

1. $h(x)$ is odd when either $g(x)$ or $f(x)$ is odd and the other even;
2. $h(x)$ is even when $g(x)$ and $f(x)$ are either both even or both odd.

So when $f(\theta)$ is even:

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\cos n\theta\, d\theta, \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = 0.
\end{aligned}$$

When $f(\theta)$ is odd:

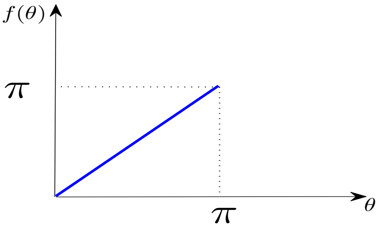
$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = 0, \\
b_n &= \frac{2}{\pi}\int_{0}^{\pi} f(\theta)\sin n\theta\, d\theta.
\end{aligned}$$

Odd & Even functions

NB: Odd functions are expressed solely in terms of sins and even functions in terms of cosines

Periodic Extensions and Half Range Series

One of the conditions of Fourier Series is that the function must be periodic, which implies it is defined for all x. But when the function $f(x)$ is defined in a finite interval, say $0 \leq x \leq \pi$, a function $H(x)$ is constructed which is identical to $f(x)$ in $0 \leq x \leq \pi$ and is periodic. $H(x)$ known as the periodic expansion for $f(x)$.
e.g. $f(\theta) = \theta$, $0 \leq \theta \leq \pi$

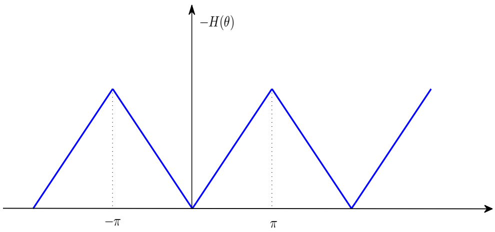


Periodic Extensions and Half Range Series

We can extend the function as an even function, e.g.,

$$H(\theta) = \begin{cases} f(-\theta) & \text{for } -\pi \le x \le 0 \\ f(\theta) & \text{for } 0 \le x \le \pi \end{cases}$$

and $H(\theta + 2\pi) = H(\theta)$



$H(\theta)$ is even periodic expansion of $f(\theta)$.
$H(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta$

Periodic Extensions and Half Range Series

where

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} \theta \cos n\theta \, d\theta = \frac{2}{\pi}\Big\{\Big[\frac{\theta \sin n\theta}{n}\Big]_0^{\pi} - \frac{1}{n}\int_0^{\pi} \sin n\theta \, d\theta\Big\} \\ &= \frac{2}{\pi}\frac{1}{n^2}\big[\cos n\theta\big]_0^{\pi} = \begin{cases} 0 & \text{if } n \text{ is even} \\ -\frac{4}{n\pi^2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Finally,

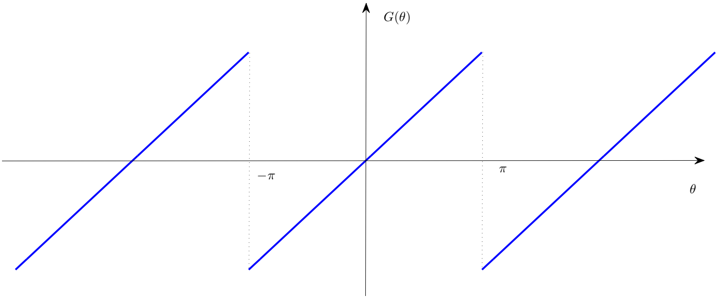
$$a_0 = \frac{2}{\pi}\int_0^{\pi} \theta \, d\theta = \pi$$

The odd periodic expansion for $f(\theta)$ is

$$G(\theta) = \begin{cases} -f(-\theta) & \text{for } -\pi \le x \le 0 \\ f(\theta) & \text{for } 0 \le x \le \pi \end{cases}$$

and $G(\theta + 2\pi) = G(\theta)$

Periodic Extensions and Half Range Series



Same as Example 1.

Fourier Series representation in arbitrary intervals

Consider a function which has period $2L$ in the interval $-L \leq x \leq L$

$$\therefore f(x+2L) = f(x) \tag{5}$$

Let us define a new variable $y = \pi x/L$ and substitute in Eq. (5), we obtain

$$f\Big[\frac{L}{\pi}(y+2\pi)\Big] = f\Big(\frac{Ly}{\pi}\Big)$$

i.e., the function $f(\frac{Ly}{\pi})$ has period 2π in the new variable y.
$f(\frac{Ly}{\pi}) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos ny + b_n \sin ny)$ where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{Ly}{\pi})dy \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{Ly}{\pi})\cos ny dy \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\frac{Ly}{\pi})\sin ny dy
\end{aligned}$$

Fourier Series representation in arbitrary intervals

So the function

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

where

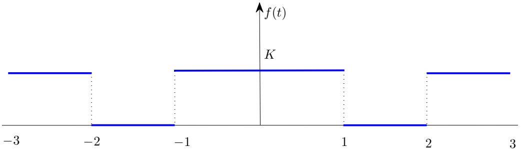
$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\,dx \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\,dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\,dx \end{aligned}$$

Fourier Series representation in arbitrary intervals: Example

Find the Fourier Series of function:

$$f(t) = \begin{cases} 0 & \text{for } -2 \leq t \leq -1 \\ K & \text{for } -1 \leq t \leq 1 \\ 0 & \text{for } 1 \leq t \leq 2 \end{cases}$$

and $T = 4$.



Fourier Series representation in arbitrary intervals: Example

$$
\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\int_{-1}^{1} K dt = K \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(t)\cos(\frac{n\pi t}{2})\,dt \\
&= \frac{1}{2}\int_{-1}^{1} K\cos(\frac{n\pi t}{2})\,dt \\
&= \frac{2K}{n\pi}\sin\frac{n\pi}{2}
\end{aligned}
$$

$a_n = 0$ when n is even and $a_n = \frac{2K}{n\pi}$ for $n = 1, 5, 9, ...$ and $a_n = -\frac{2K}{n\pi}$ for $n = 3, 7, ...$ Thus,

$$f(t) = \frac{K}{2} + \frac{2K}{\pi}\Big[\cos\frac{\pi t}{2} - \frac{1}{3}\cos\frac{3\pi t}{2} + \frac{1}{5}\cos\frac{5\pi t}{2} + ...\Big]$$

Fourier Series representation in arbitrary intervals: Example