Modelling & Analysis II
Partial Differential Equations 1

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Most physical phenomena (in a continuous media) can be described in terms of fields *i.e.* as a function of position and time *e.g.* temperature, velocity, magnetism, *etc.*

So $T = T(x, y, z, t)$, $\boldsymbol{v} = \boldsymbol{v}(x, y, z, t)$, *etc.*

The way a physical quantity represented by a field is related from point to point or time to time is by a partial differential equation (PDE). The PDE is the mathematical statement of the way in which laws of nature govern the phenomenon.

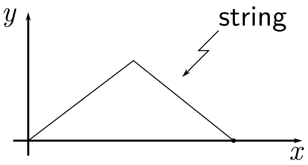
Example

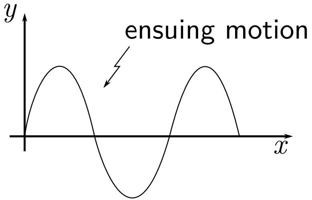
By applying the principle of conservation of energy (a law of nature) to a conducting medium we derived the Heat Equation

$$\frac{\partial T}{\partial t} = \frac{K}{c\rho} \nabla^2 T,$$

where K is conductivity, c is specific heat capacity, and ρ is density. The Heat Equation governs the temperature distribution in a conducting medium. We also derived the Laplace's equation $\nabla^2 \phi = 0$ which governs steady state heat conduction and inviscid fluid flow.

The Wave Equation





Another important equation is the wave equation, which governs the motion of a vibrating string or membrane. Consider a flexible elastic string fixed at two end points being plucked and we are interested in the ensuing motion of the string.

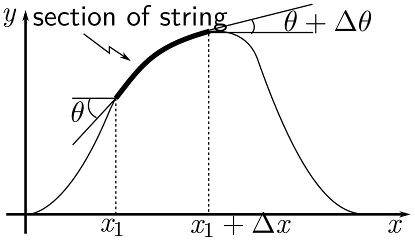
In order to obtain the governing equation for the displacement, let us assume;

(i) The string has uniform density ρ (mass/unit length).

(ii) The string is elastic but offers no resistance to bending, so the tension in the string always acts tangentially.

(iii) Gravitational effects & air friction can be ignored.

i.e. we are considering small transverse vibrations of a flexible string under large tension.



Now consider a segment of string between x_1 and $x_1 + \Delta x$. Applying Newton's second law to this segment and considering vertical forces only;

$$T(x_1 + \Delta x)\sin(\theta + \Delta\theta) - T(x_1, t)\sin(\theta) = \rho \Delta x \frac{\partial^2 y}{\partial t^2}(\bar{x}, t), \quad (1)$$

where $\bar{x}$ is the centre of mass of the segment. Then;

$$\frac{T(x_1 + \Delta x)\sin(\theta + \Delta\theta) - T(x_1, t)\sin(\theta)}{\Delta x} = \rho \frac{\partial^2 y}{\partial t^2}(\bar{x}, t).$$

Let $T_v(x_1, t)$ represent the vertical component of $T(x_1, t)$, so;

$$\frac{T_v(x_1 + \Delta x) - T_v(x_1, t)}{\Delta x} = \rho \frac{\partial^2 y}{\partial t^2}(\bar{x}, t)$$

$$\therefore \frac{\partial T_v}{\partial x_1} = \rho \frac{\partial^2 y}{\partial t^2}, \qquad (2)$$

as $\Delta x \to 0$ and $\bar{x} \to x_1$. Considering the horizontal component only;

$$T(x_1 + \Delta x, t)\cos(\theta + \Delta\theta) - T(x_1, t)\cos(\theta) = 0$$

$$\therefore T_h(x_1 + \Delta x, t) - T_h(x_1, t) = 0,$$

i.e.

$$\frac{\partial T_h}{\partial x_1} = 0.$$

But

$$T_v = T_h \tan(\theta) \Rightarrow T_h \frac{\partial y}{\partial x_1}. \tag{3}$$

Substituting (3) into (2) we get;

$$\begin{aligned}
\frac{\partial}{\partial x_1}\left(T_h \frac{\partial y}{\partial x_1}\right) &= \rho \frac{\partial^2 y}{\partial t^2} \\
\therefore\ T_h \frac{\partial^2 y}{\partial x_1^2} &= \rho \frac{\partial^2 y}{\partial t^2} \\
\therefore\ \frac{\partial^2 y}{\partial x_1^2} &= \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2}, \qquad (4)
\end{aligned}$$

where a^2 is T_h/ρ. Since x_1 is an arbitrary point on the string, Eq. (4) can be rewritten as a general form,

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 y}{\partial t^2} \tag{5}$$

This is the one dimensional Wave Equation.

The Solution of the Wave Equation

The Wave equation is;

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{6}$$

Let us look for solutions of the form;

$$u(x,t) = X(x)\,T(t).$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\mathrm{d}^2 X}{\mathrm{d}x^2}\,T(t) \qquad \& \qquad \frac{\partial^2 u}{\partial t^2} = \frac{\mathrm{d}^2 T}{\mathrm{d}t^2}X(x),$$

$$= X''(x)\,T(t) \qquad \& \qquad = T''(t)X(x).$$

Hence, (6) becomes;

$$X''(x)\,T(t) = \frac{1}{c^2}\,T''(t)X(x)$$
$$\therefore \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)}$$

i.e. the right hand side is a function of t and the left hand side only a function of x;

$$\therefore \frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \text{ a constant } = \alpha$$

Now we need to consider different cases.

Case 1 - $\alpha > 0$
Let $\alpha = \lambda^2$

$$\begin{aligned} X'' &= \lambda^2 X, \quad \& \quad T'' = \lambda^2 T c^2 \\ \therefore X(x) &= Ae^{\lambda x} + Be^{-\lambda x}, \quad \& \quad T(t) = De^{\lambda t} + Ee^{-\lambda t} \end{aligned}$$

where A, B, D & E are arbitrary constants, which are determined from boundary conditions.

Case 2 - $\alpha = 0$
Let $\alpha = 0$

$$X(x) = Ax + B, \quad \& \quad T(t) = Dt + E$$

Case 3 - $\alpha < 0$
Let $\alpha = -\lambda^2$

$$\begin{aligned} X'' &= -\lambda^2 X, \quad \& \quad T'' = -\lambda^2 T c^2 \\ \therefore X(x) &= A\cos(\lambda x) + B\sin(\lambda x), \quad \& \quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct) \end{aligned}$$

Boundary Conditions

Consider a stretched string fixed at both ends, with zero initial velocity *i.e.*

(i) $u(0, t) = 0$

(ii) $u(L, t) = 0$

(iii) $\frac{\partial}{\partial t} u(x, 0) = 0$

(iv) $u(x, 0) = f(x)$ where $f(x)$ is the initial displacement

Let us see which solution is suitable for the problem.

Case 1 - $\alpha > 0$

From boundary condition (i);

$$\begin{aligned} u(0,t) &= 0 \Rightarrow X(0)\,T(t) = 0, \\ \therefore X(0) &= 0 \Rightarrow A + B = 0 \end{aligned}$$

From boundary condition (ii);

$$\begin{aligned} u(L,t) &= 0 \Rightarrow X(L)\,T(t) = 0, \\ \therefore X(L) &= 0 \Rightarrow Ae^{\lambda L} + Be^{-\lambda L} = 0. \end{aligned}$$

Since $e^{\lambda L} - e^{-\lambda L} \neq 0$, then to satisfy the condition we need to have

$$A = B = 0$$

i.e. this leads to a trivial solution.

Case 2 - $\alpha = 0$

From boundary condition (i);

$$\begin{aligned} u(0,t) &= 0 \Rightarrow X(0)\,T(t) = 0, \\ \therefore B &= 0. \end{aligned}$$

From boundary condition (ii);

$$\begin{aligned} u(L,t) &= 0 \Rightarrow X(L)\,T(t) = 0, \\ \therefore AL + B &= 0 \Rightarrow A = 0, \end{aligned}$$

i.e. this leads to a trivial solution.

Case 3 - $\alpha < 0$

$$X(x) = A\cos(\lambda x) + B\sin(\lambda x).$$

From boundary condition (i);

$$\begin{aligned} u(0,t) &= 0 \Rightarrow X(0)\,T(t) = 0 \\ \therefore X(0) &= 0 \Rightarrow A = 0 \end{aligned}$$

From boundary condition (ii);

$$\begin{aligned} u(L,t) &= 0 \Rightarrow X(L)\,T(t) = 0 \\ \therefore X(L) &= 0 \Rightarrow B\sin(\lambda L) = 0. \end{aligned}$$

Now either $B = 0$, which results in a trivial solution again, or $\sin(\lambda L) = 0$. Hence,

$$\sin(\lambda L) = 0 \Rightarrow \lambda L = n\pi,$$

where λ is known as the *eigenvalue* of the problem. So finally,

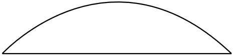
$$
\begin{aligned}
X(x) &= B\sin\left(\frac{n\pi x}{L}\right) \\
T(t) &= D\cos\left(\frac{n\pi ct}{L}\right) + E\sin\left(\frac{n\pi ct}{L}\right)
\end{aligned}
$$

alternatively;

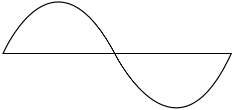
$$
\begin{aligned}
u_n(x,t) &= \sin\left(\frac{n\pi x}{L}\right)\left[BD\cos\left(\frac{n\pi ct}{L}\right) + BE\sin\left(\frac{n\pi ct}{L}\right)\right] \\
&= \sin\left(\frac{n\pi x}{L}\right)\left[D_n\cos\left(\frac{n\pi ct}{L}\right) + E_n\sin\left(\frac{n\pi ct}{L}\right)\right],
\end{aligned}
$$

where $D_n = BD$ & $E_n = BE$ and $n = 1, 2, 3,$ *etc.* . The displacement $u_n(x,t)$ is referred to as the n^{th} eigenfunction or the n^{th} normal mode of the vibrating string.

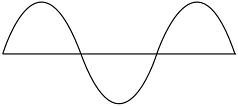
Normal modes



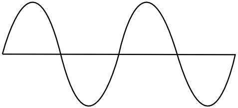
$n = 1:$ frequency $= \frac{c}{2L}$ = fundamental



$n = 2:$ frequency $= \frac{c}{L}$



$n = 3:$ frequency $= \frac{3c}{2L}$



$n = 4:$ frequency $= \frac{2c}{L}$

Now from initial condition (iii):

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,0) &= 0 \\
\therefore X(x)\,T'(t) &= 0 \Rightarrow T'(0) = 0 \\
T'(t) &= -\frac{n\pi c}{L} D_n \sin\left(\frac{n\pi ct}{L}\right) + \frac{n\pi c}{L} E_n \cos\left(\frac{n\pi ct}{L}\right) \\
\therefore T'(0) &= 0 \Rightarrow E_n = 0 \\
u_n(x,t) &= D_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right).
\end{aligned}
$$

Compare this to simple harmonic motion where;

$$
\begin{aligned}
y &= A \sin \omega t \\
\textit{i.e. } D_n \sin\left(\frac{n\pi x}{L}\right) &= \text{amplitude of oscillation,} \\
\text{Period of oscillation} &= \frac{2\pi}{\omega} = \frac{2\pi}{(n\pi c)/L} = \frac{2L}{nc}.
\end{aligned}
$$

So the n^{th} normal mode vibrates with a period $2L/nc$ seconds which corresponds to a frequency of $nc/2L$ cycles per second. Since $c^2 = T_h/\rho$ then the frequency is equal to

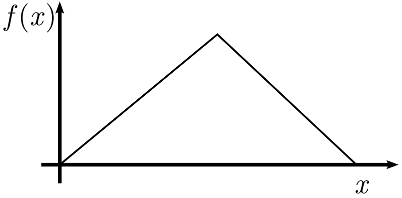
$$\frac{n}{2L}\left(\frac{T_h}{\rho}\right)^{1/2}.$$

So if the string is vibrating in a normal mode, its pitch may be sharpened (frequency increased) by either decreasing L or increasing T_h (the tension).

Finally we need to satisfy the final initial condition;

$$u(x,0) = f(x).$$

If the string is plucked in the middle, then $f(x)$ could look like the sketch opposite.



Clearly, a single normal mode of oscillation cannot satisfy the initial condition. So we have to use the superposition principle.

Theorem

If u_1 and u_2 are any solutions of a linear homogeneous Partial Differential Equation in some region then

$$u = c_1 u_1 + c_2 u_2,$$

where c_1 and c_2 are constants and are also a solution of that equation in that region.

Definition

A partial differential equation is said to be linear if it is of the first degree in the field variables and its partial derivatives, *e.g.*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} \qquad \text{is linear but}$$

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial t}\right)^2 \qquad \text{is non-linear.}$$

Definition

A linear partial differential equation is said to be homogenous if every term in the equation contains either the field variable or one of its derivatives, *e.g.*

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \qquad \text{is homogenous, but}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + 3 \qquad \text{is inhomogeneous.}$$

Proof.

Consider $\partial^2 u/\partial x^2 = K^{-1}\partial u/\partial t$. Suppose u_1 and u_2 are solutions, then;

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{1}{K}\frac{\partial u_1}{\partial t} \qquad \& \qquad \frac{\partial^2 u_2}{\partial x^2} = \frac{1}{K}\frac{\partial u_2}{\partial t}.$$

Now

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}\left(u_1 c_1 + u_2 c_2\right) &= \frac{\partial^2}{\partial x^2}\left(c_1 u_1\right) + \frac{\partial^2}{\partial x^2}\left(c_2 u_2\right) \\
&= c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial^2 u_2}{\partial x^2} \\
&= c_1 \left(\frac{1}{K}\frac{\partial u_1}{\partial t}\right) + c_2 \left(\frac{1}{K}\frac{\partial u_2}{\partial t}\right) \\
&= \frac{1}{K}\frac{\partial}{\partial t}\left(c_1 u_1 + c_2 u_2\right).
\end{aligned}$$

□

Now since the wave equation is a linear homogeneous partial differential equation, we can use the superposition principle, *i.e.* we can say that

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\
&= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right),
\end{aligned}
$$

i.e. $u(x,t)$, which is the sum of all harmonics, is also a solution of the P.D.E.
Now applying the initial condition $u(x,0) = f(x)$ we get;

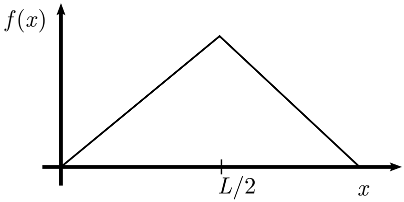
$$u(x,0) = f(x) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right),$$

i.e. this is a half-range Fourier series, which can be solved to find D_n, *i.e.*

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x.$$

The string is plucked in the middle according to:

$$f(x) = \begin{cases} x & \text{for } 0 \leq x \leq L/2 \\ L - x & \text{for } L/2 \leq x \leq L. \end{cases}$$



$$\begin{aligned} \therefore D_n &= \frac{2}{L}\int_0^{L/2} x \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x + \frac{2}{L}\int_{L/2}^{L} (L - x) \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x \\ &= \frac{4L}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \\ \therefore u(x, t) &= \sum_{n=1}^{\infty} \frac{4L}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi ct}{L}\right). \end{aligned}$$

This gives the displacement of the string.