Modelling & Analysis II
Fourier Series

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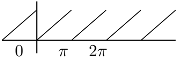
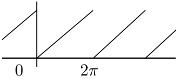
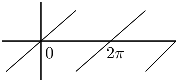
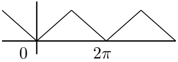
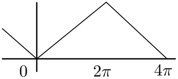
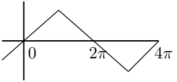
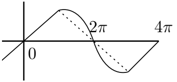
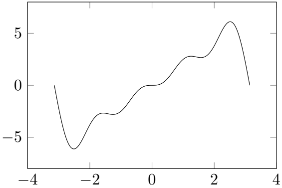
	Example	Period	Periodic function
1		π	$\frac{1}{2} - \frac{1}{\pi}\left(\sin 2x + \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x \ldots\right)$
2		2π	$1 - \frac{2}{\pi}\left(\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right)$
3		2π	$\frac{2}{\pi}\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right)$
4		2π	$\frac{1}{2} - \frac{1}{\pi^2}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right)$
5		4π	$1 - \frac{8}{\pi^2}\left(\cos\frac{1}{2}x + \frac{1}{3^2}\cos\frac{3}{2}x - \frac{1}{5^2}\cos\frac{5}{2}x + \ldots\right)$
6		4π	$\frac{8}{\pi^2}\left(\sin\frac{1}{2}x - \frac{1}{3^2}\sin\frac{3}{2}x + \frac{1}{5^2}\sin\frac{5}{2}x - \ldots\right)$
7		4π	$\frac{8}{\pi^2}\left(\frac{4}{3}\sin\frac{1}{2}x + \frac{1}{45}\sin\frac{3}{2}x - \frac{4}{525}\sin\frac{5}{2}x + \ldots\right)$

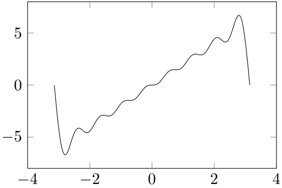
Partial Summation

$$f(x) = 2x, \qquad -\pi \le x \le \pi$$

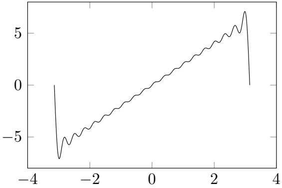
$$S_4(x) = \sum_{n=1}^{4} \frac{4}{n}(-1)^{n+1} \sin nx$$



$$S_8(x) = \sum_{n=1}^{8} \frac{4}{n}(-1)^{n+1} \sin nx$$

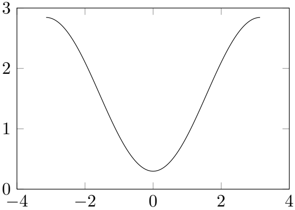


$$S_{19}(x) = \sum_{n=1}^{19} \frac{4}{n}(-1)^{n+1} \sin nx$$

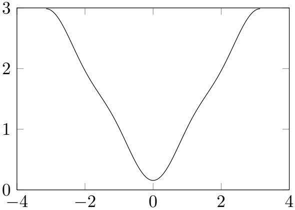


$$f(x) = |x|, \qquad -\pi \le x \le \pi$$

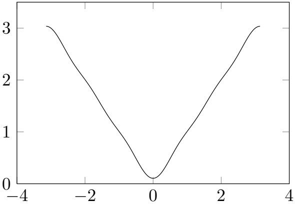
$$S_1(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x$$



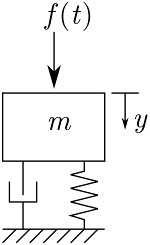
$$S_2(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x$$



$$S_3(x) = \frac{\pi}{2} - \frac{4}{\pi}\cos x - \frac{4}{9\pi}\cos 3x - \frac{4}{25\pi}\cos 5x$$



Example



A body of mass 1 kg is attached to a rigid support by a spring of modulus 36 kg/s^2 and a dashpot, with damping constant 0.05 kg/s. The system is kept in motion by an external force $f(t)$ Newtons which is periodic with period $2T$ and is given by:

$$f(t) = f(t+2T) = \begin{cases} C(t+T) & \text{in } -T \leq t < 0; \\ Ct & 0 \leq t < T. \end{cases}$$

Find the steady state solution.

Solution

The governing equation is

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 0.05\frac{\mathrm{d}y}{\mathrm{d}t} + 36y = f(t), \tag{1}$$

where $y(t)$ is the displacement in metres. Now

$$y(t) = \underbrace{U(t)}_{\text{C.F. (or transient)}} + \underbrace{V(t)}_{\text{P.I. (or steady state solution)}}$$

$$\begin{aligned} U(t) &= \text{C.F.} = \text{solution of homogeneous equation} \\ &= e^{-\alpha t}\left[Ae^{\sqrt{\alpha^2\omega^2 t}} + Be^{-\sqrt{\alpha^2\omega^2}t}\right], \end{aligned}$$

where $\alpha = 0.05$ and $\omega^2 = 36$. This solution tends to zero after sufficiently long time.

To calculate the steady state solution $V(t)$ we first need to express the forcing function $f(t)$ by a Fourier Series:

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right],$$

where:

$$\begin{aligned}
a_0 &= \frac{1}{T}\int_{-T}^{T} f(t)\mathrm{d}t = \frac{1}{T}\int_{-T}^{0} C(t+T)\mathrm{d}t + \frac{1}{T}\int_{0}^{T} Ct\mathrm{d}t \\
&= \frac{C}{T}\left[\frac{t^2}{2} + Tt\right]_{-T}^{0} + \frac{C}{T}\left[\frac{t^2}{2}\right]_{0}^{T} \\
&= \frac{CT}{2} + \frac{CT}{2} = CT \\
a_n &= \frac{1}{T}\int_{-T}^{T} f(t)\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t \\
&= \frac{C}{T}\int_{-T}^{0} \underbrace{(t+T)\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t}_{\mathrm{I}} + \frac{C}{T}\int_{0}^{T} \underbrace{t\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t}_{\mathrm{II}}.
\end{aligned}$$

Using integration by parts:

$$\mathsf{I} = \left[\frac{(t+T)}{p}\cancel{\sin}(pt)\right]_{-T}^{0} - \int_{-T}^{0} \frac{1}{p}\sin(pt)\mathrm{d}t$$

where $p = \frac{n\pi}{T}$.

$$\therefore \mathsf{I} = \left[\frac{1}{p^2}\cos(pt)\right]_{-T}^{0}.$$

Similarly:

$$\begin{aligned}\mathsf{II} &= \left[\frac{t}{p}\cancel{\sin}(pt)\right]_{0}^{T} - \int_{0}^{T} \frac{1}{p}\sin(pt)\mathrm{d}t \\ &= \left[\frac{1}{p^2}\cos(pt)\right]_{0}^{T}\end{aligned}$$

$$\mathsf{I} + \mathsf{II} = \frac{1}{p^2}[1 - \cos(pt) + \cos(pt) - 1] = 0$$

$$\therefore a_n = 0$$

$$b_n = \frac{C}{T}\left[\int_{-T}^{0} (t + T)\sin\left(\frac{n\pi t}{T}\right)\mathrm{d}t + \int_{0}^{T} t\cos\left(\frac{n\pi t}{T}\right)\mathrm{d}t\right]$$

$$\therefore b_n = \begin{cases} -\frac{2CT}{n\pi} & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd.} \end{cases}$$

$$\therefore f(t) = \frac{CT}{2} - \frac{2CT}{\pi} \sum_{n=2,4,6\ldots}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi t}{T}\right). \tag{2}$$

Now, by superposition, let

$$V(t) = V_0 + \sum_{n=1}^{\infty} V_n(t), \tag{3}$$

where

$$V_n = A_n\cos\left(\frac{n\pi t}{T}\right) + B_n\sin\left(\frac{n\pi t}{T}\right). \tag{4}$$

Differentiating (4) and substituting into (1) we get:

$$\cos\left(\frac{n\pi t}{T}\right) : -\left(\frac{n\pi}{T}\right)^2 A_n + 0.05\left(\frac{n\pi}{T}\right) B_n + 36A_n = 0$$

$$\sin\left(\frac{n\pi t}{T}\right) : -\left(\frac{n\pi}{T}\right)^2 B_n - 0.05\left(\frac{n\pi}{T}\right) A_n + 36B_n = b_n$$

$$\text{constant} : V_0 = \frac{CT}{72}$$

$$\therefore A_n = \frac{-0.05\frac{n\pi}{T} b_n}{\left(\frac{n^2\pi^2}{T^2} - 36\right)^2 + (0.05\frac{n\pi}{T})^2}$$

$$B_n = \frac{-\left(\frac{n^2\pi^2}{T^2} - 36\right) b_n}{\left(\frac{n^2\pi^2}{T^2} - 36\right)^2 + (0.05\frac{n\pi}{T})^2}$$

$$V(t) = \frac{CT}{72} + \sum_{n=2,4,6...}^{\infty} \left[A_n \cos(\frac{n\pi t}{T}) + B_n \sin(\frac{n\pi t}{T})\right].$$

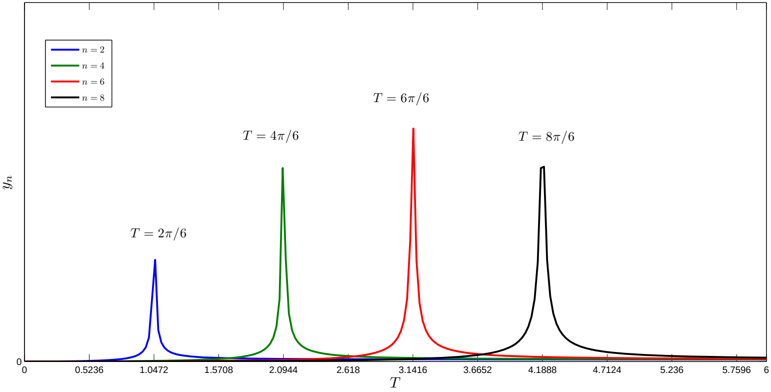
But if $\tan(\theta_n) = w$ and $\cos(\theta_n) = 1/\sqrt{1+w^2}$ then

$$V(t) = \frac{CT}{72} + \sum_{n=1}^{\infty} y_n \cos\left(\frac{n\pi t}{T} - \theta_n\right),$$

where $y_n = (A_n^2 + B_n^2)^{1/2}$ and $\theta_n = \arctan(B_n/A_n)$. The amplitude of vibration is y_n, which is given by:

$$y_n = \frac{b_n}{\sqrt{\left(\frac{n^2\pi^2}{T^2} - 36\right)^2 + (0.05\frac{n\pi}{T})^2}}, \qquad \text{for } n = 2, 4, 6, \ldots$$

when $\frac{n\pi}{T} \approx 6$ then the denominator becomes very small, so the ratio y_n/b_n becomes very large. These harmonics are known as the resonance frequencies of the system.



$n=2$
$n=4$
$n=6$
$n=8$
$T=2\pi/6$
$T=4\pi/6$
$T=6\pi/6$
$T=8\pi/6$
y_n
0
0
0.5236
1.0472
1.5708
2.0944
2.618
3.1416
3.6652
4.1888
4.7124
5.236
5.7596
6
T