# Nonlinear Attitude Control of Spacecrafts


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***Abstract*—In this article, we consider the attitude tracking control problem for rigid spacecraft with bounded external disturbances. Two nonlinear control problems are presented, with one being Global exponential stability, something which is not shown by linearized or simple nonlinear control laws and other being $\mathcal{L}_2$ gain stable. Both the proposed controllers are continuous and can achieve predefined-time predefined-bounded stability. That is, the attitude tracking errors are driven to a predefined-bounded region around the origin within a predefined time, which can be set as a tuning parameter during the controller design, independently of initial conditions Finally, numerical simulations are carried out to evaluate the performance of the proposed control law.**


## I. Introduction

Attitude control of rigid spacecraft has been studied by many researchers in the literature . In particular, finite-time attitude control has attracted a significant amount of attention from researchers in virtue of its ability to provide better disturbance rejection property as well as higher precision attitude control performance. The design approaches for finite-time attitude control schemes may include the terminal sliding mode method [6], [7], the technique of adding a power integrator and the homogeneity theorem. It should be mentioned that the time of convergence resulted in finite-time control schemes is heavily dependent on initial conditions. In the presence of some bounded disturbance, the system might not converge in some finite time. For the purpose of achieving superior robustness and tracking performance, nonlinear attitude tracking control should be used in lieu of linear control.

## II. Control 1: Robust Nonlinear Tracking Control Law with Global Exponential Stability

A new robust nonlinear tracking control law for attitude control of a spacecraft with large uncertainty is presented, which guarantees both global exponential convergence to the desired attitude trajectory and bounded tracking errors (in the sense of finite-gain $L_p$ stability and ISS) in the presence of uncertainties and disturbances. The benefits of this new attitude tracking control law include superior robustness due to no feedforward cancellation and straightforward extensions to integral control and various attitude representations such as MRPs and SO(3). Therefore, the proposed robust nonlinear tracking control law is designed to exploit the benefit of no feedforward cancellation while achieving superior tracking performance in the presence of large modeling uncertainties, measurement errors, and actuator saturations.

### A. System Dynamics

We will use modified Rodriguez parameter(MRP) to model spacecraft dynamics.

$$\dot{q} = Z(q)\omega = \frac{1}{2}\left(\frac{(1-q^Tq)}{2}I_3 + \tilde{q} + qq^T\right)\omega \tag{1}$$

$$J\dot{\omega} = (J\omega)\times\omega + Bu + d_{ext} \tag{2}$$

where where $\omega \in \mathbb{R}^3$ is the angular velocity of the spacecraft in a body-fixed frame, $\tilde{} \in \mathbb{R}^{3\times 3}$ denoted skew-symmetric matrix associated with corresponding vector, $u \in \mathbb{R}^{n_t}$ is the outputs of $n_t$ actuators,$B \in \mathbb{R}^{3\times n_t}$ be the corresponding control influence matrix, $J \in \mathbb{R}^{3\times 3}$ is the inertia matrix about the centre of mass, $d_{ext} \in \mathbb{R}^3$ is the bounded external disturbance, $q \in \mathbb{R}^3$ denotes the MRPs representing the spacecraft attitude with respect to an inertial frame. For reference, MRP are related to Euler angle representation as

$$q(t) = \eta(t)\tan\theta(t)) \tag{3}$$

where $\theta \in [0, 2\pi)$ with $\eta$ and $\theta$ being the Euler eigenaxis and eigenangle, respectively.
Suppose the desired attitude and angular velocity are denoted by $q_d$ and $\omega_d$ respectively. Then tracking error are calculated are:

$$q_e = q \otimes q_d = \frac{q_d(q^Tq-1) + q(1-q_d^Tq_d - 2\tilde{q}_dq)}{1+q_d^Tq_d + q^Tq + 2q_d^Tq} \tag{4}$$

$$\omega_e = \omega - C(q_e)\omega_d = I_3 + \frac{8(\tilde{q}_e)^2 - 4(1-q_e^Tq_e)\tilde{q}_e}{(1+q_e^Tq_e)^2} \tag{5}$$

Finally the error dynamics of a spacecraft (used for tracking) can be written as

$$\dot{q}_e = Z(q_e)\omega_e \tag{6}$$

$$\begin{aligned} J\dot{\omega}_e &= (J\omega)\times\omega - JC(q_e)\dot{\omega}_d + (J\omega_e)\times C(q_e)\omega_d + Bu + d_{ext} \\ &= f(q_e, \omega, \omega_d, \dot{\omega}_d) + Bu + d_{ext} \end{aligned} \tag{7}$$

Let $q_d$ denote the desired attitude orientation of the stabilized system. The attitude control objective is to stabilize the system, in the presence of uncertain physical parameters, bounded disturbances, measurement errors, and actuator saturations, such that for some appropriate $\epsilon_{trans} > 0$, $\epsilon_{ss} > 0$, $T >> 0$

$$\|\boldsymbol{\omega}(t)\|_2 \leq \varepsilon_{\text{trans}}, \qquad \forall\ t > 0$$

$$\|\boldsymbol{q}(t) - \boldsymbol{q}_{\text{final}}\|_2 \leq \varepsilon_{\text{ss}}, \qquad \forall\ t > T$$

The transient error bound $\epsilon_{trans}$ is imposed on the angular velocity $\omega(t)$ to ensure that the system is always within the technological capability of the sensors and actuators onboard the spacecraft. It is desired that after time T, the system should achieve the desired attitude orientation $q_d$ as shown in the steady-state condition. If the system has to hold its attitude within the given steady-state error bound $\epsilon_{ss}$, then the desired angular velocity $\omega_d$ of the stabilized system should be sufficiently close to 0 rad/s.

### B. Control law

For the given desired attitude trajectory $q_d(t)$, and positive–definite constant matrices $K_r \in \mathbb{R}^{3\times 3}$ and $\Lambda_r \in \mathbb{R}^{3\times 3}$, we define the following control law:

$$u_c = \hat{J}\dot{\omega}_r - S\left(\hat{J}\hat{\omega}\right)\omega_r K_r\left(\hat{\omega} - \omega_r\right) \tag{8}$$

where,

$$\omega_r = \mathbf{Z}^{-1}(\hat{q})\dot{q}_d(t) + \mathbf{Z}^{-1}(\hat{q})\Lambda_r\left(q_d(t) - \hat{q}\right) \tag{9}$$

where, $\omega_r$ is the desired reference trajectory, S represents the skew-symmetric matrix of $\hat{J}\hat{w}$ and $\hat{(.)}$ is the known part. For presenting the proof of stability for given control law, the following lemmas are used:

**Lemma 1.** Contraction Theory: We consider a smooth nonlinear nonautonomous system:

$$\dot{x}(t) = f(x(t), t) \tag{10}$$

A virtual displacement $\delta x$ is defined as an infinitesimal displacement at fixed time, and $\Theta(x,t)$ is a smooth coordinate transformation of the virtual displacement such that $\delta z = \Theta\delta x$. Then, if there exists a positive $\lambda$ and a uniformly $M(x,t) = \Theta(x,t)^T\Theta(x,t)$ positive–definite metric such that

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\quad &\left(\delta z^T\delta z\right) = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\delta\boldsymbol{x}^T\boldsymbol{M}(\boldsymbol{x},t)\delta\boldsymbol{x}\right)\\ &= \delta\boldsymbol{x}^T\left(\dot{\boldsymbol{M}} + \left(\tfrac{\partial f}{\partial\boldsymbol{x}}\right)^T\boldsymbol{M} + \boldsymbol{M}\tfrac{\partial f}{\partial\boldsymbol{x}}\right)\delta\boldsymbol{x}\\ &\leq -2\lambda\delta\boldsymbol{x}^T\boldsymbol{M}(\boldsymbol{x},t)\delta\boldsymbol{x}\end{aligned} \tag{11}$$

then all system trajectories converge exponentially fast to a single trajectory regardless of the initial conditions $(\delta x, \delta z \to 0)$ at a rate of $\lambda$ (i.e., contracting), and $\lambda$ is the largest eigenvalue of the symmetric part of $\left(\dot{\Theta} + \Theta\frac{\partial f}{\partial x}\right)\Theta^{-1}$.

We define the $\mathcal{L}_p$ norm in the extended space $\mathcal{L}_{pe}, p \in [1,\infty]$ as follows:

$$\|(u)\tau\|\ \mathcal{L}p = \left(\int_0^\tau \|\boldsymbol{u}(t)\|_2^p\ \mathrm{d}t\right)^{1/p} < \infty, \quad p \in [1,\infty), \tag{12}$$

$$\|(u)\tau\|\ \mathcal{L}_\infty = \sup_{t\geq 0}\|(u(t))\tau\|_2 < \infty \tag{13}$$

where $(\boldsymbol{u})_\tau$ is a truncation of $u(t)$, i.e., $(u(t))_\tau = 0$ for $t \geq \tau, \tau \in [0,\infty)$ whereas $(u(t))_\tau = u(t)$ for $0 \leq t \leq \tau$.

**Lemma 2.** Robust contraction and link to $\mathcal{L}_p$ stability and ISS:
Let $P_1(t)$ be a solution of the contracting system [Eq.(10)], globally exponentially tending to a single trajectory at a contraction rate of $\lambda$. Equation (10) is now perturbed as:

$$\dot{x}(t) = f(x(t), t) + d(x(t), t) \tag{14}$$

and $P_2(t)$ denotes the trajectory of Eq. (14). Then, the smallest path integral (i.e., distance) $R(t) = \int_{P_1}^{P_2}\|\delta z(t)\|_2 = \int_{P_1}^{P_2}\|\Theta(x,t)\delta x(t)\|_2$, $\forall t \geq 0$ exponentially converges to the following error ball:

$$\lim_{t\to\infty} R(t) \leq \sup_{x,t}\frac{\|\Theta(x,t)d(x(t),t)\|_2}{\lambda}$$

with $\Theta d \in \mathcal{L}_\infty$. Furthermore, if $d(x(t),t) \in \mathcal{L}_{\mathrm{pe}}$, then Eq. (14) is finite-gain $\mathcal{L}p$ stable with $p \in [1,\infty]$ for an output function $y = h(x,d,t)$ with $\int_{Y_1^2}^{Y_2}\|\delta y\|2 \leq \eta_1\int_{P_1}^{P_2}\|\delta x\|_2 + \eta_2\|d\|_2, \exists\eta_1,\eta_2 \geq 0$, because

$$\begin{aligned}\left\|\left(\int_{Y_1}^{Y_2}\|\delta y\|_2\right)_\tau\right\|_{\mathcal{L}p} &\leq \frac{\eta_1\zeta R(0)}{\sqrt{\lambda_{\min}(\boldsymbol{M})}}\\ &+ \left(\frac{\eta_1}{\lambda} + \eta_2\right)\frac{\|(\Theta d)_\tau\|_{\mathcal{L}p}}{\sqrt{\lambda\min(\boldsymbol{M})}}, \quad \forall\tau \in [0,\infty)\end{aligned} \tag{15}$$

where $Y_1(t)$ and $Y_2(t)$ denote the output trajectories of the original contracting system and its perturbed system, respectively, and $\zeta = 1$ if $p = \infty$ or $\zeta = 1/(\lambda p)^{1/p}$ if $p \in [1,\infty)$. The perturbed system [Eq. (14)] is also input-to-state stable.

### C. Proof of Stability

We obtain the closed-loop dynamics by substituting $u_c$ from (8) in (2) as:

$$\begin{aligned}&J\dot{\omega}_e - S\left(J\hat{\omega}\right)\omega_e + K_r\omega_e\\ &\qquad = \underbrace{\left[d_{\text{res}} + \Delta J\dot{\omega}_r - S\left(\Delta J\hat{\omega}\right)\omega_r\right]}\, d_{\text{res},2}\end{aligned} \tag{16}$$

where $\omega_e = \hat{\omega} - \omega_r$ and $d_{res}$ is the resultant disturbance. The virtual dynamics of y derived from (12) without $d_{res,2}$ is given as:

$$\begin{aligned}&J\dot{y} - S\left(J\hat{\omega}\right)y + K_r y\\ &\qquad = 0\end{aligned} \tag{17}$$

After we obtain the dynamics of the infinitesimal displacement at fixed time, $\delta_y$ from Eq. (15), we perform the squared-length analysis (from Lemma 1):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta y^T J \delta y\right) = -2\delta y^T \boldsymbol{K}_r \delta y \leq \frac{-2\lambda_{\min}\left(K_r\right)}{\lambda_{\max}\left(J\right)}\left(\delta y^T \boldsymbol{J} \delta \boldsymbol{y}\right) \quad (18)$$

Hence, it follows from the contraction analysis (Lemma 1) that all system trajectories of Eq. (13) converge exponentially fast to a single trajectory at a rate of $\frac{[\lambda_{min}(K_r)]}{[\lambda_{max}(J)]}$. Hence,the control law indeed globally exponentially stabilizes the closed-loop system without the resultant disturbance $d_{res,2}$.

In the presence of bounded resultant disturbance $\boldsymbol{d}_{\text{res, }2}$, it follows from Lemma 2 in the Appendix that

$$\lim_{t\to\infty}\int_0^{\omega_e}\|\delta\boldsymbol{y}\|_2 \leq \frac{\lambda_{\max}\left(\boldsymbol{J}_{\text{tot}}^{B_{\text{CM}}}\right)}{\lambda_{\min}\left(\boldsymbol{K}_r\right)\lambda_{\min}\left(\boldsymbol{J}_{\text{tot}}^{B_{CM}}\right)}\sup\|\boldsymbol{d}_{\text{res }2}\|_2 \quad (19)$$

Hence, the dynamics of the closed-loop system is bounded in the presence of bounded resultant disturbance $\boldsymbol{d}_{res,2}$. We now prove that convergence of $\boldsymbol{\omega}_e \to 0$ implies convergence of the system's trajectory to the desired trajectory ($\hat{\boldsymbol{q}} \to \boldsymbol{q}_d$). It follows from the definition of $\boldsymbol{w}_r$, that

$$\boldsymbol{\omega}_e = \mathbf{Z}^{-1}(\hat{\boldsymbol{q}})\left(\dot{\hat{\boldsymbol{q}}} - \dot{\boldsymbol{q}}_d\right) + \mathbf{Z}^{-1}(\hat{\boldsymbol{q}})\boldsymbol{\Lambda}_r\left(\hat{\boldsymbol{q}} - \boldsymbol{q}_d\right) = \mathbf{Z}^{-1}(\hat{\boldsymbol{q}})\left(\dot{\boldsymbol{q}}_e + \boldsymbol{\Lambda}_r\boldsymbol{q}_e\right) \quad (20)$$

where $\boldsymbol{q}_e = (\hat{\boldsymbol{q}} - \boldsymbol{q}_d)$. In the absence of $\boldsymbol{\omega}_e$, all system trajectories of $\delta\boldsymbol{q}_e$. will converge exponentially fast to a single trajectory ($\delta\boldsymbol{q}_e \to 0$) with a rate of $\lambda_{\min}\left(\boldsymbol{\Lambda}_r\right)$, where the virtual displacement $\delta\boldsymbol{q}_e$ is an infinitesimal displacement at fixed time. In the presence of $\boldsymbol{\omega}_\epsilon$, it follows from Lemma 2 that

$$\begin{aligned}\lim_{t\to\infty}\int_0^{\boldsymbol{q}_e}\|\delta\boldsymbol{q}e\|_2 &\leq \frac{1}{\lambda_{\min}\left(\boldsymbol{\Lambda}_r\right)}\sup_t\|\boldsymbol{Z}(\hat{\boldsymbol{q}})\boldsymbol{\omega}_e\|_2 \\ &\leq \frac{\lambda_{\max}\left(\boldsymbol{J}_{tot}^{B_{CM}}\right)}{\lambda_{\min}\left(\boldsymbol{\Lambda}_r\right)\lambda_{\min}\left(\boldsymbol{K}_r\right)\lambda_{\min}\left(\boldsymbol{J}_{\text{tot}}^{B_{CM}}\right)} \\ &\quad \left(\sup_t \sigma_{\text{man}}(\boldsymbol{Z}(\hat{\boldsymbol{q}}))\right)\left(\sup_t\|\boldsymbol{d}_{\text{res, }2}\|_2\right) \quad (21)\end{aligned}$$

Hence, we have shown, by constructing a hierarchically combined closed-loop system of $\boldsymbol{\omega}_e$and $\boldsymbol{q}_e$, that the attitude trajectory $\boldsymbol{q}$ will globally exponentially converge to a bounded error ball around the desired trajectory $\boldsymbol{q}_d(t)$. Moreover, it follows from Lemma 2 in that this control law is finite-gain $\mathcal{L}_p$ stable and input-to-state stable. Hence, the control gains $\boldsymbol{K}_r$, and $\boldsymbol{\Lambda}_r$, can be designed such that the error bounds $\varepsilon_{\text{tram}}$ and $\varepsilon_{ss}$ are satisfied.

The desired attitude trajectory $\boldsymbol{q}_d(t)$ can be any reference trajectory that we would like the system to track.

## III. Control 2: The Robust Nonlinear $H_\infty$ State-Feedback Control

There has been lot of Interest in robust H control in the last two decade. Interpreting nonlinear H control in terms of dissipativity and differential games where the solution is related to an adequate Hamilton-Jacobi inequality. The Hamilton-Jacobi partial differential inequality in linear systems is well known to reduce to the Riccati inequality, which may be solved quickly using efficient numerical algorithms. In the nonlinear case, however, there is yet no systematic numerical approach for solving this partial differential inequality. To this end, various approaches have been proposed to solve the Hamilton-Jacobi inequality numerically. One of the suggested methods is a Taylor series expansion of the storage function, in an iterative fashion, numerically efficient solution remains an unsolved issue. A recent computational relaxation based on the sum of squares (SOS) decomposition for multivariable polynomials and semidefinite programming provides potentially effective ways for the analysis and synthesis of nonlinear systems. The new computationally tractable analysis methodology provides a new way of searching for SOS decomposition to relax the original problem. As a result, a convex parametrization of the nonlinear H control problem was derived in based on a pair of positive definite matrix functions. Prempain formulated the $\mathcal{L}_2$-gain analysis problem for polynomial nonlinear systems as a convex state-dependent LMI, which can be recasted as a SOS optimization problem. This approach was shown promising to overcome the numerical difficulty in solving the Hamilton-Jacobi inequality and provides an analytic solution at the same time. Motivated by all of these developments, we propose a computational scheme for solving the robust nonlinear dynamic output-feedback design problems for a class of affine nonlinear systems with presence of uncertainties which are unmodelled, matched and unstructured.

### A. System Dynamics

Consider a rigid body spacecraft which rotates around its center of mass under the influence of control and perturbations torques. Let $\mathcal{B}$ denote a spacecraft body frame, i.e., a Cartesian coordinates frame with the origin at the center of mass. Let $\mathcal{R}$ denote the Earth Centered Earth Inertial reference frame (ECEI). Let q denote the quaternion of rotation from $\mathcal{R}$ to $\mathcal{B}$, with vector part e and scalar part q and $\omega$ denote the angular velocity of $\mathcal{B}$ with respect to $\mathcal{R}$ expressed in $\mathcal{B}$. The rotational dynamics and kinematics of the rigid body spacecraft are governed by the following differential equations where $\mathcal{J}$

$$\frac{d}{dt}\begin{bmatrix}\omega \\ \mathbf{e} \\ q\end{bmatrix} = \begin{bmatrix}-J^{-1}[\omega\times]J\omega \\ \frac{1}{2}\left(qI_3 + [\mathbf{e}\times]\right)\omega \\ -\frac{1}{2}\mathbf{e}^T\omega\end{bmatrix} + \begin{bmatrix}J^{-1} \\ \mathbf{0}_{4\times 3}\end{bmatrix}\mathbf{T}_b$$

denotes the spacecraft tensor of inertia matrix in $\mathcal{B}$ , $[\omega\times]$

denotes the cross-product matrix related to $\omega$, and $T_b$ is the vector of total external torques applied to the spacecraft, i.e.

$$T_b = u_b + \omega_b \tag{22}$$

where $u_b$ denote the $3 \times 1$ vector of control torques and $w_b$ denote the $3 \times 1$ vector of disturbance torques.

### *B. The Robust Nonlinear $H_\infty$ problem*

Considering the following system $\Sigma$:

$$\dot{x} = f(x) + g_1(x)w + g_2(x)u \tag{23}$$

$$y = x \tag{24}$$

$$z = h_1(x) + k_{12}(x)u \tag{25}$$

The $H_\infty$ control problem is described as finding a controller K(x) which produces a control input such that in the closed-loop configuration satisfies,

$$\int_0^\infty ||z(t)||^2 dt \leq \gamma^2 \left[ ||x_0||^2 + \int_0^\infty ||w(t)||^2 dt \right] \tag{26}$$

then we can say that the closed loop system has an $\mathcal{L}_2 - gain \leq \gamma$. Furthermore, the closed-loop system should be stable. For state-space system to be dissipative with respect to the supply rate s, there should exist a storage function S, such that,

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t1} s(w(t), z(t)) dt \tag{27}$$

This equation is called the dissipative inequality. Now, by choosing a suppy rate:

$$s(w, z) = \frac{1}{2}\gamma^2 ||w||^2 - \frac{1}{2}||z||^2, \gamma \geq 0 \tag{28}$$

$\Sigma$ is dissipative is dissipative with respect to this supply rate if and only if there exists S $\geq 0$ such that for all $t_1 \geq t_0$, $x(t_0)$ and $u$ valid the following:

$$\frac{1}{2}\int_{t_0}^{t_1} \left(\gamma^2 ||w||^2 - ||z||^2\right) dt \geq S(x(t_1)) - S(x(t_0)) \tag{29}$$

If the system $\Sigma$ is dissipative with respect to the suppy rate $s(w, z)$ then it follows the system $\Sigma$ has $\mathcal{L}_2 - gain \geq \gamma$. We will consider a storage functions $S$ as $C^1$ functions. By letting $t_1 \to t_0$ we see that Eq. (16) is equivalent to:

$$S_x \dot{x} \leq s(w, z(x, u)), \ \forall x, u \tag{30}$$

Here, $S_x(x)$ denotes the vector of partial derivatives $S_x(x) = \left(\frac{\partial S}{\partial x_1}(x), ..., \frac{\partial S}{\partial x_n}(x)\right)$.Furthermore, one can establish a direct link between dissipativity and Lyapunov stability.

Now if we assume $x^* \in \mathcal{X}$ to be a local minimum of $S$, then it will be a stable equilibrium of the unforced system $\dot{x} = f(x)$ with Lyapunov function $V(x) = S(x) - S(x^*) \geq 0$, for $x$ around $x^*$. Incorporating $V(x)$ dissipation inequility becomes as follows:

$$V_x(f(x) + g_1(x)w + g_2(x)u) - \frac{1}{2}\gamma^2 ||w||^2 + \frac{1}{2}||z(x, u)||^2 \leq 0 \tag{31}$$

Differenciating above inequality with respect to $w$, we get,

$$V_x * g_1(x) - \gamma^2 w^T \tag{32}$$

Assume maximum value of w to be w*. Equating above differentiation to zero, we get,

$$V_x * g_1(x) - \gamma^2 w*^T = 0 \tag{33}$$

$$V_x * g_1(x) = \gamma^2 w*^T \tag{34}$$

$$w*^T = \frac{1}{\gamma^2} V_x * g_1(x) \tag{35}$$

$$w* = \frac{1}{\gamma^2} g_1^T V_x^T \tag{36}$$

Now, differenciating above inequality with respect to $u$, we get,

$$V_x * g_2(x) + z^T \frac{\partial z}{\partial u} \tag{37}$$

Assume minimum value of u to be u*. Equating above differentiation to zero, we get,

$$V_x * g_2(x) + z^T \frac{\partial z}{\partial u} = 0 \tag{38}$$

Here,

$$z^T = h_1^T + u^T k_{12}^T \tag{39}$$

$$\frac{\partial z}{\partial u} = k_{12} \tag{40}$$

Putting above values in eq. (30), we get,

$$V_x * g_2(x) + (h_1^T + u *^T k_{12}^T) k_{12} = 0 \tag{41}$$

$$V_x * g_2(x) + (h_1^T k_{12} + u *^T k_{12}^T k_{12}) = 0 \tag{42}$$

$$V_x * g_2(x) + (h_1^T k_{12} + u *^T ||k_{12}||) = 0 \tag{43}$$

Assuming $||k_{12}||$ = 1 and $h_1^T k_{12}$ = 0, we get,

$$V_x * g_2(x) + u*^T = 0 \tag{44}$$

$$u*^T = -V_x * g_2(x) \tag{45}$$

$$u* = -g_2^T V_x^T \tag{46}$$

Substituting $u = u*$ and $w = w*$ in inequality (23), yields the Hamilton Jacobi inequality (HJI):

$$V_x f(x) + \frac{V_x}{2}\left(\frac{1}{\gamma^2} g_1(x) g_1(x)^T - g_2(x) k_{12}(x)^T k_{12} g_2(x)^T\right) V_x^T + \frac{1}{2} h_1(x)^T h_1(x) \leq 0 \tag{47}$$

Thus, if there exists a $V \geq 0$ which satisfies the above inequality, then the system $\Sigma$ has an $\mathcal{L}_2 - gain \leq \gamma$. Therefore, sufficient condition for a system to have $\mathcal{L}_2 - gain$ is the existence of a controller $u(x) = K(x)$ which renders a dissipative closed loop system. By taking $t_0 = 0$ and assuming that $V(x(0)) \leq \gamma^2 ||x(0)||^2$ then the dissiptivity implies that $\mathcal{L}_2 - gain \leq \gamma$.

### C. The Robust Nonlinear $H_\infty$ State-Feedback Problem

Considering the following system:

$$\dot{x} = f(x) + \Delta f(x,\theta,t) + g_1(x)w + [g_2(x) + \Delta g_2(x,\theta,t)]u \tag{48}$$

$$y = x \tag{49}$$

$$z = h_1(x) + k_{12}(x)u \tag{50}$$

where $\Delta f(x,\theta,t), \Delta g_2(x,\theta,t) \in \Xi$ are unknown functions and $\theta \in \Theta \subset \mathbb{R}^2$ are system parameters. Now, we have to find a control function $u * (x) = K(x)$ such that the closed loop system has locally $\mathcal{L}_2 - gain \leq \gamma > 0$ with internal-stability from the disturbance signal $w$ to the output $z$.
The admissible uncertainties of the system $\Xi$ and $\Theta$ belong to the following sets:

$$\begin{aligned} \Xi = \{\Delta f, \Delta g_2 | \Delta f(x,\theta,t) = H_1(x)F(x,\theta,t)E_1(x), \\ \Delta g_2(x,\theta,t) = g_2(x)F(x,\theta,t)E_2(x), \\ ||F(x,\theta,t)||^2 \leq 1 \forall x \in \chi, \theta \in \Theta, t \in \mathbb{R}\} \end{aligned} \tag{51}$$

$$\Theta = \{0 \leq \theta \leq |\theta_u|\} \tag{52}$$

Considering the dissipation inquality,

$$V_x(x)\left(f(x) + \Delta f(x,\theta,t) + g_1(x)w + [g_2(x) + \Delta g_2(x,\theta,t)]\, u\right) - \frac{1}{2}\gamma^2 w^T w + \frac{1}{2}\left(h_1^T(x)h_1(x) + u^T k_{12}^T(x)k_{12}(x)u\right) \leq 0 \tag{53}$$

while maximizing with respect to w results in,

$$w^* = \frac{1}{\gamma^2} g_1^T(x)V_r^T(x) \tag{54}$$

while minimizing with respect to u results in

$$u^* = -R_2^{-1}(x)\left[2I + \frac{1}{4}E_2^T(x)E_2(x)\right] g_2^T(x)V_x^T(x) \tag{55}$$

where $R_2 = k_{12}^T(x)k_{12}(x)$. Substituting into the dissipation inequality results in the following Hamilton Jacobi inequality,

$$\begin{aligned} &V_x(x)\left(f(x) + H_1(x)F(x,\theta,t)E_1(x)\right) + \\ &\quad \frac{1}{2\gamma^2}V_x(x)g_1(x)g_1^T(x)V_x^T(x) + \frac{1}{2}h_1^T(x)h_1(x) \\ &\quad - \frac{1}{2}V_x(x)\left[g_2(x) + g_2(x)F(x,\theta,t)E_2(x)\right] \\ &R_2^{-1}(x)\left[g_2^T(x) + E_2^T(x)F^T(x,\theta,t)g_2^T(x)\right]V_x^T(x) \leq 0 \end{aligned} \tag{56}$$

## IV. Simulations

1) For Robust Nonlinear Tracking Control Law with GES, we have the simulations for attitude MRP q(t), tracking some fuel-optimum reference trajectory presented in the paper in the absence of measurement errors and modeling uncertainties.

2) In Fig. 2 For The Robust Nonlinear $H_\infty$ State-Feedback Control, the closed-loop attitude quaternion during two orbit cycles are simulated after the impact of a 1 [gr] particle, is described as a impulse function of 1.5 [Nm] with a duration of 0.1 [sec] which acts as $\omega_b$ disturbance torque considering the rotational dynamics and kinematics of a rigid body spacecraft by the differential equations governed by the quaternion dynamics(Eq. 22)

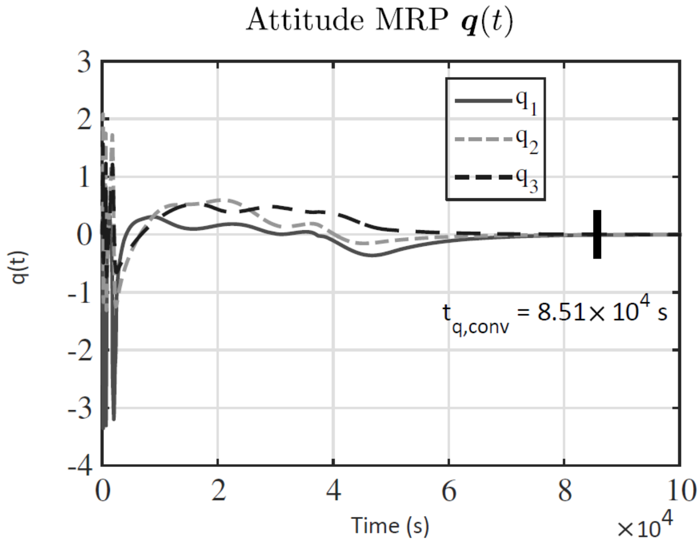


Fig. 1. Robust Nonlinear Tracking Control Law with GES

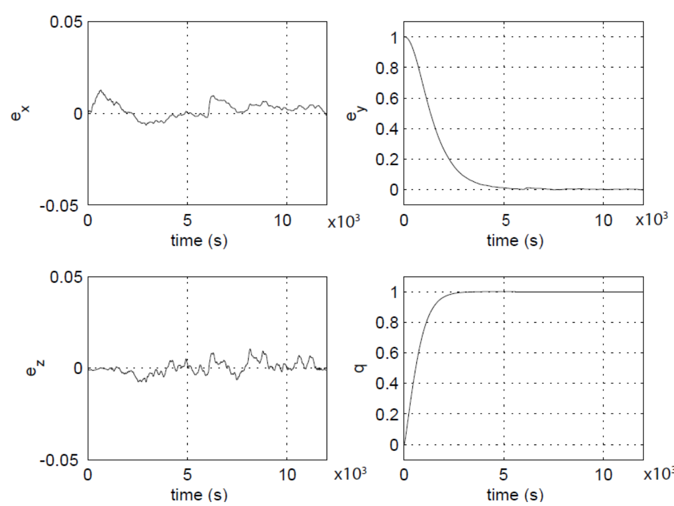


Fig. 2. The Robust Nonlinear $H_\infty$ State-Feedback Control

## V. Comparison

The two papers present nonlinear control strategy for the attitude control of the spacecraft. The two control laws are ROBUST NONLINEAR TRACKING CONTROL LAW WITH GLOBAL EXPONENTIAL STABILITY(Control 1) and THE ROBUST NONLINEAR $H_\infty$ STATE-FEEDBACK CONTROL(Control 2). Paper with Control 1 deals with a spacecraft carrying a large object for which control like robust $H_\infty$ control (Control 2) are used for attitude control of spacecraft with uncertainties and disturbances that use exact feed-forward cancellation, similar to feedback linearization, exhibit a large resultant disturbance torque due to unprecedentedly large modeling uncertainties of the captured object. Control 1 do not have a feedforward cancellation term experience a much smaller resultant disturbance torque. The control and the system presented in the first paper(with control 1) can be considered as an attitude control problem with a large disturbance torque while the simulations for paper 2 (with control 2) considers a smaller disturbance torque.

## References


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