Assignment 6

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Download all python codes from:

https://github.com/varenya27/AI1103/blob/main/ Assignment6/codes

and all latex-tikz codes from:

https://github.com/varenya27/AI1103/blob/main/ Assignment6/main.tex

1 Problem

Let Y_1 denote the first order statistic in a random sample of size n from a distribution that has the pdf,

$$f(x) = \begin{cases} e^{-(x-\theta)} & \text{when } \theta < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Obtain the distribution of $Z_n = n(Y_1 - \theta)$

2 Solution

The first order statistic for any sample is the the minimum of the given sample. In order to calculate the distribution for Z_n , we need the cumulative distribution function F(x):

$$F(x) = \int_{-\infty}^{x} f(t) dt$$

$$= \int_{-\infty}^{\theta} 0 dt + \int_{\theta}^{x} e^{\theta - t} dt$$
(2.0.1)

$$= \int_{-\infty}^{\theta} 0 dt + \int_{\theta}^{\Lambda} e^{\theta - t} dt \qquad (2.0.2)$$

$$= \left(e^{\theta - t}\right)_{x}^{\theta}$$

$$= 1 - e^{\theta - x}$$

$$(2.0.3)$$

$$=1-e^{\theta-x} \tag{2.0.4}$$

$$= 1 - f(x) \tag{2.0.5}$$

This gives,

$$F(x) = \begin{cases} 1 - e^{-(x-\theta)} & \text{when } \theta < x < \infty \\ 0 & \text{otherwise} \end{cases}$$
 (2.0.6)

Also,

$$F(x) = \Pr\left(X \le x\right) \tag{2.0.7}$$

$$\implies 1 - F(x) = \Pr(X > x) \tag{2.0.8}$$

$$Y_1 = min\{X_1, X_2, ... X_n\}$$
 (2.0.9)

Distributions for $Z_n = n(Y_1 - \theta)$ are required; let $f_{Z_n}(z)$ and $F_{Z_n}(z)$ be the pdf and cdf for Z_n respectively. The cdf can be calculated as:

$$F_{Z_n}(z) = \Pr(n(Y_1 - \theta) \le z)$$
 (2.0.10)

$$= \Pr\left(Y_1 \le \frac{z}{n} + \theta\right) \tag{2.0.11}$$

$$= 1 - \Pr\left(Y_1 > \frac{z}{n} + \theta\right)$$
 (2.0.12)

Since Y_1 is a minimum order statistic, every term in the random sample must be included in (2.0.12). Let $\left(\frac{z}{n} + \theta\right) = z'$:

$$F_{Z_n} = 1 - \Pr(X_1 > z', X_2 > z', ..., X_n > z')$$

$$= 1 - \prod_{i=1}^{n} \Pr(X_i > z')$$

$$= 1 - (1 - F(z'))^n$$

$$\implies F_{Z_n}(z) = 1 - \left(1 - F\left(\frac{z}{n} + \theta\right)\right)^n \qquad (2.0.13)$$

$$=1-\left(f\left(\frac{z}{n}+\theta\right)\right)^{n} \tag{2.0.14}$$

The expression for the cdf can thus be written as:

$$F_{Z_n}(z) = \begin{cases} 1 - e^{-n(\frac{z}{n} + \theta - \theta)} & \text{when } \theta < \frac{z}{n} + \theta < \infty \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 - e^{-z} & \text{when } 0 < z < \infty \\ 0 & \text{otherwise} \end{cases}$$
(2.0.15)

Using the cdf in (2.0.13) to calculate the pdf:

$$f_{Z_n}(z) = \frac{d}{dz} F_{Z_n}(z)$$

$$= \frac{d}{dz} \left(1 - \left(1 - F \left(\frac{z}{n} + \theta \right) \right)^n \right)$$

$$= -n \left(1 - F \left(\frac{z}{n} + \theta \right) \right)^{n-1} \times \frac{d}{dz} \left(1 - F \left(\frac{z}{n} + \theta \right) \right)$$

$$= -n \times \left(f \left(\frac{z}{n} + \theta \right) \right)^{n-1} \times \left(\frac{-1}{n} \right) \times \left(f \left(\frac{z}{n} + \theta \right) \right)$$

$$= \left(f \left(\frac{z}{n} + \theta \right) \right)^n$$
(2.0.16)

Putting the expression for f(x) in (2.0.16):

$$f_{Z_n}(z) = \begin{cases} e^{n(\theta - (\frac{z}{n} + \theta))} & \text{when } 0 < z < \infty \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} e^{-z} & \text{when } 0 < z < \infty \\ 0 & \text{otherwise} \end{cases}$$
(2.0.17)

The plots for the cdf in (2.0.15) and the pdf in (2.0.17) are shown in Fig. 0 and Fig. 0 respectively:

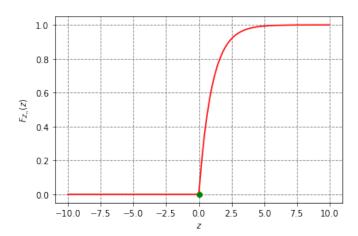


Fig. 0: cdf of Z_n

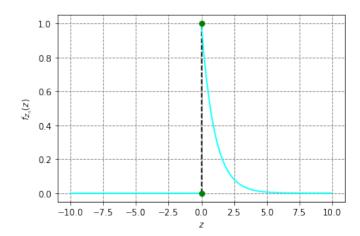


Fig. 0: pdf of Z_n