

Assignment 3

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Download all python codes from:

<https://github.com/varenya27/AI1103/blob/main/Assignment3/codes>

and all latex-tikz codes from:

<https://github.com/varenya27/AI1103/blob/main/Assignment3/main.tex>

PROBLEM

Let (X,Y) be the coordinates of a point chosen at random inside the disc $x^2 + y^2 \leq r^2$ where $r \geq 0$. The probability that $Y \geq mX$ is

- (a) $\frac{1}{2r}$ (c) $\frac{1}{2}$
 (b) $\frac{1}{2^m}$ (d) $\frac{1}{2^{r+m}}$

SOLUTION

We know that the point (X,Y) satisfies the equation

$$x^2 + y^2 \leq r^2 \quad (0.0.1)$$

Let a random variable $Z \in \{0, 1\}$ denote the possible outcomes of the experiment

Equation satisfied by (X,Y)	Z
$y - mx < 0$	0
$y - mx \geq 0$	1

TABLE I: Outcome of the Experiment

The coordinates (X,Y) can be parametrized as follows:

$$X = a \sin \theta \quad (0.0.2)$$

$$Y = a \cos \theta \quad (0.0.3)$$

where $a \in [0, r]$ and $\theta \in [0, 2\pi]$.

$$Y \geq mX \quad (0.0.4)$$

$$\implies a \sin \theta \geq ma \cos \theta \quad (0.0.5)$$

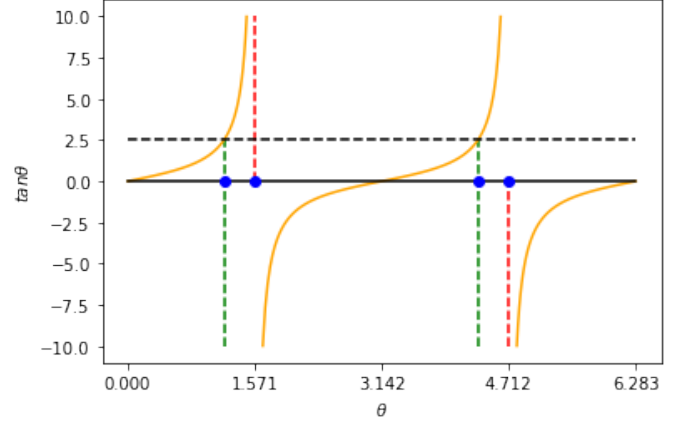


Fig. 0: $\tan \theta$ with $m = 2.5$

This gives two cases for an arbitrary value of m (as seen in the figure above):

- 1) when $\theta \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$, from the figure,

$$\tan \theta \geq m \quad (0.0.6)$$

$$\implies \theta \in [\tan^{-1} m, \pi/2] \quad (0.0.7)$$

- 2) similarly, when $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\tan \theta \leq m \quad (0.0.8)$$

$$\implies \theta \in [\pi/2, \pi + \tan^{-1} m] \quad (0.0.9)$$

$$\therefore \theta \in [\tan^{-1} m, \pi + \tan^{-1} m] \quad (0.0.10)$$

θ will have a uniform probability distribution function:

$$f(\theta) = \begin{cases} 0 & \text{if } \theta < 0 \\ \frac{1}{2\pi} & \text{if } 0 \leq \theta \leq 2\pi \\ 0 & \text{if } \theta > 2\pi \end{cases}$$

The shaded region of the figure below represents

the required probability.

$$\begin{aligned} \Pr(\arctan m \leq \theta \leq \tan^{-1} m + \pi) \\ = \int_{\tan^{-1} m}^{\pi + \tan^{-1} m} f(\theta) d\theta \end{aligned} \quad (0.0.11)$$

$$= \int_{\tan^{-1} m}^{\pi + \tan^{-1} m} \frac{1}{2\pi} d\theta \quad (0.0.12)$$

$$= \frac{\pi}{2\pi} \quad (0.0.13)$$

$$= \frac{1}{2} \quad (0.0.14)$$

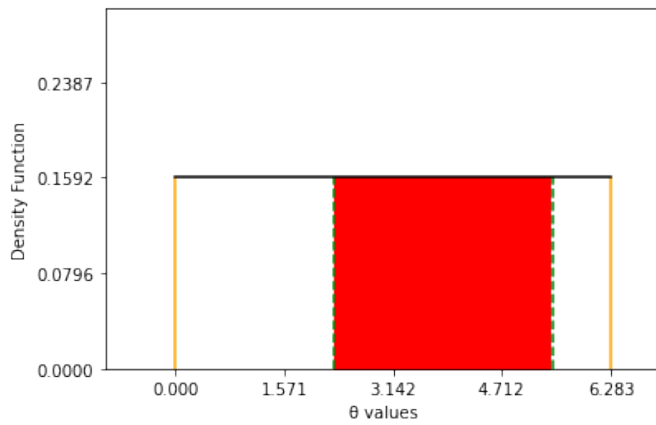


Fig. 0: Distribution function of θ

\therefore option (c) is correct.