Assignment 3

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Download all python codes from:

https://github.com/varenya27/AI1103/blob/main/Assignment3/codes

and all latex-tikz codes from:

https://github.com/varenya27/AI1103/blob/main/Assignment3/main.tex

Problem

Let (X,Y) be the coordinates of a point chosen at random inside the disc $x^2 + y^2 \leq r^2$ where $r \geq 0$. The probability that $Y \geq mX$ is

(a) $\frac{1}{2^r}$

(b) $\frac{1}{2^m}$

(c) $\frac{1}{2}$

(d) $\frac{1}{2^{r+m}}$

Solution

We know that the point (X, Y) satisfies the equation

$$x^2 + y^2 \leq r^2 \tag{0.0.1}$$

Let a random variable $Z \in \{0, 1\}$ denote the possible outcomes of the experiment

Equation satisfied by (X,Y)	Z
$y - mx < 0$	0
$y - mx \geq 0$	1

TABLE I: Outcome of the Experiment

The coordinates (X, Y) can be parametrized as follows:

$$X = a \sin\theta \tag{0.0.2}$$
$$Y = a \cos\theta \tag{0.0.3}$$

where $a \in [0, r]$ and $\theta \in [0, 2\pi]$.

$$Y \geq mX \tag{0.0.4}$$
$$\implies a \sin\theta \geq ma \cos\theta \tag{0.0.5}$$

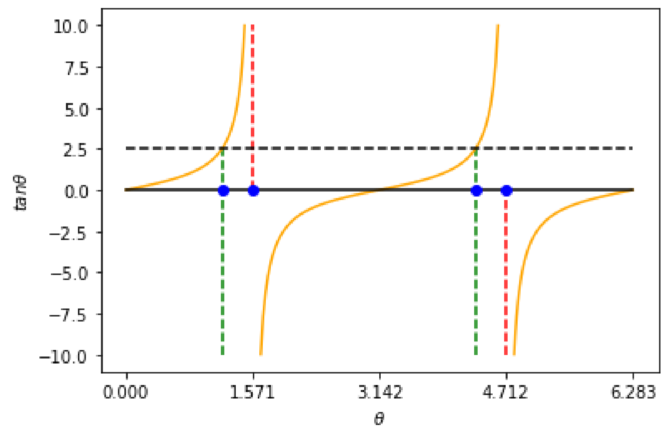

Fig. 0: $tan\theta$ with $m = 2.5$

This gives two cases,

1) when $\theta \in [0, \frac{\pi}{2}] \cup [\frac{3\pi}{2}, 2\pi]$

$$\tan\theta \geq m \tag{0.0.6}$$
$$\implies \theta \in [\tan^{-1} m, \pi/2] \tag{0.0.7}$$

2) when $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$

$$\tan\theta \leq m \tag{0.0.8}$$
$$\implies \theta \in [\pi/2, \pi + \tan^{-1} m] \tag{0.0.9}$$

$$\therefore \theta \in [\tan^{-1} m, \pi + \tan^{-1} m] \tag{0.0.10}$$

θ will have a uniform probability distribution function:

$$f(\theta) = \begin{cases} 0 & \text{if } \theta < 0 \\ \frac{1}{2\pi} & \text{if } 0 \leq \theta \leq 2\pi \\ 0 & \text{if } \theta > 2\pi \end{cases}$$

The shaded region of the figure below represents the required probability.

$$\Pr\left(\arctan m \le \theta \le \tan^{-1} m + \pi\right)$$

$$= \int_{\tan^{-1} m}^{\pi+\tan^{-1} m} f(\theta)\, d\theta \quad (0.0.11)$$

$$= \int_{\tan^{-1} m}^{\pi+\tan^{-1} m} \frac{1}{2\pi}\, d\theta \quad (0.0.12)$$

$$= \frac{\pi}{2\pi} \quad (0.0.13)$$

$$= \frac{1}{2} \quad (0.0.14)$$

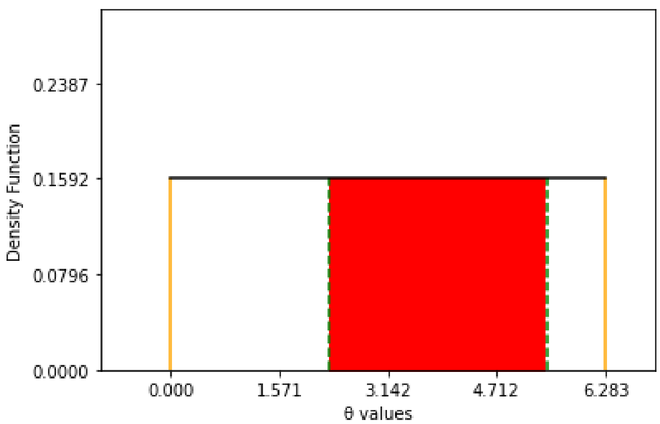

Fig. 0: Distribution function of θ

$\therefore$ option (c) is correct.