Assignment 6

Varenya Upadhyaya EP20BTECH11026

Download all python codes from:

https://github.com/varenya27/AI1103/blob/main/Assignment6/codes

and all latex-tikz codes from:

https://github.com/varenya27/AI1103/blob/main/Assignment6/main.tex

1 Problem

Let Y_1 denote the first order statistic in a random sample of size n from a distribution that has the pdf,

$$f(x) = \begin{cases} e^{-(x-\theta)} & \text{when } \theta < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Obtain the distribution of $Z_n = n(Y_1 - \theta)$

2 Solution

The first order statistic for any sample is the the minimum of the given sample. In order to calculate the distribution for Z_n, we need the the cumulative distribution function $F(x)$:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt \tag{2.0.1}$$
$$= \int_{-\infty}^{\theta} 0\,dt + \int_{\theta}^{x} e^{\theta-t}\,dt \tag{2.0.2}$$
$$= \left(e^{\theta-t}\right)_{x}^{\theta} \tag{2.0.3}$$
$$= 1 - e^{\theta-x} \tag{2.0.4}$$
$$= 1 - f(x) \tag{2.0.5}$$

This gives,

$$F(x) = \begin{cases} 1 - e^{-(x-\theta)} & \text{when } \theta < x < \infty \\ 0 & \text{otherwise} \end{cases} \tag{2.0.6}$$

Also,

$$F(x) = \Pr(X \leq x) \tag{2.0.7}$$
$$\implies 1 - F(x) = \Pr(X > x) \tag{2.0.8}$$
$$Y_1 = min\{X_1, X_2, ...X_n\} \tag{2.0.9}$$

Distributions for $Z_n = n(Y_1 - \theta)$ are required; let $f_{Z_n}(z)$ and $F_{Z_n}(z)$ be the pdf and cdf for Z_n respectively. The cdf can be calculated as:

$$F_{Z_n}(z) = \Pr(n(Y_1 - \theta \leq z) \tag{2.0.10}$$
$$= \Pr\left(Y_1 \leq \frac{z}{n} + \theta\right) \tag{2.0.11}$$
$$= 1 - \Pr\left(Y_1 \; > \frac{z}{n} + \theta\right) \tag{2.0.12}$$

Since Y_1 is a minimum order statistic, every term in the random sample must be included in (2.0.12). Let $\left(\frac{z}{n} + \theta\right) = z'$:

$$F_{Z_n} = 1 - \Pr(X_1 > z', X_2 > z', ..., X_n > z')$$
$$= 1 - \prod_{i=1}^{n} \Pr(X_i > z')$$
$$= 1 - (1 - F(z'))^n$$
$$\implies F_{Z_n}(z) = 1 - \left(1 - F\left(\frac{z}{n} + \theta\right)\right)^n \tag{2.0.13}$$
$$= 1 - \left(f\left(\frac{z}{n} + \theta\right)\right)^n \tag{2.0.14}$$

The expression for the cdf can thus be written as:

$$F_{Z_n}(z) = \begin{cases} 1 - e^{-n(\frac{z}{n}+\theta-\theta)} & \text{when } \theta < \frac{z}{n} + \theta < \infty \\ 0 & \text{otherwise} \end{cases}$$
$$= \begin{cases} 1 - e^{-z} & \text{when } 0 < z < \infty \\ 0 & \text{otherwise} \end{cases} \tag{2.0.15}$$

Using the cdf in (2.0.13) to calculate the pdf:

$$
\begin{aligned}
f_{Z_n}(z) &= \frac{d}{dz} F_{Z_n}(z) \\
&= \frac{d}{dz}\left(1 - \left(1 - F\left(\frac{z}{n} + \theta\right)\right)^n\right) \\
&= -n\left(1 - F\left(\frac{z}{n} + \theta\right)\right)^{n-1} \times \frac{d}{dz}\left(1 - F\left(\frac{z}{n} + \theta\right)\right) \\
&= -n \times \left(f\left(\frac{z}{n} + \theta\right)\right)^{n-1} \times \left(\frac{-1}{n}\right) \times \left(f\left(\frac{z}{n} + \theta\right)\right) \\
&= \left(f\left(\frac{z}{n} + \theta\right)\right)^n
\end{aligned} \tag{2.0.16}
$$

Putting the expression for $f(x)$ in (2.0.16):

$$
\begin{aligned}
f_{Z_n}(z) &= \begin{cases} e^{n\left(\theta - \left(\frac{z}{n} + \theta\right)\right)} & \text{when } 0 < z < \infty \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} e^{-z} & \text{when } 0 < z < \infty \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{2.0.17}
$$

The plots for the cdf in (2.0.15) and the pdf in (2.0.17) are shown in Fig. 0 and Fig. 0 respectively:

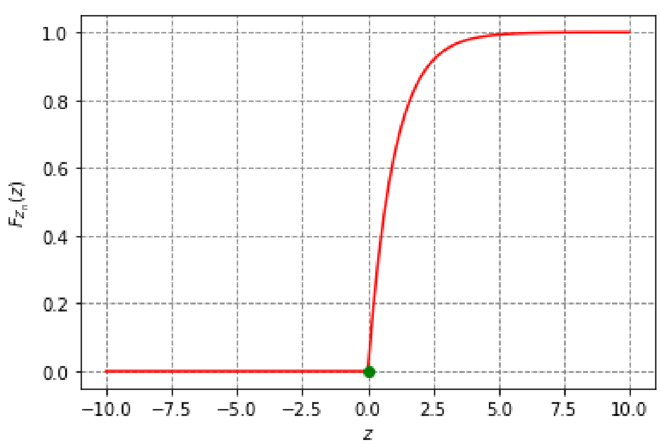

Fig. 0: cdf of Z_n

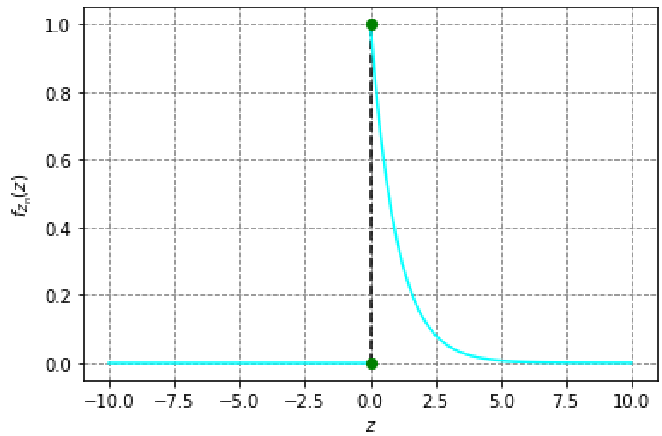

Fig. 0: pdf of Z_n