

# Fourier Series

Varenya Upadhyaya

## CONTENTS

1	<b>Periodic Function</b>	1
2	<b>Fourier Series</b>	1
3	<b>Fourier Transform</b>	3
4	<b>Filter</b>	5
5	<b>Filter Design</b>	5

**Abstract**—This manual provides a simple introduction to Fourier Series

## 1 PERIODIC FUNCTION

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \quad (1.1)$$

1.1 Plot  $x(t)$ .

**Solution:** The following code plots Fig. 1.1

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/1
_1.py
```

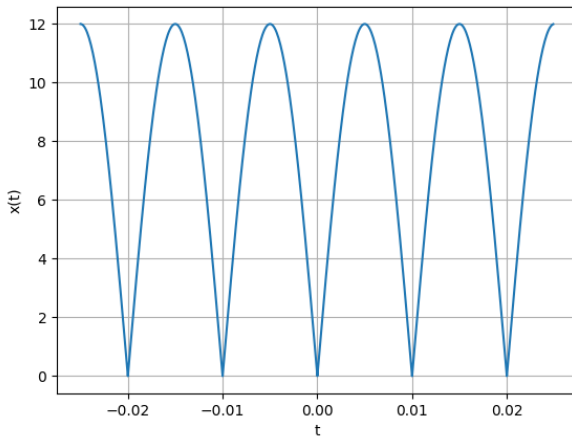


Fig. 1.1:  $x(t)$

1.2 Show that  $x(t)$  is periodic and find its period.  
Let  $T$  be the period of  $x(t)$

$$x(t + T) = A |\sin(2\pi f_0 (t + T))| \quad (1.2)$$

$$= A |\sin(2\pi f_0 t + 2\pi f_0 T)| \quad (1.3)$$

$$= A |\sin(2\pi f_0 t)|, \quad 2\pi f_0 T = \pi \quad (1.4)$$

$$\Rightarrow T = \frac{1}{2f_0} \quad (1.5)$$

## 2 FOURIER SERIES

Consider  $A_0 = 12$  and  $f_0 = 50$  for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.1)$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} dt \quad (2.2)$$

**Solution:**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.3)$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} dt \quad (2.4)$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-n)f_0 t} dt \quad (2.5)$$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0} \delta(k - n) \quad (2.6)$$

$$= \frac{1}{f_0} c_n \quad (2.7)$$

$$\Rightarrow c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} dt \quad (2.8)$$

2.2 Find  $c_k$  for (1.1)

**Solution:**

$$c_k = 50 \int_{-0.01}^{0.01} 12 |\sin(2\pi 50t)| e^{-j100\pi kt} dt \quad (2.9)$$

$$= 600 \int_{-0.01}^{0.01} |\sin(100\pi t)| \cos(100\pi kt) dt$$

$$+ j600 \int_{-0.01}^{0.01} |\sin(100\pi t)| \sin(100\pi kt) dt \quad (2.10)$$

$$= 600 \int_0^{0.01} 2 \sin(100\pi t) \cos(100\pi kt) dt \quad (2.11)$$

$$= 600 \int_0^{0.01} (\sin(100\pi(k+1)t)) dt$$

$$- 600 \int_0^{0.01} (\sin(100\pi(k-1)t)) dt \quad (2.12)$$

$$= 6 \frac{1 + (-1)^k}{\pi} \left( \frac{1}{k+1} - \frac{1}{k-1} \right) \quad (2.13)$$

$$= \begin{cases} \frac{24}{\pi(1-k^2)} & \text{even } k \\ 0 & \text{odd } k \end{cases} \quad (2.14)$$

2.3 Verify (2.1) using python.

**Solution:** The following code plots Fig. 2.3

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/2
_3.py
```

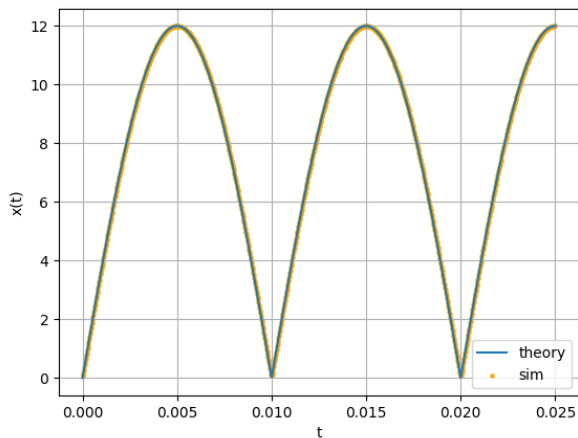


Fig. 2.3:  $x(t)$  from the fourier series

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \quad (2.15)$$

and obtain the formulae for  $a_k$  and  $b_k$ .

**Solution:**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.16)$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{j2\pi k f_0 t} + c_{-k} e^{-j2\pi k f_0 t} \quad (2.17)$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$

$$+ \sum_{k=0}^{\infty} j(c_k - c_{-k}) \sin(2\pi k f_0 t) \quad (2.18)$$

Hence, for  $k \geq 0$ ,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \quad (2.19)$$

$$= \begin{cases} 50 \int_{-0.01}^{0.01} x(t) dt & k = 0 \\ 100 \int_{-0.01}^{0.01} x(t) \cos(100\pi kt) dt & k > 0 \end{cases} \quad (2.20)$$

$$b_k = \frac{c_k - c_{-k}}{j} = 100 \int_{-0.01}^{0.01} x(t) \sin(100\pi kt) dt \quad (2.21)$$

2.5 Find  $a_k$  and  $b_k$  for (1.1)

**Solution:** Clearly  $x(t)$  is even

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \quad (2.22)$$

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k f_0 t} \quad (2.23)$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{j2\pi k f_0 t} = x(t) \quad (2.24)$$

$$\Rightarrow c_k = c_{-k} \quad (2.25)$$

thus for  $k \geq 0$ ,

$$a_k = \begin{cases} \frac{24}{\pi} & k = 0 \\ \frac{48}{\pi(1-k^2)} & k > 0, k \text{ even} \\ 0 & \text{otherwise} \end{cases} \quad (2.26)$$

$$b_k = 0 \quad (2.27)$$

2.6 Verify (2.15) using python.

**Solution:** The following code plots Fig. 2.6

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/2
_6.py
```

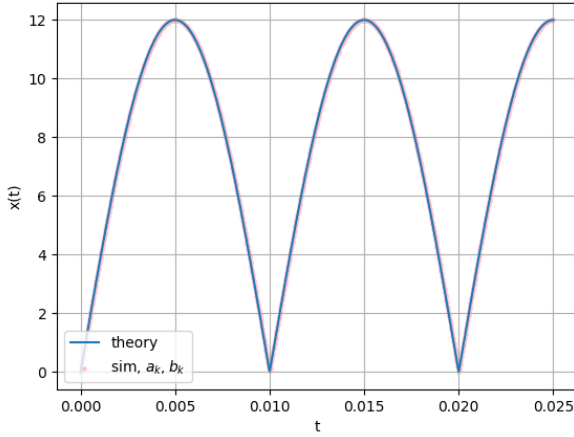


Fig. 2.6:  $x(t)$  from the fourier series using  $a_k, b_k$

### 3 FOURIER TRANSFORM

3.1

$$\delta(t) = 0, \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (3.2)$$

3.2 The Fourier Transform of  $g(t)$  is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t} dt \quad (3.3)$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \quad (3.4)$$

$$(3.5)$$

**Solution:**

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t} dt \quad (3.6)$$

$$= \int_{-\infty}^{\infty} g(t') e^{-j2\pi f (t' + t_0)} dt' \quad (3.7)$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(t') e^{-j2\pi f t'} dt' \quad (3.8)$$

$$= G(f) e^{-j2\pi f t_0} \quad (3.9)$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.10)$$

**Solution:** using the inverse fourier transform:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t} df \quad (3.11)$$

$$\Rightarrow g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi f t} dt \quad (3.12)$$

$$\Rightarrow g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t} dt \quad (3.13)$$

$$\Rightarrow G(t) \xleftrightarrow{\mathcal{F}} g(-f) \quad (3.14)$$

3.5  $\delta(t) \xleftrightarrow{\mathcal{F}} ?$

**Solution:**

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt \quad (3.15)$$

$$= 1 \quad (3.16)$$

3.6  $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$

**Solution:**

$$e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t} dt \quad (3.17)$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi (f + f_0) t} dt \quad (3.18)$$

$$= \delta(f + f_0) \quad (3.19)$$

3.7  $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$

**Solution:**

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \cos(2\pi f_0 t) e^{-j2\pi f t} dt \quad (3.20)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) e^{-j2\pi f t} dt \quad (3.21)$$

$$= \frac{1}{2} (\delta(f - f_0) + \delta(f + f_0)) \quad (3.22)$$

3.8 Find the Fourier Transform of  $x(t)$  and plot it. Verify using python.

**Solution:**

$$x(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k e^{-j2\pi k f_0 t} dt \quad (3.23)$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(k f_0 + f) \quad (3.24)$$

$$= \sum_{k=-\infty}^{\infty} \frac{24}{\pi(1-4k^2)} \delta(2k f_0 + f) \quad (3.25)$$

The following code plots Fig. 3.8

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/3
_8.py
```

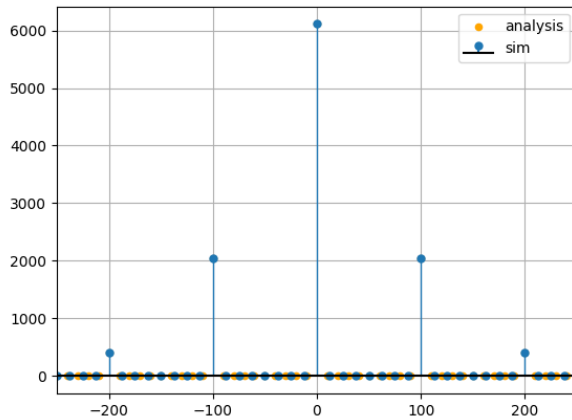


Fig. 3.8: fourier transform of  $x(t)$

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(f) \quad (3.26)$$

Verify using python.

**Solution:**

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \text{rect}(t) e^{-j2\pi f t} dt \quad (3.27)$$

$$= \int_{-0.5}^{0.5} e^{-j2\pi f t} dt \quad (3.28)$$

$$= \frac{j}{2\pi f} (e^{-j\pi f} - e^{j\pi f}) \quad (3.29)$$

$$= \frac{\sin \pi f}{\pi f} \quad (3.30)$$

$$= \text{sinc}(f) \quad (3.31)$$

The following code plots Fig. 3.9

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/3
_9.py
```

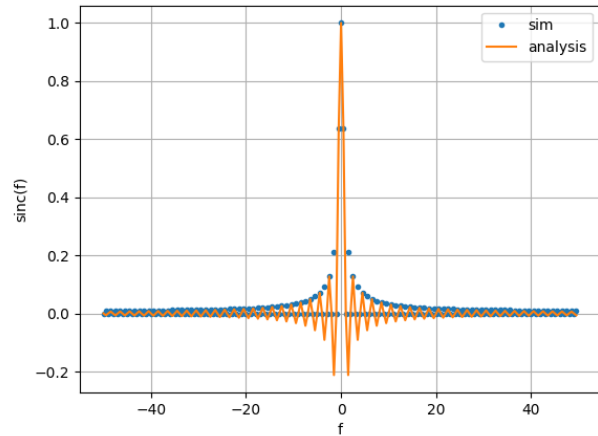


Fig. 3.9: fourier transform of  $\text{rect}(t)$

3.10  $\text{sinc}(t) \xleftrightarrow{\mathcal{F}} ?$ . Verify using python.

**Solution:** Using the inverse fourier transform and the fact that  $\text{rect}(f)$  is even:

$$\text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(-f) = \text{rect}(f) \quad (3.32)$$

The following code plots Fig. 3.10

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/3
_10.py
```

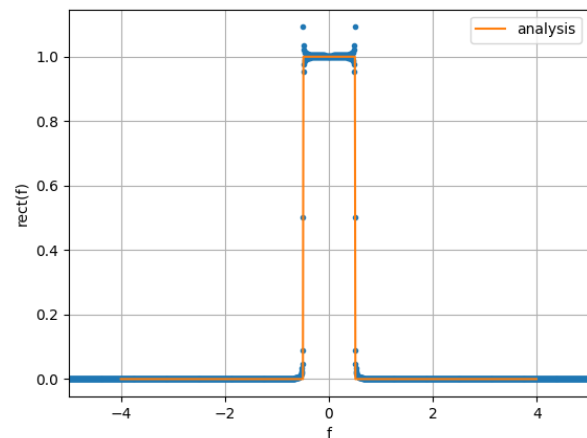


Fig. 3.10: fourier transform of  $\text{sinc}(t)$

## 4 FILTER

4.1 Find  $H(f)$  which transforms  $x(t)$  to DC 5V.

**Solution:** Let  $y(t)$  represent the 5V DC output.  $H(f)$  should be a low pass (to ensure DC) filter. With  $f_0 = 50\text{Hz}$ :

$$H(f) = \text{rect}\left(\frac{f}{2f_0}\right) \quad (4.1)$$

Since the output is 5V

$$\frac{24}{\pi}H(f) = 5\text{rect}\left(\frac{f}{2f_0}\right) \quad (4.2)$$

$$\Rightarrow H(f) = \frac{5\pi}{24}\text{rect}\left(\frac{f}{2f_0}\right) \quad (4.3)$$

4.2 Find  $h(t)$ . Applying the inverse fourier transform:

$$h(t) = \int_{-\infty}^{\infty} H(f)e^{j2\pi ft} df \quad (4.4)$$

$$= \frac{5\pi}{24} \int_{-\infty}^{\infty} \text{rect}\left(\frac{f}{2f_0}\right) e^{j2\pi ft} df \quad (4.5)$$

$$= \frac{5\pi f_0}{12} \text{sinc}(2f_0 t) \quad (4.6)$$

4.3 Verify your result using through convolution. The following code plots Fig. 4.3

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/4
_3.py
```

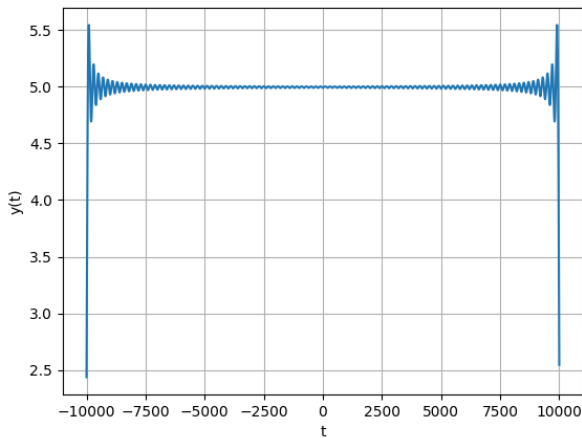


Fig. 4.3: convolution of  $h(t)$  and  $x(t)$

## 5 FILTER DESIGN

5.1 Design a Butterworth filter for  $H(f)$ .

**Solution:** For a Butterworth filter of order  $n$  and cutoff freq  $f_c$ :

$$|H_n(f)|^2 = \frac{1}{\sqrt{1 + \left(\frac{f}{f_c}\right)^{2n}}} \quad (5.1)$$

$$A = -10 \log_{10} |H(f)|^2 \quad (5.2)$$

Let the passband and stopband frequency cut-offs be 50Hz and 100 Hz respectively with the corresponding attenuations as -1db and -20dB.

$$A_p = -10 \log_{10} \left( 1 + \left( \frac{f_p}{f_c} \right)^{2n} \right) \quad (5.3)$$

$$A_s = -10 \log_{10} \left( 1 + \left( \frac{f_s}{f_c} \right)^{2n} \right) \quad (5.4)$$

$$\Rightarrow \left( \frac{f_p}{f_s} \right)^{2n} = \frac{10^{-\frac{A_p}{10}} - 1}{10^{-\frac{A_s}{10}} - 1} \quad (5.5)$$

$$(0.5)^{2n} = \frac{10^{0.1} - 1}{10^2 - 1} \quad (5.6)$$

$$4^{-n} = \frac{0.2589}{99} \quad (5.7)$$

$$n = 4.289 \quad (5.8)$$

Taking  $n = 5$  and solving for  $f_c$

$$f_{c1} = f_p \left( 10^{-\frac{A_p}{10}} - 1 \right)^{-\frac{1}{2n}} = 57.23\text{Hz} \quad (5.9)$$

$$f_{c2} = f_s \left( 10^{-\frac{A_s}{10}} - 1 \right)^{-\frac{1}{2n}} = 63.16\text{Hz} \quad (5.10)$$

$$\Rightarrow f_c \approx 60\text{Hz} \quad (5.11)$$

5.2 Design a Chebyshev filter for  $H(f)$ .

**Solution:** For a Chebyshev filter of order  $n$ , ripple  $\epsilon$ , cutoff freq  $f_c$ :

$$|H_n(f)|^2 = \frac{1}{\sqrt{1 + \epsilon^2 C_n^2 \left( \frac{f}{f_c} \right)}} \quad (5.12)$$

where  $C_n = \cosh^{-1}(n \cosh x)$  The chebyshev polynomial is given as:

$$c_n(x) = \begin{cases} \cos(n \cos^{-1} x) & |x| \leq 1 \\ \cosh(n \cosh^{-1} x) & \text{otherwise} \end{cases} \quad (5.13)$$

For a passband frequency cutoff  $f_p = f_c$ , stopband freq  $f_s$ , attenuation  $A_s$  and p-p ripple

$\delta$ :

$$\delta = 10 \log_{10} (1 + \epsilon^2) \quad (5.14)$$

$$\Rightarrow \epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \quad (5.15)$$

For  $f_s > f_p$ :

$$A_s = -10 \log_{10} \left( 1 + \epsilon^2 c_n^2 \left( \frac{f_s}{f_p} \right) \right) \quad (5.16)$$

$$\Rightarrow c_n \left( \frac{f_s}{f_p} \right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \quad (5.17)$$

$$\Rightarrow n = \frac{\cosh^{-1} \left( \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \right)}{\cosh^{-1} \left( \frac{f_s}{f_p} \right)} \quad (5.18)$$

Thus with  $f_p = 60\text{Hz}$ ,  $f_s = 100\text{Hz}$ ,  $\delta = 0.5\text{dB}$ ,  $A_s = -20\text{dB}$ , we can calculate  $n$  and  $\epsilon$  as follows:

$$\epsilon = \sqrt{10^{\frac{0.5}{10}} - 1} = 0.35 \quad (5.19)$$

$$n = \frac{\cosh^{-1} \left( \frac{\sqrt{10^{-\frac{20}{10}} - 1}}{0.35} \right)}{\cosh^{-1} \left( \frac{100}{60} \right)} = 3.68 \approx 4 \quad (5.20)$$

5.3 Design a circuit for your Butterworth filter.

**Solution:** For a butterworth filter of order  $n$ ,  $C_k, L_k$  value for  $\omega = 1$

$$C_1 = C_5 = 0.618F \quad (5.21)$$

$$C_3 = 2F \quad (5.22)$$

$$L_2 = L_4 = 1.618H \quad (5.23)$$

$$C'_k = \frac{C_k}{\omega_c} \quad (5.24)$$

$$L'_k = \frac{L_k}{\omega_c} \quad (5.25)$$

Taking  $f_c = 60\text{ Hz}$ ,

$$f_c = 60 \Rightarrow \omega_c = 120\pi \quad (5.26)$$

$$C'_1 = C'_5 = 1.64\text{mF} \quad (5.27)$$

$$L'_2 = L'_4 = 4.29\text{mH} \quad (5.28)$$

$$C'_3 = 5.31\text{mF} \quad (5.29)$$

$$(5.30)$$

The L-C network is shown in Fig. 5.3. The following code plots Fig. 5.3

```
wget https://raw.githubusercontent.com/varennya27/EE3900/master/charger/codes/5_3.cir
```

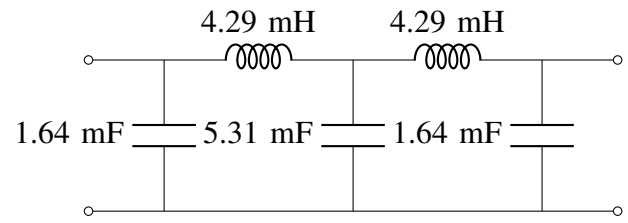


Fig. 5.3: L-C Butterworth Filter

```
com/varennya27/EE3900/master/charger/codes/5_3.cir
```

```
wget https://raw.githubusercontent.com/varennya27/EE3900/master/charger/codes/5_3.py
```

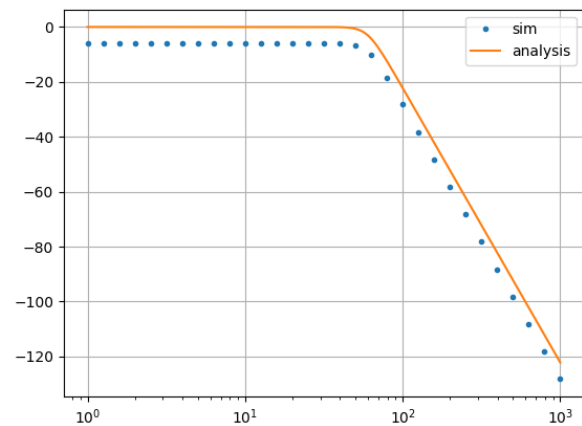


Fig. 5.3: Butterworth filter sim

5.4 Design a circuit for your Chebyshev filter. **Solution:**

For order 3 and 0.5 dB ripple, following a similar procedure as in the Butterworth filter taking  $f_c = 50\text{Hz}$ ,

$$C'_1 = 4.43\text{mF} \quad (5.31)$$

$$L'_2 = 3.16\text{mH} \quad (5.32)$$

$$C'_3 = 6.28\text{mF} \quad (5.33)$$

$$L'_4 = 2.23\text{mH} \quad (5.34)$$

The L-C network is shown in Fig. 5.4. The following code plots Fig. 5.4

```
wget https://raw.githubusercontent.com/varennya27/EE3900/master/charger/codes/5_4.cir
```

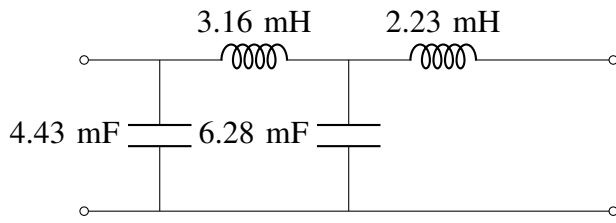


Fig. 5.4: L-C Chebyshev Filter

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/5
_4.py
```

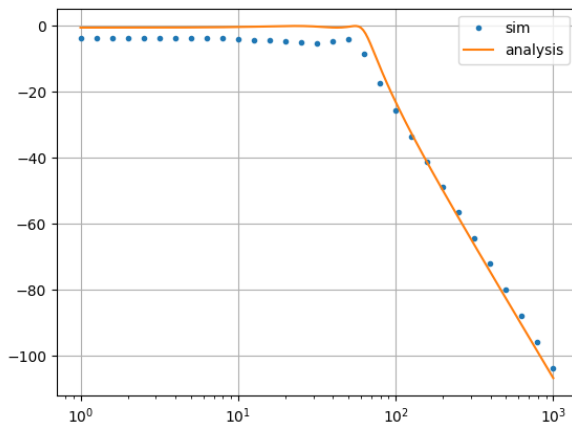


Fig. 5.4: Chebyshev filter sim

### 5.5 Design a digital Butterworth filter

**Solution:** The following code plots Fig. 5.5 and Fig. 5.5

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/charger/codes/5
_final.py
```

Fig 5.5 shows the plot with the transfer functions  $H(f)$  along with  $X(f)$  and  $Y(f)$ .  $y(t)$  is calculated as:

$$Y(f) = H(f) \times X(f) \quad (5.35)$$

$$y(t) \xleftrightarrow{\mathcal{F}} Y(f) \quad (5.36)$$

Doing the computation gives  $y(t)$  as seen in Fig. 5.5

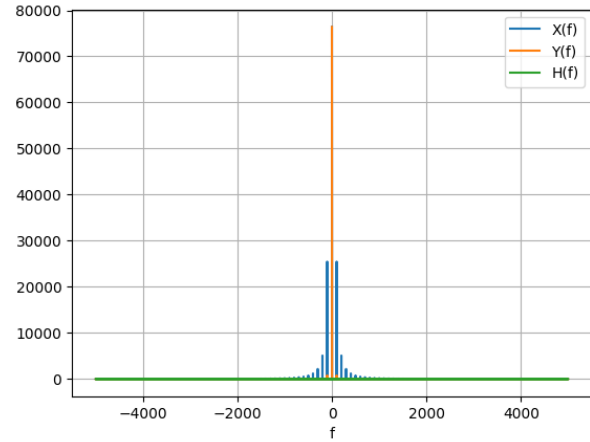


Fig. 5.5: Transfer function with i/o functions

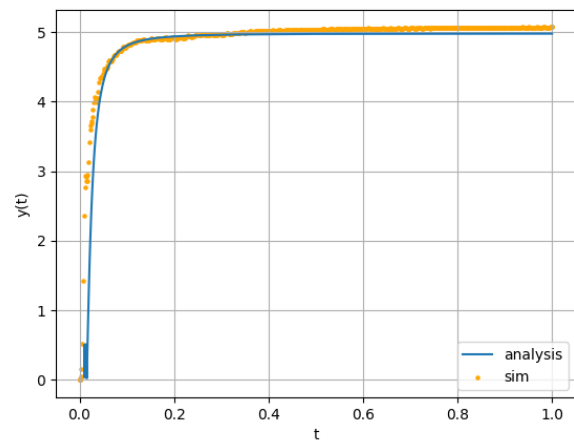


Fig. 5.5: DC 5V output