Pingala Series

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Abstract—This manual provides a simple introduction to Transforms

1 JEE 2019

Let α, β be the roots of the polynomial $x^2 - x - 1$

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1

$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3

$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: The following code verifies the above results:

```
wget https://raw.githubusercontent.com/varenya27/
    EE3900/master/pingala/codes/1.py
```

The first three results are true while the last one is false.

2 PINGALA SERIES

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^+(z) = \sum_{n=0}^{\infty} x(n)z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

Fig. 2.2: x(n)

2.2 The *Pingala* series is generated using the difference equation

$$x(n + 2) = x(n + 1) + x(n), \quad x(0) = x(1) = 1, n \geq 0 \tag{2.2}$$

Generate a stem plot for $x(n)$.
Solution: The following code plots **??**

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/pingala/codes/2
    _2.py
```

2.3 Find $X^+(z)$.

Solution:

$$x(n + 2) = x(n + 1) + x(n)$$
$$(2.3)$$

$$\implies \mathcal{Z}^+\{x(n + 2)\} = \mathcal{Z}^+\{x(n + 1)\} + \mathcal{Z}^+\{x(n)\}$$
$$(2.4)$$

$$\implies z^2 X^+(z) - z^2 - z = zX^+(z) - z + X^+(z)$$
$$(2.5)$$

$$\implies X^+(z)(z^2 - z - 1) = z^2 \qquad (2.6)$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \qquad (2.7)$$

$$= \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})}$$
$$(2.8)$$

ROC: $|z| > \frac{1 + \sqrt{5}}{2}$

2.4 Find $x(n)$.

$$X^+(z) = \frac{1}{(1 - \alpha z^{-1})(1 - \beta z^{-1})} \qquad (2.9)$$

$$= \left(\frac{1}{(\alpha - \beta)z^{-1}}\right)\left(\frac{1}{1 - \alpha z^{-1}} - \frac{1}{1 - \beta z^{-1}}\right)$$
$$(2.10)$$

$$= \frac{1}{z^{-1}(\alpha - \beta)}\left(\sum_{n=0}^{\infty}(\alpha z^{-1})^n - (\beta z^{-1})^n\right)$$
$$(2.11)$$

$$= \sum_{n=0}^{\infty}\frac{\alpha^n - \beta^n}{\alpha - \beta}z^{-n+1} \qquad (2.12)$$

$$= \sum_{n=1}^{\infty}\frac{\alpha^n - \beta^n}{\alpha - \beta}z^{-n+1} \qquad (2.13)$$

$$= \sum_{n=0}^{\infty}\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}z^{-n} \qquad (2.14)$$

$$\implies x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) \qquad (2.15)$$

$$= a_{n+1}u(n) \qquad (2.16)$$

2.5 Sketch

$$y(n) = x(n - 1) + x(n + 1), \quad n \geq 0 \quad (2.17)$$

Solution: The following code plots **??**

wget https://raw.githubusercontent.com/
varenya27/EE3900/master/pingala/codes/2
_5.py

2.6 Find $Y^+(z)$.

Fig. 2.5: y(n)

Solution:

$$\mathcal{Z}^+\{y(n)\} = \mathcal{Z}^+\{x(n + 1)\} + \mathcal{Z}^+\{x(n - 1)\}$$
$$(2.18)$$

$$Y^+(z) = zX^+(z) - z + z^{-1}Z^+(z) \qquad (2.19)$$

$$= (z + z^{-1})\frac{1}{1 - z^{-1} - z^{-2}} - z \quad (2.20)$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \qquad (2.21)$$

ROC for the above z transform will again be $|z| > \frac{1 + \sqrt{5}}{2}$

2.7 Find $y(n)$.

Solution:

$$Y^+(z) = X^+(z) + \frac{2}{(z-\alpha)(z-\beta)} \quad (2.22)$$

$$= X^+(z) + \frac{2}{\alpha-\beta}\left(\frac{1}{1-\alpha z^{-1}} + \frac{1}{1-\beta z^{-1}}\right) \quad (2.23)$$

$$= X^+(z) + \frac{2}{\alpha-\beta}\left(\sum_{n=0}^{\infty}(\alpha^n + \beta^n)z^{-n}\right) \quad (2.24)$$

$$\implies y(n) = x(n) + 2u(n)\frac{\alpha^n + \beta^n}{\alpha - \beta} \quad (2.25)$$

$$= \frac{\alpha^{n+1} + 2\alpha^n - \beta^{n+1} + 2\beta^n}{\alpha - \beta}u(n) \quad (2.26)$$

$$= \frac{\alpha^n(\alpha+1) + \alpha^n - \beta^n(\beta+1) + \beta^n}{\alpha - \beta}u(n) \quad (2.27)$$

$$= \frac{\alpha^{n+2} - \beta^{n+2} - \alpha\beta(\alpha^n + \beta^n)}{\alpha - \beta}u(n) \quad (2.28)$$

$$= \frac{(\alpha-\beta)(\alpha^{n+1} + \beta^{n+1})}{\alpha - \beta}u(n) \quad (2.29)$$

$$\implies y(n) = \alpha^{n+1} + \beta^{n+1} \quad (2.30)$$

Solution: Using the definition of $x(n)$ from (**??**)

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1} \quad (3.2)$$

$$= \sum_{k=0}^{n-1} x(n) \quad (3.3)$$

$$= \sum_{k=0}^{n-1} x(k) + x(n) \times 0 \quad (3.4)$$

$$= \sum_{k=0}^{n-1} x(k)u(n-1-k) + x(n) \times u(-1) \quad (3.5)$$

$$= \sum_{k=0}^{n} x(k)u(n-1-k) \quad (3.6)$$

$$= x(n) * u(n-1) \quad (3.7)$$

3.2 Show that

$$a_{n+2} - 1, \quad n \geq 1 \quad (3.8)$$

can be expressed as

$$[x(n+1) - 1]u(n) \quad (3.9)$$

Solution: The above expression can be written for $n \geq 0$ as:

$$a_{n+1} - 1, \quad n \geq 0 \quad (3.10)$$

$$= x(n) - 1, \quad n \geq 0 \quad (3.11)$$

$$= (x(n) - 1)u(n) \quad (3.12)$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10}X^+(10) \quad (3.13)$$

3 Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(n) = x(n) * u(n-1) \quad (3.1)$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^{k+1}} \tag{3.14}$$

$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} \tag{3.15}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{x(k)}{10^k} \tag{3.16}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} x(k) 10^{-k} \tag{3.17}$$

$$= \frac{X^+(10)}{10} \tag{3.18}$$

3.4 Show that

$$\alpha^n + \beta^n, \quad n \geq 1 \tag{3.19}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right) u(n) \tag{3.20}$$

and find $W(z)$.

Solution: Replacing n with $n+1$:

$$\alpha^{n+1} + \beta^{n+1}, \quad n \geq 0 \tag{3.21}$$

$$= (\alpha^{n+1} + \beta^{n+1}) u(n) = w(n) \tag{3.22}$$

The z-transform can be computed as follows:

$$W(z) = \sum_{n=-\infty}^{\infty} w(n) z^{-n} \tag{3.23}$$

$$= \sum_{n=0}^{\infty} \alpha^{n+1} z^{-n} + \beta^{n+1} z^{-n} \tag{3.24}$$

$$= \alpha \sum_{n=0}^{\infty} \alpha^n z^{-n} + \beta \sum_{n=0}^{\infty} \beta^n z^{-n} \tag{3.25}$$

$$= \frac{\alpha}{1 - \alpha z^{-1}} + \frac{\beta}{1 - \beta z^{-1}} \tag{3.26}$$

$$= \frac{\alpha + \beta - 2\alpha\beta z^{-1}}{1 + \alpha\beta z^{-2} - (\alpha + \beta) z^{-1}} \tag{3.27}$$

$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.28}$$

ROC: $|z| > \frac{1 + \sqrt{5}}{2}$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10} Y^+(10) \tag{3.29}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^{k+1}} \tag{3.30}$$

$$= \sum_{k=0}^{\infty} \frac{x(k)}{10^{k+1}} \tag{3.31}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} \frac{y(k)}{10^k} \tag{3.32}$$

$$= \frac{1}{10} \sum_{k=0}^{\infty} y(k) 10^{-k} \tag{3.33}$$

$$= \frac{Y^+(10)}{10} \tag{3.34}$$

3.6 Solve the JEE 2019 problem.

Solution:

a) From (??)

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.35}$$

Taking the positive z transform on the RHS:

$$\mathcal{Z}\{x(n) * u(n-1)\} = X^+(z) z^{-1} \frac{1}{1 - z^{-1}} \tag{3.36}$$

$$= \frac{z^{-1}}{(1 - z^{-1} - z^{-2})(1 - z^{-1})} \tag{3.37}$$

$$= z \left(\frac{1}{1 - z^{-1} - z^{-2}} - \frac{1}{1 - z^{-1}} \right) \tag{3.38}$$

$$= z \sum_{n=0}^{\infty} (x(n) - 1) z^{-n} \tag{3.39}$$

$$= \sum_{n=0}^{\infty} (x(n) - 1) z^{-n+1} \tag{3.40}$$

$$= \sum_{n=0}^{\infty} (x(n+1) - 1) z^{-n} \tag{3.41}$$

Taking the Inverse Z transform on (??):

$$x(n) * u(n-1) = (x(n+1) - 1) u(n) \tag{3.42}$$

$$\implies \sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{3.43}$$

b) From (**??**)

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{X^+(10)}{10} \qquad (3.44)$$

$$= \frac{100}{100 - 10 - 1} \times \frac{1}{10} \qquad (3.45)$$

$$= \frac{10}{89} \qquad (3.46)$$

c) Using (**??**), (**??**), (**??**) we get:

$$b_n = a_{n+1} + a_{n-1} \qquad (3.47)$$

$$= x(n) + x(n-2) \qquad (3.48)$$

$$= y(n-1) \qquad (3.49)$$

$$= \alpha^n + \beta^n \qquad (3.50)$$

d) From (**??**)

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{Y^+(10)}{10} \qquad (3.51)$$

$$= \frac{100 + 20}{100 - 10 - 1} \times \frac{1}{10} \qquad (3.52)$$

$$= \frac{12}{89} \neq \frac{8}{89} \qquad (3.53)$$

Thus, options (a),(b) and (c) are correct