

# Circuits and Transforms

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**Abstract**—This manual provides a simple introduction to Transforms

## 1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

## 2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu C$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu C$ .

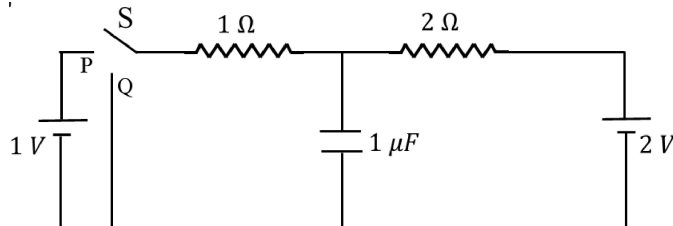
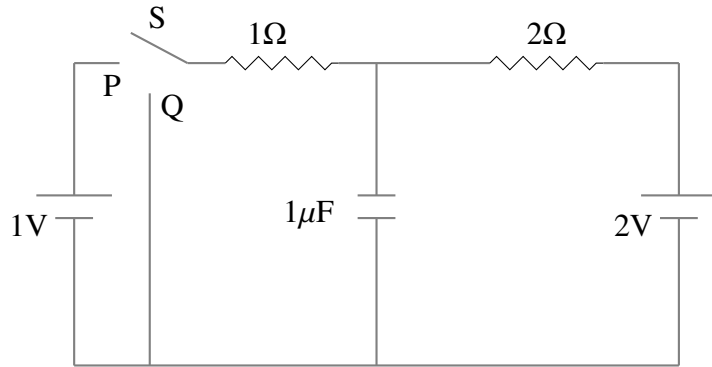


Fig. 2.1

2. Draw the circuit using latex-tikz.

**Solution:** The circuit is drawn below:



3. Find  $q_1$ .

**Solution:** After a long time, the capacitor starts to behave like an open switch, which means that no current will flow through the capacitor. Assume that the circuit is grounded at the negative terminals of the battery and current in the circuit is  $i$ . Applying KVL in the loop:

$$1 + i + 2i - 2 = 0 \quad (2.1)$$

$$\Rightarrow i = \frac{1}{3}A \quad (2.2)$$

$$\frac{q_1 \mu}{C} = 1 + \frac{1}{3} \quad (2.3)$$

$$\Rightarrow q_1 = \frac{4}{3} \quad (2.4)$$

4. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$  and find the ROC.

**Solution:**

$$\mathcal{L}\{u(t)\}(s) = \int_{-\infty}^{\infty} u(t)e^{-st} dt \quad (2.5)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.6)$$

$$= \frac{1}{s} \quad (2.7)$$

ROC for the above will be  $Re(s) > 0$

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{H}} L \frac{1}{s+a}, \quad a > 0 \quad (2.8)$$

and find the ROC.

**Solution:**

$$\mathcal{L}\{e^{-at}u(t)\}(s) = \int_{-\infty}^{\infty} u(t)e^{-(s+a)t} dt \quad (2.9)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.10)$$

$$= \frac{1}{s+a} \quad (2.11)$$

ROC:  $\text{Re}(s) > -a$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

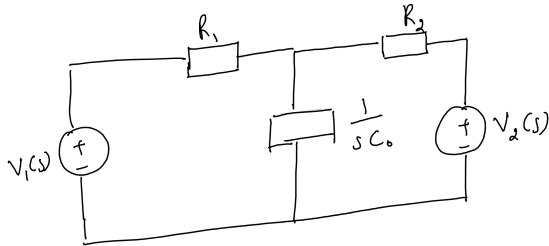


Fig. 2.2

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \quad (2.12)$$

$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \quad (2.13)$$

Find the voltage across the capacitor  $V_{C_0}(s)$ .

**Solution:**

$$V_1(s) = \frac{1}{s} \quad (2.14)$$

$$V_2(s) = \frac{2}{s} \quad (2.15)$$

Assume that the bottom of the circuit is grounded. Applying KCL at the middle junction:

$$\frac{V_1(s) - V_{C_0}(s)}{R_1} + \frac{V_2(s) - V_{C_0}(s)}{R_2} = V_{C_0}(s)sC_0 \quad (2.16)$$

$$V_{C_0}(s) \left( sC_0 + \frac{1}{R_1} + \frac{1}{R_2} \right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2} \quad (2.17)$$

$$\Rightarrow V_{C_0}(s) = \frac{2R_1 + R_2}{sR_1R_2} \frac{R_1R_2}{R_1 + R_2 + sC_0R_1R_2} \quad (2.18)$$

$$\Rightarrow V_{C_0}(s) = \frac{2R_1 + R_2}{s(R_1 + R_2 + sC_0R_1R_2)} \quad (2.19)$$

7. Find  $v_{C_0}(t)$ . Plot using python.

**Solution:** (2.19) can be split into partial fractions as:

$$\frac{2R_1 + R_2}{s(R_1 + R_2 + sC_0R_1R_2)} = \frac{A}{s} + \frac{B}{R_1 + R_2 + sC_0R_1R_2} \quad (2.20)$$

$$\Rightarrow A = \frac{2R_1 + R_2}{R_1 + R_2}, B = -R_1R_2C_0 \frac{2R_1 + R_2}{R_1 + R_2} \quad (2.21)$$

Therefore,

$$V_{C_0}(s) = \frac{2R_1 + R_2}{R_1 + R_2} \left( \frac{1}{s} - \frac{1}{\frac{R_1 + R_2}{R_1R_2C_0} + s} \right) \quad (2.22)$$

Applying an inverse Laplace transform on both sides gives:

$$v_{C_0}(t) = \frac{2R_1 + R_2}{R_1 + R_2} u(t) \left( 1 - e^{-\frac{R_1 + R_2}{R_1R_2C_0}t} \right) \quad (2.23)$$

Plugging in the values:

$$v_{C_0}(t) = \frac{4}{3} u(t) \left( 1 - e^{-\frac{3}{2 \times 10^{-6}}t} \right) \quad (2.24)$$

$$= \frac{4}{3} u(t) \left( 1 - e^{-1.5t \times 10^{-6}} \right) \quad (2.25)$$

The following code plots Fig. 2.3

```
wget https://raw.githubusercontent.com/
varenaya27/EE3900/master/ckt-sig/codes/2
_7.py
```

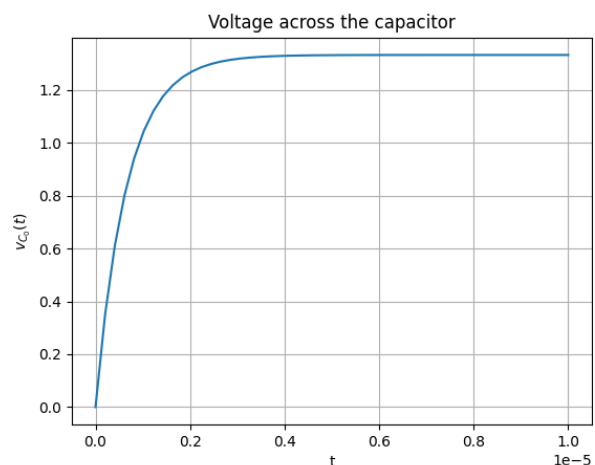


Fig. 2.3: Capacitor voltage

8. Verify your result using ngspice.

**Solution:** The following code simulates the circuit and plots Fig. 2.4:

wget https://raw.githubusercontent.com/varenya27/EE3900/master/ckt-sig/codes/2\_8.cir

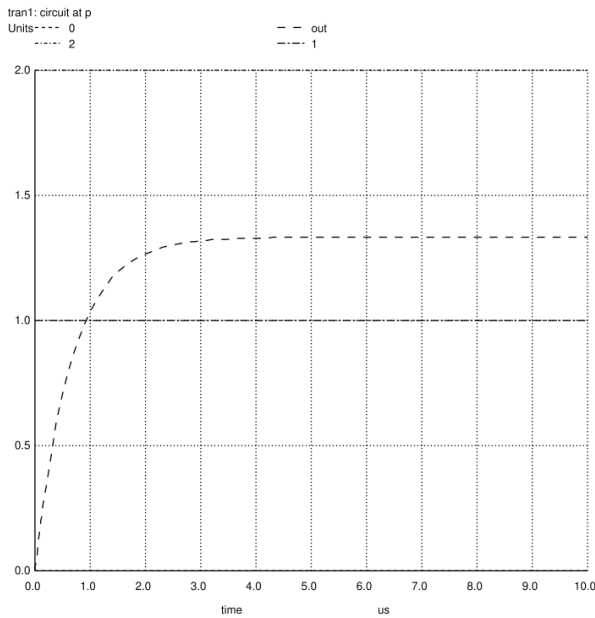


Fig. 2.4: Capacitor voltage in ngspice

9. Obtain Fig. 2.2 using the equivalent differential equation.

**Solution:** Using KVL in the two separate loops and assuming currents  $i_1, i_2$  in the loops such

that  $dq/dt = i_1 + i_2$

$$1 - i_1 = \frac{q}{C} = 2 - 2i_2 \quad (2.26)$$

$$\Rightarrow 2 - 2i_2 + 2 - 2i_1 = q_1/\mu + 2q_1/\mu \quad (2.27)$$

$$\Rightarrow 4 - 2\frac{dq_1}{dt} = 3q_1/\mu \quad (2.28)$$

$$\Rightarrow 2 - 1.5q_1/\mu = \frac{dq_1}{dt} \quad (2.29)$$

$$\Rightarrow \int_0^{q_1} \frac{dq_1}{2 - 1.5q_1/\mu} = \int_0^t dt \quad (2.30)$$

$$\Rightarrow \ln\left(\frac{2 - 1.5q_1/\mu}{2}\right) = 1.5t \times 10^6 \quad (2.31)$$

$$\Rightarrow q_1 = \frac{4}{3}(1 - e^{1.5t \times 10^6}) \quad (2.32)$$

### 3 INITIAL CONDITIONS

1. Find  $q_2$  in Fig. 2.1.

**Solution:** The capacitor acts like an open switch. Let  $i$  be the current in the circuit.

$$2 = 3i \quad (3.1)$$

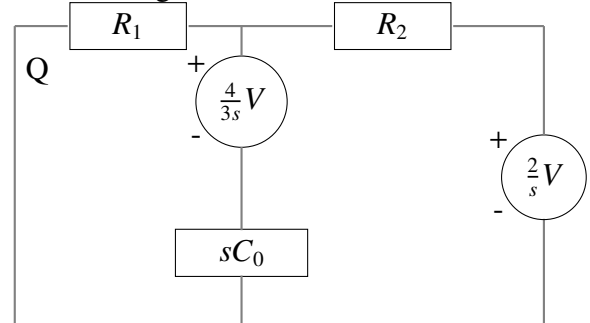
$$\Rightarrow i = \frac{2}{3} \quad (3.2)$$

$$\frac{q_2}{C} = 1 \times i = \frac{2}{3} \quad (3.3)$$

$$\Rightarrow q_2 = \frac{2}{3} \quad (3.4)$$

2. Draw the equivalent  $s$ -domain resistive circuit when  $S$  is switched to position  $Q$ . Use variables  $R_1, R_2, C_0$  for the passive elements. Use latex-tikz.

**Solution:** The resistive circuit figure is drawn below



3.  $V_{C_0}(s) = ?$

**Solution:** Assuming the base is grounded, ap-

plying KCL at the middle junction:

$$\frac{2/s - V_{C_0}(s)}{R_2} = \left( V_{C_0}(s) - \frac{4}{3s} \right) sC_0 + \frac{V_{C_0}(s)}{R_1} \quad (3.5)$$

$$\frac{2}{sR_2} + \frac{4C_0}{3} = V_{C_0}(s) \left( sC_0 + \frac{1}{R_1} + \frac{1}{R_2} \right) \quad (3.6)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{sC_0 + \frac{1}{R_1} + \frac{1}{R_2}} \quad (3.7)$$

$$(3.8)$$

4.  $v_{C_0}(t) = ?$  Plot using python.

$$V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{sC_0 + \frac{1}{R_1} + \frac{1}{R_2}} \quad (3.9)$$

$$= \frac{2}{R_2 C_0} \left( \frac{1}{s} \times \frac{1}{s + \frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \right) + \frac{4/3}{s + \frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \quad (3.10)$$

$$= \frac{2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( \frac{1}{s} - \frac{1}{s + \frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \right) + \frac{4/3}{s + \frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \quad (3.11)$$

Applying the inverse Laplace on both sides:

$$v_{C_0}(t) = \frac{2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( 1 - e^{-\frac{t}{C_0 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)}} \right) u(t) + \frac{4}{3} e^{-\frac{t}{C_0 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)}} u(t) \quad (3.12)$$

Putting in  $R_1 = 1, R_2 = 2, C_0 = 10^{-6}$ :

$$v_{C_0}(t) = \frac{2}{3} \left( 1 - e^{-1.5t \times 10^6} \right) u(t) + \frac{4}{3} e^{-1.5t \times 10^6} u(t) \quad (3.13)$$

$$= \frac{2}{3} \left( 1 + e^{-1.5t \times 10^6} \right) u(t) \quad (3.14)$$

The following code plots Fig. 3.1

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/ckt-sig/codes/3
_4.py
```

5. Verify your result using ngspice.

**Solution:** The following code simulates the circuit and plots Fig. 3.2:

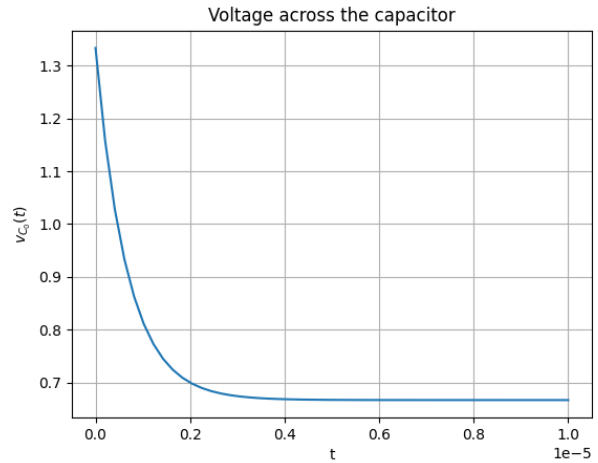


Fig. 3.1: Capacitor voltage

```
wget https://raw.githubusercontent.com/
varenya27/EE3900/master/ckt-sig/codes/3
_5.cir
```

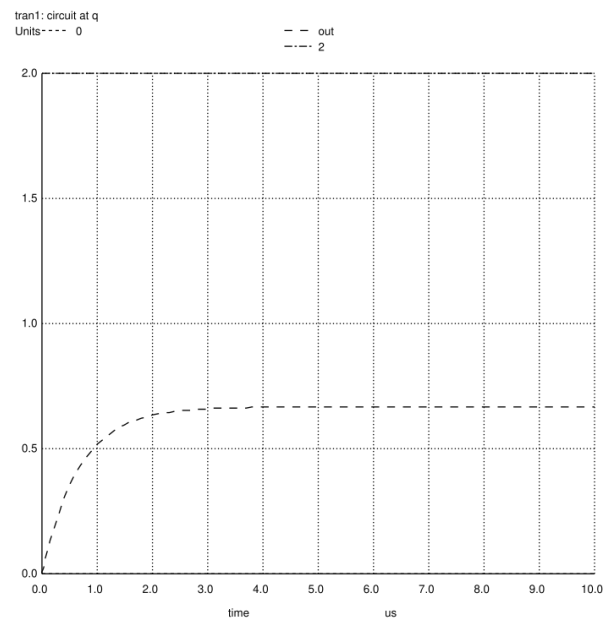


Fig. 3.2: Capacitor voltage in ngspice

6. Find  $v_{C_0}(0^-)$ ,  $v_{C_0}(0^+)$  and  $v_{C_0}(\infty)$ .

**Solution:** at  $t = 0^-$ , S is connected to P

$$v_{C_0}(0^-) = \frac{4}{3}V \quad (3.15)$$

From (3.14)

$$v_{C_0}(0^+) = \frac{4}{3}V \quad (3.16)$$

$$v_{C_0}(\infty) = \frac{2}{3}V \quad (3.17)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

#### 4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.

**Solution:**

Let  $i_1$  and  $i_2$  be the currents i through  $R_1$  and the capacitor resp. Current through  $R_2$  will thus be  $i_1 + i_2$ .

$$i_1 + 2(i_1 + i_2) = 2 \quad (4.1)$$

$$\Rightarrow i_1 = \frac{2}{3}(1 - i_2) \quad (4.2)$$

$$(4.3)$$

Applying KVL in the loop with  $R_2$  and the capacitor

$$\frac{q}{C} + 2\frac{2 + i_2}{3} = 2 \quad (4.4)$$

$$\frac{q}{C} + \frac{2}{3}\frac{dq}{dt} = \frac{2}{3} \quad (4.5)$$

$$\dot{q} + \frac{3q}{2C} = 12C\frac{dq}{dt} = 2C - 3q \quad (4.6)$$

$$\frac{dq}{2C - 3q} = \frac{dt}{2C} \quad (4.7)$$

$$\ln\left(\frac{2C - 3q}{2C}\right) = -\frac{3t}{2C} \quad (4.8)$$

$$\Rightarrow q = \frac{2}{3}(1 - e^{-1.5t \times 10^6})\mu C \quad (4.9)$$

2. Find  $H(s)$  considering the output voltage at the capacitor.

**Solution:** The input and output voltages can be

expressed as functions of time as follows:

$$V_{in}(t) = 2u(t) \quad (4.10)$$

$$V_{out}(t) = \frac{2}{3}\left(1 - e^{-1.5t \times 10^6}\right)u(t) \quad (4.11)$$

$$\mathcal{L}\{V_{in}(t)\} = \frac{2}{s} \quad (4.12)$$

$$\mathcal{L}\{V_{out}\} = \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s + 1.5e6}\right) \quad (4.13)$$

$$H(s) = \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s + 1.5e6}\right) / \frac{2}{s} \quad (4.14)$$

$$= \frac{1e6}{2(s + 1.5e6)} \quad (4.15)$$

3. Plot  $H(s)$ . What kind of filter is it?  
4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.16)$$

5. Find  $H(z)$ .  
6. How can you obtain  $H(z)$  from  $H(s)$ ?