Fourier Series

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2 FOURIER SERIES

Abstract—**This manual provides a simple introduction to Fourier Series**

1 PERIODIC FUNCTION

Let

$$x(t) = A_0|\sin(2\pi f_0 t)| \tag{1.1}$$

1.1 Plot $x(t)$.

Solution: The following code plots Fig. **??**

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/1
    _1.py
```
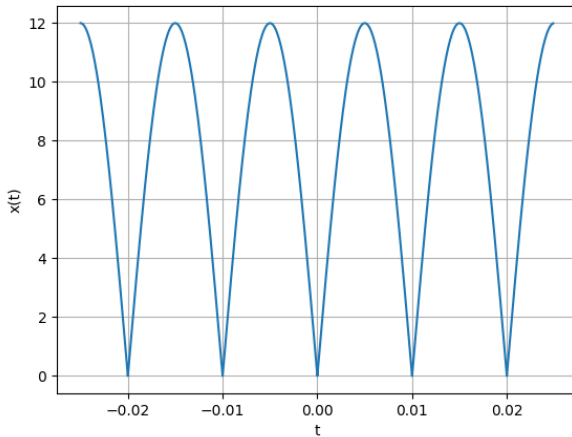

Fig. 1.1: x(t)

1.2 Show that $x(t)$ is periodic and find its period.

Let T be the period of $x(t)$

$$x(t + T) = A|\sin(2\pi f_0(t + T))| \tag{1.2}$$
$$= A|\sin(2\pi f_0 t + 2\pi f_0 T)| \tag{1.3}$$
$$= A|\sin(2\pi f_0 t)|, \quad 2\pi f_0 T = \pi \tag{1.4}$$
$$\implies T = \frac{1}{2f_0} \tag{1.5}$$

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t} \, dt \tag{2.2}$$

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.3}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi n f_0 t} dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} dt \tag{2.4}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-n)f_0 t} dt \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0}\delta(k-n) \tag{2.6}$$

$$= \frac{1}{f_0}c_n \tag{2.7}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi n f_0 t} \, dt \tag{2.8}$$

2.2 Find c_k for (**??**)

Solution:

$$c_k = 50 \int_{-0.01}^{0.01} 12|\sin(2\pi 50t)|e^{-\jmath 100\pi kt}\, dt \quad (2.9)$$

$$= 600 \int_{-0.01}^{0.01} |\sin(100)|\cos(100\pi kt)\, dt$$

$$+ \jmath 600 \int_{-0.01}^{0.01} |\sin(100\pi t)|\sin(100\pi kt)\, dt \tag{2.10}$$

$$= 600 \int_{0}^{0.01} 2\sin(100\pi t)\cos(100\pi kt)\, dt \tag{2.11}$$

$$= 600 \int_{0}^{0.01} (\sin(100\pi(k+1)t))\, dt$$

$$- 600 \int_{0}^{0.01} (\sin(100\pi(k-1)t))\, dt \tag{2.12}$$

$$= 6\frac{1+(-1)^k}{\pi}\left(\frac{1}{k+1} - \frac{1}{k-1}\right) \tag{2.13}$$

$$= \begin{cases} \frac{24}{\pi(1-k^2)} & \text{even } k \\ 0 & \text{odd } k \end{cases} \tag{2.14}$$

2.3 Verify (**??**) using python.
Solution: The following code plots Fig. **??**

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/2
    _3.py
```
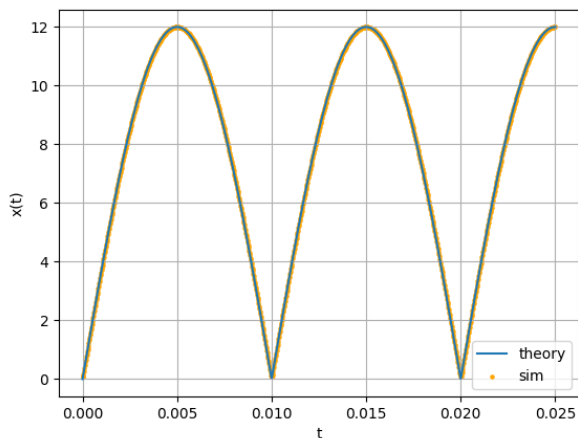

Fig. 2.3: x(t) from the fourier series

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \jmath \sin 2\pi k f_0 t) \tag{2.15}$$

and obtain the formulae for a_k and b_k.
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.16}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.17}$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$

$$+ \sum_{k=0}^{\infty} \jmath (c_k - c_{-k}) \sin(2\pi k f_0 t) \tag{2.18}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.19}$$

$$= \begin{cases} 50 \int_{-0.01}^{0.01} x(t)\, dt & k = 0 \\ 100 \int_{-0.01}^{0.01} x(t)\cos(100\pi kt)\, dt & k > 0 \end{cases} \tag{2.20}$$

$$b_k = \frac{c_k - c_{-k}}{\jmath} = 100 \int_{-0.01}^{0.01} x(t)\sin(100\pi kt)\, dt \tag{2.21}$$

2.5 Find a_k and b_k for (**??**)
Solution: Clearly x(t) is even

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.22}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.23}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} = x(t) \tag{2.24}$$

$$\implies c_k = c_{-k} \tag{2.25}$$

thus for $k \geq 0$,

$$a_k = \begin{cases} \frac{24}{\pi} & k = 0 \\ \frac{48}{\pi(1-k^2)} & k > 0,\ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.26}$$

$$b_k = 0 \tag{2.27}$$

2.6 Verify (**??**) using python.
Solution: The following code plots Fig. **??**

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/2
    _6.py
```
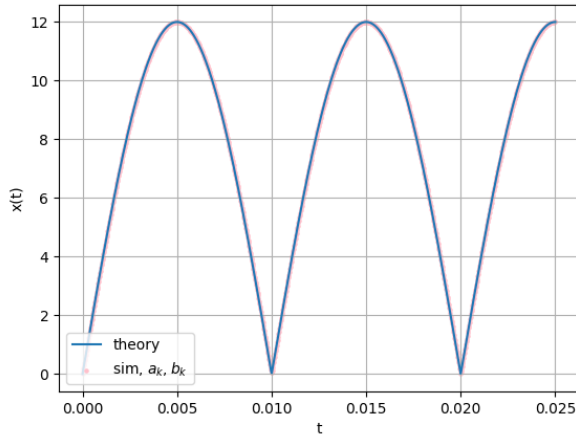

Fig. 2.6: x(t) from the fourier series using a_k, b_k

3 FOURIER TRANSFORM

3.1
$$\delta(t) = 0, \quad t \neq 0 \quad (3.1)$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad (3.2)$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt \quad (3.3)$$

3.3 Show that

$$g(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi ft_0} \quad (3.4)$$
$$(3.5)$$

Solution:

$$g(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}\, dt \quad (3.6)$$

$$= \int_{-\infty}^{\infty} g(t')e^{-j2\pi f(t' + t_0)}\, dt' \quad (3.7)$$

$$= e^{-j2\pi ft_0} \int_{-\infty}^{\infty} g(t')e^{-j2\pi ft'}\, dt' \quad (3.8)$$

$$= G(f)e^{-j2\pi ft_0} \quad (3.9)$$

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \quad (3.10)$$

Solution: using the inverse fourier transform:

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df \quad (3.11)$$

$$\implies g(f) = \int_{-\infty}^{\infty} G(t)e^{j2\pi ft}\, dt \quad (3.12)$$

$$\implies g(-f) = \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\, dt \quad (3.13)$$

$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \quad (3.14)$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow}?$
Solution:

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\, dt \quad (3.15)$$

$$= 1 \quad (3.16)$$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow}?$
Solution:

$$e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t}e^{-j2\pi ft}\, dt \quad (3.17)$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi(f + f_0)t}\, dt \quad (3.18)$$

$$= \delta(f + f_0) \quad (3.19)$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow}?$
Solution:

$$\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \cos(2\pi f_0 t)e^{-j2\pi ft}\, dt \quad (3.20)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{2j\pi f_0 t} + e^{-2j\pi f_0 t}\right)e^{-j2\pi ft}\, dt \quad (3.21)$$

$$= \frac{1}{2}\left(\delta(f - f_0) + \delta(f + f_0)\right) \quad (3.22)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution:

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \, dt \qquad (3.23)$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(k f_0 + f) \qquad (3.24)$$

$$= \sum_{k=-\infty}^{\infty} \frac{24}{\pi(1 - 4k^2)} \delta(2k f_0 + f) \qquad (3.25)$$

The following code plots Fig. **??**
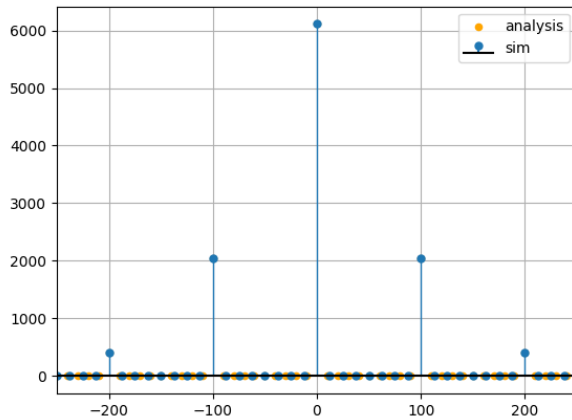
Fig. 3.8: fourier transform of $x(t)$

3.9 Show that

$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}(t) \qquad (3.26)$$

Verify using python.
Solution:

$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \text{rect}(t) \, e^{-\jmath 2\pi f t} \, dt \qquad (3.27)$$

$$= \int_{-0.5}^{0.5} e^{-\jmath 2\pi f t} \, dt \qquad (3.28)$$

$$= \frac{\jmath}{2\pi f} \left(e^{-\jmath \pi f} - e^{\jmath \pi f} \right) \qquad (3.29)$$

$$= \frac{\sin \pi f}{\pi f} \qquad (3.30)$$

$$= \text{sinc}(f) \qquad (3.31)$$

The following code plots Fig. **??**
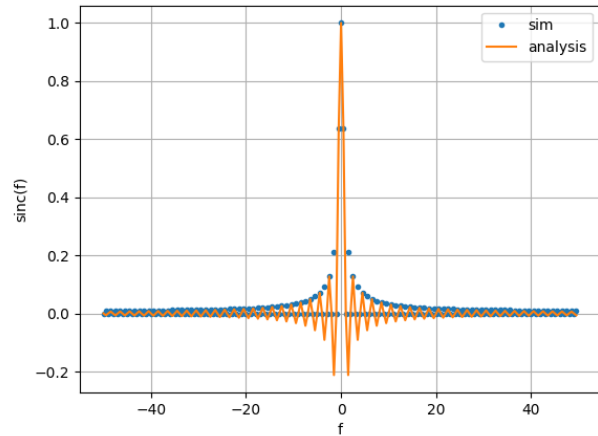
Fig. 3.9: fourier transform of $rect(t)$

3.10 $\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow}$?. Verify using python.
Solution: Using the inverse fourier transform and the fact that $\text{rect}(f)$ is even:

$$\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(-f) = \text{rect}(f) \qquad (3.32)$$

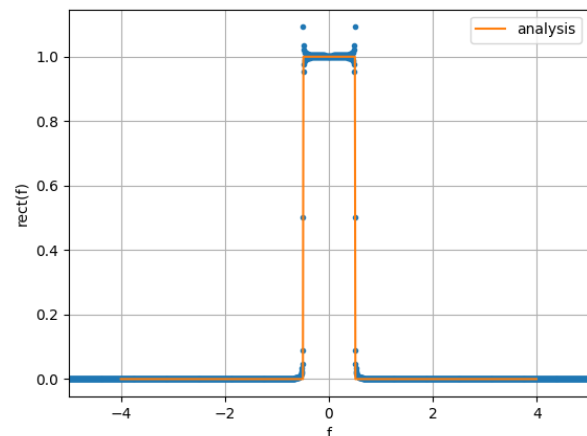The following code plots Fig. **??**

Fig. 3.10: fourier transform of $sinc(t)$

4 FILTER

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: Let $y(t)$ represent the 5V DC output. $H(f)$ should be a low pass (to ensure DC) filter. With $f_0 = 50Hz$:

$$H(f) = \text{rect}\left(\frac{f}{2f_0}\right) \quad (4.1)$$

Since the output is 5V

$$\frac{24}{\pi}H(f) = 5\text{rect}\left(\frac{f}{2f_0}\right) \quad (4.2)$$

$$= \frac{5\pi}{24}\text{rect}\left(\frac{f}{2f_0}\right) \quad (4.3)$$

4.2 Find $h(t)$. Applying the inverse fourier transform:

$$h(t) = \int_{\infty}^{\infty} H(f)e^{j2\pi ft}\,df \quad (4.4)$$

$$= \frac{5\pi}{24}\int_{\infty}^{\infty} \text{rect}\left(\frac{f}{2f_0}\right)e^{j2\pi ft}\,df \quad (4.5)$$

$$= \frac{5\pi f_0}{12}\text{sinc}(2f_0t) \quad (4.6)$$

4.3 Verify your result using through convolution. The following code plots Fig. **??**

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/4
    _3.py
```
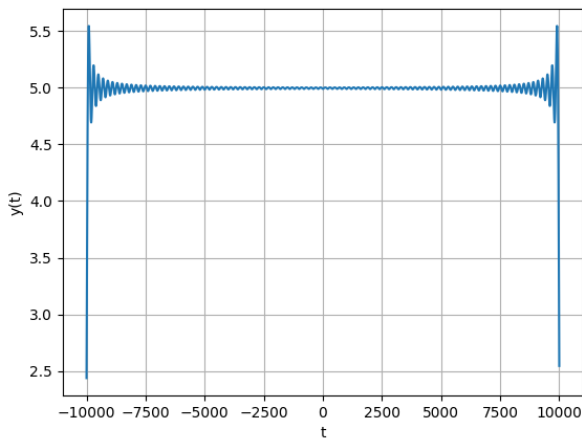

Fig. 4.3: convolution of $h(t)$ and $x(t)$

5 FILTER DESIGN

5.1 Design a Butterworth filter for $H(f)$.

5.2 Design a Chebyschev filter for $H(f)$.
5.3 Design a circuit for your Butterworth filter.
5.4 Design a circuit for your Chebyschev filter.