Fourier Series

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Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1 Periodic Function

Let

$$x(t) = A_0 |\sin(2\pi f_0 t)| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution: The following code plots Fig. 1.1

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/1
    _1.py
```

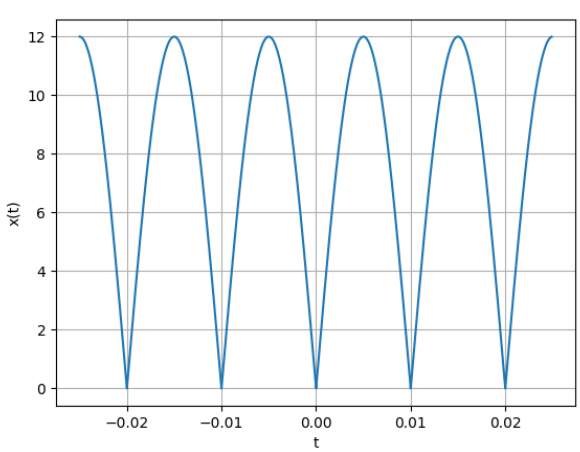

Fig. 1.1: x(t)

1.2 Show that $x(t)$ is periodic and find its period. Let T be the period of $x(t)$

$$x(t+T) = A|sin(2\pi f_0(t+T))| \tag{1.2}$$
$$= A|sin(2\pi f_0 t + 2\pi f_0 T)| \tag{1.3}$$
$$= A|\sin(2\pi f_0 t)|, \quad 2\pi f_0 T = \pi \tag{1.4}$$
$$\implies T = \frac{1}{2f_0} \tag{1.5}$$

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.3}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t} dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi (k-n) f_0 t} dt \tag{2.4}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{\jmath 2\pi (k-n) f_0 t} dt \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0} \delta(k-n) \tag{2.6}$$

$$= \frac{1}{f_0} c_n \tag{2.7}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt \tag{2.8}$$

2.2 Find c_k for (1.1)
Solution:

$$c_k = 50 \int_{-0.01}^{0.01} 12|\sin(2\pi 50 t)| e^{-\jmath 100\pi k t} \, dt \tag{2.9}$$

$$= 600 \int_{-0.01}^{0.01} |\sin(100)| \cos(100\pi k t) \, dt$$
$$+ \jmath 600 \int_{-0.01}^{0.01} |\sin(100\pi t)| \sin(100\pi k t) \, dt \tag{2.10}$$

$$= 600 \int_{0}^{0.01} 2 \sin(100\pi t) \cos(100\pi k t) \, dt \tag{2.11}$$

$$= 600 \int_{0}^{0.01} (\sin(100\pi(k+1)t)) \, dt$$
$$- 600 \int_{0}^{0.01} (\sin(100\pi(k-1)t)) \, dt \tag{2.12}$$

$$= 6 \frac{1 + (-1)^k}{\pi} \left(\frac{1}{k+1} - \frac{1}{k-1} \right) \tag{2.13}$$

$$= \begin{cases} \frac{24}{\pi(1-k^2)} & \text{even } k \\ 0 & \text{odd } k \end{cases} \tag{2.14}$$

2.3 Verify (2.1) using python.
Solution: The following code plots Fig. 2.3

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/2
    _3.py
```

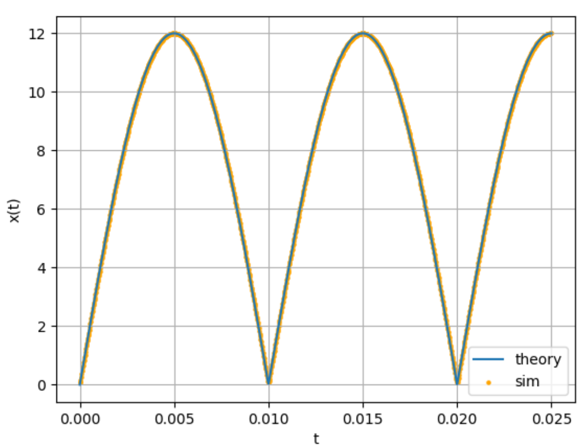

Fig. 2.3: x(t) from the fourier series

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \jmath \sin 2\pi k f_0 t) \tag{2.15}$$

and obtain the formulae for a_k and b_k.
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.16}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.17}$$

$$= c_0 + \sum_{k=1}^{\infty} (c_k + c_{-k}) \cos(2\pi k f_0 t)$$
$$+ \sum_{k=0}^{\infty} \jmath (c_k - c_{-k}) \sin(2\pi k f_0 t) \tag{2.18}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.19}$$

$$= \begin{cases} 50 \int_{-0.01}^{0.01} x(t) \, dt & k = 0 \\ 100 \int_{-0.01}^{0.01} x(t) \cos(100\pi k t) \, dt & k > 0 \end{cases} \tag{2.20}$$

$$b_k = \frac{c_k - c_{-k}}{\jmath} = 100 \int_{-0.01}^{0.01} x(t) \sin(100\pi k t) \, dt \tag{2.21}$$

2.5 Find a_k and b_k for (1.1)
Solution: Clearly x(t) is even

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.22}$$

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.23}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} = x(t) \tag{2.24}$$

$$\implies c_k = c_{-k} \tag{2.25}$$

thus for $k \geq 0$,

$$a_k = \begin{cases} \frac{24}{\pi} & k = 0 \\ \frac{48}{\pi(1-k^2)} & k > 0, \ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.26}$$

$$b_k = 0 \tag{2.27}$$

2.6 Verify (2.15) using python.
Solution: The following code plots Fig. 2.6

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/2
    _6.py
```

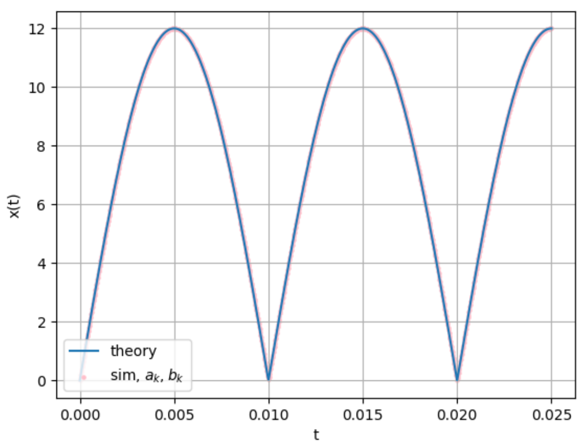

Fig. 2.6: x(t) from the fourier series using a_k, b_k

3 Fourier Transform

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-j2\pi f t_0} \tag{3.4}$$

$$\tag{3.5}$$

Solution:

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-j2\pi f t}\, dt \tag{3.6}$$

$$= \int_{-\infty}^{\infty} g(t') e^{-j2\pi f (t' + t_0)}\, dt' \tag{3.7}$$

$$= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} g(t') e^{-j2\pi f t'}\, dt' \tag{3.8}$$

$$= G(f) e^{-j2\pi f t_0} \tag{3.9}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.10}$$

Solution: using the inverse fourier transform:

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi f t}\, df \tag{3.11}$$

$$\implies g(f) = \int_{-\infty}^{\infty} G(t) e^{j2\pi f t}\, dt \tag{3.12}$$

$$\implies g(-f) = \int_{-\infty}^{\infty} G(t) e^{-j2\pi f t}\, dt \tag{3.13}$$

$$\implies G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.14}$$

3.5 $\delta(t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t}\, dt \tag{3.15}$$

$$= 1 \tag{3.16}$$

3.6 $e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$e^{-j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} e^{-j2\pi f_0 t} e^{-j2\pi f t}\, dt \tag{3.17}$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi (f + f_0) t}\, dt \tag{3.18}$$

$$= \delta(f + f_0) \tag{3.19}$$

3.7 $\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} ?$
Solution:

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \cos(2\pi f_0 t) e^{-j2\pi f t}\, dt \tag{3.20}$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left(e^{2j\pi f_0 t} + e^{-2j\pi f_0 t} \right) e^{-j2\pi f t}\, dt \tag{3.21}$$

$$= \frac{1}{2} \left(\delta(f - f_0) + \delta(f + f_0) \right) \tag{3.22}$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution:

$$x(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t}\, dt \tag{3.23}$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(k f_0 + f) \tag{3.24}$$

$$= \sum_{k=-\infty}^{\infty} \frac{24}{\pi(1-4k^2)} \delta(2k f_0 + f) \tag{3.25}$$

The following code plots Fig. 3.8

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/3
    _8.py
```

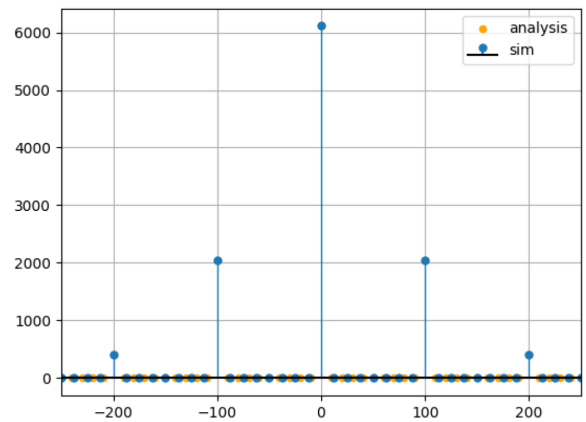

Fig. 3.8: fourier transform of $x(t)$

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(t) \tag{3.26}$$

Verify using python.
Solution:

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \text{rect}(t)\, e^{-\jmath 2\pi f t}\, dt \tag{3.27}$$

$$= \int_{-0.5}^{0.5} e^{-\jmath 2\pi f t}\, dt \tag{3.28}$$

$$= \frac{\jmath}{2\pi f}\left(e^{-\jmath\pi f} - e^{\jmath\pi f}\right) \tag{3.29}$$

$$= \frac{\sin \pi f}{\pi f} \tag{3.30}$$

$$= \text{sinc}(f) \tag{3.31}$$

The following code plots Fig. 3.9

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/3
    _9.py
```

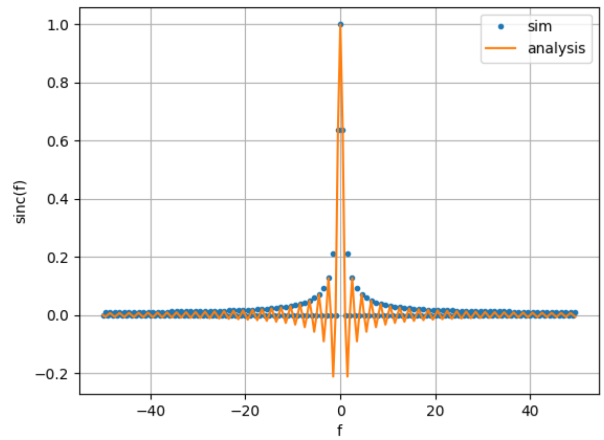

Fig. 3.9: fourier transform of $rect(t)$

3.10 $\text{sinc}(t) \xleftrightarrow{\mathcal{F}} ?$. Verify using python.
Solution: Using the inverse fourier transform and the fact that $\text{rect}(f)$ is even:

$$\text{sinc}(t) \xleftrightarrow{\mathcal{F}} \text{rect}(-f) = \text{rect}(f) \tag{3.32}$$

The following code plots Fig. 3.10

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/3
    _10.py
```

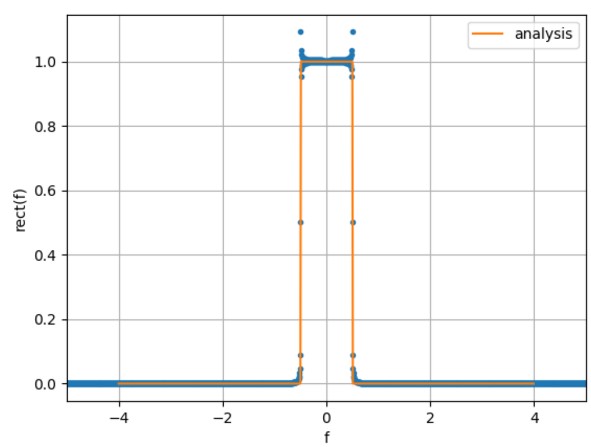

Fig. 3.10: fourier transform of $sinc(t)$

4 Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: Let $y(t)$ represent the 5V DC output. $H(f)$ should be a low pass (to ensure DC) filter. With $f_0 = 50Hz$:

$$H(f) = \text{rect}\left(\frac{f}{2f_0}\right) \tag{4.1}$$

Since the output is 5V

$$\frac{24}{\pi}H(f) = 5\text{rect}\left(\frac{f}{2f_0}\right) \tag{4.2}$$

$$\implies H(f) = \frac{5\pi}{24}\text{rect}\left(\frac{f}{2f_0}\right) \tag{4.3}$$

4.2 Find $h(t)$. Applying the inverse fourier transform:

$$h(t) = \int_{\infty}^{\infty} H(f)e^{\j2\pi ft}\, df \tag{4.4}$$

$$= \frac{5\pi}{24}\int_{\infty}^{\infty} \text{rect}\left(\frac{f}{2f_0}\right)e^{\j2\pi ft}\, df \tag{4.5}$$

$$= \frac{5\pi f_0}{12}\text{sinc}\left(2f_0 t\right) \tag{4.6}$$

4.3 Verify your result using through convolution. The following code plots Fig. 4.3

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/4
    _3.py
```

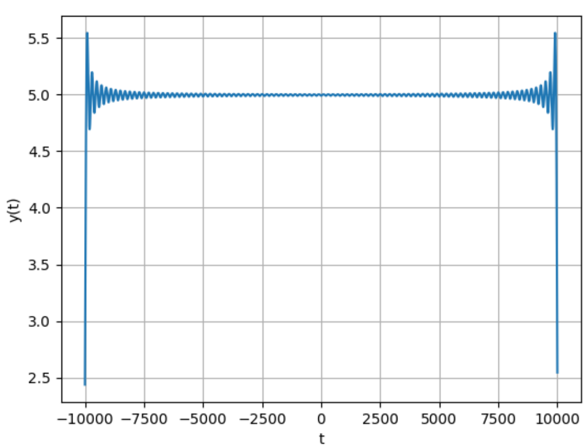

Fig. 4.3: convolution of $h(t)$ and $x(t)$

5 Filter Design

5.1 Design a Butterworth filter for $H(f)$.
Solution: For a Butterworth filter of order n and cutoff freq f_c:

$$|H_n(f)|^2 = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2n}}} \tag{5.1}$$

$$A = -10\log_{10}|H(f)|^2 \tag{5.2}$$

Let the passband and stopband fruquency cutoffs be 50Hz and 100 Hz respectively with the corresponding attenuations as -1db and -20dB.

$$A_p = -10log_{10}\left(1+\left(\frac{f_p}{f_c}\right)^{2n}\right) \tag{5.3}$$

$$A_s = -10log_{10}\left(1+\left(\frac{f_s}{f_c}\right)^{2n}\right) \tag{5.4}$$

$$\implies \left(\frac{f_p}{f_s}\right)^{2n} = \frac{10^{-\frac{-A_p}{10}}-1}{10^{-\frac{-A_s}{10}}-1} \tag{5.5}$$

$$(0.5)^{2n} = \frac{10^{0.1}-1}{10^2-1} \tag{5.6}$$

$$4^{-n} = \frac{0.2589}{99} \tag{5.7}$$

$$n = 4.289 \tag{5.8}$$

Taking $n = 5$ and solving for f_c

$$f_{c_1} = f_p\left(10^{-\frac{A_p}{10}}-1\right)^{-\frac{1}{2n}} = 57.23Hz \tag{5.9}$$

$$f_{c_2} = f_s\left(10^{-\frac{A_s}{10}}-1\right)^{-\frac{1}{2n}} = 63.16Hz \tag{5.10}$$

$$\implies f_c \approx 60Hz \tag{5.11}$$

5.2 Design a Chebyschev filter for $H(f)$.
Solution: For a Chebyshev filter of order n, ripple ϵ, cutoff freq f_c:

$$|H_n(f)|^2 = \frac{1}{\sqrt{1+\epsilon^2 c_n^2\left(\frac{f}{f_c}\right)}} \tag{5.12}$$

where $C_n = cosh^{-1}(ncoshx)$ The chebyshev polynomial is given as:

$$c_n(x) = \begin{cases}\cos(n\cos^{-1}x) & |x| \le 1\\ \cosh(n\cosh^{-1}x) & \text{otherwise}\end{cases} \tag{5.13}$$

For a passband frequency cuttoff $f_p = f_c$, stopband freq f_s, attenuation A_s and p-p ripple

δ:

$$\delta = 10\log_{10}\left(1+\epsilon^2\right) \tag{5.14}$$

$$\implies \epsilon = \sqrt{10^{\frac{\delta}{10}} - 1} \tag{5.15}$$

For $f_s > f_p$:

$$A_s = -10\log_{10}\left(1+\epsilon^2 c_n^2\left(\frac{f_s}{f_p}\right)\right) \tag{5.16}$$

$$\implies c_n\left(\frac{f_s}{f_p}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}} - 1}}{\epsilon} \tag{5.17}$$

$$\implies n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_p}\right)} \tag{5.18}$$

Thus with $f_p = 60Hz, f_s = 100Hz, \delta = 0.5dB, A_s = -20dB$, we can calculate n and ϵ as follows:

$$\epsilon = \sqrt{10^{\frac{0.5}{10}} - 1} = 0.35 \tag{5.19}$$

$$n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{100}{10}}-1}}{0.35}\right)}{\cosh^{-1}\left(\frac{100}{60}\right)} = 3.68 \approx 4 \tag{5.20}$$

5.3 Design a circuit for your Butterworth filter. **Solution:** For a butterworth filter of order n, C_k, L_k value for $\omega = 1$

$$C_1 = C_5 = 0.618F \tag{5.21}$$

$$C_3 = 2F \tag{5.22}$$

$$L_2 = L_4 = 1.618H \tag{5.23}$$

$$C_k' = \frac{C_k}{\omega_c} \tag{5.24}$$

$$L_k' = \frac{L_k}{\omega_c} \tag{5.25}$$

Taking $f_c = 60$ Hz ,

$$f_c = 60 \implies \omega_c = 120\pi \tag{5.26}$$

$$C_1' = C_5' = 1.64mF \tag{5.27}$$

$$L_2' = L_4' = 4.29mH \tag{5.28}$$

$$C_3' = 5.31mF \tag{5.29}$$

$$\tag{5.30}$$

The L-C network is shown in Fig. 5.3. The following code plots Fig. 5.3

```
wget https://raw.githubusercontent.
    com/varenya27/EE3900/master/
    charger/codes/5_3.cir
```

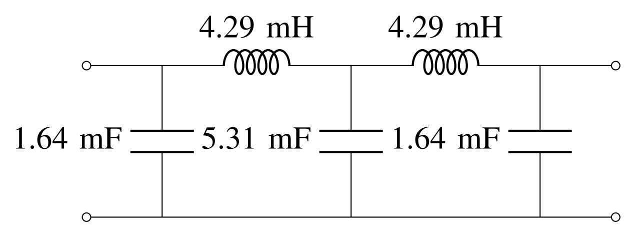

Fig. 5.3: L-C Butterworth Filter

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/5
    _3.py
```

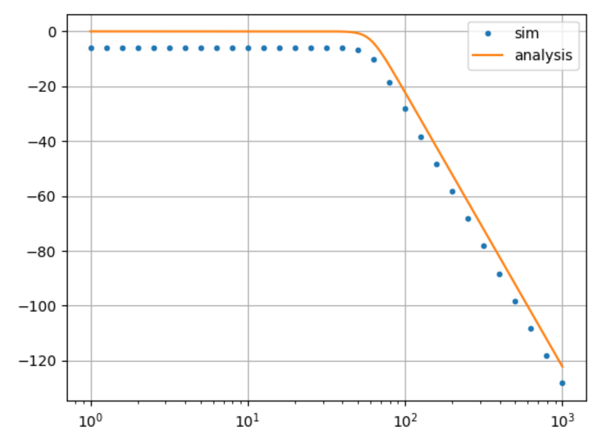

Fig. 5.3: Butterworth filter sim

5.4 Design a circuit for your Chebyshev filter. **Solution:** For order 3 and 0.5 dB ripple, following a similar procedure as in the Butterworth filter taking $f_c = 50Hz$,

$$C_1' = 4.43mF \tag{5.31}$$

$$L_2' = 3.16mH \tag{5.32}$$

$$C_3' = 6.28mF \tag{5.33}$$

$$L_4' = 2.23mH \tag{5.34}$$

The L-C network is shown in Fig. 5.4. The following code plots Fig. 5.4

```
wget https://raw.githubusercontent.
    com/varenya27/EE3900/master/
    charger/codes/5_4.cir
```

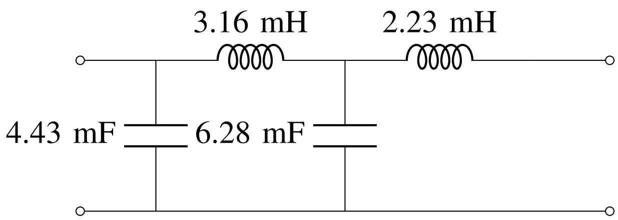

Fig. 5.4: L-C Chebyshev Filter

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/5
    _4.py
```

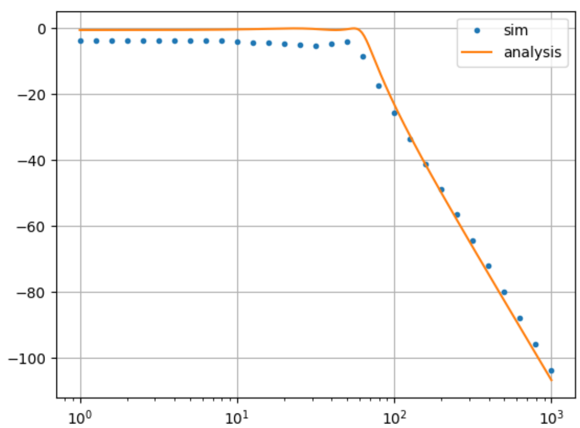

Fig. 5.4: Chebyshev filter sim

5.5 Design a digital Butterworth filter

Solution: The following code plots Fig. **??** and Fig. **??**

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/charger/codes/5
    _final.py
```

Fig **??** shows the plot with the transfer functions $H(f)$ along with $X(f)$ and $Y(f)$. $y(t)$ is calculated as:

$$Y(f) = H(f) \times X(f) \tag{5.35}$$

$$y(t) \xleftrightarrow{\mathcal{F}} Y(f) \tag{5.36}$$

Doing the computation gives $y(t)$ as seen in Fig. **??**

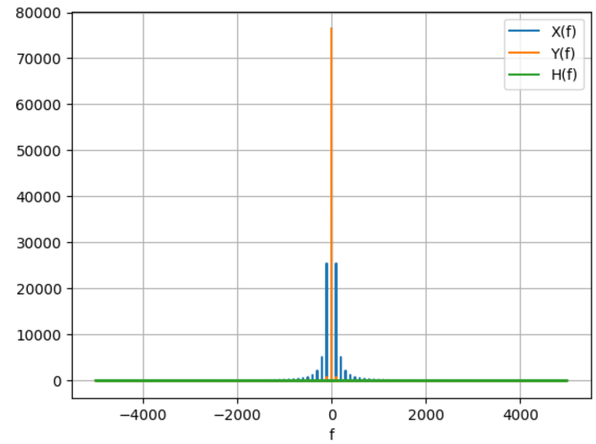

Fig. 5.5: Transfer function with i/o functions

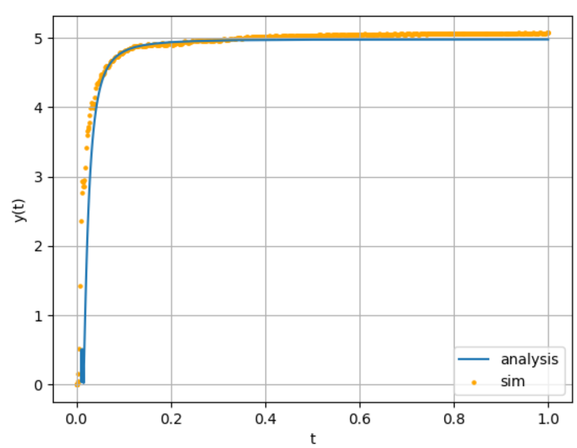

Fig. 5.5: DC 5V output