Circuits and Transforms

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***Abstract*—This manual provides a simple introduction to Transforms**

1 Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2 Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

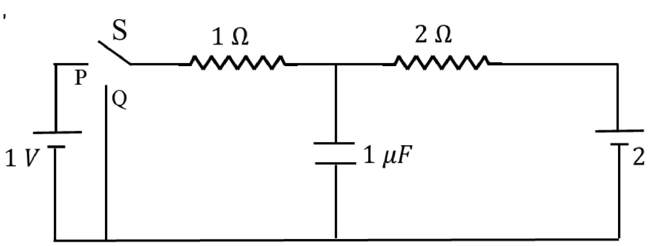
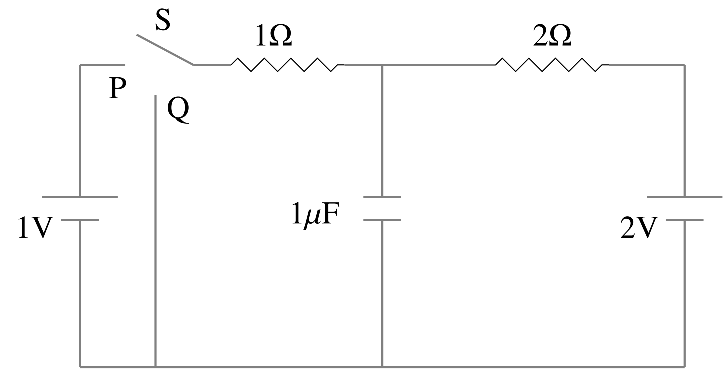


Fig. 2.1

2. Draw the circuit using latex-tikz.
Solution: The circuit is drawn below:



3. Find q_1.
Solution: After a long time, the capacitor starts to behave like an open switch, which means that no current will flow through the capacitor. Assume that the circuit is grounded at the negative terminals of the battery and current in the circuit is i. Applying KVL in the loop:

$$1 + i + 2i - 2 = 0 \tag{2.1}$$
$$\implies i = \frac{1}{3}A \tag{2.2}$$
$$\frac{q_1\mu}{C} = 1 + \frac{1}{3} \tag{2.3}$$
$$\implies q_1 = \frac{4}{3} \tag{2.4}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution:

$$\mathcal{L}\{u(t)\}(s) = \int_{-\infty}^{\infty} u(t)e^{-st}dt \tag{2.5}$$
$$= \int_{0}^{\infty} e^{-st}dt \tag{2.6}$$
$$= \frac{1}{s} \tag{2.7}$$

ROC for the above will be $Re(s) > 0$

5. Show that

$$e^{-at}u(t) \overset{\mathcal{H}}{\longleftrightarrow} L\frac{1}{s+a}, \quad a > 0 \tag{2.8}$$

and find the ROC.

Solution:

$$\mathcal{L}\{e^{-at}u(t)\}(s) = \int_{-\infty}^{\infty} u(t)e^{-(s+a)t}dt \tag{2.9}$$

$$= \int_{0}^{\infty} e^{-(s+a)t}dt \tag{2.10}$$

$$= \frac{1}{s+a} \tag{2.11}$$

ROC: $Re(s) > -a$

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

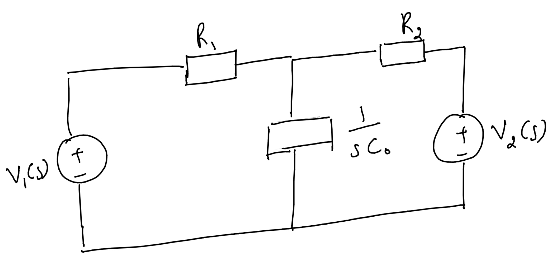


Fig. 2.2

$$u(t) \xleftrightarrow{\mathcal{H}} LV_1(s) \tag{2.12}$$

$$2u(t) \xleftrightarrow{\mathcal{H}} LV_2(s) \tag{2.13}$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$V_1(s) = \frac{1}{s} \tag{2.14}$$

$$V_2(s) = \frac{2}{s} \tag{2.15}$$

Assume that the bottom of the circuit is grounded. Applying KCL at the middle junction:

$$\frac{V_1(s) - V_{C_0}(s)}{R1} + \frac{V_2(s) - V_{C_0}(s)}{R2} = V_{C_0}(s)sC_0 \tag{2.16}$$

$$V_{C_0}(s)\left(sC_0 + \frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V_1(s)}{R_1} + \frac{V_2(s)}{R_2} \tag{2.17}$$

$$\implies V_{C_0}(s) = \frac{2R_1 + R2}{sR_1R_2} \frac{R_1R_2}{R_1 + R_2 + sC_0R_1R_2} \tag{2.18}$$

$$\implies V_{C_0}(s) = \frac{2R_1 + R_2}{s\left(R_1 + R_2 + sC_0R_1R_2\right)} \tag{2.19}$$

7. Find $v_{C_0}(t)$. Plot using python.

Solution: (2.19) can be split into partial fractions as:

$$\frac{2R_1 + R_2}{s\left(R_1 + R_2 + sC_0R_1R_2\right)} = \frac{A}{s} + \frac{B}{R_1 + R_2 + sC_0R_1R_2} \tag{2.20}$$

$$\implies A = \frac{2R_1 + R2}{R_1 + R_2}, B = -R_1R_2C_0\frac{2R_1 + R_2}{R_1 + R_2} \tag{2.21}$$

Therefore,

$$V_{C_0}(s) = \frac{2R_1 + R_2}{R_1 + R_2}\left(\frac{1}{s} - \frac{1}{\frac{R_1+R_2}{R_1R_2C_0} + s}\right) \tag{2.22}$$

Applying an inverse Laplace transform on both sides gives:

$$v_{C_0}(t) = \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\frac{R_1+R_2}{R_1R_2C_0}t}\right) \tag{2.23}$$

Plugging in the values:

$$v_{C_0}(t) = \frac{4}{3}u(t)\left(1 - e^{-\frac{3}{2\times10^{-6}}t}\right) \tag{2.24}$$

$$= \frac{4}{3}u(t)\left(1 - e^{-1.5t\times10^{-6}}\right) \tag{2.25}$$

The following code plots Fig. 2.3

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/ckt−sig/codes/2
    _7.py
```

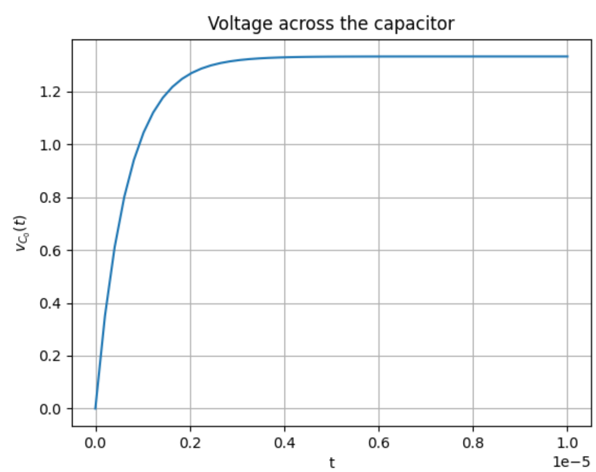


Fig. 2.3: Capacitor voltage

8. Verify your result using ngspice.

Solution: The following code simulates the circuit and plots Fig. 2.4:

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/ckt−sig/codes/2
    _8.cir
```

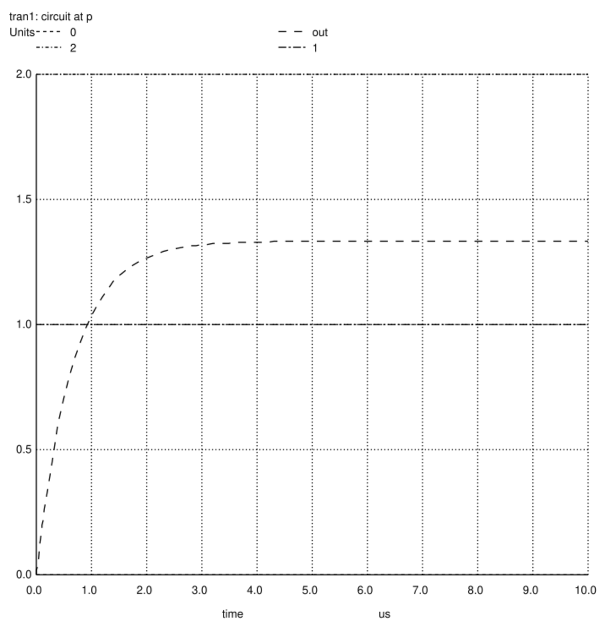


Fig. 2.4: Capacitor voltage in ngspice

9. Obtain Fig. 2.2 using the equivalent differential equation.
Solution: Using KVL in the two separate loops and assuming currents i_1, i_2 in the loops such that $dq/dt = i_1 + i_2$

$$1 - i_1 = \frac{q}{C} = 2 - 2i_2 \tag{2.26}$$

$$\implies 2 - 2i_2 + 2 - 2i_1 = q_1/\mu + 2q_1/\mu \tag{2.27}$$

$$\implies 4 - 2\frac{dq_1}{dt} = 3q_1/\mu \tag{2.28}$$

$$\implies 2 - 1.5q_1/\mu = \frac{dq_1}{dt} \tag{2.29}$$

$$\implies \int_0^{q_1} \frac{dq_1}{2 - 1.5q_1/\mu} = \int_0^t dt \tag{2.30}$$

$$\implies ln(\frac{2 - 1.5q_1/\mu}{2}) = 1.5t \times 10^6 \tag{2.31}$$

$$\implies q_1 = \frac{4}{3}(1 - e^{1.5t\times10^6}) \tag{2.32}$$

3 Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: The capacitor acts like an open switch. Let i be the current in the circuit.

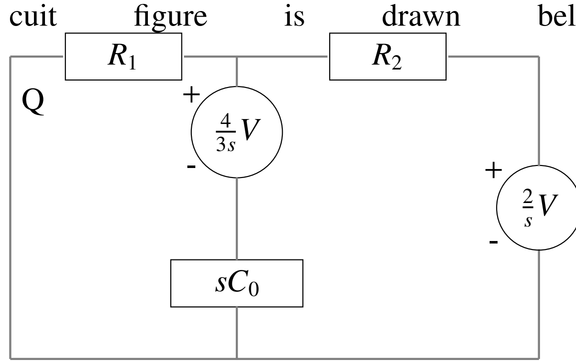
$$2 = 3i \tag{3.1}$$

$$\implies i = \frac{2}{3} \tag{3.2}$$

$$\frac{q_2}{C} = 1 \times i = \frac{2}{3} \tag{3.3}$$

$$\implies q_2 = \frac{2}{3} \tag{3.4}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.
Solution: The resistive circuit figure is drawn below



3. $V_{C_0}(s) = ?$
Solution: Assuming the base is grounded, ap-

plying KCL at the middle junction:

$$\frac{2/s - V_{C_0}(s)}{R_2} = \left(V_{C_0}(s) - \frac{4}{3s}\right)sC_0 + \frac{V_{C_0}(s)}{R_1} \tag{3.5}$$

$$\frac{2}{sR_2} + \frac{4C_0}{3} = V_{C_0}(s)\left(sC_0 + \frac{1}{R_1} + \frac{1}{R_2}\right) \tag{3.6}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{sC_0 + \frac{1}{R_1} + \frac{1}{R_2}} \tag{3.7}$$

$$\tag{3.8}$$

4. $v_{C_0}(t) = ?$ Plot using python.

$$V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{sC_0 + \frac{1}{R_1} + \frac{1}{R_2}} \tag{3.9}$$

$$= \frac{2}{R_2C_0}\left(\frac{1}{s} \times \frac{1}{s + \frac{1}{C_0}\left(\frac{1}{R_1}\frac{1}{R_2}\right)}\right) + + \frac{4/3}{s + \frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)} \tag{3.10}$$

$$= \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\right) + \frac{4/3}{s + \frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)} \tag{3.11}$$

Applying the inverse Laplace on both sides:

$$v_{C_0}(t) = \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{\frac{-t}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\right)u(t) + \frac{4}{3}e^{\frac{-t}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}u(t) \tag{3.12}$$

Putting in $R_1 = 1, R_2 = 2, C_0 = 10^{-6}$:

$$v_{C_0}(t) = \frac{2}{3}\left(1 - e^{-1.5t\times10^6}\right)u(t) + \frac{4}{3}e^{-1.5t\times10^6}u(t) \tag{3.13}$$

$$= \frac{2}{3}\left(1 + e^{-1.5t\times10^6}\right)u(t) \tag{3.14}$$

The following code plots Fig. 3.1

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/ckt-sig/codes/3
    _4.py
```

5. Verify your result using ngspice.

Solution: The following code simulates the circuit and plots Fig. 3.2:

Fig. 3.1: Capacitor voltage

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/ckt-sig/codes/3
    _5.cir
```

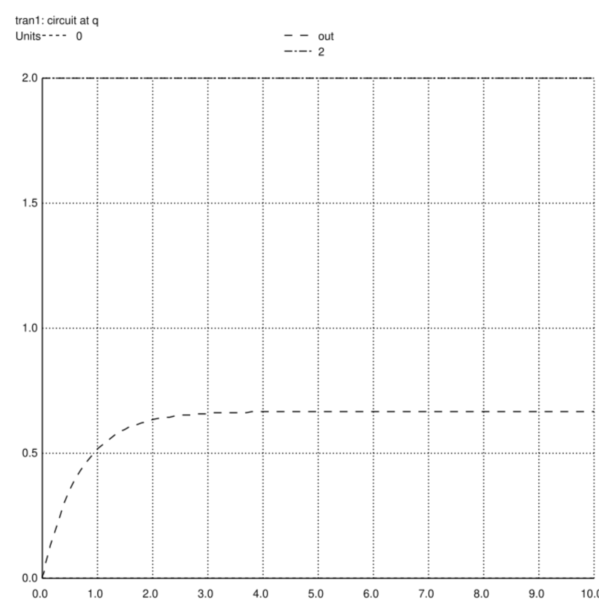


Fig. 3.2: Capacitor voltage in ngspice

6. Find $v_{C_0}(0-)$, $v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
   Solution: at $t = 0^-$, S is connected to P

$$v_{C_0}(0^-) = \frac{4}{3}V \tag{3.15}$$

   From (3.14)

$$v_{C_0}(0^+) = \frac{4}{3}V \tag{3.16}$$

$$v_{C_0}(\infty) = \frac{2}{3}V \tag{3.17}$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equation.

4 Bilinear Transform

1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation.
   Solution:
   Applying KCL at the Capacitor Junction:

$$\frac{V_2 - V_C}{R_2} = \frac{dq}{dt} + \frac{V_C}{R_1} \tag{4.1}$$

$$V_2 - V_C = 2C\dot{V}_C + 2V_C \tag{4.2}$$

$$\implies C\dot{V}_C + \frac{3}{2}V_C = \frac{V_2}{2} \tag{4.3}$$

2. Find $H(s)$ considering the ouput voltage at the capacitor.
   Solution: The input and output voltages can be expressed as functions of time as follows:

$$V_{in}(t) = 2u(t) \tag{4.4}$$

$$V_{out}(t) = \frac{2}{3}\left(1 - e^{-1.5t\times10^6}\right)u(t) \tag{4.5}$$

$$\mathcal{L}\{V_{in}(t)\} = \frac{2}{s} \tag{4.6}$$

$$\mathcal{L}\{V_{out}\} = \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s + 1.5e6}\right) \tag{4.7}$$

$$H(s) = \frac{2}{3}\left(\frac{1}{s} - \frac{1}{s + 1.5e6}\right)/\frac{2}{s} \tag{4.8}$$

$$= \frac{1e6}{2\left(s + 1.5e6\right)} \tag{4.9}$$

3. Plot $H(s)$. What kind of filter is it?
   Solution: The following code simulates the circuit and plots Fig. 4.1:

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/ckt−sig/codes/4
    _3.py
```

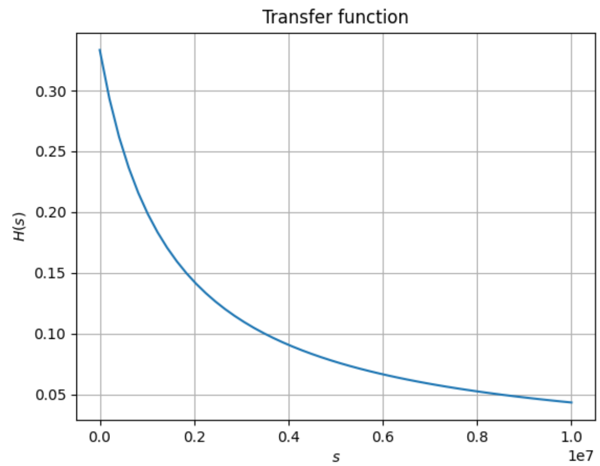


Fig. 4.1: Transfer function

   $H(s)$ looks like a low-pass filter

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \tag{4.10}$$

   Solution: Using (4.3) and applying the trapezoidal rule:

$$\int_a^b f(x)dx = \frac{1}{2}(b-a)\left(f(b) + f(a)\right) \tag{4.11}$$

$$C\dot{V}_c + \frac{3V_c}{2} = \frac{V_2}{2} \tag{4.12}$$

   Applying the limits from n to n+1

$$C\int_{v_c(n)}^{v_c(n+1)} dV_c + \int_n^{n+1} \frac{3V_c}{2}dt = \int_n^{n+1} u(t)dt \tag{4.13}$$

$$\implies C\left(v_c(n+1) - v_c(n)\right) + \frac{3}{4}\left(v_c(n+1) + v_c(n)\right) = \frac{u(n+1) + u(n)}{2} \tag{4.14}$$

$$v_c(n+1)\left(C + 0.75\right) = v_c(n)\left(C - 0.75\right) + \frac{u(n+1) + u(n)}{2} \tag{4.15}$$

5. Find $H(z)$.

$$H(z) = \frac{V_c(z)}{V_2(z)} \tag{4.16}$$

where

$$v_2(n) = 2u(n) \tag{4.17}$$

$$\implies V_2(z) = \frac{2}{1 - z^{-1}} \tag{4.18}$$

ROC: $|z| > 1$ and

$$v_c(n+1)\left(C + \frac{3}{4}\right) - v_c(n)\left(C - \frac{3}{4}\right) = \frac{u(n+1) + u(n)}{2} \tag{4.19}$$

$$zV_c(z)\left(C + \frac{3}{4}\right) - V_c(z)\left(C - \frac{3}{4}\right) = \frac{z+1}{2\left(1 - z^{-1}\right)} \tag{4.20}$$

$$\implies V_c(z) = \frac{2(1+z)}{(1 - z^{-1})((4C+3)z - 4C + 3)} \tag{4.21}$$

Plugging in the two results in (4.16):

$$H(z) = \frac{z+1}{(4C+3)z - 4C + 3} \tag{4.22}$$

with $C = 10^{-6}$ and ROC $|z| > 1$

6. How can you obtain $H(z)$ from $H(s)$?
   Solution: $H(z)$ can be obtained using the Bi-linear Transform.

$$s \longrightarrow \frac{2}{T}\frac{z-1}{z+1} \tag{4.23}$$

where T is the integration step size, ($T = 1$). $H(s)$ can be written in with $C = 10^{-6}$ from (4.9) as:

$$H(z) = \frac{0.5}{2C\frac{z-1}{z+1} + 1.5} \tag{4.24}$$

$$= \frac{z+1}{4C(z-1) + 3(z+1)} \tag{4.25}$$

$$= \frac{z+1}{(4C+3)z - 4C + 3} \tag{4.26}$$

Clearly, (4.22) and (4.26) are the same.

7. Plot the output in all four different forms.
   Solution:

$$V_c(z) = H(z)V_2(z) \tag{4.27}$$

$$= \frac{z+1}{(4C+3)z - 4C + 3} \times \frac{2}{1 - z^{-1}} \tag{4.28}$$

$$= \frac{2}{3}\frac{1}{1 - z^{-1}} - \frac{2}{(4C+3) - (4C-3)z^{-1}} \tag{4.29}$$

Taking the inverse Z-transform on both sides

$$v_c(n) = \frac{2}{3}u(n) - \frac{2}{4C+3}u(n)\left(\frac{4C-3}{4C+3}\right)^n \tag{4.30}$$

Since $C << 1$, we get the final form:

$$v_c(n) = \frac{2}{3}u(n)\left(1 - \left(\frac{4C-3}{4C+3}\right)^n\right) \tag{4.31}$$

Using (4.15), (4.31), (3.14) and the ngpsice simulations, we get 4.2. While plotting the sampling rate was taken as $T = 10^{-7}$, which means that n followed $n = \{1e-7, 2e-7, 3e-7\cdots\}$

The following code plots Fig. 4.2

```
wget https://raw.githubusercontent.com/
    varenya27/EE3900/master/ckt−sig/codes/4
    _4.py
```

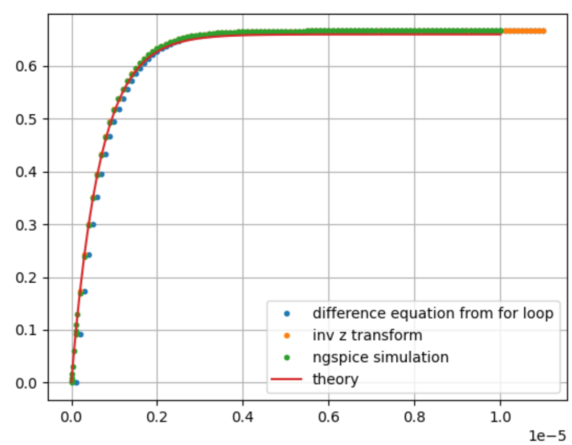


Fig. 4.2: $v_c(n)$ using different methods