The Emergence of GR from String Theory

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Review of GR & Important terms

- Einstein’s (geometric) theory for gravity
- Describes how matter (stress energy tensor) curves spacetime (metric/Reimann tensor) and how spacetime governs trajectories.
- Minkowski (or flat) Metric:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

- General metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

Review of GR & Important terms

- Christoffel Symbol:

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} (\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})$$

- Reimann Curvature Tensor:

$$R^{\mu}_{\sigma\nu\rho} = \Gamma^{\mu}_{\rho\sigma,\nu} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\nu\lambda}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\mu}_{\rho\lambda}\Gamma^{\lambda}_{\nu\sigma}$$

- Einstein's Field Equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Point Particle Dynamics

- Action for a point particle:

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}$$

- Action in terms of a general affine parameter

$$S = -m \int d\lambda \sqrt{-\eta_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda}}$$

- The additional parameter in this definition comes with a gauge symmetry so the overall degrees of freedom are the same
- The upshot of this definition: Poincaré symmetry $X^\mu \rightarrow X'^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu$

Point Particle Dynamics - Equations of Motion

- Minimizing the action gives the equations of motion for the particle,

$$\frac{\partial \vec{p}}{\partial t} = 0,$$

$$\vec{p} = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}}^2}}$$

- Alternative action (to simplify the square root term):

$$S = \frac{1}{2} \int d\tau \left(e^{-1} \dot{X}^2 - m^2 e \right)$$

Point Particle Dynamics - Equations of Motion

- EoM for the added parameter (Lagrange multiplier)

$$-e^{-2}\dot{X}^2 - m^2 = 0$$

$$\implies e = m^{-1}\sqrt{-\dot{X}^2}$$

- A special definition of e gives,

$$S = -\frac{1}{2}\int d\tau\sqrt{-g_{\tau\tau}}\left(g^{\tau\tau}\dot{X}^2 + m^2\right)$$

which essentially looks like we introduced gravity to the particle's wordline

String Theory Formalism

- Instead of point particles, assume that matter is made up of strings
- String coordinates: (τ, σ)
- External Coordinates (in D dimensions): $(X^0, X^1 \ldots X^{D-1})$
- As the string moves through spacetime, it traces out a worldsheet (just like the point particle traced out a worldline)
- The worldsheet is identified by the external (or target space) coordinates
- Can be thought of as a D-component vector field living on the string, any change in the field implies motion in the target space

Actions

- induced metric (pullback of the Minkowski metric):

$$\gamma_{\alpha\beta} = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}$$

- determinant of this metric is,

$$\gamma = (\dot{X} \cdot \dot{X})(X' \cdot X') - (\dot{X} \cdot X')^2$$

- area of the worldsheet traced out by the string is then,

$$A = \int d^2\sigma \sqrt{-\gamma} = \int d^2\sigma \sqrt{(\dot{X} \cdot \dot{X})(X' \cdot X') - (\dot{X} \cdot X')^2}$$

Nambu Goto Action

- the action is proportional to the worldsheet area

$$S = -T \int d^2\sigma \sqrt{-\gamma} = -T \int d^2\sigma \sqrt{(\dot{X} \cdot \dot{X})(X' \cdot X') - (\dot{X} \cdot X')^2}$$

- T is a proportionality constant (and can be thought of as the tension in the string)
- Symmetries of this action:
    - Poincaré symmetry
    - Gauge invariance of the internal coordinates

Equations of motion

- Lagrangian for the system:

$$L = -T\sqrt{(\dot{X}\cdot\dot{X})(X'\cdot X') - (\dot{X}\cdot X')^2}$$

- Equations of motion

$$\frac{\partial}{\partial\tau}\left(\frac{\partial L}{\partial\dot{X}^{\mu}}\right) + \frac{\partial}{\partial\sigma}\left(\frac{\partial L}{\partial X'^{\mu}}\right) = 0$$

$$\implies \frac{\partial}{\partial\tau}\left(-T\frac{(\dot{X}\cdot X')X'_{\mu} - X'^2\dot{X}_{\mu}}{\sqrt{(\dot{X}\cdot\dot{X})(X'\cdot X') - (\dot{X}\cdot X')^2}}\right) + \frac{\partial}{\partial\sigma}\left(-T\frac{(\dot{X}\cdot X')\dot{X}_{\mu} - \dot{X}^2X'_{\mu}}{\sqrt{(\dot{X}\cdot\dot{X})(X'\cdot X') - (\dot{X}\cdot X')^2}}\right) = 0$$

Polyakov Action: introduce a new field

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

- $g^{\alpha\beta}$: dynamical worldsheet metric
- this sets up the image of a 2dimensional gravitational field introduced to the worldsheet
- since this new quantity is an independent variable, it comes with its own equations of motion (to ensure that the net degrees of freedom of the system remain constant)

Symmetries of the Polyakov Action

1. Poincaré symmetry: $X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu$
2. Gauge invariance:

$$X^\mu(\sigma) \to \tilde{X}^\mu(\tilde{\sigma}) = X^\mu(\sigma)$$

$$g_{\alpha\beta}(\sigma) \to \frac{\partial\sigma^\lambda}{\partial\tilde{\sigma}^\alpha}\frac{\partial\sigma^\rho}{\partial\tilde{\sigma}^\beta} g_{\lambda\rho}(\sigma)$$

3. Conformal Symmetry: $g_{\alpha\beta} \to e^{2\phi(\sigma)} g_{\alpha\beta}$

Conformal symmetry shows up since we're dealing with 2 dimensions (the determinant and the metric term cancel out any additional factor introduced). As a consequence, we introduce the conformal gauge where we set this metric to Minkowski

Equations of Motion

- updated action:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial_\alpha X \cdot \partial^\alpha X$$

- minimizing this form of the action gives:

$$\partial_\alpha \partial^\alpha X_\mu = \Box X = 0$$

- varying the action wrt dynamical metric allows us to introduce the stress-energy tensor

$$T_{\alpha\beta} = -\frac{2}{T\sqrt{-g}}\frac{\partial S}{\partial g^{\alpha\beta}}$$

Constraints and Lightcone Coordinates

- imposing the equation of motion for the metric yields physical constraints:

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^2 + X'^2) = 0$$
$$T_{01} = T_{10} = \dot{X}X' = 0$$

- the system is now a free wave solution with 2 constraints
- introduce a new coordinate system:

$$\sigma^+ = \tau + \sigma$$
$$\sigma^- = \tau - \sigma$$

Lightcone Coordinates

- equation of motion in LC coordinates

$$\partial_-\partial_+ X^\mu = 0$$

$$X^\mu(\tau, \sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$$

- where the left and right moving waves can be expanded as:

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\sigma^-}$$

Constraints in LC Coordinates

- the physical constraints in LC coordinates become

$$(\partial_+ X)^2 = (\partial_- X)^2 = 0$$

- each of these evaluate to

$$\alpha' \sum_n \tilde{L}_n e^{-in\sigma^+} = 0 \qquad \alpha' \sum_n L_n e^{-in\sigma^-} = 0$$

- where we introduce the Virasoro generators

$$\tilde{L}_n = \frac{1}{2} \sum_m \tilde{\alpha}_{n-m} \cdot \tilde{\alpha}_m \qquad L_n = \tfrac{1}{2} \textstyle\sum_m \alpha_{n-m} \cdot \alpha_m$$

Constraints in LC Coordinates

- the zeroth mode constraint for the left and right modes gives rise to the mass-shell constraint

$$M = \frac{4}{\alpha'} \sum_{n>0} \alpha_n \cdot \alpha_{-n} = \frac{4}{\alpha'} \sum_{n>0} \tilde{\alpha}_n \cdot \tilde{\alpha}_{-n}$$

- when we quantize, the mode amplitudes will be promoted to level operators
- rewriting this expression in the quantized setting will also allow us to come up with the mass spectrum of the string, where different excited states correspond to different fields/particles

Quantization

- we could either quantize the system first (ie. promote everything to an operator and build the Fock space) and then deal with the constraints [Covariant Quantization] or do it in the reverse order [Lightcone Quantization]
- The issue with Covariant Quantization is that the Fock space ends up having ghosts (or negative norm states) which then need to be decoupled from the theory using the physical constraints
- In LC quantization, since we dealt with the constraints, there are no ghosts

Lightcone Coordinates (in the target space)

- introduce a new coordinate system by picking out the temporal and one spatial dimension

$$X^{\pm} = \sqrt{\frac{1}{2}}(X^0 \pm X^{D-1})$$

- rewriting the solution:

$$X^+ = X_L^+(\sigma^+) + X_R^+(\sigma^-)$$
$$X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-)$$

- next, we introduce the lightcone gauge to fix X^+ and then solve for X^- using the physical constraints

Lightcone Coordinates (in the target space)

- Fixing a lightcone gauge:

$$X_L^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^+ \quad , X_R^+ = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+ \sigma^-$$

$$X^+ = x^+ + \alpha' p^+ \left(\frac{\sigma^+ + \sigma^-}{2} \right) = x^+ + \alpha' p^+ \tau$$

- This fix aligns the X^+ coordinate with the worldsheet time and x^+ simply represents shifts in τ.
- The solutions for X^- then come from solving the physical constraints:

$$\partial_- X_R^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i \qquad \partial_+ X_L^- = \frac{1}{\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i$$

Lightcone Coordinates (in the target space)

- using these constraints, the terms in the general mode expansion can be determined

$$X_L^-(\sigma^+) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}\tilde{\alpha}_n^- e^{-in\sigma^+}$$

$$X_R^-(\sigma^-) = \frac{1}{2}x^- + \frac{1}{2}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n}\alpha_n^- e^{-in\sigma^-}$$

- comparing the zeroth term in the two settings gives the mass-shell condition

$$M^2 = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha_{-n}^i \alpha_n^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i$$

Commutation Relations

- we can now promote our variables to operators and write down the corresponding commutation relations

$$[x^i, p^j] = i\delta^{ij} \quad [x^-, p^+] = -i$$
$$[\alpha_n^i, \alpha_m^j] = [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = n\delta^{ij}\delta_{n+m,0}$$

- the commutation relation for x^+ and p^- is similar to that for t and H in non-relativistic quantum mechanics

$$[x^+, p^-] = -i$$

Building the Fock Space

- define a ground state (with some momentum *p*)

$$\hat{p}^{\mu} |0;p\rangle = p^{\mu} |0;p\rangle \quad , \alpha_i^n |0;p\rangle = \tilde{\alpha}^i |0;p\rangle = 0$$

- the corresponding spectrum can be built using the annihilation and creation operators
- writing down the mass spectrum in this framework is tricky because of

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$$M^2 = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - a\right) = \frac{4}{\alpha'}\left(\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}_{-n}^i\tilde{\alpha}_n^i - a\right)$$

Mass Spectrum

- to evaluate the unknown constant, we assume a scenario without the constant and then perform normal ordering to see what the additional factor looks like
- this introduces a divergent sum into the mix $\frac{D-2}{2}\sum_{n>0} n$
- to evaluate this we use renormalization by introducing a UV-cutoff for the system $$\sum_{n>0} n \to \sum_{n>0} n e^{-\epsilon n}$$
- simplifying the sum using a GP and then isolating the divergence in terms of the new parameter allows us to renormalize it away and then impose the limit tending to zero

Mass Spectrum

- with the system renormalized, the resultant mass-shell condition is

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right)$$

- ground state: negative mass squared particle called the tachyon

$$M^2 = -\frac{D-2}{6\alpha'}$$

- first excited state:

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right)$$

First Excited State

- first excited state defined by $\alpha_{-1}^{i}\tilde{\alpha}_{-1}^{j}\left|0;p\right\rangle$ which implies D-2 degrees of freedom
- this is clear since i and j run from 1 to D-2 (the 0 and D-1 components were fixed by the lightcone gauge and the physical constraints)
- the only way this can happen is if the corresponding particles are massless, since massive particles have D-1 degrees of freedom in D dimensions

$$M^2 = \frac{4}{\alpha'}\left(1 - \frac{D-2}{24}\right) = 0$$

$$\implies D = 26$$

First Excited State

$$G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(X) \ .$$

- the first excited state can be split into 3 irreducible representations
  1. traceless symmetric (spacetime metric)
  2. antisymmetric (Kalb-Ramond field)
  3. traceless scalar (dilation)
- the particle associated with the traceless symmetric field is the graviton (massless, spin 2)
- if we assume the spacetime metric to be a perturbation around the Minkowski metric, and truncate the corresponding action to the quadratic order, we can promote everything to an operator and arrive at a description of gravity as the quantization of the graviton

First Excited State

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

- this however brings back the same issues with covariant quantization we saw earlier - ghosts in the Fock space
- the action is however invariant under a gauge transformation, and this symmetry is used to decouple the negative norm states from the physical processes
- the gauge symmetry ensures that the theory as a result of this field will obey diffeomorphism invariance
- the theory resulting from massless spin 2 fields is general relativity

Polyakov action in curved spacetime

- to derive Einstein's field equations, consider the general Polyakov action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma G_{\mu\nu} \partial_\alpha X^\mu \partial^\alpha X^\nu$$

- expanding a classical solution, $X^\mu(\sigma) = x^\mu + \sqrt{\alpha'} Y^\mu(\sigma)$
- classically, this action gives rise to a conformally invariant theory, but to ensure that this holds true after quantization, we introduce the beta functional for the metric,

$$\beta_{\mu\nu}(G) \sim \mu \frac{\partial G_{\mu\nu}}{\partial \mu},$$

Polyakov action in curved spacetime

- since we introduce UV cutoffs to regulate divergences while quantizing a classical system, the resultant physical quantities oftend depend on the scale of a given process (in our case, this scale will be the length scale of the string)
- however, this dependence breaks the conformal invariance and since that is a gauge symmetry we need to explore the scenario where there is no such dependence
- introduce Reimann Normal Coordinates such that, for the expansion of X, the metric follows,

$$G_{\mu\nu}(X) = \delta_{\mu\nu} - \frac{\alpha'}{3} R_{\mu\lambda\nu\kappa}(x) Y^\lambda Y^\kappa$$

Polyakov action in curved spacetime

- in the expression for the propagator under these new coordinates, there comes a divergence which is resolved by introducing a UV-cutoff.
- the cutoff is renormalized away by the following transformations

$$Y^\mu \to Y^\mu + \frac{\alpha'}{6\epsilon} R^\mu_\nu Y^\nu$$

$$G_{\mu\nu} \to G_{\mu\nu} + \frac{\alpha'}{\epsilon} R_{\mu\nu}$$

- this gives the beta functional for the metric which must be set to zero

$$\beta_{\mu\nu}(G) = \frac{\alpha'}{\epsilon} R_{\mu\nu} = 0$$
$$\implies R_{\mu\nu} = 0$$

Thank you!

References

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- David Berenstein, String Theory Lecture Notes

- Leo Brewin, Reimann Normal Coordinates Notes