# PHY 565* - Advanced Topics in GR

9/17/2024

Announcements & Reminders

- Project topic selection – start thinking about it now
  - Due by Friday, 9/27
  - If you have a topic not covered in my list of suggestions, email me early so that we can discuss it

- “Homework” – please try the exercises I throw into the lectures. If you get stuck, talk to me or email me!
  - I’ve started to compile a PDF of solutions to some of the exercises from class. These solutions are a mix of my own as well as solutions that students send me
  - You can find the solutions by going to the link at the top of the Homework column in the lecture schedule
  - This file will be updated as more exercises are solved

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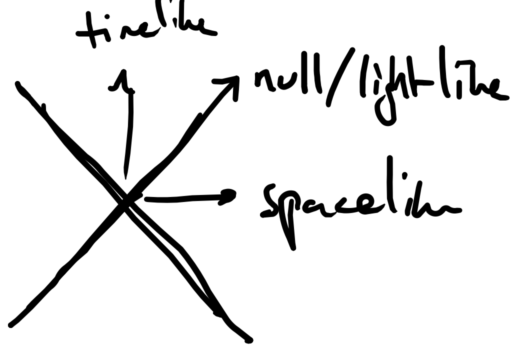
Last Time

- Minkowski spacetime
    - Spacetime points and translations
    - Minkowski metric and causal structure $\eta[b,c]$




    - Lorentz transformations $\eta[\Lambda b, \Lambda c] = \eta[b,c]$
    - Inertial frames $\{e_a\}$
    - Inertial index notation $\eta[e_a, e_b] = \eta_{ab} = \begin{bmatrix} -1 & & 0 \\ & 1 & \\ 0 & & \ddots \end{bmatrix}$

- Higher tensors and a small sprinkling of representation theory
    - Some errors I emailed about…

# Rep. Theory Revisited

* Focus on rotations in D dimensions : $SO(D)$

$S \to$ special [det = 1]

$O \to$ orthogonal $\underline{R^T = R^{-1}}$

↳ Defining rep/vector rep

↳ $R = D \times D$ matrix $= \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$

↳ $\vec{v}$ is a D-dim vect $= \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$

$\Big\}$ $R\vec{v} = \vec{v}'$ ← rotated vector

$\underbrace{\qquad\qquad}$

$\underline{R^i{}_j v^j = v^{i\prime}}$

* This rep. has dimension D

# ANTISYMMETRIC 2-TENSOR REP

Note: $D^2$ independent components

* Any (2,0) tensor $T^{ij} \to R^i{}_k R^j{}_l T^{kl}$

[example: $v^i w^j \to R^i{}_k v^k R^j{}_l w^l = R^i{}_k R^j{}_l (v^k w^l)$]

* Antisymmetric (2,0) $T^{[ij]} = \frac{1}{2}\left(T^{ij} - T^{ji}\right)$

(turns out to be irreducible)

Dimensionality: diagonal components are all zero

↳ antisymmetry means only have to specify the upper-triangular components: $\frac{D(D-1)}{2}$

# A coincidence? (No)

* The dimensionality of the anti-sym. rep = number of parameters you need to specify a rotation

Consider an infinitesimal rotation $\delta R^i{}_j = \delta^i_j + \varepsilon \omega^i{}_j$

$\varepsilon^2 \approx 0$

* Recall: $R^T R = \mathbb{I} \rightsquigarrow \underbrace{(R^T)^i{}_k}_{R_k{}^i} R^k{}_j = \delta^i_j \Big\} R_k{}^i R^k{}_j = \delta^i_j$

· Substitute $\delta R^i{}_j$ into the orthogonality condition!

$$\delta R_k{}^i \, \delta R^k{}_j = \delta^i_j$$

$$\underbrace{\left(\delta^i_k + \varepsilon \omega_k{}^i\right)\left(\delta^k_j + \varepsilon \omega^k{}_j\right)} = \delta^i_j$$

Exercise: Show that the above left-hand side

$$= \delta^i_j + \varepsilon\left(\omega_j{}^i + \omega^i{}_j\right) + \varepsilon^2\left(\cdots\right)$$

We find:

$$\cancel{\delta^i{}_j} + \varepsilon\left(\omega_j{}^i + \omega^i{}_j\right) + \underbrace{\varepsilon^2(\cdots)}_{\approx 0} = \cancel{\delta^i{}_j}$$

Conclusion: $\omega^i{}_j = -\omega_j{}^i \quad \Longleftrightarrow \quad \omega^{ij} = -\omega^{ji}$

↳ The parameters that define infinitesimal rotations form antisym. tensors!

[Aside: Antisymmetric tensors are intimately related to angular momentum...]

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Today

- Spacetime particle mechanics
- Current (or flux) densities ← if time permits

# Relativistic Particle Mechanics

* Spacetime interval : Given an infinitesimal spacetime displacement

$dx^a$

$$ds^2 \equiv \text{Spacetime interval} \equiv \eta_{ab} dx^a dx^b = \overbrace{\eta_{00}}^{-1} dt^2 + \overbrace{\delta_{ij} dx^i dx^j}^{d\vec{x} \cdot d\vec{x}}$$

* $dx^a$ timelike $\Leftrightarrow ds^2 < 0$
* $dx^a$ spacelike $\Leftrightarrow ds^2 > 0$
* $dx^a$ lightlike $\Leftrightarrow ds^2 = 0$

examples $dx^0 = dt \quad dx^3 = dz \neq 0$

(all others are 0)

$\hookrightarrow$ timelike case : $|dz| < |dt| \}$ $\left|\frac{dz}{dt}\right| < 1$

$\hookrightarrow$ spacelike case : $|dz| > |dt|$

$\hookrightarrow$ $|dz| = |dt| \rightsquigarrow \left|\frac{dz}{dt}\right| = 1$

## MASSIVE PARTICLES:

A particle that can be brought to rest [equivalently it has a rest frame] is called <u>massive</u>

Spacetime momentum in the rest frame takes a standard form $K^a$ $\begin{cases} K^i = 0 \\ K^0 = m \equiv \text{rest mass} \end{cases}$

* We can transform to an arbitrary inertial frame: $P^a = \Lambda^a{}_b K^b$

$\boxed{\text{Exercise: Show } \eta_{ab} P^a P^b = P_a P^a = -m^2}$

# PROPER TIME FOR MASSIVE PARTICLES

* A time interval $\Delta\tau$ that elapses in the particle's rest frame is called the particle's PROPER TIME interval

↳ This quantity can be computed from any arbitrary inertial frame by noting that the infinitesimal proper time $d\tau$ is part of an infinitesimal displacement $d\bar{X}^a$ $\begin{cases} d\bar{X}^i = 0 \\ d\bar{X}^0 = d\tau \end{cases}$

↳ $$\underbrace{-d\bar{X}^a \, d\bar{X}^b \, \eta_{ab}}_{\text{invariant combination}} = d\tau^2$$

$$dx^a = \Lambda^a{}_b \, d\bar{X}^b$$

$$d\tau^2 = \underbrace{-\eta_{ab} \, dx^a \, dx^b}_{ds^2}$$

* Overall proper time that elapses

$$\Delta\tau = \int_{\tau_i}^{\tau_f} d\tau = \int_{\lambda_i}^{\lambda_f} d\lambda \sqrt{-\eta_{ab} \frac{dx^a(\tau(\lambda))}{d\lambda} \frac{dx^b(\tau(\lambda))}{d\lambda}}$$

where $\frac{dx^a(\tau(\lambda))}{d\lambda} = \frac{dx^a}{d\tau} \frac{d\tau}{d\lambda}$

and $\lambda$ is a parameter that increases monotonically with $\tau$

↳ $\lambda$ can be chosen for convenience. One choice is $\lambda = t$

Exercises: * Show $d\tau = \dfrac{dt}{\gamma}$

$$\gamma \equiv \frac{1}{\sqrt{1-v^2}}$$

Lorentz/gamma factor

$$\underbrace{v^i = \frac{dx^i}{dt}}_{\text{Newtonian velocity of a particle}}$$

* Show that if we define the spacetime velocity

$$u^a \equiv \frac{dx^a}{d\tau}$$

a) This transforms as a spacetime vector

b) $u^0 = \gamma \quad u^i = \gamma v^i$

c) $u_a u^a = -1$

Exercise: Given that the spacetime momentum is $p^a \equiv m u^a$

Show that $p^0 = m\gamma$ $\qquad$ $p^i = m\gamma v^i$

$p^0 = E$ ← relativistic energy

$p^i$ ← relativistic momentum components

# Massless Particles

* If we take the limit as $m \to 0$ our momentum components look somewhat pathological

↪ Loophole: We can simultaneously send $\gamma \to \infty$ such that $m\gamma = E = \text{fixed}$

↪ momentum stays well defined $p^0 = E$ $p^i = Ev^i$

*Notice $\gamma \to \infty$ as $v = |\vec{v}| \to 1$

$$\boxed{\text{Exercise: Show that in this limit } |\vec{p}| = E}$$