PHY 565 - GR Project Descriptions

1 No-Hair Theorems

A no-hair theorem states that black hole solutions to Einstein's field equations only depend on a small number of parameters—much smaller than all the possible configurations of energy-momentum that could have led to a given solution. If we specialize to situations involving gravitation and electromagnetism only then we have, for example, the theorem

> Stationary, asymptotically flat black hole solutions in the presence of electromagnetic fields are completely determined by mass, electric and magnetic charge, and angular momentum.

For this project, you should aim to explain what black hole no-hair theorems are, what assumptions they are based on, and some of the technical elements that are used to formulate and prove these sorts of theorems. You should also describe any counter-examples that may exist and how they get around the theorems.

1.1 Important Example to Discuss

One relatively simple version of a no-hair theorem is the proof that the Schwarzschild black hole is the only static, spherically symmetric solution to Einstein's equations in a vacuum. Your report should delve into this case in detail.

1.2 Useful Reference Material

Sean Carroll's book *Spacetime and Geometry*, Chapter 6.

Robert Wald's book *General Relativity*, Chapter 12, particularly the discussion at the end of the chapter and the given references.

2 Penrose-Hawking Singularity Theorems

Einstein first published his equations governing gravitation in 1915. Within that same year, the first exact solution to these equations was discovered by Karl Schwarzschild—a remarkable achievement both due to the fact that Einstein's field equations are quite complicated and nonlinear (Einstein thought it would be impossible to find exact solutions!) and that Schwarzschild found the solution while stationed with the German army on the Russian front during World War I. Schwarzschild sadly died of an autoimmune disorder that he developed while serving his time in the army, less than half a year after sending his work to Einstein.

Schwarzschild's solution describes a nontrivial gravitational field in a vacuum, but there is a point at which the physical spacetime curvature diverges known as a singularity. Einstein and others tried to dismiss such divergences as mathematical artifacts that would no longer obtain for more complicated, realistic descriptions of gravitational collapse. The hope that such singularities could be banished from realistic physics was shattered by Roger Penrose and Stephen Hawking when they proved their singularity theorems in the 1960's.

The aims of this project are **(a)** to explain the rigorous definition of singularities, **(b)** to describe the idea behind the singularity theorems and aspects of their technical formulation, and **(c)** to describe some of the key tools that go into proving singularity theorems including the assumptions about energy conditions and the Raychaudhuri equation and its derivation.

As some ideas for extensions, you could look at what happens when some of the assumptions that lead to the singularity theorems are weakened (particularly energy conditions) and discuss the sorts of exotic matter that must be hypothesized to get around the theorems.

2.1 Key Technical Examples

Your report should include calculations involving the Schwarzschild solution and possibly other black hole solutions that demonstrate some of the key tools that are used to prove the singularity theorems.

2.2 Useful References

The Wikipedia entry on Penrose-Hawking singularity theorems is a good place to start. Some of the key technical tools are well-described in Sean Carroll's book *Spacetime and Geometry.*

3 Emergence of General Relativity From String Theory

3.1 Introduction to Classical String Theory

String theory is based on the idea that instead of treating particles as point-like we should instead think of them as having a one-dimensional, string-like structure. These notions of point-like versus string-like should be thought of as arising from classical limits of underlying quantum theories, but the comparison is most straightforward at the classical level.

First we can look at relativistic point particles in the absence of any background gravitational fields. A massive point-like particle sweeps out a worldline as time passes,

$$\Delta\tau = \int d\tau = \int dt\sqrt{1 - |d\vec{x}/dt|^2},$$

where $\vec{x}$ and t are the coordinates associated with the particle according to some inertial frame. One can show that the relativistic physics of a free particle arises from a principle of maximal proper time, leading to the condition that the particle's acceleration is zero in the absence of external forces,

$$\frac{d^2x^a}{dt^2} = 0.$$

The solutions to the equation above describe straight line motion.

A string however, is a spatially spread out object that sweeps out a *worldsheet*. The intrinsic geometry of the worldsheet can be described by a *worldsheet metric* $\gamma_{\alpha\beta}$ where $\alpha, \beta = 0, 1$ are worldsheet coordinate indices and σ^α are intrinsic worldsheet coordinates with $\tau \equiv \sigma^0$, which can be thought of as something akin to the proper time of the string, and $\sigma \equiv \sigma^1$ the intrinsic spatial coordinate along the spatial length of the string. The intrinsic worldsheet area is thus

$$\Delta A = \int d^2\sigma\sqrt{-\gamma},$$

where $\gamma \equiv \mathrm{Det}(\gamma_{\alpha\beta})$.

To make contact with spacetime, we imagine an inertial observer who tracks the different points of the string using inertial coordinate functions $X^a(\tau, \sigma)$ with spacetime index $a = 0, 1, 2, 3, \ldots, d$, and d being the spatial dimension of spacetime. Then the matrices of partial derivatives

$$\frac{\partial X^a}{\partial\sigma^\beta}, \text{ and, } \frac{\partial\sigma^\beta}{\partial X^a}$$

map spacetime tensor components to *induced* worldsheet tensor components,

$$T^{\alpha}{}_{\beta} = \frac{\partial \sigma^{\alpha}}{\partial X^{b}} T^{b}{}_{a} \frac{\partial X^{a}}{\partial \sigma^{\beta}}.$$

The metric on the worldsheet *induced* by the spacetime Minkowski metric is thus

$$\begin{aligned} \gamma_{\alpha\beta} &= \eta_{ab} \frac{\partial X^{a}}{\partial \sigma^{\alpha}} \frac{\partial X^{b}}{\partial \sigma^{\beta}} \\ &= \frac{\partial X}{\partial \sigma^{\alpha}} \cdot \frac{\partial X}{\partial \sigma^{\beta}}, \end{aligned}$$

where X is the spacetime position vector and the dot in the last expression is the spacetime inner product defined by the Minkowski metric.

Exercise 1. Show that

$$(\gamma_{\alpha\beta}) = \begin{pmatrix} \dot{X} \cdot \dot{X} & \dot{X} \cdot X' \\ X' \cdot \dot{X} & X' \cdot X' \end{pmatrix},$$

where

$$\dot{X} \equiv \frac{\partial X}{\partial \tau}, \qquad X' \equiv \frac{\partial X}{\partial \sigma},$$

and show that the worldsheet metric determinant is given by

$$\gamma = (\dot{X} \cdot \dot{X})(X' \cdot X') - (\dot{X} \cdot X')^2.$$

Thus, explain that in the inertial spacetime coordinates, the spacetime area of the worldsheet is

$$\Delta A = \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X} \cdot \dot{X})(X' \cdot X')}.$$

3.2 The Project Goals

The discussion above is a basic introduction to relativistic string theory in a classical (non-quantum) context. The goals of this project are to:

- Describe other classical actions that are used for the quantization of string theory
- Describe how the spacetime metric shows up from the perspective of the classical worldsheet
- Discuss the emergence of Einstein's field equations in a vacuum from the quantization of the worldsheet theory

Note that the last point goes beyond the scope of what we cover in the course, so the discussion should not necessarily include an actual, rigorous derivation of the result. You should identify the key ingredients that go into the calculation, and provide a conceptual understanding of the role played by renormalization.

3.3 Useful References

There is a very nice paper by Huggett and Vistarini, *Deriving General Relativity From String Theory*, which lays out some of the key conceptual and technical points. I highly recommend reading this and using it as a guide for how to structure your own explanation. You can find the paper at `http://philsci-archive.pitt.edu/11116/1/Huggett-Vistarini.pdf`.

For more in-depth detail, you can take a look at Polchinski's textbook *String Theory Vol. 1*.

4 Rindler Geometry and Unruh Radiation

5 Vaidya Spacetime and More Realistic Black Holes

Vaidya spacetime is a close cousin of the black hole spacetime famously described by the Schwarzschild metric:

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{dr^2}{\left(1 - \frac{2M}{r}\right)} - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

In Vaidya spacetime, the mass M is a function of something related to time, meaning that it can be used to model matter that collapses to form a black hole. It can also be used to model the reverse process, of a black hole radiating away mass and evaporating.

The goal in this project is to compute physical details about this spacetime including the curvature, the motion of different sorts of test particles, and the energy-momentum distribution that corresponds to the Vaidya metric.

5.1 Useful References

`https://arxiv.org/pdf/2103.08340.pdf`

6 The ADM Formalism for General Relativity

7 Charged Black Holes and Supersymmetry

7.1 Useful References

Journey into a charged black hole: https://jila.colorado.edu/∼ajsh/insidebh/rn.html

Intro to charged black holes: https://jila.colorado.edu/∼ajsh/bh/rn.html

8 Massless Spin-2 Quantum Particles and the Equivalence Principle

9 Gravitational Waves

10 Numerical Methods in General Relativity

11 Simulations of Jumping Into Black Holes

Journey Inside Black Holes: https://jila.colorado.edu/∼ajsh/insidebh/index.html

12 Cosmology