Advanced G.R. Project Outline: 3+1 Decomposition and Surrogate Modeling

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1 Einstein's Field Equations

Einstein's field equations (EFE) are the starting point for everything in general relativity. They are a set of 10 coupled partial differential equations, which solve for the metric of space-time $g_{\mu\nu}$,

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = 8\pi T_{\mu\nu} \tag{1}$$

where the RHS has all the information about the matter in the system, and the LHS has the information about how the fabric of spacetime curves. It's as the famous quote goes, "matter tells spacetime how to curve, spacetime tells matter how to move."

The condensed form of the EFE does not do justice to the degree of complexity of the equations themselves. So on expanding the curvature terms in EFE in terms of the metric, the Ricci tensor is,

$$\begin{aligned} R_{\mu\nu} =& \partial_\alpha \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)\right) - \partial_\nu \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\alpha} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\alpha}\right)\right) \\ &+ \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\lambda} + \partial_\lambda g_{\beta\mu} - \partial_\beta g_{\mu\lambda}\right)\right)\left(\frac{1}{2}g^{\rho\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right)\right) \\ &- \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\nu g_{\beta\lambda} + \partial_\lambda g_{\beta\nu} - \partial_\beta g_{\nu\lambda}\right)\right)\left(\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\rho} + \partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho}\right)\right) \end{aligned} \tag{2}$$

and, the Ricci Scalar is,

$$\begin{aligned} R = g^{\mu\nu}\Bigg(& \partial_\alpha \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu}\right)\right) - \partial_\nu \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\alpha} + \partial_\alpha g_{\beta\mu} - \partial_\beta g_{\mu\alpha}\right)\right) \\ &+ \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\mu g_{\beta\lambda} + \partial_\lambda g_{\beta\mu} - \partial_\beta g_{\mu\lambda}\right)\left(\frac{1}{2}g^{\rho\sigma}\left(\partial_\nu g_{\sigma\rho} + \partial_\rho g_{\sigma\nu} - \partial_\sigma g_{\nu\rho}\right)\right)\right) \\ &- \left(\frac{1}{2}g^{\alpha\beta}\left(\partial_\nu g_{\beta\lambda} + \partial_\lambda g_{\beta\nu} - \partial_\beta g_{\nu\lambda}\right)\left(\frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\sigma\rho} + \partial_\rho g_{\sigma\mu} - \partial_\sigma g_{\mu\rho}\right)\right)\right)\Bigg) \end{aligned} \tag{3}$$

This makes it clearer as to why EFE are considered difficult to solve. Wave-like solutions of the EFE that escape to infinity are called gravitational waves (GWs), which can be detected by the Laser Interferometer Gravitational Wave Observatory (LIGO). The detection and analysis of these signals via the aforementioned instrument is carried out via the process of Bayesian-inference. This process involves matching (computing the noise weighted overlap of) a timeseries of observed data with known hypothesized GW signal models, and then calculating posteriors of a set parameters being in the observed signal.

All the information about the source is contained within the energy momentum tensor $T^{\mu\nu}$. A few examples for sources of GWs are; coalescing compact objects (black-holes or neutron-stars), supernovae explosions, spinning asymmetric compact objects, the cosmic-microwave-background. However, currently LIGO searches focus only on the first item in that list, Compact Binary Coalescences (CBCs).

2 Numerical Relativity and an Introduction to the 3+1 Decomposition

Numerical Relativity (NR) is a prescription of solving the Einstein's field equations exactly on a computer [1]. But it is not as simple as putting 1, 2 and 3 on a discreet grid while approximating the derivative operators with finite difference approximations. The equations first need to be case into a form that can be put on a computer in a way such that we evolve a known system.

The 3 + 1 decomposition (or ADM formalism, named after the physicists Arnowitt, Deser, and Misner) is a foundational approach to splitting spacetime into a series of three-dimensional slices. This method enables the study of the evolution of the geometry of space over time, breaking down Einstein's equations into a set of initial-value (Cauchy) problems. In this decomposition, spacetime is foliated into hypersurfaces Σ_t labeled by a time parameter t. Depending on the definition of t, one can interpret these foliations as the spacial domain at a particular time.

Spacetime Decomposition:

Let the four-dimensional spacetime manifold $\mathcal{M}$ be equipped with a metric $g_{\mu\nu}$, which we decompose into a series of three-dimensional spacelike hypersurfaces. The $3+1$ decomposition splits the spacetime metric as follows:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where:

- α is the **lapse function** determining the rate of advance of time between slices,
- β^i is the **shift vector** controlling the relative movement of coordinates within the slice, and
- γ_{ij} is the **induced 3-metric** on each spatial slice.

The lapse and shift separate the temporal and spatial components of the metric, allowing us to express dynamics on a spatial slice.

When this form of the metric is substituted into EFE, the result is a set of constraint equations and evolution equations. The former are equations that are required to hold at all evolution steps, whereas the latter evolve the system in time.

Extrinsic Curvature and Constraint Equations:

The extrinsic curvature K_{ij} describes how the hypersurfaces Σ_t are embedded in the four-dimensional spacetime. It is defined as:

$$K_{ij} = -\frac{1}{2\alpha} \left(\partial_t \gamma_{ij} - \mathcal{L}_\beta \gamma_{ij} \right),$$

where $\mathcal{L}_\beta$ is the Lie derivative along the shift vector β^i. The extrinsic curvature K_{ij} captures the rate of change of the metric γ_{ij} in time and is central to the evolution equations.

The Einstein equations split into two sets in this formalism:

1. **Hamiltonian Constraint:**

$$\mathcal{H} = R + K^2 - K_{ij} K^{ij} - 16\pi\rho = 0,$$

   where R is the Ricci scalar of the 3-metric γ_{ij}, $K = \gamma^{ij} K_{ij}$ is the trace of the extrinsic curvature, and ρ represents the energy density as seen by an observer moving orthogonally to Σ_t.

2. **Momentum Constraint:**

$$\mathcal{M}^i = D_j \left(K^{ij} - \gamma^{ij} K \right) - 8\pi S^i = 0,$$

   where D_j denotes the covariant derivative associated with γ_{ij}, and S^i is the momentum density.

Evolution Equations:

The remaining components of the Einstein equations provide the time evolution of γ_{ij} and K_{ij}. These are given by:

1. **Evolution of the 3-metric:**

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \mathcal{L}_\beta \gamma_{ij}.$$

2. **Evolution of the Extrinsic Curvature:**

$$\partial_t K_{ij} = -D_i D_j \alpha + \alpha \left(R_{ij} + K K_{ij} - 2K_{ik} K^k{}_j \right) + \mathcal{L}_\beta K_{ij} - 8\pi\alpha \left(S_{ij} - \frac{1}{2} \gamma_{ij} (S - \rho) \right),$$

   where R_{ij} is the Ricci tensor of the 3-metric γ_{ij}, S_{ij} is the spatial stress tensor, and $S = \gamma^{ij} S_{ij}$.

Importance of the $3+1$ Decomposition:

The $3+1$ decomposition framework is particularly useful in numerical relativity, as it reformulates Einstein's field equations into a system of partial differential equations that are suitable for numerical simulation. This enables the study of dynamic, strong-field spacetimes, such as those near black holes or in merging neutron star binaries; and also the study of far-field objects like gravitational waves.

This approach enables the formulation of initial value problems in general relativity, allowing us to compute the evolution of gravitational fields in complex spacetimes, for which analytical solutions do not exist in general.

The problem is that these set of equations, though can be put on a numerical grid and evolved, are not numerically stable. And hence physicists have worked on approaches that modify the set of equations in the ADM formalism to be numerically stable. Which can then be put on a computer and solved numerically.

3 Why Surrogate Models

The gravitational wave can computed by extracting the metric at several extraction spheres at varying finite radii and then extrapolating it to future null-infinity. Hence NR simulations, in principle, give a way to generate exact waveforms for a given set of parameters (masses, spins, eccentricity, ...).

NR simulations, though extremely accurate, are extremely computationally expensive, for e.g. it can take a few weeks to months for a single BBH simulation to finish on a supercomputer. Hence it is infeasible to run a simulation for a dense enough set of parameters, for tasks like creating a template bank for parameter estimation, which typically require datasets with roughly a million waveform signals. Just a back-of-the-envelope calculation tells one that it would take ~ 400 years for a single such dataset to get generated via NR.

This necessitates **surrogate models**, which are fast(er) models that are trained on existing waveforms at certain parameter values, and can give almost NR-accurate waveforms for parameter values inside the training parameter space ([2]). It can very roughly be seen as an interpolation tool. A lot of LIGO data analysis relies of surrogate models, for e.g. [3] is the the standard way of using Effective One Body models in LIGO. Also, the LIGO-Virgo-Kagra collaboration used the NR surrogates as the main model for the first intermediate-mass BH discovery.

4 Future topics

- Variants of the 3+1 decomposition of EFE will potentially be studied in detail (like the B.S.S.N formalism), which is a prescription to recast EFE into a numerically tractable (hyperbolic) form, which can then be solved numerically.
- Study the algorithm to build surrogates. It is a 4-step process, and all of them will be studied in detail.
    1. waveform decomposition and preprocessing
    2. basis reduction
    3. empirical interpolation
    4. parametric fitting

References

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