PHY 565* - Advanced Topics in GR

9/12/2024

Announcements & Reminders

- Project topic selection – start thinking about it now
  - Due by Friday, 9/27
  - If you have a topic not covered in my list of suggestions, email me early so that we can discuss it

- “Homework” – please try the exercises I throw into the lectures. If you get stuck, talk to me or email me!

- Reminder: Next week we start to meet via zoom every lecture until Oct 17th

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Last Time

- Euclidean spaces and their symmetries
  - Cartesian frames and coordinates
- ~~Extension to Minkowski spacetime~~
- ~~A very small sprinkling of representation theory~~

A little bit on Euclidean (SO(D)) tensors

* $SO(D) \leftarrow$ Group of rotations in D dimensions

  ↳ vector representation (defining) $\Big\}$ $D \times D$ Special orthogonal matrices $R^T = R^{-1}$

  ↳ $R^T R = R R^T = \mathbb{I}$

  $\det[R] = 1$

  $\boxed{\text{↳ Exercise: check ~~Show that~~ dim is } \frac{D(D-1)}{2}}$

* trivial representation: All elements of $SO(D) \longrightarrow 1$

  [1D rep.]

Consider antisymmetric matrices

* The representations consist of <u>tensors</u>

$\hookrightarrow$ $T^{ijk\ \ \cdots}{}_{lmn\cdots}$

* Upstairs indices transform via R : $\tilde{v}^i = R^i{}_j v^j$ $\rightsquigarrow$ $R\begin{pmatrix}\vdots\end{pmatrix}$

* Downstairs indices transform via $R^{-1} = R^T$: $\tilde{v}_i = R_i{}^j v_j$

$\downarrow$

$\underbrace{(\cdots)R}_{\text{equivalent}}$

$\left(R^T\begin{pmatrix}\vdots\end{pmatrix}\right)^T$

* Raising and lowering convention:

$$v_i = \delta_{ij} v^j \qquad v^i = \delta^{ij} v_j$$

If you have a 2-index tensor T^{ij}

$$\hookrightarrow \quad T^{ij} \longrightarrow R^i{}_k R^j{}_l T^{kl}$$

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Today

- Minkowski spacetime
  - Spacetime points and translations
  - Minkowski metric and causal structure
  - Lorentz transformations
  - Inertial frames
  - Inertial index notation

- Higher tensors and a small sprinkling of representation theory

- Spacetime particle physics?

MINKOWSKI SPACETIME

* Extend our Euclidean space discussion by adding a temporal dimension

Spatial points $\mathcal{P}$ $\longrightarrow$ Spacetime points $\mathcal{E}$ (events)

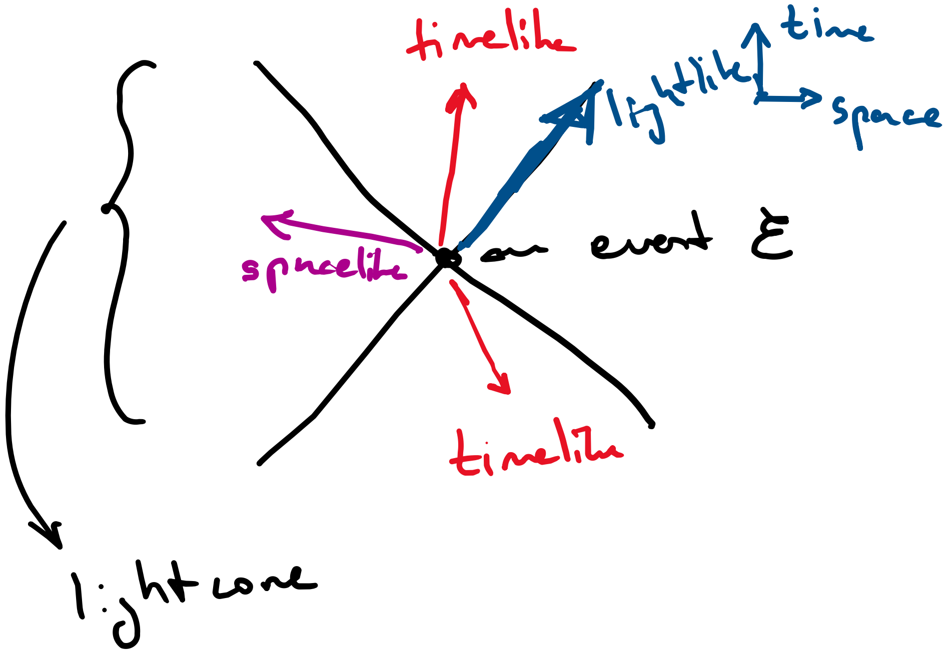
* Spatial translations $\vec{b}$ $\longrightarrow$ Spacetime translations $\mathbf{b}$

Spacetime vectors (index free notation)

* Euclidean metric $\delta[\ ,\ ]$ $\longrightarrow$ Minkowski metric $\eta[\ ,\ ]$

CAUSAL STRUCTURE OF SPACETIME

- η encodes causal structure



$\eta[b,b] > 0$: SPACELIKE

$\eta[b,b] < 0$: timelike

$\eta[b,b] = 0$: lightlike / null

LORENTZ TRANSFORMATIONS! Λ

$$\eta[\Lambda b, \Lambda c] = \eta[b, c]$$

INERTIAL FRAMES

* Pseudo-O.N. basis $\{\underset{\text{spacetime vectors}}{e_a}\}$: $\eta[e_a, e_b] = \underbrace{\eta_{ab} = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix}}_{\text{Minkowski metric components}}$

  a = inertial spacetime index $= 0, i$, $i = 1, \ldots, D$

* Inertial frame : Join an origin point $\mathcal{E}_0$ to the basis

  ↳ Minkowski position vector $\underset{\text{spacetime vector}}{x} \equiv \underset{\text{Minkowski coordinates of our inertial frame}}{x^a}\, \underset{\text{basis vectors}}{e_a}$, $\underbrace{x^a(\mathcal{E}_0) = 0}$

Components of Lorentz Transformations

$$\tilde{e}_b = \Lambda e_b = e_a \underbrace{\Lambda^a{}_b}$$

components of Λ in the basis $\{e_a\}$

* Raising and lowering convention: $V_a = \eta_{ab} V^b$

$$V^a = \eta^{ab} V_b$$

* Example: $\Lambda_a{}^b = \eta_{ac}\,\eta^{bd}\,\Lambda^c{}_d$

* Metric invariance: $\eta[\Lambda e_c, \Lambda e_d] = \eta[e_a \Lambda^a{}_c, e_b \Lambda^b{}_d]$

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$$\underbrace{\eta[e_c, e_d]}_{\eta_{cd}} = \eta[e_a, e_b]\Lambda^a{}_c \Lambda^b{}_d$$
$$= \eta_{ab}\Lambda^a{}_c \Lambda^b{}_d$$

Conclude: $\eta_{cd} = \eta_{ab}\Lambda^a{}_c \Lambda^b{}_d$

* EXERCISE: Show that the raising and lowering conventions and the metric invariance condition imply $(\Lambda^{-1})^a{}_b = \Lambda_b{}^a$.

A LITTLE
REP. THEORY FOR LORENTZ SYMMETRY

* Lorentz transformations form a group $O(\underset{\text{+'s}}{D},\underset{\text{-'s}}{1})$
  - ↳ special Lorentz transformations $SO(D,1) \leftarrow \det(\Lambda)=1$
    - ↳ preserve orientation in time of spacetime vectors and spatial orientation
    - * And possible complications [Exercise go read about orthochronous component of $SO(D,1)$]

* trivial rep: $\Lambda \longrightarrow 1$

- vector / defining rep : $\quad v^a \rightsquigarrow \Lambda^a{}_b v^b$

- higher tensor reps
  - ↳ e.g. $\quad (2,0)$ rep $\quad T^{ab} \longrightarrow \Lambda^a{}_c \Lambda^b{}_d T^{cd}$

    $(0,2)$ rep $\quad T_{ab} \longrightarrow \Lambda_a{}^c \Lambda_b{}^d T_{cd}$

* Irreducible reps cannot be broken down into smaller dimensional components

Example of irreducible reps:

* Trace part of $(2,0)$ tensor: $T = \eta_{ab} T^{ab} = T^a{}_a$

$\hookrightarrow$ 1D rep [equiv to trivial] $\rightarrow$ 1D rep

* Antisymmetric rep: $T^{[ab]} \equiv \frac{1}{2}\left(T^{ab} - T^{ba}\right) \rightarrow \frac{(D+1)D}{2}$

* Symmetric traceless rep: $T^{(ab)} \equiv \frac{1}{2}\left(T^{ab} + T^{ba}\right) \rightarrow \frac{(D+1)(D+2)}{2} - 1$

$\hookrightarrow T^{\langle ab\rangle} = T^{(ab)} - \frac{1}{(D+1)} T \eta^{ab}$