PHY 565* - Advanced Topics in GR

9/10/2024

Announcements & Reminders

- My contact information:
  - Name: Prof. David Kagan
  - Email: dkagan@umassd.edu
  - Office: SENG 203-D
  - Hours: M/W 11 – 12; also by appointments…or just coming to chat with me!

- Syllabus and course schedule at at https://dkagan.sites.umassd.edu

- **In-person versus Zoom-based classes**
  - I've posted my best guesses about when we will have classes via Zoom versus in person. You can find the details on the course lecture schedule
  - We will meet in person this week
  - Starting next week, we will shift to Zoom until 10/17, when we will meet in person
  - We'll then basically alternate, with Tuesdays mostly via Zoom and Thursdays in person

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Last Time

- Conceptual review of GR
  - Inertial frames postulate
  - Relativity postulate
  - Signal limit / speed of light / interaction rate postulate
    - These postulates (locally) lead to Lorentz symmetry
  - Homogeneity of time and space → conservation of energy and momentum → rest mass
    - (Postulate that tachyons aren't allowed…)
  - Gravitational field postulate (to explain Newtonian gravity)
    - Acts on gravitational mass (not assumed related to rest mass)

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Last Time Continued

- Conceptual review of GR
  - Postulate strong equivalence principle: uniform gravitational fields are indistinguishable from pseudoaccelerations in non-inertial frames
    - Implies weak equivalence principle (gravitational mass related to rest mass)
    - Extend to non-uniform fields by working locally
  - Postulate minimal coupling: SR holds locally (gravitational fields can be removed locally from other interactions)
  - Math: differentiable manifolds naturally describe spacetime that locally looks like SR, but patched together nontrivially → encode gravity in spacetime manifold curvature
  - Signal speed limits → curvature changes locally in a relativistic way
    - Changes of curvature described by wave equation
    - Postulate Einstein Field Equations → simplest (?) equations that yield these behaviors in a way that respects symmetry under general changes of coordinates

TERM PROJECTS [PLEASE READ SYLLABUS!]

↳ List of topics (Step 1: choose one or propose your own by end of month)

↳ **Step 2:**
- Make an outline
- Share w/ me & your advising group
- Advising meeting

↳ **Step 3:**
- Write up project
- Record presentation, share with me and group
- Final advising meeting

Some possible questions

* Typical to conflate $\underbrace{\text{reference frames}}_{\text{more physical}}$ w/ $\underbrace{\text{coordinate choices}}_{\text{more arbitrary}}$

  → Explore the distinction more carefully

* Are there more fundamental principles / postulates that give rise to EFE's

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Today

- Euclidean spaces and their symmetries
    - Cartesian frames and coordinates
- Extension to Minkowski spacetime
- A very small sprinkling of representation theory

EUCLIDEAN SPACE

* Euclidean space is a set of points and a vector space of translations
  ↪ if P is a point and $\vec{b}$ is a translation vector
  then $Q = P + \vec{b}$ is another point
* The vectors themselves satisfy: $\alpha \vec{v} + \beta \vec{w}$ → this is also a vector.
  for real α & β
* Vectors come with a Euclidean metric or inner product

EUCLIDEAN INNER PRODUCT $\quad \delta[\vec{b}, \vec{c}]$ = a real number

↪ $\delta[\vec{b}, \vec{b}] = 0$ only if $\vec{b} = \vec{0}$

and all the other standard axioms for inner product

The magnitude of $\vec{b}$: $|\vec{b}|^2 = \delta[\vec{b}, \vec{b}]$

The angle between $\vec{b}$ & $\vec{c}$ $\quad \cos\theta_{\vec{b}\vec{c}} = \dfrac{\delta[\vec{b}, \vec{c}]}{|\vec{b}|\,|\vec{c}|}$

↪ Note that $\vec{b} \cdot \vec{c} \equiv \delta[\vec{b}, \vec{c}]$

Rotational Symmetry

* Rotations act linearly on vectors, preserving angles and magnitudes

$$\vec{b}' = R\vec{b}$$

rotation R acts on vector $\vec{b}$ yielding vector $\vec{b}'$

$$\delta[R\vec{b}, R\vec{c}] = \delta[\vec{b}, \vec{c}]$$

* In addition to rotations, REFLECTION transformations also satisfy the above relation

Bases & Frames

- An orthonormal basis $\{\vec{e}_i\}$ $i = 1, \dots, D$
  (ON)
  dimension of the Euclidean space

  Satisfies:

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$$

Kronecker δ-symbol

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Components of Euclidean metric in an ON basis

* CARTESIAN FRAME: ON basis $\{\vec{e}_i\}$ joined to a origin point $\mathcal{O}$

CARTESIAN COORDINATES :

Position vector

can be assigned

to any point

in our Euclidean

space

[rooted at origin]

cartesian

position coordinates

$$\vec{x} = \sum_{i=1}^{D} x^i \vec{e}_i = x^i \vec{e}_i = x^l \vec{e}_l$$

E.S.C

i is a

dummy index

can be

replaced

with any

other spatial

index

(i, j, k, l, m, n)

Rotation components

* Act on basis with rotation: $\vec{e}_j{}' = R\vec{e}_j$

$$\boxed{\text{Exercise: } \{\vec{e}_j{}'\} \text{ forms an ON basis}}$$

* Since R is linear: $R\vec{e}_j = \vec{e}_i \underbrace{R^i{}_j}_{\text{components of } R \text{ in } \{\vec{e}_i\}}$

$$\boxed{\text{* Exercise: } \delta[\vec{e}_i, R\vec{e}_j] = \delta_{ik}R^k{}_j \underset{\uparrow}{=} R_{ij}}$$

lowering convention $\rightsquigarrow$ $\left\{ \begin{array}{l} v_i = \delta_{ij} v^j \\ v^i = \delta^{ij} v_j \end{array} \right.$

Defining Relation for $\left\{\begin{matrix}\text{Reflections}\\ \text{Rotations}\end{matrix}\right\}$ in Component Form

$$\delta[R\vec{b}, R\vec{c}] = \delta[\vec{b}, \vec{c}]$$

↳ let $\vec{b}$ & $\vec{c}$ be basis vectors:

$$\delta[R\vec{e}_i, R\vec{e}_j] = \delta[\vec{e}_k R^k{}_i, \vec{e}_l R^l{}_j]$$

$$= \delta[\vec{e}_k, \vec{e}_l] R^k{}_i R^l{}_j$$

$$\delta[\vec{e}_i, \vec{e}_j] = \delta_{ij} \quad = \delta_{kl} R^k{}_i R^l{}_j$$

Conclusion

$$\delta_{ij} = \delta_{kl} R^k{}_i R^l{}_j$$

Exercise: Matrix notation

↳ If we associate $R^i{}_j$ with components of a matrix R

Show that the defining property of reflections & rotations

$$\delta_{ij} = \delta_{kl} R^k{}_i R^l{}_j$$

is equivalent to $R^T R = R R^T = \mathbb{I}$ identity

where $(R^T)^i{}_j = R_j{}^i = \delta_{jk} \delta^{il} R^k{}_l$