The Emergence of General Relativity from String Theory

Varenya Upadhyaya

General Relativity

General Relativity is a geometric theory that relates the curvature of spacetime to the matter and radiation via the Einstein Field Equations. The equations are given as,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \tag{1}$$

When there are no source terms (matter) present, the resultant vacuum Einstein equations become,

$$R_{\mu\nu} = 0 \tag{2}$$

A more detailed introduction will be included in the report. All in all, we will try to build up to the fact that (2), and subsequently (1) can be derived by quantizing string theory.

String Theory Formalism

String Theory is built on the idea that instead of point particles, matter is made up of objects with one spatial and one temporal dimension – strings. These strings can be thought of as spacetime objects that come with internal coordinates (τ, σ) with the D-dimensional external space having coordinates X^μ. As the string moves through spacetime, it sweeps out a worldsheet, which can be identified by assigning the external coordinates $(X^0, X^1 \ldots X^{D-1})$ to each internal coordinate (τ, σ). This can be thought of as a D-component vector field on the string, such that any change in this field would correspond to some motion in the external (or target) space.

Nambu-Goto and Polyakov Actions

For a string with no external forces (ie. a flat external spacetime), the induced metric $g_{\alpha\beta}$ is,

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} \tag{3}$$

where $\alpha, \beta = 0, 1$. The simplest invariant action for a classical relativistic string is

$$S = -T\int dA = -T\int d^2\sqrt{-g} = -T\int d^2\sigma\sqrt{-det\left(\eta_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta}\right)} \tag{4}$$

The term inside the the square root can be rewritten as,

$$g_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial \sigma^\alpha}\frac{\partial X^\nu}{\partial \sigma^\beta} = \frac{\partial X}{\partial \sigma^\alpha}\cdot\frac{\partial X}{\partial \sigma^\beta} \tag{5}$$

$$(g_{\alpha\beta}) = \begin{pmatrix} \frac{\partial X}{\partial \sigma^0}\cdot\frac{\partial X}{\partial \sigma^0} & \frac{\partial X}{\partial \sigma^0}\cdot\frac{\partial X}{\partial \sigma^1} \\ \frac{\partial X}{\partial \sigma^1}\cdot\frac{\partial X}{\partial \sigma^0} & \frac{\partial X}{\partial \sigma^1}\cdot\frac{\partial X}{\partial \sigma^1} \end{pmatrix} = \begin{pmatrix} \dot{X}\cdot\dot{X} & \dot{X}\cdot X' \\ X'\cdot\dot{X} & X'\cdot X' \end{pmatrix} \tag{6}$$

This leaves us with the Nambu-Goto action,

$$S_{NG} = -T \int d^2\sigma \sqrt{(\dot{X} \cdot X')^2 - (\dot{X} \cdot \dot{X})(X' \cdot X')} \tag{7}$$

T here represents the tension in the string. Introducing an 'internal' metric $\gamma_{\alpha\beta}$ on the string world sheet gives the Polyakov action,

$$S_\sigma = -\frac{T}{2} \int d^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} \tag{8}$$

More on the actions and the internal metric in the report.

Deriving GR from String Theory

The following points will be expanded on (conceptually and mathematically) in the report:

1. Classical String Theory
   - (a) The corresponding Hamiltonian for (8) can be minimized with respect to the X^μ's. These wave solutions represent the general solutions for closed strings.
   - (b) The amplitudes $(\alpha_n, \tilde{\alpha}_n)$ of the modes of strings are used to introduce Virasoro Generators $L_m, \tilde{L}_m$ which give the stress energy tensors of the 2-D string spacetime.
   - (c) At scales larger than the characteristic string lengths, the strings appear as point particles; the corresponding four-momentum dot product paired with the Virasoro constraints gives a relation between the Mass and the Hamiltonian of the string ($H \propto M^2$)
2. Quantizing String Theory
   - (a) X^μ is a field on the 2-D Minkowski spacetime or the string worldsheet
   - (b) The equal time commutation relations give us a way to think of the $(\alpha, \tilde{\alpha})$'s as raising and lowering operators.
   - (c) This will show that the quantized mass spectrum is discrete – giving rise to massless particles (photons, gravitons) and negative tachyon modes.
   - (d) For string tensions, the modes can reproduce masses for the meson family however since the the appropriate tension for quantum gravity is much higher, observed particles aren't theorized to be mode excitations
   - (e) The Virasoro generators for the quantum scenario can be retrieved by replacing the α's in the classical case with operators
3. Quantizing with a general metric in the action
   - (a) The Polyakov action for a general Lorentzian metric is,

     $$S_\sigma = -\frac{T}{2} \int d^2\sigma \sqrt{-\gamma}\gamma^{\alpha\beta}G_{\mu\nu}\frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta} \tag{9}$$
   - (b) From the string's PoV, the action gives rise to a 2D QFT (non linear σ model) in which $G_{\mu\nu}$ is a self coupling for the X^μ field
   - (c) This QFT is renormalizable and this renormalization introduces a length dependence, described in terms of β functions. The β-function for $G_{\mu\nu}$ is given as $\beta_G = R_{\mu\nu} + O(1/T)$ [1].
   - (d) Since string theory has conformal symmetry, there can be no dependence on the length (length scale here is wrt the string metric). The β function vanishes, leaving $R_{\mu\nu} = 0$; EFE in vacuum.
4. Generalizing the result
   - (a) Conformal invariance of string theory led to the vacuum EFE.
   - (b) Assuming, for instance, a Yang-Mills field in the target space and rewriting the action, will now give slightly modified β-functions (there will be a term corresponding to the energy-momentum tensor of the Yang-Mills field). Setting $\beta_G = 0$ then gives rise to the EFEs[1]

[1] This might be slightly beyond the scope of this project but I will try to explain this more systematically in the actual report

Bibliography

[1] D. Friedan. "Nonlinear Models in $2 + \epsilon$ Dimensions". In: *Phys. Rev. Lett.* 45 (13 Sept. 1980), pp. 1057–1060. DOI: 10.1103/PhysRevLett.45.1057. URL: https://link.aps.org/doi/10.1103/PhysRevLett.45.1057.