Classical and Quantum Optics

Assignment-1 Answers

Varghese Reji

Problem 1

(a) We know that,

$$\mathcal{F}\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdyg(x,y)e^{-2\pi i(f_X x+f_Y y)} \tag{1}$$

Then, we can write,

$$\mathcal{F}\mathcal{F}\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df_X df_Y\ e^{-2\pi i(f_X x+f_Y y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\ g(x',y')e^{-2\pi i(f_X x'+f_Y y')}$$

By interchanging the order of integration,

$$\mathcal{F}\mathcal{F}\{g(x,y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\ g(x',y') \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df_X df_Y e^{-2\pi i(f_X(x'+x)+f_Y(y'+y))} \tag{2}$$

But we have the identity,

$$\delta(x-x') = \frac{1}{2\pi}\int_{-infty}^{\infty} dpe^{ip(x-x')} \tag{3}$$

Using this, we can write 2 as

$$\begin{aligned}\mathcal{F}\mathcal{F}\{g(x,y)\} =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\ g(x',y')\delta(x+x',y+y') \\ =& g(-x,-y)\end{aligned} \tag{4}$$

So, we can write,

$$\mathcal{F}\mathcal{F}\{g(x,y)\} = g(-x,-y)$$

Similerly,

$$\begin{aligned}\mathcal{F}^{-1}\mathcal{F}^{-1}\{g(x,y)\} =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df_X df_Y\ e^{2\pi i(f_X x+f_Y y)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\ g(x',y')e^{2\pi i(f_X x'+f_Y y')} \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\ g(x',y') \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df_X df_Y e^{2\pi i(f_X(x'+x)+f_Y(y'+y))} \\ =& \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx'dy'\ g(x',y')\delta(x+x',y+y') \\ =& g(-x,-y)\end{aligned}$$

Then,

$$\mathcal{F}\mathcal{F}\{g(x,y)\} = \mathcal{F}^{-1}\mathcal{F}^{-1}\{g(x,y)\} = g(-x,-y)$$

(b) We can write the convolution of two functions as

$$g(t) \otimes h(t) = \int_{-\infty}^{\infty} d\tau \; g(\tau)h(t-\tau) \tag{5}$$

So, the given fucntion can be written as

$$\mathcal{F}\{g(x,y)\} \otimes \mathcal{F}\{h(x,y)\} = G(f_X, f_Y) \otimes H(f_X, f_Y)$$

Using equation 5, we can write this as,

$$G(f_X, f_Y) \otimes H(f_X, f_Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dF_X dF_Y G(F_X, F_Y)H(f_X - F_X, f_Y - F_Y) \tag{6}$$

Let us take the inverse transform of this.

$$\begin{aligned}
&\mathcal{F}^{-1}\{G(f_X, f_Y) \otimes H(f_X, f_Y)\} \\
=&\mathcal{F}^{-1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dF_X dF_Y G(F_X, F_Y)H(f_X - F_X, f_Y - F_Y) \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dF_X dF_Y G(F_X, F_Y)\mathcal{F}^{-1}\{H(f_X - F_X, f_Y - F_Y)\} \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dF_X dF_Y G(F_X, F_Y)\exp\left(2\pi i\left[F_X x + F_Y y\right]\right) \\
&\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} df_X df_Y H(f_X - F_X, f_Y - F_Y)\exp\left(2\pi i\left[(f_X - F_X)x + (f_Y - F_Y)y\right]\right) \\
=&\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dF_X dF_Y G(F_X, F_Y)h(x,y) \\
=&g(x,y)h(x,y)
\end{aligned}$$

$\therefore$

$$\mathcal{F}\{g(x,y)f(x,y) = \mathcal{F}\{g(x,y)\} \otimes \mathcal{F}\{h(x,y)\}$$

(c)

$$\mathcal{F}\{\nabla^2 g(x,y)\} = -4\pi^2(f_x^2 + f_y^2)\mathcal{F}\{g(x,y)\}$$

We know that:

$$\int_V \left(\psi\vec{\nabla}^2\phi + \vec{\nabla}\psi\cdot\vec{\nabla}\phi\right) dV = \oint_S \psi\vec{\nabla}\phi\cdot d\vec{S} \tag{7}$$

Let us take $\psi = e^{-2\pi i(f_x x + f_y y)}$ and $\phi = g(x,y)$.

But here, we can assume that value of g is really small at large x and y. Gradiance of it also can be neglected. So the RHS of equ 7 will be 0 when we take the closed integral along a large closed surface which includes the volume of integration in LHS.

Then, we can write,

$$\begin{aligned}
\int_V \psi\vec{\nabla}^2\phi dV &= -\int_V \vec{\nabla}\psi\cdot\vec{\nabla}\phi dV \\
&= \int_V \phi\vec{\nabla}^2\psi dV
\end{aligned} \tag{8}$$

Then, we can write,

$$
\begin{aligned}
\mathcal{F}(\nabla^2 g(x,y)) &= \int_{-\infty}^{\infty} dxdy\ e^{-2\pi i(f_X x + f_X x)} \nabla^2 g(x,y) \\
&= \int_{-\infty}^{\infty} dxdy\ g(x,y) \nabla^2 e^{-2\pi i(f_X x + f_X x)} \\
&= -4\pi^2 (f_X^2 + f_Y^2) \int_{-\infty}^{\infty} dxdy\ g(x,y) \nabla^2 e^{-2\pi i(f_X x + f_X x)} \\
&= -4\pi^2 (f_X^2 + f_Y^2) \mathcal{F}\{g(x,y)\}
\end{aligned}
$$

Problem 2

(a) Given $g_R(r) = \delta(r - r_0)$.

$$g(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(r)e^{-2\pi i(ux+vy)}dxdy$$

Let us take,

$$\begin{aligned} x + iy =& re^{i\theta} \\ u + iv =& \rho e^{i\phi} \end{aligned}$$

Then,

$$\begin{aligned} x =& r\cos\theta \\ y =& r\sin\theta \\ r =& \sqrt{x^2+y^2} \\ u =& \rho\cos\phi \\ v =& \rho\sin\phi \\ \rho =& \sqrt{u^2+v^2} \end{aligned}$$

Then, we can write,

$$\begin{aligned} g(\rho) =& \int_0^{\infty}\int_0^{2\pi} f(r)e^{-2\pi i r\rho(\cos\phi\cos\theta+\sin\phi\sin\theta)}rdrd\theta \\ =& \int_0^{\infty}\int_0^{2\pi} f(r)e^{-2\pi i r\rho\cos(\theta-\phi)}rdrd\theta \\ =& \int_0^{\infty}\int_{-\phi}^{2\pi-\phi} f(r)e^{-2\pi i r\rho\cos\theta}rdrd\theta \\ =& \int_0^{\infty}\int_0^{2\pi} f(r)e^{-2\pi i r\rho\cos\theta}rdrd\theta \\ =& \int_0^{\infty} f(r)\left[\int_0^{2\pi} e^{-2\pi i r\rho\cos\theta}d\theta\right]rdr \\ =& 2\pi\int_0^{\infty} f(r)J_0\left(2\pi\rho r\right)rdr \end{aligned}$$

So,

$$g(\rho) = 2\pi\int_0^{\infty} f(r)J_0\left(2\pi\rho r\right)rdr \tag{9}$$

Here, $f(r) = g_R(r) = \delta(r - r_0)$.
Then,

$$\begin{aligned} \mathcal{B}\{g_R(r)\} =& g(\rho) \\ =& 2\pi\int_0^{\infty} f(r)J_0\left(2\pi\rho r\right)rdr \\ =& 2\pi\int_0^{\infty} \delta(r-r_0)J_0\left(2\pi\rho r\right)rdr \\ =& 2\pi r_0 J_0\left(2\pi\rho r_0\right) \end{aligned}$$

$\therefore$

$$\mathcal{B}\{g_R(r)\} = 2\pi r_0 J_0\left(2\pi\rho r_0\right)$$

(b) $g_R(r) = 1 for a \leq r \leq 1$ and zero elsewhere.

We have the identity,

$$\int_0^x x' J_0(x')dx' = xJ_1(x) \tag{10}$$

So,

$$\begin{aligned}
\mathcal{B}\{g_R(r)\} =& 2\pi \int_a^1 J_0\left(2\pi\rho r\right) rdr \\
=& 2\pi \left[\int_0^1 J_0\left(2\pi\rho r\right) rdr - \int_0^a J_0\left(2\pi\rho r\right) rdr\right] \\
=& \frac{1}{2\pi\rho^2}\left[\int_0^1 J_0\left(2\pi\rho r\right)(2\pi\rho r)(2\pi\rho dr) - \int_0^a J_0\left(2\pi\rho r\right)(2\pi\rho r)(2\pi\rho dr)\right]
\end{aligned}$$

Let, $x = 2\pi\rho r$. Then, when \$r=a, $x = 2\pi\rho a$. $\Rightarrow a = \frac{x}{2\pi\rho}$. Apply same for $r = 1$. Then,

$$\begin{aligned}
\mathcal{B}\{g_R(r)\} =& \frac{1}{4\pi^2\rho^2}\left[\int_0^1 J_0\left(2\pi\rho r\right)(2\pi\rho r)(2\pi\rho dr) - \int_0^a J_0\left(2\pi\rho r\right)(2\pi\rho r)(2\pi\rho dr)\right] \\
=& \frac{1}{2\pi\rho^2}\left[\int_0^{2\pi\rho} J_0\left(x\right) xdx - \int_0^{2\pi\rho a} J_0\left(x\right) xdx\right] \\
=& \frac{1}{2\pi\rho^2}\left[2\pi\rho J_1(2\pi\rho) - 2\pi\rho a J_1(2\pi\rho a)\right] \\
=& \frac{1}{\rho}\left[J_1(2\pi\rho) - aJ_1(2\pi\rho a)\right]
\end{aligned}$$

$\therefore$

$$\mathcal{B}\{g_R(r)\} = \frac{J_1(2\pi\rho) - aJ_1(2\pi\rho a)}{\rho}$$

(c) Given, $\mathcal{B}\{g_R(r)\} = G(\rho)$.

From equ 9,

$$g(\rho) = 2\pi \int_0^\infty g_R(r) J_0\left(2\pi\rho r\right) rdr$$

Here, $g_R(r) \to g_R(ar) \Rightarrow$

$$\begin{aligned}
\mathcal{B}\{g_R(ar)\} =& 2\pi \int_0^\infty g_R(ar) J_0\left(2\pi\rho r\right) rdr \\
=& 2\pi \int_0^\infty g_R(ar) J_0\left(2\pi a\left(\frac{\rho}{a}\right) r\right)\left(\frac{ar}{a}\right)\frac{adr}{a} & (ra \to x) \\
=& \frac{1}{a^2} 2\pi \int_0^\infty g_R(a) J_0\left(2\pi\left(\frac{\rho}{a}x\right)\right) xdx \\
=& \frac{1}{a^2} G\left(\frac{\rho}{a}\right)
\end{aligned}$$

Hence,

$$\mathcal{B}\{g_R(ar)\} = \frac{1}{a^2} G\left(\frac{\rho}{a}\right)$$

(d) Fourier-Bessel transform is basically Fourier transform of 2-D functions with circular symmetry. So, we can do the FT of given function in cartesian coordinate also.

Given function is $g_R(r) = \exp\left(-\pi r^2\right) = \exp\left(-\pi(x^2+y^2)\right)$.

This is variable separable. So, we can take the Fourier transform of individuals. Then, we will get,

$$\begin{aligned}\mathcal{F}\{\exp\left(-\pi(x^2+y^2)\right) = \mathcal{F}\{\exp(-\pi x^2)\}\mathcal{F}\{\exp(-\pi y^2)\}\\ = \exp(-\pi f_X^2)\exp(-\pi f_Y^2)\\ = \exp\left(-\pi(f_X^2+f_Y^2)\right)\end{aligned}$$

But, $f_X^2 + f_Y^2 = \rho^2$

$\therefore$,

$$\mathcal{B}\{\exp(-\pi r^2)\} = \exp(-\pi\rho^2)$$

Problem 3

$$W(f,x) = \int_{-\infty}^{\infty} g(x+\frac{\xi}{2})g^*(x-\frac{\xi}{2})\exp(-j2\pi f\xi)d\xi \tag{11}$$

(a) $g(x) = \exp(j\pi\beta x^2)$

$$\begin{aligned}W(f,x) &= \int_{-\infty}^{\infty} \exp\left[j\pi\beta\left(x+\frac{\xi}{2}\right)^2\right]\exp\left[-j\pi\beta\left(x-\frac{\xi}{2}\right)^2\right]\exp(-j2\pi f\xi)d\xi \\ &= \int_{-\infty}^{\infty} \exp\left[j\pi\beta\left[\left(x+\frac{\xi}{2}\right)^2-\left(x-\frac{\xi}{2}\right)^2\right]\right]\exp(-j2\pi f\xi)d\xi\end{aligned}$$

$$\begin{aligned}\left(x+\frac{\xi}{2}\right)^2-\left(x-\frac{\xi}{2}\right)^2 =& x^2+x\xi+\frac{\xi^2}{4}-x^2+x\xi-\frac{\xi^2}{4} \\ =& 2x\xi\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}W(f,x) =& \int_{-\infty}^{\infty} \exp\left[j\pi\beta\left[\left(x+\frac{\xi}{2}\right)^2-\left(x-\frac{\xi}{2}\right)^2\right]\right]\exp(-j2\pi f\xi)d\xi \\ =& \int_{-\infty}^{\infty} \exp\left[j2\pi\beta x\xi\right]\exp(-j2\pi f\xi)d\xi \\ =& \int_{-\infty}^{\infty} \exp\left[j2\pi(\beta x - f)\xi\right]d\xi \\ =& \delta(f-\beta x)\end{aligned}$$

$\therefore$

$$W(f,x) = \delta(f-\beta x)$$

(b) $g(x) = \exp(j\pi\beta x^2)\mathrm{rect}\left(\frac{x}{2L}\right)$

$$\begin{aligned}W(f,x) &= \int_{-\infty}^{\infty} \exp\left[j\pi\beta\left(x+\frac{\xi}{2}\right)^2\right]\exp\left[-j\pi\beta\left(x-\frac{\xi}{2}\right)^2\right]\mathrm{rect}\left(\frac{x+\frac{\xi}{2}}{2L}\right)\mathrm{rect}\left(\frac{x-\frac{\xi}{2}}{2L}\right)\exp(-j2\pi f\xi)d\xi \\ &= \int_{-\infty}^{\infty} \exp\left[j2\pi\beta x\xi\right]\mathrm{rect}\left(\frac{x}{2L}+\frac{\xi}{4L}\right)\mathrm{rect}\left(\frac{x}{2L}-\frac{\xi}{4L}\right)\exp(-j2\pi f\xi)d\xi\end{aligned}$$

By the definition of rectangular function,

$$\Pi(x) = \begin{cases} 0 \text{ if} & |x| > \frac{1}{2} \\ \frac{1}{2} \text{ if} & |x| = \frac{1}{2} \\ 1 \text{ if} & |x| < \frac{1}{2}\end{cases} \tag{12}$$

By analysing it, we will get

$$\mathrm{rect}\left(\frac{x}{2L}+\frac{\xi}{4L}\right)\mathrm{rect}\left(\frac{x}{2L}-\frac{\xi}{4L}\right) = \mathrm{rect}\left(\frac{\xi}{4(L-|x|)}\right) \tag{13}$$

Then, we will get,

$$
\begin{aligned}
W(f,x) =& \int_{-\infty}^{\infty} \exp\left[j2\pi\beta x\xi\right] \mathrm{rect}\left(\frac{x}{2L}+\frac{\xi}{4L}\right) \mathrm{rect}\left(\frac{x}{2L}-\frac{\xi}{4L}\right) \exp(-j2\pi f\xi)d\xi \\
=& \int_{-\infty}^{\infty} \exp\left[j2\pi(\beta x - f)\xi\right] \mathrm{rect}\left(\frac{\xi}{4(L-|x|)}\right) d\xi \\
=& [4(L-|x|)]\mathrm{sinc}[4(L-|x|)(\beta x - f)]
\end{aligned}
$$

(c) The plots are given in figures 1 and 2.

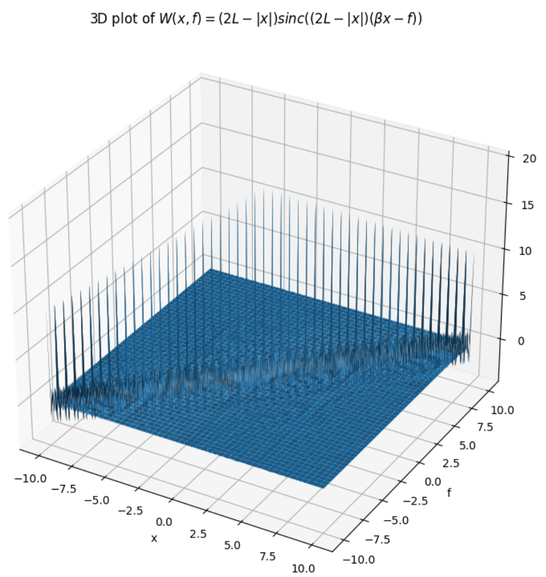

Figure 1: 3D plot of $W(f,x) = [4(L-|x|)]\mathrm{sinc}[4(L-|x|)(\beta x - f)]$

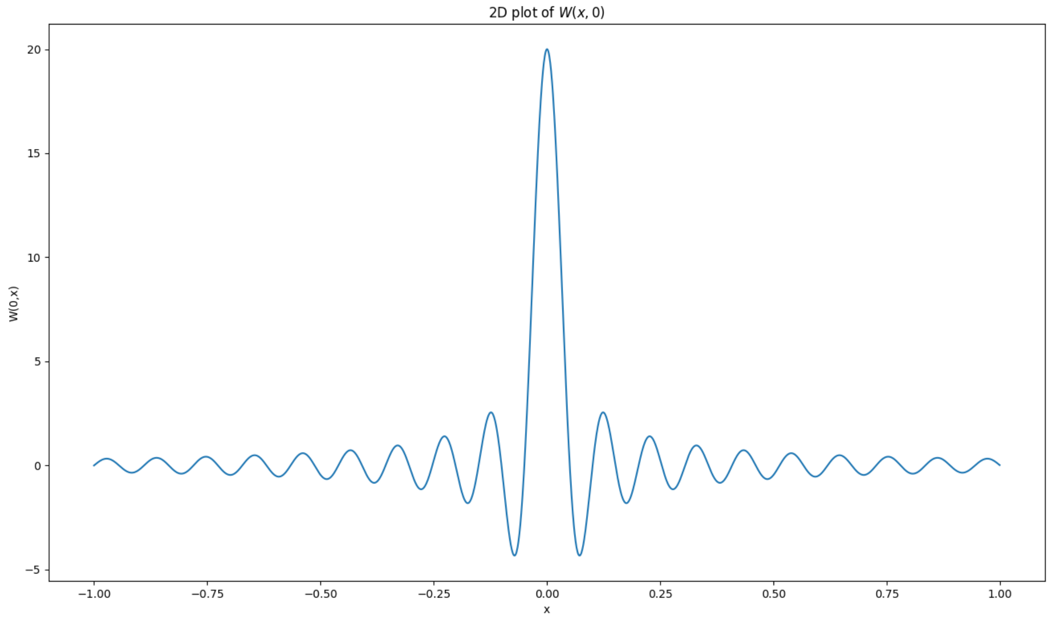

Figure 2: 2D plot of $W(0,x) = [4(L - |x|)]\text{sinc}[4(L - |x|)(\beta x)]$

Problem 4

The grating is modeled as a transmitting structure with amplitude transmittance:

$$t_A(\xi,\eta) = \frac{1}{2}\left[1 + m\cos\left(2\pi\frac{\xi}{L}\right)\right]$$

For more simplicity, we can assume that the grating structure is bounded by a square aperture of width $2w$. m representa the peak-to-peak change of amplitude transmittance across the screen and $f_0 = \frac{1}{L}$ is the spatial frequency of the grating. Using these, we can modify t_A.

$$t_A(\xi,\eta) = \frac{1}{2}\left[1 + m\cos\left(2\pi f_0 \xi\right)\right]\text{rect}\left(\frac{\xi}{2w}\right)\text{rect}\left(\frac{\eta}{2w}\right) \tag{14}$$

Let us say that the screen s normally illuminated by a unit-amplitude plane wave. The field distribution across the aperture is equal simply to t_A. To find the Fraunhofer diffraction pattern, we first take the transform of t_A.

$$\mathcal{F}\left[\frac{1}{2}\left[1 + m\cos\left(2\pi f_0\xi\right)\right]\right] = \frac{1}{2}\delta(f_X, f_Y) + \frac{m}{4}\delta(f_X + f_0, f_Y) + \frac{m}{4}\delta(f_X - f_0, f_Y)$$

$$\mathcal{F}\left[\text{rect}\left(\frac{\xi}{2w}\right)\text{rect}\left(\frac{\eta}{2w}\right)\right] = A\text{sinc}(2wf_X)\text{sinc}(2wf_Y)$$

A is the area of the aperture bounding the grating. Using the convolution theorem, we can write that, The FT of $U(\xi,\eta)$ is the product of equations that we got above.

$$\begin{aligned}\mathcal{F}\{U(\xi,\eta)\} =& A\text{sinc}(2wf_X)\text{sinc}(2wf_Y)\left[\frac{1}{2}\delta(f_X, f_Y) + \frac{m}{4}\delta(f_X + f_0, f_Y) + \frac{m}{4}\delta(f_X - f_0, f_Y)\right] \\ =& \frac{A}{2}\text{sinc}(2wf_Y)\left[\text{sinc}(2wf_X) + \frac{m}{2}\text{sinc}(2w(f_X + f_0)) + \frac{m}{2}\text{sinc}(2w(f_X - f_0))\right]\end{aligned} \tag{15}$$

Now, using the formula of diffraction pattern,

$$U(x,y) = \frac{e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{j\lambda z} \mathcal{F}\{U(\xi,\eta)\} \tag{16}$$

the amplitude distribution of our diffraction will be,

$$U(x,y) = \frac{A e^{jkz} e^{j\frac{k}{2z}(x^2+y^2)}}{2j\lambda z} \text{sinc}(2w\frac{y}{\lambda z}) \left[\text{sinc}(2w\frac{x}{\lambda z}) + \frac{m}{2}\text{sinc}(\frac{2w}{\lambda z}(x+f_0\lambda z)) + \frac{m}{2}\text{sinc}(\frac{2w}{\lambda z}(x-f_0\lambda z)) \right] \tag{17}$$

The intensity distribution will be the square of 17.

$$I(x,y) = \frac{A^2}{4\lambda^2 z^2} \text{sinc}^2(2w\frac{y}{\lambda z}) \left[\text{sinc}(2w\frac{x}{\lambda z}) + \frac{m}{2}\text{sinc}(\frac{2w}{\lambda z}(x+f_0\lambda z)) + \frac{m}{2}\text{sinc}(\frac{2w}{\lambda z}(x-f_0\lambda z)) \right]^2 \tag{18}$$

But if there are many grating periods within the aperture, then $f_0 >> \frac{1}{w}$. So, the cross term of sinc functions will be negligible. Therefore,

$$I(x,y) = \frac{A^2}{4\lambda^2 z^2} \text{sinc}^2(2w\frac{y}{\lambda z}) \left[\text{sinc}^2(2w\frac{x}{\lambda z}) + \frac{m}{2}\text{sinc}^2(\frac{2w}{\lambda z}(x+f_0\lambda z)) + \frac{m}{2}\text{sinc}^2(\frac{2w}{\lambda z}(x-f_0\lambda z)) \right] \tag{19}$$

The width of each order will be $\frac{\lambda z}{w}$. For, $x = 0$, it will be the central maximum. The one coming after that will be first order.

Problem 5

We have the equation:

$$u_-(P,t) = \int_{-\infty}^{0} U(P,\nu)\exp(j2\pi\nu t)d\nu \tag{20}$$

U(P,ν) is the Fourier spectrum of $u(P,t)$.

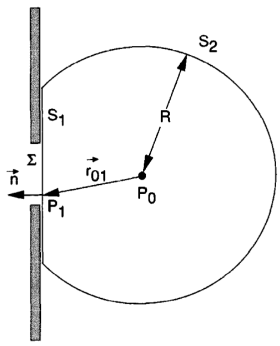

Figure 3: The surface

The figure is as showin in 3.
We have the equation:

$$u_-(P_0,t) = \iint_\Sigma \frac{\cos(\vec{n},\vec{r}_{01})}{2\pi v r_{01}} \int_{-\infty}^{\infty} -j2\pi\nu' U(P_1,-\nu')\exp\left[-j2\pi\nu'\left(t-\frac{r_{01}}{v}\right)\right] d\nu' ds \tag{21}$$

Here, the central frequency is $\bar{\nu}$, and the bandwidth $\Delta\nu$. So, the integral in the frequency is non-vanishing only in the range $\left(\bar{\nu}-\frac{\Delta\nu}{2}, \bar{\nu}+\frac{\Delta\nu}{2}\right)$. Given that $\Delta\nu << \bar{\nu}$. So, the first ν' will varie a small amount, since it is just linear in nature. We can replace that with $\bar{\nu}$. Also $\frac{1}{\Delta\nu} >> \frac{nr_{01}}{v}$. Then we can replace ν' in the exponential with the term $\frac{\nu' r_{01}}{v}$, by $\bar{nu}$. We don't know the ν' in other terms varie or that will make a huge difference in the result. Then, 21 will be,

$$\begin{aligned}
u_-(P_0,t) &= \iint_\Sigma \frac{\cos(\vec{n},\vec{r}_{01})}{2\pi v r_{01}} \int_{-\infty}^{\infty} -j2\pi\bar{\nu} U(P_1,-\nu')\exp\left[-j2\pi\nu' t\right]\exp\left[j2\pi\bar{\nu}\left(\frac{r_{01}}{v}\right)\right] d\nu' ds \\
&= -j2\pi\bar{\nu} \iint_\Sigma \exp\left[j2\pi\bar{\nu}\left(\frac{r_{01}}{v}\right)\right] \frac{\cos(\vec{n},\vec{r}_{01})}{2\pi v r_{01}} \int_{-\infty}^{\infty} U(P_1,-\nu')\exp\left[-j2\pi\nu' t\right] d\nu' ds \\
&= \frac{1}{j\bar{\lambda}} \iint_\Sigma \exp\left[j2\pi\bar{\nu}\left(\frac{r_{01}}{v}\right)\right] \frac{\cos(\vec{n},\vec{r}_{01})}{r_{01}} \int_{-\infty}^{\infty} U(P_1,-\nu')\exp\left[-j2\pi\nu' t\right] d\nu' ds \\
&= \frac{1}{j\bar{\lambda}} \iint_\Sigma \exp\left[j2\pi\bar{\nu}\left(\frac{r_{01}}{v}\right)\right] \frac{\cos(\vec{n},\vec{r}_{01})}{r_{01}} u_-(P_1,t) ds \qquad \text{(Using equation 3-55 in Goodman).}
\end{aligned}$$

But, outside sigma, we can say that $u_-(P_1,t)$ will be zero. Then, we can make the integral upto infinity. So, we will get the equation,

$$v_-(P_0,t) = \frac{1}{j\bar{\lambda}} \iint_{-\infty}^{\infty} \exp\left(j\bar{k}r_{01}\right) \frac{\cos(\vec{n},\vec{r}_{01})}{r_{01}} u_-(P_1,t) ds \tag{22}$$

Problem 6

The periodic triangular wave is given by

$$y = |x| \quad (-\pi < x \leq \pi)$$

A single wave can be written in the form,

$$y = \begin{cases} x \text{ if } 0 \leq x < \pi \\ -x \text{ if } -\pi \leq x < 0 \end{cases} \tag{23}$$

The period of wave is 2π.
We can write a periodic wave in the form,

$$g(t) = \sum_{n=-\infty}^{\infty} C_n e^{i2\pi f_0 t} \tag{24}$$

Where C_n is the Fourier coefficients and defined by the formula:

$$C_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} g(t) e^{-i2\pi n f_0 t} dt \tag{25}$$

$f_0 = \frac{1}{T}$
Here, $T = 2\pi$. $\therefore$

$$\begin{aligned}
C_n =& \frac{1}{2\pi}\left[\int_{-\pi}^{0} (-x) e^{-inx} dx + \int_{0}^{\pi} e^{-inx} dx\right] \\
=& \frac{-1 + e^{-i\pi n}(1 + i\pi n)}{n^2} + \frac{-1 + e^{i\pi n}(1 - i\pi n)}{n^2} \\
=& \frac{-1 + e^{-i\pi n}(1 + i\pi n) - 1 + e^{i\pi n}(1 - i\pi n)}{n^2} \\
=& \frac{-2 + e^{-i\pi n} + i\pi n e^{-i\pi n} + e^{i\pi n} - i\pi n e^{i\pi n}}{n^2} \\
=& \frac{-2 + 2\cos(n\pi) - 2\pi n \sin(n\pi)}{n^2} \\
=& 2\frac{(-1)^n - 1}{n^2}
\end{aligned}$$

Then, the Fourier transform of g(x) is,

$$\begin{aligned}
G(f) =& \mathcal{F}\{g(x)\} \\
=& \sum_{n=-\infty}^{\infty} 2\frac{(-1)^n - 1}{n^2} \int_{-\infty}^{\infty} e^{i2\pi(f - nf_0)x} dx \\
=& \sum_{n=-\infty}^{\infty} 2\frac{(-1)^n - 1}{n^2} \delta(f - nf_0)
\end{aligned}$$

So, the fourier transform of the given function can be written as:

$$G(f) = \sum_{n=-\infty}^{\infty} 2\frac{(-1)^n - 1}{n^2} \delta\left(f - \frac{n}{2\pi}\right)$$

The signal and it's fourier transform is shown in figure 4.

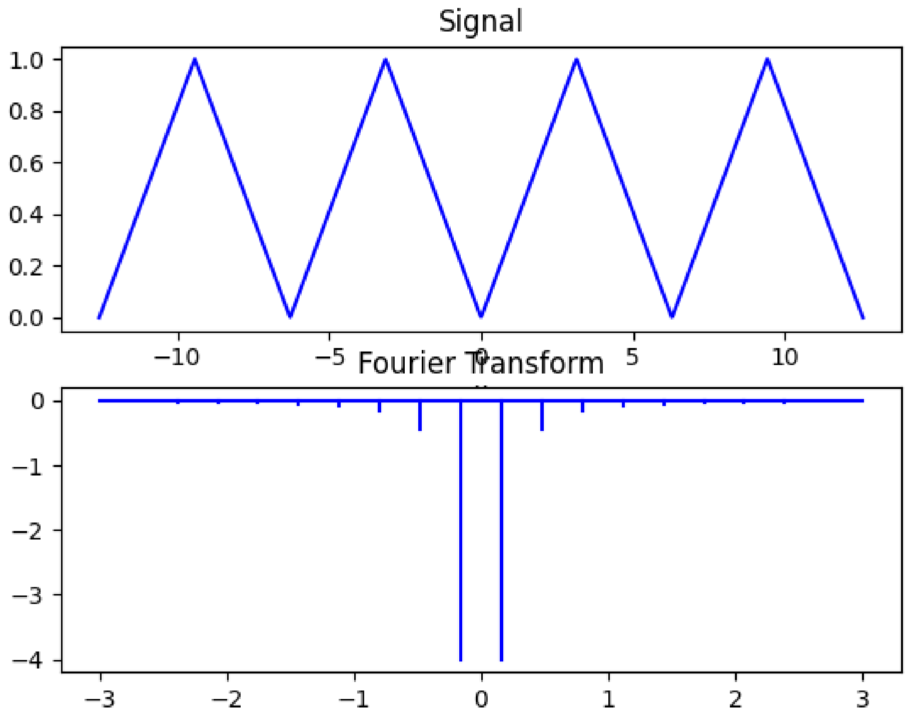

Figure 4: The signal and it's FT

Problem 7

Given function is a periodic array of δ-function which every fifth member is missing. The function as showin in figure 5

This can be considered as a wave train of group of 4 dirac delta functions. So, one complete wave can be written as:

$$\Delta(x) = \delta(x+3) + \delta(x+1) + \delta(x-1) + \delta(x-3) \tag{26}$$

The given function is a convolution of $g(x)$ with the function

$$h(x) = \sum_{n=-\infty}^{\infty} \delta(x-5n) \tag{27}$$

Now, let us write the given equation in the form of equation 24. Here, $T = 10$.
Then,

$$\begin{aligned}
C_n =& \frac{1}{10}\int_{-5}^{5} dx e^{-\frac{2\pi n x}{10}} \left[\delta(x+3) + \delta(x+1) + \delta(x-1) + \delta(x-3)\right] \\
=& \frac{1}{10}\left[e^{\frac{i6\pi n}{10}} + e^{\frac{-i6\pi n}{10}} + e^{\frac{i2\pi n}{10}} + e^{\frac{-i2\pi n}{10}}\right] \\
=& \frac{1}{5}\left[\cos\left(\frac{2\pi n}{5}\right) + \cos\left(\frac{\pi n}{5}\right)\right] \\
=& \frac{2}{5}\cos\left(\frac{3\pi n}{10}\right)\cos\left(\frac{\pi n}{10}\right)
\end{aligned}$$

$\therefore$

$$
\begin{aligned}
G(f) =& \mathcal{F}\{g(x)\} \\
=& \sum_{n=-\infty}^{\infty} \frac{2}{5} \cos\left(\frac{3\pi n}{10}\right) \cos\left(\frac{\pi n}{10}\right) \int_{-\infty}^{\infty} e^{i2\pi(f-nf_0)x} dx \\
=& \sum_{n=-\infty}^{\infty} \frac{2}{5} \cos\left(\frac{3\pi n}{10}\right) \cos\left(\frac{\pi n}{10}\right) \delta\left(f - \frac{n}{10}\right)
\end{aligned}
$$

So, the Fourier transform is:

$$
G(f) = \sum_{n=-\infty}^{\infty} \frac{2}{5} \cos\left(\frac{3\pi n}{10}\right) \cos\left(\frac{\pi n}{10}\right) \delta\left(f - \frac{n}{10}\right)
$$

The plot of signal and it's transform is shown in figure 5.

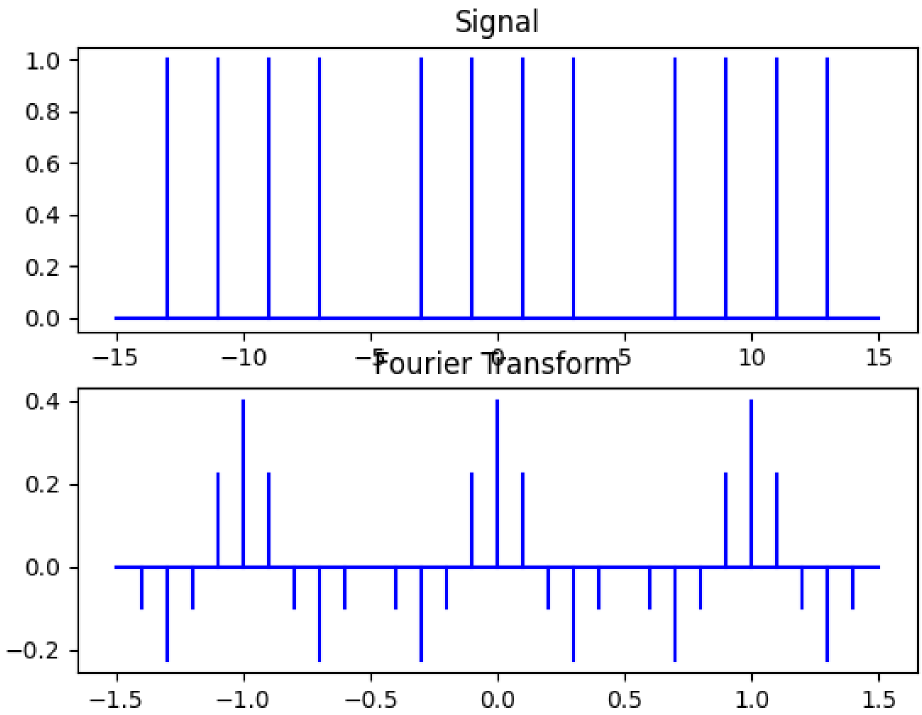

Figure 5: Signal and Frequency Spectrum

Problem 8

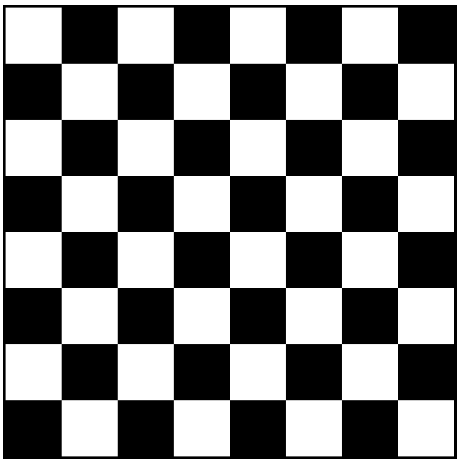

Figure 6: The Chessboard

The chess board looks as shown in figure 6.This can be represented as a convolution of a 2d square function and an array of delta functions. Let us take the center as the origin. Side of one square can be taken as 2 unit. So, the array of dirac delta functions can be written as the following.

$$\begin{aligned}\Delta(x,y) = &[\delta(x-3)+\delta(x-7)+\delta(x+1)+\delta(x+5)]\,[\delta(y-1)+\delta(y-5)+\delta(y+3)+\delta(y+7)]\\ &+[\delta(x-1)+\delta(x-5)+\delta(x+3)+\delta(x+7)]\,[\delta(y-3)+\delta(y-7)+\delta(y+1)+\delta(y+5)]\end{aligned}$$

First, let us find the fourier transform of this.

$$\begin{aligned}&\mathcal{F}\{\Delta(x,y)\} =\\ &\int_{-\infty}^{\infty} dx e^{-i2\pi f_X x}\,[\delta(x-3)+\delta(x-7)+\delta(x+1)+\delta(x+5)]\\ &\int_{-\infty}^{\infty} dy e^{-i2\pi f_Y y}\,[\delta(y-1)+\delta(y-5)+\delta(y+3)+\delta(y+7)]\\ &+\int_{-\infty}^{\infty} dx e^{-i2\pi f_X x}\,[\delta(x-1)+\delta(x-5)+\delta(x+3)+\delta(x+7)]\\ &\int_{-\infty}^{\infty} dy e^{-i2\pi f_Y y}\,[\delta(y-3)+\delta(y-7)+\delta(y+1)+\delta(y+5)]\\ &=[\exp(-i6\pi f_X)+\exp(-i14\pi f_X)+\exp(i2\pi f_X)+\exp(i10\pi f_X)]\\ &\quad[\exp(-i2\pi f_Y)+\exp(-i10\pi f_Y)+\exp(i6\pi f_Y)+\exp(i14\pi f_Y)]\\ &+[\exp(-i2\pi f_X)+\exp(-i10\pi f_X)+\exp(i6\pi f_X)+\exp(i14\pi f_X)]\\ &\quad[\exp(-i6\pi f_Y)+\exp(-i14\pi f_Y)+\exp(i2\pi f_Y)+\exp(i10\pi f_Y)]\end{aligned}$$

Then, the Fourier transform of the array can be written as:

$$\begin{aligned}\mathcal{F}\{\Delta(x,y)\} = &[\exp(-i6\pi f_X)+\exp(-i14\pi f_X)+\exp(i2\pi f_X)+\exp(i10\pi f_X)]\\ &[\exp(-i2\pi f_Y)+\exp(-i10\pi f_Y)+\exp(i6\pi f_Y)+\exp(i14\pi f_Y)]\\ &+[\exp(-i2\pi f_X)+\exp(-i10\pi f_X)+\exp(i6\pi f_X)+\exp(i14\pi f_X)]\\ &[\exp(-i6\pi f_Y)+\exp(-i14\pi f_Y)+\exp(i2\pi f_Y)+\exp(i10\pi f_Y)]\end{aligned} \tag{28}$$

Here, we can define our rectangular function as,

$$rect(x) = \begin{cases} 0 \text{ if } |x| > 1 \\ \frac{1}{2} \text{ if } |x| = 1 \\ 1 \text{ if } |x| < 1 \end{cases} \tag{29}$$

We can use the same definition for y also. Then,

$$\begin{aligned} \mathcal{F}\{rect(x)\} &= \int_{-1}^{1} dx e^{-i2\pi f_X x} \\ &= \frac{e^{-2f_X i\pi} - e^{2f_X i\pi}}{-2f_X i\pi} \\ &= \frac{\sin(2\pi f_X)}{2\pi f_X} \\ \mathcal{F}\{rect(y)\} &= \int_{-1}^{1} dy e^{-i2\pi f_Y y} \\ &= \frac{\sin(2\pi f_Y)}{2\pi f_Y} \end{aligned} \tag{30}$$

Then,

$$\mathcal{F}\{rect(x, y)\} = \text{sinc}(2\pi f_X)\text{sinc}(2\pi f_Y) \tag{31}$$

Since the chess board is the convolution of the defined array of dirac delta functions and a 2d rectangular function, using the Convolution theorem, the resultant Fourier transform will be the product of each. So, the resultant FT will be the product of 28 and 31. (Since the FT of dirac delta array is long, I am not copying it here).

$\therefore$

$$G(f_X, f_Y) = \mathcal{F}\{\Delta(x, y)\} \times \text{sinc}(2\pi f_X)\text{sinc}(2\pi f_Y) \tag{32}$$

Another Method

(This is more convincing for me)

Rotate the axis by $\frac{\pi}{4}$ in counterclockwise direction. Then, we can write

$$\begin{aligned} X &= x - y \\ Y &= x + y \end{aligned} \tag{33}$$

We will get a $\frac{1}{\sqrt{2}}$ from the rotation matrix. But we have to scale the coordinate by a factor of $\sqrt{2}$, to combensate the length. Then we will get this set of equations.

From 7, we can write the array of delta function.

$$\begin{aligned} \Delta(x, y) = & \left[\delta(X + 1) + \delta(X + 3) + \delta(X + 5) + \delta(X + 7) + \delta(X - 1) + \delta(X - 3) + \delta(X - 5) + \delta(X - 7)\right] \delta(Y) \\ & + \left[\delta(X + 1) + \delta(X + 3) + \delta(X + 5) + \delta(X - 1) + \delta(X - 3) + \delta(X - 5)\right] \left[\delta(Y - 2) + \delta(Y + 2)\right] \\ & + \left[\delta(X + 1) + \delta(X + 3) + \delta(X - 1) + \delta(X - 3) +\right] \left[\delta(Y - 2) + \delta(Y + 2) + \delta(Y - 4) + \delta(Y + 4)\right] \\ & + \left[\delta(X + 1) + \delta(X - 1) +\right] \left[\delta(Y - 2) + \delta(Y + 2) + \delta(Y - 4) + \delta(Y + 4) + \delta(Y - 6) + \delta(Y + 6)\right] \end{aligned} \tag{34}$$

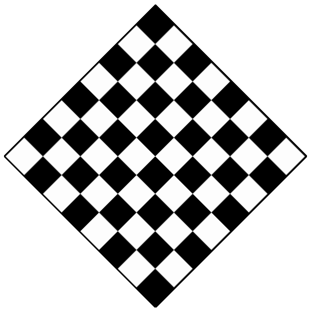

Figure 7: The Chessboard-Rotated

The Fourier Transform of this is,

$$\begin{aligned}\mathcal{F}\{\Delta(x,y)\} =& 2\left[\cos 2\pi F_X + \cos 6\pi F_X + \cos 10\pi F_X + \cos 14\pi F_X\right] \\ &+ 4\left[\cos 2\pi F_X + \cos 6\pi F_X + \cos 10\pi F_X\right]\cos 4\pi F_Y \\ &+ 4\left[\cos 2\pi F_X + \cos 6\pi F_X\right]\left[\cos 4\pi F_Y + \cos 8\pi F_Y\right] \\ &+ 4\left[\cos 2\pi F_X\right]\left[\cos 4\pi F_Y + \cos 8\pi F_Y + \cos 12\pi F_Y\right]\end{aligned} \tag{35}$$

We did a coordinate transformation to do this. So we need to do that in the Fourier space also.

$$\begin{aligned}G(f_X, f_Y) =& \iint_{-\infty}^{\infty} dxdy f(x,y) e^{-2i\pi(f_X x + f_Y y)} \\ =& \frac{1}{4}\iint_{-\infty}^{\infty} dXdY f\left(\left(\frac{X+Y}{2}\right), \left(\frac{X-Y}{2}\right)\right) e^{-2i\pi\left(f_X\left(\frac{X+Y}{2}\right) + f_Y\left(\frac{X-Y}{2}\right)\right)}\end{aligned} \tag{36}$$

From this, we can say that,

$$\begin{aligned}F_X =& \frac{f_X + f_Y}{2} \\ F_Y =& \frac{f_X - f_Y}{2}\end{aligned} \tag{37}$$

Apply these to 35,

$$\begin{aligned}\mathcal{F}\{\Delta(x,y)\} =& 8\left[\cos\pi\left(f_X+f_Y\right) + \cos 3\pi\left(f_X+f_Y\right) + \cos 5\pi\left(f_X+f_Y\right) + \cos 7\pi\left(f_X+f_Y\right)\right] \\ &+ 16\left[\cos\pi\left(f_X+f_Y\right) + \cos 3\pi\left(f_X+f_Y\right) + \cos 5\pi\left(f_X+f_Y\right)\right]\cos 2\pi\left(f_X-f_Y\right) \\ &+ 16\left[\cos\pi\left(f_X+f_Y\right) + \cos 3\pi\left(f_X+f_Y\right)\right]\left[\cos 2\pi\left(f_X-f_Y\right) + \cos 4\pi\left(f_X-f_Y\right)\right] \\ &+ 16\left[\cos\pi\left(f_X+f_Y\right)\right]\left[\cos 2\pi\left(f_X-f_Y\right) + \cos 4\pi\left(f_X-f_Y\right) + \cos 6\pi\left(f_X-f_Y\right)\right]\end{aligned} \tag{38}$$

For the squre, we can use the result 31.
The final solution will be product of both.

$$\mathcal{F}\{g(x,y)\} = \mathcal{F}\{\Delta(x,y)\} \times \mathrm{sinc}(2\pi f_X)\mathrm{sinc}(2\pi f_Y) \tag{39}$$

The plot of this is shown in figure 8.

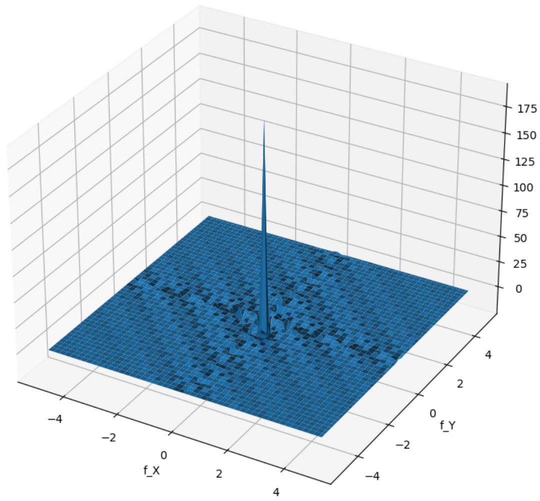

Figure 8: 3D plot of the Fourier Transform

Problem 9

The visibility ν is defined as:

$$\nu = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \tag{40}$$

And the degree of coherence can be written as:

$$\gamma(\vec{r_1}, \vec{r_2}, \tau) = \frac{< f(t) f^*(t+\tau) >}{(I_1 I_2)^{\frac{1}{2}}} \tag{41}$$

where $\tau = t_2 - t_1$, t_1 and t_2 are the time of arrival of beams from slit to the screen and $I_1 = \langle f(r_{1,t}) f^*(r_{1,t}) \rangle$

For a double slit setup, the total intensity on the screen can be written as

$$\begin{aligned} I =& I_1 + I_2 + (< f_1(t) f_2^*(t+\tau) >) + (< f_1^*(t) f_2(t+\tau) >) \\ =& I_1 + I_2 + 2\mathcal{K}(< f_1(t) f_2^*(t+\tau) >) \end{aligned} \tag{42}$$

$\mathcal{K}$ represents a phase factor between f_1 and f_2. In I_{max}, $\mathcal{K}$ is +1 and I_{min}, $\mathcal{K}$ is -1.

$\therefore$

$$I_{max} = I_1 + I_2 + 2(< f_1(t) f_2^*(t+\tau) >)$$
$$I_{min} = I_1 + I_2 - 2(< f_1(t) f_2^*(t+\tau) >)$$

Then, we can write 40 as:

$$\nu = \frac{2(< f_1(t) f_2^*(t+\tau) >)}{I_1 + I_2} \tag{43}$$

From 41, we can now write,

$$\nu = \frac{2(I_1 I_2)^{\frac{1}{2}} |\gamma(\vec{r_1}, \vec{r_2}, \tau)|}{I_1 + I_2} \tag{44}$$

This is the equation that we get which connects degree of coherence and visibility of fringes while the intensity is different. If the intensities are same, $I_1 = I_2$. Then, $I_1+I_2=2I_1$, $I_{1I2}=I_1^2$.

$\therefore$,

$$\nu = |\gamma(\vec{r_1}, \vec{r_2}, \tau)|$$

So, our result is consistant with the special case which is given in equestion.

Ref: `https://arxiv.org/pdf/1905.00917.pdf`.

⋆ ⋆ ⋆