CQO Assignment 1. Deadline: 30th October 11:59 pm

(1) Prove the following Fourier transform theorems:

(a) $\mathcal{F}\mathcal{F}\{g(x,y)\} = \mathcal{F}^{-1}\mathcal{F}^{-1}\{g(x,y)\} = g(-x,-y)$ at all points of continuity of g.

(b) $\mathcal{F}\{g(x,y)\,h(x,y)\} = \mathcal{F}\{g(x,y)\} \otimes \mathcal{F}\{h(x,y)\}$.

(c) $\mathcal{F}\{\nabla^2 g(x,y)\} = -4\pi^2(f_X^2 + f_Y^2)\mathcal{F}\{g(x,y)\}$ where ∇^2 is the Laplacian operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

(2) Prove the following Fourier-Bessel transform relations:

(a) If $g_R(r) = \delta(r - r_0)$, then

$$\mathcal{B}\{g_R(r)\} = 2\pi r_0 J_0(2\pi r_0 \rho).$$

(b) If $g_R(r) = 1$ for $a \le r \le 1$ and zero otherwise, then

$$\mathcal{B}\{g_R(r)\} = \frac{J_1(2\pi\rho) - a J_1(2\pi a\rho)}{\rho}.$$

(c) If $\mathcal{B}\{g_R(r)\} = G(\rho)$, then

$$\mathcal{B}\{g_R(ar)\} = \frac{1}{a^2} G\left(\frac{\rho}{a}\right).$$

(d) $\mathcal{B}\{\exp(-\pi r^2)\} = \exp(-\pi\rho^2)$.

(3) The *Wigner distribution function* of a one-dimensional function $g(x)$ is defined by

$$W(f,x) = \int_{-\infty}^{\infty} g(x + \xi/2)\, g^*(x - \xi/2) \exp(-j2\pi f\xi)\, d\xi$$

and is a description of the simultaneous (one-dimensional) space and spatial-frequency occupancy of a signal.

(a) Find the Wigner distribution function of the infinite-length chirp function by inserting $g(x) = \exp(j\pi\beta x^2)$ in the definition of $W(f,x)$.

(b) Show that the Wigner distribution function for the one-dimensional finite chirp

$$g(x) = \exp(j\pi\beta x^2)\,\mathrm{rect}\left(\frac{x}{2L}\right)$$

is given by

$$W(f,x) = (2L - |x|)\,\mathrm{sinc}\,[(2L - |x|)(\beta x - f)]$$

for $|x| < 2L$ and zero otherwise.

(c) plot the Wigner distribution function of the finite-length chirp for $L = 10$ and $\beta = 1$, with x ranging from -10 to 10 and f ranging from -10 to 10. To make the nature of this function clearer, also plot $W(0, x)$ for $|x| \le 1$.

(4) Calculate the Fraunhofer diffraction pattern of the amplitude grating we discussed in class for Fresnel diffraction.

(5) Consider a real nonmonochromatic disturbance $u(P, t)$ of center frequency $\bar{\nu}$ and bandwidth $\Delta\nu$. Let a related complex-valued disturbance $u_{-}(P, t)$ be defined as consisting of only the negative-frequency components of $u(P, t)$. Thus

$$u_{-}(P, t) = \int_{-\infty}^{0} U(P, \nu) \exp(j2\pi\nu t)\, d\nu$$

where $U(P, \nu)$ is the Fourier spectrum of $u(P, t)$. Assuming the geometry of Fig. 3.6 show that if

$$\frac{\Delta\nu}{\bar{\nu}} \ll 1 \qquad \text{and} \qquad \frac{1}{\Delta\nu} \gg \frac{nr_{01}}{v}$$

then

$$u_{-}(P_0, t) = \frac{1}{j\bar{\lambda}} \iint_{-\infty}^{\infty} u_{-}(P_1, t) \frac{\exp(j\bar{k}r_{01})}{r_{01}} \cos(\vec{n}, \vec{r}_{01})\, ds$$

where $\bar{\lambda} = v/\bar{\nu}$ and $\bar{k} = 2\pi/\bar{\lambda}$. In the above equations, n is the refractive index of the medium and v is the velocity of propagation.

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(6) Derive the Fourier transform of a periodic triangular wave, defined in one period as $y = |x|$ $(-\pi < x \leq \pi)$.

(7) A periodic array of δ-function has every fifth member missing. What is its Fourier transform?

(8) What is the Fraunhofer diffraction pattern of a mask in the form of a chessboard with opaque and transparent squares?

(9) When two light waves of equal intensity interfere, the visibility of the interference fringes is equal to the degree of coherence Derive an equivalent relationship when the two interfering waves have differing intensities I_1 and I_2.