(4.1) In the Einstein analysis we assume that the light radiation has a broad spectrum compared to the transition line. Let us now consider the contrary situation in which the spectral width of the light beam is much smaller than the linewidth of the transition. This is the kind of situation that occurs when a narrow-band laser beam interacts with an atom, either inside a laser cavity or externally.

(a) Explain why it is appropriate to write the spectral energy intensity of the beam as:

$$u(\omega') = u_\omega \delta(\omega' - \omega),$$

where ω is the angular frequency of the beam, u_ω is its energy density in $\mathrm{J\,m^{-3}}$, and $\delta(x)$ is the Dirac delta function.

(b) Let us assume that the frequency dependence of the absorption probability follows the spectral lineshape function $g_\omega(\omega)$. This implies that the Einstein B coefficients will also vary with frequency. Explain why it is appropriate to write the frequency dependence of the Einstein B_{12} coefficient as:[1]

$$B_{12}(\omega') = \frac{g_2}{g_1} \frac{\pi^2 c^3}{\hbar n^3 \omega'^3} \frac{1}{\tau} g_\omega(\omega'),$$

where g_1 and g_2 are the lower and upper level degeneracies, n is the refractive index of the medium, and τ is the radiative lifetime of the upper level.

(c) Hence show that the total absorption rate defined as

$$W_{12} = N_1 \int_0^\infty B_{12}(\omega') u(\omega') \,\mathrm{d}\omega'$$

is given by:

$$W_{12} = N_1 \frac{g_2}{g_1} \frac{\pi^2 c^3}{\hbar n^3 \omega^3 \tau} u_\omega g_\omega(\omega).$$

(d) Repeat the argument to show that the total stimulated-emission rate is given by:

$$W_{21} = N_2 \frac{\pi^2 c^3}{\hbar n^3 \omega^3 \tau} u_\omega g_\omega(\omega).$$

(4.11) Consider an atom interacting with a monochromatic beam of light with angular frequency ω, where ω is close to a transition frequency ω_0.

(a) Use the results of Exercise (4.1) to show that the net rate of downward transitions (defined as the stimulated-emission rate less the absorption rate) is given by:

$$W_{21}^{\mathrm{net}} = \frac{\pi^2 c^3}{\hbar n^3 \omega^3 \tau} u_\omega g_\omega(\omega) \Delta N,$$

where $\Delta N = N_2 - (g_2/g_1)N_1$, u_ω is the energy per unit volume of the beam, and $g_\omega(\omega)$ is the spectral lineshape function.

(b) Show that optical intensity is given by $I = u_\omega c/n$.

(c) Consider a unit area of beam propagating in the $+z$-direction through the medium. Show that the incremental increase in the intensity $\mathrm{d}I$ in a length element $\mathrm{d}z$ is given by:

$$\mathrm{d}I = W_{21}^{\mathrm{net}} \hbar\omega \,\mathrm{d}z.$$

(d) Hence show that the gain coefficient is given by eqn 4.43.

$$\gamma(\omega) = \frac{\lambda^2}{4n^2\tau} \Delta N g_\omega(\omega), \qquad (4.43)$$

(4.12) Calculate the fraction of the energy of a 00-mode laser beam with beam radius w within a distance w from the beam centre.

(4.13) A helium–neon laser consists of a laser tube of length 0.3 m with mirrors bonded to the end of the tube. The output coupler has a reflectivity of 99%. The laser operates on the 632.8 nm transition of neon (relative atomic mass 20.18), which has an Einstein A coefficient of $3.4 \times 10^6\ \mathrm{s}^{-1}$. The tube runs at 200 °C and the laser transition is Doppler-broadened. On the assumption that the only loss in the cavity is through the output coupler, that the average refractive index is equal to unity, and that the laser operates at the line centre, calculate:

(a) the gain coefficient in the laser tube;

(b) the population inversion density.

11.1 Estimate the Doppler and collision line widths of emission from H_2O molecules at $\lambda = 0.5\,\mu\mathrm{m}$, at 300 K and atmospheric pressure. Assume the collision cross-section to be the same as the geometrical size of the molecule.

11.2 Monochromatic light is scattered at 90° from a cell containing 10^{-16} g particles in suspension at 300 K. Estimate the coherence time and linewidth of the scattered light.

14.2 Several output modes of a laser, indicated by the small integer n which lies between, say, +5 and −5, are represented by the waves

$$E_n = a\exp\{-\mathrm{i}[(\omega_0 + n\omega_1)t + \phi_n]\}. \qquad (15.8)$$

where ω_1 is the mode-spacing frequency. To illustrate mode-locking, calculate the wave resulting from superposition of these modes when (a) ϕ_n is a random variable and (b) all $\phi_n = 0$. (It is convenient to do this by computer.)

Calculate and plot the intensity profile across the diameter of a laser spot of the TEM00 and TEM 10 modes emanating from a resonator with circular symmetry. Next, do the same for a resonator with rectangular symmetry. Here, plot along the X axis. You may assume any numerical values for geometric parameter for the resonators if needed.

14.5 A material has six energy levels A to F at 2, 1.9, 1.7, 1.6, 1.1 and 0.4 eV above the ground state, G. The time-constants for the various possible transitions in nanoseconds are shown in Fig. 15.12. Suggest possible lasers working with this material, and give the pump and output wavelengths of each one.

Figure 15.12 Energy scheme for lasing medium. The arrows show transition times in nanoseconds.