Classical and Quantum Optics

Assignment-2 Answers

Varghese Reji

Problem 1

A beam with a photon flux of 1000 photons s[-1] is incident on a detector with a quantum efficiency of 20%. If the time interval of the counter is set to 10s, calculate the average and standard deviation of the photocount number for the followin g scenarios:

(a) the light has Poissonian statistics;

(b) the light has super-Poissonian statistics with Δ n=2× Δ n_{Poisson} ;

(c) the light is in a photon number state.

Answer

ϕ = 1000/s, η=20%, t=10s

(a) $\bar{n} = \frac{L\phi}{c}$

But, $L = ct \Rightarrow \bar{n} = t\phi$

Then, $\bar{n} = 10000$.$

$\Delta n = \sqrt{n} = 100$

The photocount number is given by,

$$(\Delta N)^2 = \eta^2(\Delta n)^2 + \eta(1-\eta)\bar{n}$$

$\Rightarrow$,

$$\Delta N = 44.721$$

.

$$\bar{N} = \eta\bar{n} = 2000$$

(b) Given $\Delta n = 2 \times \Delta n_{Poisson}$. i.e., $\Delta n = 89.442$

We know that, for super poissonian statistics,

$$(\Delta n)^2 = \bar{n} + \bar{n}^2$$

.

$\Rightarrow$

$$\bar{n} + \bar{n}^2 = 4\bar{n}^2_{Poiss} = 8000$$

By solving this,

$$\bar{n} = 88.944$$

$$\Delta N = 18.28, \bar{N} = 17.79$$

(c) In phton number state, $\Delta n = 0$. That means, there is no variation from mean value. Then, $\bar{n} = 10000$

$\Rightarrow$

$$\bar{N} = 2000, \Delta N = 40$$

Problem 2

Calculate the values of g$^{(2)}(0)$ for a monochromatic light wave with a square wave intensity modulation of $\pm$ 20%.

Answer

$$g^(2)(0) = \frac{\langle I(t)^2 \rangle}{\langle I(t) \rangle^2}$$

Since the intensity modulation is A=0.2, let us write,

$$I(t) = I_0(1 + 0.2\Theta(T/2))$$

Θ is a square pulse.

$$\langle I(t) \rangle^2 = I_0^2$$

$$\langle I(t)^2 \rangle = \frac{I_0^2}{T} \int_0^T (1 + 0.2\Theta(T/2))^2$$

Then,

$$\begin{aligned} g^{(2)}(0) &= \frac{1}{T} \int_0^T dt(1 + 0.2\Theta(T/2))^2 \\ &= \frac{1}{T} \left[\int_0^T dt + 0.4 \int_0^{T/2} dt + 0.04 \int_0^{T/2} dt \right] \\ &= 1 + 0.2 + 0.02 \\ &= 1.22 \end{aligned}$$

Then,

$$g^(2)(0) = 1.22$$

Problem 3

The 632.8 nm line of a neon discharge lamp is Doppler-broadened with a linewidth of 1.5GHz. Sketch the second-order correlation function g$^{(2)}(\tau)$ for τ in range 0-1 ns.

Answer

The coherence time is given by the formula

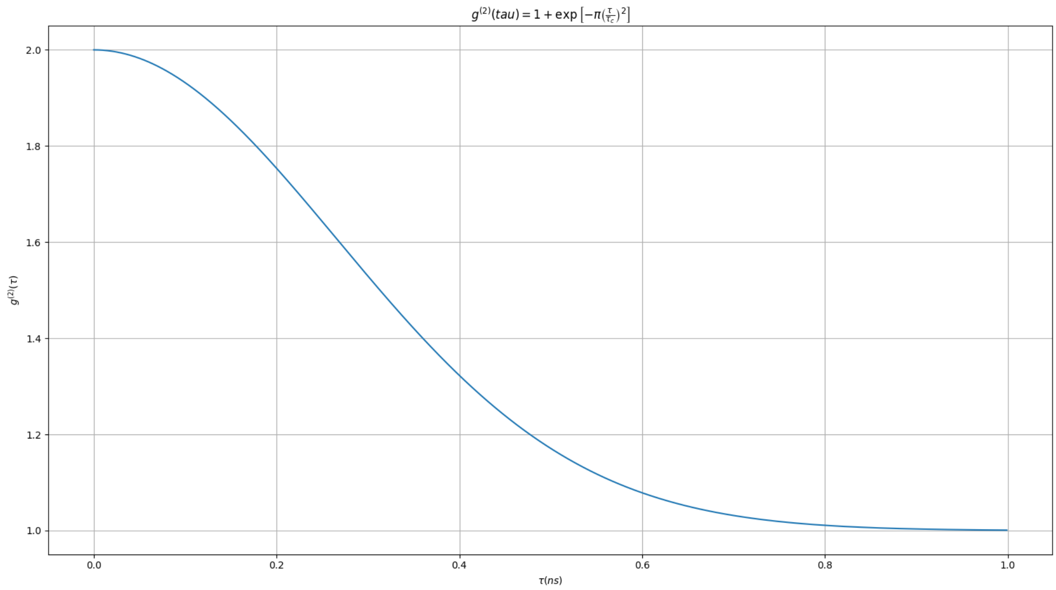
$$\tau_c = \frac{\lambda^2}{c\delta\lambda}$$

Given, $\Delta\nu = 1.5GHz$.
And, $d\lambda = \frac{c}{\lambda^2}d\nu \Rightarrow \tau_c = \frac{1}{d\nu}$
$\therefore$

$$\tau_c = 6.67e - 10s = .667ns$$

From Mark Fox,

$$g^{(2)}(\tau) = 1 + \exp\left[-\pi\left(\frac{\tau}{\tau_c}\right)^2\right]$$



Problem 4

For the coherent states $|\alpha\rangle$ with α=5, calculate

(a) the mean photon number;

(b) the standard deviation in the photon number;

(c) the quantum uncertinity in the optical phase.

Answer

(a) $\alpha = 5$. Then, $\bar{n} = |\alpha|^2 = 25$

(b) $\Delta n = |\alpha| = 5$

(c) $\Delta\phi = \frac{\text{uncertinity diameter}}{\alpha} = \frac{1/2}{5} = \frac{1}{10}$

Problem 5

A ruby laser operating at 693 nm emits pulses of energy 1mJ. Calculate the quantum uncertinity in the phase of the laser light.

Answer

$\lambda = 693nm, E = 1mJ.$

Then,

$$n = 3.486 \times 10^{15}$$

n := 3.486e15
a := $\mathrm{n}^{0.5}$ =⟩ 59042357.6765
dp := 0.5 / a =⟩ 8.46849651126e-9
Then,

$$\Delta\phi = \frac{\text{uncertinity diameter}}{\alpha} = \frac{1/2}{59042357.6765} = 8.468 \times 10{-}9$$

Problem 6

For the coherent state $|\alpha\rangle$ with α=$|\alpha|\mathrm{e}^{\mathrm{i}\phi}$, show that $\langle\alpha|\hat{X}_1|\alpha\rangle = |\alpha|\cos\phi$ and $\langle\alpha|\hat{X}_2|\alpha\rangle = |\alpha|\sin\phi$. Show further that $\Delta X_1 = \Delta X_2 = \frac{1}{2}$.

Answer

$$\hat{X}_1 = \frac{1}{2}(\hat{a}^\dagger + \hat{a}) \qquad \hat{X}_2 = \frac{i}{2}(\hat{a}^\dagger - \hat{a})$$

$$\hat{a}|n\rangle = (n)^{\frac{1}{2}}|n-1\rangle \qquad \hat{a}^\dagger|n\rangle = (n+1)^{\frac{1}{2}}|n+1\rangle$$

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{\frac{1}{2}}}|n\rangle$$

$$
\begin{aligned}
\langle\alpha|\hat{X}_1|\alpha\rangle &= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|(\hat{a}^\dagger+\hat{a})|m\rangle \\
&= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\left[(m+1)^{\frac{1}{2}}\langle n|m+1\rangle+m^{\frac{1}{2}}\langle n|m-1\rangle\right] \\
&= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\left[(m+1)^{\frac{1}{2}}\delta_{n,m+1}+m^{\frac{1}{2}}\delta_{n,m-1}\right] \\
&= \exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}(m+1)^{\frac{1}{2}}\delta_{n,m+1}+\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}m^{\frac{1}{2}}\delta_{n,m-1}\right] \\
&= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\frac{\alpha*^{m+1}}{((m+1)!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}(m+1)^{\frac{1}{2}}+\frac{\alpha^{m-1}}{((m-1)!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}m^{\frac{1}{2}}\right] \\
&= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\frac{\alpha*^{m+1}\alpha^m}{m!}+\frac{\alpha^{m-1}\alpha^m}{((m-1)!)}\right] \\
&= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\alpha*\frac{\alpha^{2m}}{m!}+\alpha\frac{\alpha^{2(m-1)}}{((m-1)!)}\right] \\
&= \frac{1}{2}\left[\alpha*+\alpha\right] \\
&= \frac{|\alpha|}{2}(e^{i\phi}+e^{-i\phi}) \\
&= |\alpha|\cos\phi
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha|\hat{X}_1|\alpha\rangle &= \frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|(\hat{a}^\dagger-\hat{a})|m\rangle \\
&= i\frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\left[(m+1)^{\frac{1}{2}}\langle n|m+1\rangle-m^{\frac{1}{2}}\langle n|m-1\rangle\right] \\
&= i\frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\left[(m+1)^{\frac{1}{2}}\delta_{n,m+1}-m^{\frac{1}{2}}\delta_{n,m-1}\right] \\
&= i\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}(m+1)^{\frac{1}{2}}\delta_{n,m+1}-\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}m^{\frac{1}{2}}\delta_{n,m-1}\right] \\
&= i\frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\frac{\alpha*^{m+1}}{((m+1)!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}(m+1)^{\frac{1}{2}}-\frac{\alpha^{m-1}}{((m-1)!)^{\frac{1}{2}}}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}m^{\frac{1}{2}}\right] \\
&= i\frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\frac{\alpha*^{m+1}\alpha^m}{m!}-\frac{\alpha^{m-1}\alpha^m}{((m-1)!)}\right] \\
&= i\frac{1}{2}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\left[\alpha*\frac{\alpha^{2m}}{m!}-\alpha\frac{\alpha^{2(m-1)}}{((m-1)!)}\right] \\
&= i\frac{1}{2}\left[\alpha*-\alpha\right] \\
&= i\frac{|\alpha|}{2}(e^{-i\phi}-e^{i\phi}) \\
&= |\alpha|\sin\phi
\end{aligned}
$$

$$
\hat{X}_1^2 = \frac{1}{4}(\hat{a}^\dagger+\hat{a})^2 = \frac{1}{4}(\hat{a}^\dagger\hat{a}^\dagger+\hat{a}\hat{a}+\hat{a}^\dagger\hat{a}+\hat{a}\hat{a}^\dagger)
$$

$$\hat{X}_2^2 = \frac{-1}{4}(\hat{a}^\dagger - \hat{a})^2 = \frac{1}{4}(-\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)$$

Using the commutator $[\hat{a}, \hat{a}^\dagger] = 1, \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$
$\Rightarrow$

$$\hat{X}_1^2 = \frac{1}{4}(\hat{a}^\dagger + \hat{a})^2 = \frac{1}{4}(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + 1)$$
$$\hat{X}_2^2 = \frac{-1}{4}(\hat{a}^\dagger - \hat{a})^2 = \frac{1}{4}(-\hat{a}^\dagger\hat{a}^\dagger - \hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + 1)$$

$$\langle\alpha|\hat{X}_1^2|\alpha\rangle = \frac{1}{4}\exp\left(-|\alpha|^2\right)\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|(\hat{a}^\dagger\hat{a}^\dagger + \hat{a}\hat{a} + 2\hat{a}^\dagger\hat{a} + 1)|m\rangle$$

$$\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|m\rangle = \sum_{m=0}^{\infty}\frac{|\alpha|^{2m}}{m!} = \exp\left(|\alpha|^2\right)$$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|\hat{a}^\dagger\hat{a}|m\rangle &= \sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}m\langle n|m\rangle \\
&= \sum_{m=1}^{\infty}\frac{|\alpha|^{2m}}{(m-1)!} = |\alpha|^2\exp\left(|\alpha|^2\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|\hat{a}\hat{a}|m\rangle &= \sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}(m(m-1))^{\frac{1}{2}}\langle n|m-2\rangle \\
&= \sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}(m(m-1))^{\frac{1}{2}}\delta_{n,m-2} \\
&= \sum_{m=2}^{\infty}\frac{\alpha *^{m-2}\,\alpha^m}{((m-2)!)} \\
&= \alpha^2\sum_{m=2}^{\infty}\frac{|\alpha|^{2(m-2)}}{((m-2)!)} \\
&= \alpha^2\exp\left(|\alpha|^2\right)
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}\langle n|\hat{a}^\dagger\hat{a}^\dagger|m\rangle &= \sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}((m+1)(m+2))^{\frac{1}{2}}\langle n|m+2\rangle \\
&= \sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}((m+1)(m+2))^{\frac{1}{2}}\delta_{n,m+2} \\
&= \sum_{n=0}^{\infty}\frac{\alpha*^n}{(n!)^{\frac{1}{2}}}\sum_{m=0}^{\infty}\frac{\alpha^m}{(m!)^{\frac{1}{2}}}((m+1)(m+2))^{\frac{1}{2}}\delta_{n-2,m} \\
&= \sum_{n=2}^{\infty}\frac{\alpha*^n}{((n-2)!)^{\frac{1}{2}}}\frac{\alpha^{n-2}}{((n-2)!)^{\frac{1}{2}}} \\
&= \alpha *^2\sum_{n=2}^{\infty}\frac{|\alpha|^{2n-4}}{((n-2)!)} \\
&= \alpha *^2\exp\left(|\alpha|^2\right)
\end{aligned}$$

$$
\begin{aligned}
\langle\alpha|\hat{X}_1^2|\alpha\rangle &= \frac{1}{4}(\alpha*^2 + \alpha^2 + 2|\alpha|^2 + 1) \\
&= \frac{1}{4}(\alpha*^2 + \alpha^2 + 2\alpha*\alpha + 1) \\
&= \frac{1}{4}((\alpha + \alpha*)^2 + 1) \\
&= \alpha^2\cos^2\phi + \frac{1}{4}
\end{aligned}
$$

Similerly,

$$
\begin{aligned}
\langle\alpha|\hat{X}_2^2|\alpha\rangle &= \frac{1}{4}(-\alpha*^2 - \alpha^2 + 2|\alpha|^2 + 1) \\
&= \frac{1}{4}(-\alpha*^2 - \alpha^2 + 2\alpha*\alpha + 1) \\
&= \frac{1}{4}(-(\alpha - \alpha*)^2 + 1) \\
&= \alpha^2\sin^2\phi + \frac{1}{4}
\end{aligned}
$$

Then,

$$
\begin{aligned}
\Delta\hat{X}_1^2 &= \alpha^2\cos^2\phi + \frac{1}{4} - \alpha^2\cos^2\phi = \frac{1}{4} \\
\Delta\hat{X}_1^2 &= \alpha^2\sin^2\phi + \frac{1}{4} - \alpha^2\sin^2\phi = \frac{1}{4}
\end{aligned}
$$

Then,

$$
\Delta\hat{X}_1 = \Delta\hat{X}_2 = \frac{1}{2}
$$

Problem 7

Prove that for two coherent states $|\alpha\rangle$ and $|\beta\rangle$,

$$
|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha - \beta|^2)
$$

Answer

Since both $|\alpha\rangle$ and $|\beta\rangle$ are coherent states,

$$
|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{(n!)^{\frac{1}{2}}}|n\rangle \qquad |\beta\rangle = \exp\left(-\frac{|\beta|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\beta^m}{(m!)^{\frac{1}{2}}}|m\rangle
$$

$$
\begin{aligned}
\langle\alpha|\beta\rangle &= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n\beta^m}{(n!)^{\frac{1}{2}}(m!)^{\frac{1}{2}}}\langle n|m\rangle \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^n\beta^m}{(n!)^{\frac{1}{2}}(m!)^{\frac{1}{2}}}\delta_{nm} \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha*\beta)^n}{(n!)} \\
&= \exp\left(-\frac{|\alpha|^2+|\beta|^2}{2}\right)\exp\left(\alpha*\beta\right) \\
&= \exp\left(-\frac{|\alpha|^2-2\alpha*\beta+|\beta|^2}{2}\right) \\
&= \exp\left(-\frac{|\alpha-\beta|^2}{2}\right)
\end{aligned}
$$

Then, while taking the square of this,

$$
|\langle\alpha|\beta\rangle|^2 = \exp(-|\alpha-\beta|^2)
$$