Exercise Sheet 2

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1 Continues Maps

If g is continues at x = a $\Longrightarrow \lim_{x\to a} g(x) = g(a)$, (1)
$a) If\ f\ is\ continues\ at\ g = g(a) \Longrightarrow \lim_{g\to g(a)} f(g) = f(g(a)) = (f\circ g)(a)$, (2)
$\lim_{g(x)\to g(a)} f(g(x)) = (f\circ g)(a)$,
$From$ (1) and (2) $\Longrightarrow \lim_{x\to a} f(g(x)) = (f\circ g)(a) \Longrightarrow \lim_{x\to a}(f\circ g)(x) = (f\circ g)(a) \Longrightarrow$
$f\circ g$ *is continues.*

$b)$ *We need to prove that,* $\forall (x_n) : N \to X,\ x_n \to x$
$\forall\ \varepsilon > 0\ \exists\ N \in N;\ n \geq N \Longrightarrow ||x_n - x|| < \varepsilon$,
by the triangle inequality $\exists\ N \in;\ n \geq N \Longrightarrow ||x_n|| - ||x|| < \varepsilon \Longrightarrow ||x_n|| \to ||x||$

2 Differentiable functions

a) We need to prove that $\frac{d}{dx}\ x^n = nx^{n-1}$.

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \quad *((x+h)^n = \tbinom{n}{0}x^n + \tbinom{n}{1}x^{n-1}h + \tbinom{n}{2}x^{n-2}h^2 + \cdots + \tbinom{n}{n}h^n) \\
&= \lim_{h\to 0}\frac{x^n + nx^{n-1}h + P(x,h)\cdot h^2 - x^n}{h} \\
&= \lim_{h\to 0} nx^{n-1} + P(x,h)\cdot h \\
&= nx^{n-1}.
\end{aligned}$$

where P(x,h) is some polynomial in x and h.

2.1 Differentiation Rules

a) $(\alpha f + \beta g)'(a) = \alpha f'(a) + \beta g'(a)$
I think we can prove this if we will prove Sum and Constant times properties of differentiation.
*Proof of Sum/Difference of Two Functions
$(f(x) \pm g(x))' = f'(x) \pm g'(x)$

$$\begin{aligned}
(f(x) + g(x))' &= \lim_{h\to 0}\frac{f(x+h) + g(x+h) - (f(x) + g(x))}{h} \\
&= \lim_{h\to 0}\frac{f(x+h) - f(x) + g(x+h) - g(x)}{h} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} + \lim_{h\to 0}\frac{g(x+h) - g(x)}{h} \\
&= f'(x) + g'(x).
\end{aligned}$$

*Proof of Constant Times a Function:
$(cf(x))' = \lim\limits_{h\to 0} \frac{cf(x+h)-cf(x)}{h} = c\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = cf'(x)$.

$b) Product Rule:$

$(f\,g)' = \lim\limits_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} = \lim\limits_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} = \lim\limits_{h\to 0} \frac{f(x+h)(g(x+h)-g(x))}{h} + \lim\limits_{h\to 0} \frac{g(x)(f(x+h)-f(x))}{h} = \lim\limits_{h\to 0} f(x+h)\, \frac{g(x+h)-g(x)}{h} + \lim\limits_{h\to 0} g(x)\, \frac{f(x+h)-f(x)}{h} = \left(\lim\limits_{h\to 0} f(x+h)\right)\left(\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}\right) + \left(\lim\limits_{h\to 0} g(x)\right)\left(\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}\right),$

$\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h} = g'(x) \qquad \lim\limits_{h\to 0} g(x) = g(x)$
$\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x) \qquad \lim\limits_{h\to 0} f(x+h) = f(x)$

$$(f\,g)' = f(x)\,g'(x) + g(x)\,f'(x)$$

$c) Quotientrule: \left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$

$\left(\frac{f}{g}\right)' = \lim\limits_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h} = \lim\limits_{h\to 0} \frac{1}{h}\ \frac{f(x+h)g(x)-f(x)g(x+h)}{g(x+h)g(x)} = \lim\limits_{h\to 0} \frac{1}{h}\ \frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{g(x+h)g(x)} = \lim\limits_{h\to 0} \frac{1}{g(x+h)g(x)}\ \frac{f(x+h)g(x)-f(x)g(x)+f(x)g(x)-f(x)g(x+h)}{h} = \lim\limits_{h\to 0} \frac{1}{g(x+h)g(x)}\ \left(\frac{f(x+h)g(x)-f(x)g(x)}{h} + \frac{f(x)g(x)-f(x)g(x+h)}{h}\right) = \lim\limits_{h\to 0} \frac{1}{g(x+h)g(x)}\ \left(g(x)\, \frac{f(x+h)-f(x)}{h} - f(x)\, \frac{g(x+h)-g(x)}{h}\right) = \frac{1}{\lim\limits_{h\to 0} g(x+h)\, \lim\limits_{h\to 0} g(x)}\ \left(\left(\lim\limits_{h\to 0} g(x)\right)\left(\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}\right) - \left(\lim\limits_{h\to 0} f(x)\right)\left(\lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}\right)\right) = \frac{1}{g(x)g(x)}\ (g(x)\,f'(x) - f(x)\,g'(x)) = \frac{f'\,g - f\,g'}{g^2}.$

$d) First\ proving:$
$\frac{\mathrm{d}}{\mathrm{d}x}[x^x] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{x\ln(x)}\right] = \mathrm{e}^{x\ln(x)} \frac{\mathrm{d}}{\mathrm{d}x}[x\ln(x)] = \mathrm{e}^{x\ln(x)}(\frac{\mathrm{d}}{\mathrm{d}x} \leq ft[x\cdot\ln(x) + \frac{\mathrm{d}}{\mathrm{d}x}[\ln(x)]) = \mathrm{e}^{x\ln(x)}(\ln(x) + x\cdot\dfrac{1}{x}) = \mathrm{e}^{x\ln(x)}(\ln(x)+1) = x^x(\ln(x)+1)$

$Second:$
$\lim_{h\to 0} \frac{(x+h)^{x+h}-x^x}{h} = \lim_{h\to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}h}((x+h)^{x+h}-x^x)}{\frac{\mathrm{d}}{\mathrm{d}h}h} = \lim_{h\to 0} \frac{(x+h)^{x+h}(1+\ln(x+h))}{1} = \lim_{h\to 0}(x+h)^{x+h} \cdot \lim_{h\to 0}(1+\ln(x+h)) = \lim_{h\to 0}(x+h)^{\lim_{h\to 0} x+h} \cdot \lim_{h\to 0}(1+\ln(\lim_{h\to 0}(x+h))) = (x+0)^{x+0}\cdot(1+\ln(x+0)) = x^x(\ln(x)+1)$

2.2 Chin Rule

$\frac{d}{dx}[f(u)] = \frac{d}{du}[f(u)]\frac{du}{dx}$

First method:
u'$(x) = \lim\limits_{h\to 0} \frac{u(x+h)-u(x)}{h}$
$\lim\limits_{h\to 0}\left(\frac{u(x+h)-u(x)}{h} - u'(x)\right) = \lim\limits_{h\to 0} \frac{u(x+h)-u(x)}{h} - \lim\limits_{h\to 0} u'(x) = u'(x) - u'(x) = 0$
$v(h) = \begin{cases} \dfrac{u(x+h)-u(x)}{h} - u'(x) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$
$u(x+h) = u(x) + h(v(h) + u'(x))\,(1)$
w$(k) = \begin{cases} \dfrac{f(z+k)-f(z)}{k} - f'(z) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0 \end{cases}$

$f\left(z+k\right)=f\left(z\right)+k\left(w\left(k\right)+f'\left(z\right)\right)(2)$

f $\frac{d}{dx}\left[f\left[u\left(x\right)\right]\right]=\lim_{h\to0}\frac{f\left[u\left(x+h\right)\right]-f\left[u\left(x\right)\right]}{h}$
$\left[u\left(x+h\right)\right]-f\left[u\left(x\right)\right]=f\left[u\left(x\right)+h\left(v\left(h\right)+u'\left(x\right)\right)\right]-f\left[u\left(x\right)\right]$
$=f\left[u\left(x\right)\right]+h\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right)-f\left[u\left(x\right)\right]$
$=h\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right)$

$\frac{d}{dx}\left[f\left[u\left(x\right)\right]\right]=\lim_{h\to0}\frac{h\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right)}{h}$
$=\lim_{h\to0}\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right)$

$\lim_{h\to0}k=\lim_{h\to0}h\left(v\left(h\right)+u'\left(x\right)\right)=0$

$\lim_{h\to0}w\left(k\right)=w\left(\lim_{h\to0}k\right)=w\left(0\right)=0$

$\frac{d}{dx}\left[f\left[u\left(x\right)\right]\right]=\lim_{h\to0}\left(v\left(h\right)+u'\left(x\right)\right)\left(w\left(k\right)+f'\left[u\left(x\right)\right]\right)$
$=u'\left(x\right)f'\left[u\left(x\right)\right]$
$=f'\left[u\left(x\right)\right]\frac{du}{dx}$

$Second\ method:$
$f(g(x+h))=f(g(x))+\frac{d}{dx}g(x)\cdot\frac{d}{dx}+r(||h||)$
$where\ r\to0$

3 L'Hospital's rule

$\frac{f(x+h)}{g(x+h)}=\frac{\frac{f(x+h)-f(x)}{h}}{\frac{g(x+h)-g(x)}{h}}\approx\frac{f'(x)}{g'(x)}$,
$f(x)=g(x)=0$,

$\lim_{h\to0}\frac{f(x+h)}{g(x+h)}=\frac{f'(x)}{g'(x)}$.

4 Directional Derivative

a) $\frac{d^2}{dxdy}$ f(x,y) = 0 , but

$\nabla f(x,y)=\begin{bmatrix}2x & -2y\end{bmatrix}^T$,
$f(1,0)=\begin{bmatrix}2\\0\end{bmatrix}$,
$\nabla f(x,y)u=\begin{bmatrix}2\\0\end{bmatrix}\begin{bmatrix}\frac{\sqrt{3}}{2}\frac{1}{2}\end{bmatrix}=\sqrt{3}$
b)
$\lim_{(x,y)\to+0}f(x)\neq\lim_{(x,y)\to-0}f(x)$

5 Gradient

a) grad(f(a))$\cdot\vec{v}$ = $|$grad$(f(a))||\vec{v}|$cos(θ)

$b)\quad\nabla f(x,y)=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)=(-3sin(3x+2y),-2sin(3x+2y))$,

$\|\nabla f(x,y)\|=\sqrt{3sin(3x+2y)^2+2sin(3x+2y)^2}$,

$\|\nabla f(\frac{\pi}{6},-\frac{\pi}{8})\|=\sqrt{\frac{5}{2}}$

$-1 = D_{\vec{u}}\, f(\frac{\pi}{6}, -\frac{\pi}{8}) = (a, b) \cdot \nabla f(\frac{\pi}{6}, -\frac{\pi}{8}) = (a, b) \cdot (0, \sqrt{\frac{5}{2}}) = \sqrt{\frac{5}{2}} b,$

$b = -\sqrt{\frac{2}{5}} \Longrightarrow a = \pm\sqrt{\frac{3}{5}},$

$D_{\vec{u}}(-\sqrt{\frac{2}{5}}, \pm\sqrt{\frac{3}{5}})$