Financial Mathematics
Home Assignment 1

1. Consider a risky asset with $S_0 = \$40$ at time $t = 0$ and one of two outcomes at $t = 1$: $S_1^u = \$50$ and $S_1^d = \$20$. Consider a contingent claim C at $t = 1$ given by

$$C = \begin{cases} \$30, & S_1 = S_1^u, \\ \$0, & S_1 = S_1^d. \end{cases}$$

Assume the issuer charges \$10 at $t = 0$ for the contract above. The whole \$10 premium is invested in a portfolio at $t = 0$ so that the portfolio reaches the target \$30 if $S_1 = S_1^u$ at $t = 1$ (assume $r = 0$).

1. Compute the portfolio (a, b) (shares a in the stock and cash b in bank) starting from \$10 that attains \$30 if $S_1 = S_1^u$.
2. Does this portfolio replicate the given contract (i.e., also produces \$0 if $S_1 = S_1^d$)?
3. Compute the loss incurred by the issuer if $S_1 = S_1^d$.
4. Is the price \$10 fair? If not, is it greater than fair, or less?

$S_0 = 40\$$; $S_1^u = 50\$ \mid C = 30\$$; $S_1^d = 20\$ \mid C = 0\$$

1) Introduce portfolio $\pi(G; H)$.
Then, assuming we do not take S_1^d into account now,

$$\begin{cases} V_0^\pi = G + H \cdot S_0 \\ V_1^\pi = G(1+r) + H \cdot S_1^u. \end{cases}$$

For $r = 0$ we have

$$\begin{cases} G + 40H = 10 \\ G + 50H = 30 \end{cases} \longrightarrow \begin{cases} G = -70 \\ H = 2 \end{cases} \longrightarrow \pi(-70; 2).$$

2) Check $\omega = \omega_d$ to see if $\pi(-70; 2)$ replicates C:

$G(1+r) + H \cdot S_1^d = -70 + 2 \cdot 20 = -30 \neq 0 \Rightarrow$
$\Rightarrow$ it does not.

3) $\underbrace{V_1^\pi(\omega_d)}_{\text{portfolio value}} - \underbrace{C(\omega_d)}_{\text{claim payment}} = -30 - 0 = -30.$

4) Fair price of a claim is the value of replicating portfolio at $t = 0$.

1. Find repl. portfolio:
$$\begin{cases} G + 50H = 30 \\ G + 20H = 0 \end{cases} \longrightarrow \begin{cases} G = -20 \\ H = 1. \end{cases}$$
2. Find its value at $t = 0$:
$-20 + 1 \cdot 40 = 20 > 10 \Rightarrow$ Answer: Not fair; Less.

2. Consider a T-step binomial model where $S_t = S_{t-1}\xi_t$, with

$$\xi_t = \begin{cases} 1 + u, & \text{with probability } p, \\ 1 + d, & \text{with probability } 1 - p, \end{cases} \qquad \text{all } \xi_t \text{ i.i.d.}, \ S_0 = \text{const}, \ \mathcal{F}_t = \sigma(\xi_1, \dots, \xi_t).$$

independ. and identically distributed.

1. Is (S_t) a martingale w.r.t. (with resp. to) the physical measure P? If not, compute $\mathbb{E}^P[S_t \mid \mathcal{F}_{t-1}]$ explicitly.
2. Is (S_t) a martingale w.r.t. a risk-neutral measure $\mathbb{Q}$? If not, compute $\mathbb{E}^{\mathbb{Q}}[S_t \mid \mathcal{F}_{t-1}]$. State the condition on $q := \mathbb{Q}(\xi_t = 1 + u)$ such that the discounted price $\tilde{S}_t = S_t/(1+r)^t$ is a $\mathbb{Q}$-martingale.

1) $\mathbb{E}^P[S_t \mid \mathcal{F}_{t-1}] = \mathbb{E}^P[S_{t-1}\xi_t \mid \mathcal{F}_t] = S_{t-1}\mathbb{E}^P[\xi_t \mid \mathcal{F}_t] =$
$= S_{t-1}\mathbb{E}^P[\xi_t] = S_{t-1}\big(p(1+u) + (1-p)(1+d)\big) =$
$= S_{t-1}\big(p(u-d) + d + 1\big) \Rightarrow$
$\Rightarrow$ for $\mathbb{E}^P[S_t \mid \mathcal{F}_{t-1}] = S_{t-1}$ to hold we need

$p(u-d) + d + 1 = 1 \longrightarrow p = \dfrac{d}{d-u}$ – the only value of p under which (S_t) is a martingale w.r.t. P $\Rightarrow$ in general (S_t) is not martingale w.r.t. P.

2) Let $\xi_t = \begin{cases} 1+u, & \text{with prob. } q \\ 1+d, & \text{with prob } 1-q. \end{cases}$

Then analgously, $\mathbb{E}^Q[S_t \mid \mathcal{F}_{t-1}] = S_{t-1}\big(q(u-d) + d + 1\big)$. (*)
Compute q s.t. $\tilde{S}_t = \dfrac{S_t}{(1+r)^t}$ is a Q-martingale:

$\mathbb{E}^Q[\tilde{S}_t \mid \mathcal{F}_{t-1}] = \tilde{S}_{t-1} \quad | \cdot (1+r)^t$
$\mathbb{E}^Q[(1+r)^t\tilde{S}_t \mid \mathcal{F}_{t-1}] = (1+r)^t\tilde{S}_{t-1};$
$\mathbb{E}^Q[S_t \mid \mathcal{F}_{t-1}] = (1+r)S_{t-1} \Rightarrow$ by (*) we have
$(1+r)S_{t-1} = \big(q(u-d) + d + 1\big)S_{t-1} \Rightarrow$
$\Rightarrow q(u-d) + d = r \Rightarrow q = \dfrac{r-d}{u-d}.$ [condition on q s.t. disc. price is a Q-martingale]

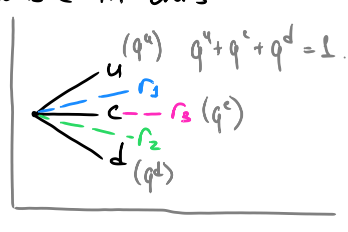
Additionally, $\left.\begin{matrix} 0 < q < 1 \\ u > d \text{ [naturally]} \end{matrix}\right| \Rightarrow d < r < u.$

Substituting into (*) get
$\mathbb{E}^Q[S_t \mid \mathcal{F}_{t-1}] = (1+r)S_{t-1} \Rightarrow$
$\Rightarrow \exists!\, r = 0$ ((S_t) is a martingale w.r.t. Q) $\Rightarrow$
$\Rightarrow$ Answer: no.

3. Consider a one-step *trinomial* market with risky returns $d < c < u$ and risk-free return $r \in (d, u)$.

1. Is this market free of arbitrage? Is it complete? Explain briefly.
2. Is it possible to hedge an arbitrary claim (C_u, C_c, C_d) with distinct payoffs? Why or why not?

1) The NA condition $(d < r < u)$ is applicable in this case:



Consider r_1:
WLOG treat c and d as a single state $D = \{c, d\}$ {we are indifferent}, so we simplify to binomial model with $D < r_1 < u$, wich is NA condition.
Consider $r_3 = c \Rightarrow d < r_3 < u \Rightarrow$ NA.
Consider r_2 (analagously to r_1):
$U = \{u, c\} \Rightarrow d < r_2 < U \Rightarrow$ NA.
In other words, $\forall r \in (d, u)$ we cannot guarantee arbitrage opportunities, bec. there still exist both prob. of being better of / worse of.

Thus, free of arbitrage. Not complete, see 2) (*).

2) Is it possible to replicate $C(\omega)$, $\omega = \{u, c, d\}$ with $\pi(G, H)$?

$$\begin{cases} (1+r)G + (1+u)S_0 \cdot H = C_u \\ (1+r)G + (1+c)S_0 \cdot H = C_c \\ (1+r)G + (1+d)S_0 \cdot H = C_d \end{cases} \xrightarrow[\text{form}]{\text{augm.}} \left[\begin{array}{cc|c} 1+r & (1+u)S_0 & C_u \\ 1+r & (1+c)S_0 & C_c \\ 1+r & (1+d)S_0 & C_d \end{array}\right] \xrightarrow{\text{RREF}}$$

$$\xrightarrow{\text{RREF}} \begin{array}{c} \begin{array}{cc} G & H \end{array} \\ \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \end{array} \Rightarrow \text{no solutions} \Rightarrow$$

$\Rightarrow$ it is not possible to hedge arbitrary claim.
Thus, the market is not complete (*) (not every claim can be replicated).

[Это очень странно. Если утв. "Part.True", то оно вообще-то False, и все границы между ними субъективны].

4. Decide whether each statement is **True**, **False**, or **Partially True** (justify in 1–3 sentences):

1. In a binomial model, $d < r = u$ (or $d = r < u$) implies absence of arbitrage and market completeness. FALSE
2. In a binomial model, $d = r = u$ implies $\overbrace{\text{absence of arbitrage}}^{\text{YES}}$ and $\overbrace{\text{market completeness}}^{\text{NO}}$. PART. TRUE
3. In a trinomial model, market completeness cannot be guaranteed. TRUE

1) $r = d \Rightarrow q = \dfrac{d-d}{u-d} = 0 \notin (0; 1) \Rightarrow$ arbitrage occurs:
in any outcome the contract is no less profitable than risk-free $\Rightarrow$ borrow at r, buy the contract.
$r = u \Rightarrow q = 1 \notin (0; 1) \Rightarrow$ arbitrage occurs:
bank is no less profitable than the contract $\Rightarrow$ sell the contract, place the cash at r.
Arbitrage exist $\Rightarrow$ market not complete.

2) Asset and bank return are determined and equivalent, thus they may be considered undistinguishable $\Rightarrow$ there is just no room for arbitrage.

$$\begin{cases} (1+r)G + S_u \cdot H = C(\omega_u) \\ (1+r)G + S_d \cdot H = C(\omega_d) \\ S_u = S_d = (1+r)S_0 \end{cases}$$

$$\begin{cases} (1+r)(G + S_0H) = C(\omega_u) \\ (1+r)(G + S_0H) = C(\omega_d) \end{cases}$$

$C(\omega_u) = C(\omega_d)$, which is obviously not the case in general $\Rightarrow$
$\Rightarrow$ not any contract can be replicated $\Rightarrow$ not complete.

3) If we assume 2 basic instr. in the market it is true.
Just check out № 3 part 2).

5. Consider a two-step binomial model with per-step returns $u = +30\%$, $d = -10\%$. The risk-free rate per step is $r = 3\%$. The initial price is $S_0 = 120$, maturity $T = 2$. Consider an **American put** with strike $K = 110$ and immediate exercise payoff $X_t = (K - S_t)^+$ at time t.

1. Find the no-arbitrage price V_0 of the American put.
2. Describe the optimal early-exercise policy at each node (at $t = 1$ and, if relevant, at $t = 0$).
3. Compute a one-step replicating hedge (G_0, H_0) at time 0 (bank cash G_0, shares H_0) and verify it on both branches at $t = 1$:

$$V_1^{(u)} = G_0(1+r) + H_0\, S_1^{(u)}, \qquad V_1^{(d)} = G_0(1+r) + H_0\, S_1^{(d)}.$$

$r = 3\%$.
(0) $S_0 = 120$
(1) $S_u = 1{,}3 \cdot 120 = 156$; $S_d = 0{,}9 \cdot 120 = 108$.
(2) $S_{uu} = 1{,}3 \cdot 156 = 202{,}8$; $S_{ud} = 0{,}9 \cdot 156 = 140{,}4$; $S_{du} = 1{,}3 \cdot 108 = 140{,}4$; $S_{dd} = 0{,}9 \cdot 108 = 97{,}2$

Verify NA-cond. for both steps: $d = -0{,}1 < r = 0{,}03 < u = 0{,}3$.
For terminal states $V_t = X_t$. Thus, at $t = 2$ we have

$V_{uu} = X_{uu} = (110 - 202{,}8)^+ = 0;$
$V_{ud} = X_{ud} = (110 - 140{,}4)^+ = 0;$
$V_{du} = X_{du} = (110 - 140{,}4)^+ = 0;$
$V_{dd} = X_{dd} = (110 - 97{,}2)^+ = 12{,}8.$

For interm. states $V_t = \max\left\{X_t; \dfrac{1}{1+r}\left(\mathbb{E}^Q[V_{t+1} \mid \mathcal{F}_{t-1}]\right)\right\}.$

$q = \dfrac{r-d}{u-d} = \dfrac{0{,}13}{0{,}4} = \dfrac{13}{40}$ – risk-neutral probability.

At $t = 1$:

$\dfrac{1}{1+r}\mathbb{E}^Q[V_2 \mid \mathcal{F}_1 \wedge S_1 = S_u] = \dfrac{1}{1{,}03}\left(\dfrac{13}{40}X_{uu} + \dfrac{27}{40}X_{ud}\right) = 0 \Rightarrow$
$\Rightarrow V_u = \max\{(110 - 156)^+; 0\} = 0.$
Indifferent between early exercising and holding.

$\dfrac{1}{1+r}\mathbb{E}^Q[S_2 \mid \mathcal{F}_1 \wedge S_1 = S_d] = \dfrac{1}{1{,}03}\left(\cancel{\dfrac{13}{40}X_{du}} + \dfrac{27}{40}X_{dd}\right) =$
$= \dfrac{864}{103} \Rightarrow V_d = \max\left\{(110 - 108)^+, \dfrac{864}{103}\right\} = 8\dfrac{40}{103} \simeq 8{,}388.$
Holding is optimal.
Overall, it is optimal to hold the option at $t = 1$.

At $t = 0$:

$\dfrac{1}{1+r}\mathbb{E}^Q[S_1 \mid \mathcal{F}_0] = \dfrac{1}{1{,}03}\left(\cancel{\dfrac{13}{40}V_u} + \dfrac{27}{40}V_d\right) = \dfrac{27 \cdot 8{,}388}{1{,}03 \cdot 40} \simeq$
$\simeq 5{,}497 \Rightarrow V_0 = \max\{(110 - 120)^+, 5{,}497\} = 5{,}497.$
Early exercise is not optimal.

3)
$$\begin{cases} \overset{1+r}{1{,}03}\,G_0 + \overset{S_u}{156} \cdot H_0 = \overset{V_u}{0} \\ \underset{}{\overset{1+r}{1{,}03}}\,G_0 + \overset{S_d}{108} \cdot H_0 = \overset{V_d}{8{,}388} \end{cases} \longrightarrow \begin{cases} G_0 \simeq 26{,}467 \\ H_0 = -0{,}17475 \end{cases} \Rightarrow$$

$\Rightarrow \pi = (26{,}467; -0{,}17475)$ – one-step replicating hedge.

Verify: $1{,}03 \cdot 26{,}467 + 156 \cdot (-0{,}17475) \simeq 0 = V_u$ ✓
$1{,}03 \cdot 26{,}467 + 108(-0{,}17475) \simeq 8{,}388 = V_d$ ✓