Homework 1

September 30, 2022

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I discussed the problems on a very high level with Kristóf Szabó (22-956-445).

1. (Optional)

2. (a)

$$\begin{aligned}
\log \mathbb{E} e^{\lambda X_i} &= \log \mathbb{E} \sum_{j=0}^{\infty} \frac{(\lambda X_i)^j}{j!} \text{ by Taylor expansion} \\
&= \log(\mathbb{E}[1] + \mathbb{E}[\lambda X_i] + \sum_{j=2}^{\infty} \frac{\mathbb{E}[X_i^j]\lambda^j}{j!}) \\
&= \log(1 + 0 + \sum_{j=2}^{\infty} \frac{\mathbb{E}[X_i^j]\lambda^j}{j!}) \\
&\leq \log(1 + \sum_{j=2}^{\infty} \frac{\mathbb{E}[|X_i^j|]\lambda^j}{j!}) \\
&\leq \log(1 + \sum_{j=2}^{\infty} \frac{\mathbb{E}[|X_i^2|b^{j-2}]\lambda^j}{j!}) \ |X| \text{ is bounded by } b \\
&= \log(1 + \sum_{j=2}^{\infty} \frac{b^{j-2}\mathbb{E}[X_i^2]\lambda^j}{j!}) \\
&= \log(1 + \sum_{j=2}^{\infty} \frac{b^{j-2}\sigma^2\lambda^j}{j!}) \\
&= \log(1 + \frac{\sigma^2}{b^2} \sum_{j=2}^{\infty} \frac{(b\lambda)^j}{j!}) \\
&= \log(1 + \frac{\sigma^2}{b^2} (\sum_{j=0}^{\infty} \frac{(b\lambda)^j}{j!} - 1 - b\lambda)) \\
&= \log(1 + \frac{\sigma^2}{b^2}(e^{b\lambda} - 1 - b\lambda)) \\
&\leq \frac{\sigma^2}{b^2}(e^{b\lambda} - 1 - b\lambda) \text{ because} 1 + x \leq e^x \text{ and taking logs, using non-negativity of } \mathbb{E}[e^{\lambda X_i}] \\
&= \frac{\sigma^2\lambda^2}{(\lambda b)^2}(e^{b\lambda} - 1 - b\lambda) \text{ getting the (redundant?) form required by the question}
\end{aligned}$$

(b)

$$
\begin{aligned}
P(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \delta) &\leq P(e^{\lambda \sum_{i=1}^{n} \frac{1}{n} X_i} \geq e^{\lambda\delta}) \text{ for any positive } \lambda \text{ due to monotonicity of exp} \\
&\leq \frac{\mathbb{E}[e^{\lambda \sum_{i=1}^{n} \frac{1}{n} X_i}]}{e^{\lambda\delta}} \text{ by Markov's inequality} \\
&= \frac{\prod_{i=1}^{n} \mathbb{E}[e^{\lambda(\frac{1}{n} X_i)}]}{e^{\lambda\delta}} \text{ by independence} \\
&\leq \frac{\prod_{i=1}^{n} e^{\frac{(\frac{\sigma_i}{n})^2}{(\frac{b}{n})^2}(e^{\frac{b\lambda}{n}} - 1 - \frac{b\lambda}{n})}}{e^{\lambda\delta}} \text{ using part a) with } \frac{X_i}{n} \text{ instead of } X_i \\
&= \frac{e^{\frac{\frac{\sigma^2}{n}}{(\frac{b}{n})^2}(e^{\frac{b\lambda}{n}} - 1 - \frac{b\lambda}{n})}}{e^{\lambda\delta}} \text{combining sigmas} \\
&= \frac{e^{\frac{n\sigma^2}{b^2}(e^{\frac{b\lambda}{n}} - 1 - \frac{b\lambda}{n})}}{e^{\lambda\delta}} \\
&= e^{\frac{n\sigma^2}{b^2}(e^{\frac{b\lambda}{n}} - 1 - \frac{b\lambda}{n}) - \lambda\delta}
\end{aligned}
$$

Minimising this expression wrt λ we get the optimum at

$$\lambda = \frac{n \log(\frac{\delta b}{\sigma^2} + 1)}{b}$$

Substituting it back, we get

$$
\begin{aligned}
P(\frac{1}{n}\sum_{i=1}^{n} X_i \geq \delta) &\leq e^{\frac{n\sigma^2}{b^2}(e^{\frac{b \frac{n\log(\frac{\delta b}{\sigma^2}+1)}{b}}{n}} - 1 - \frac{b\frac{n\log(\frac{\delta b}{\sigma^2}+1)}{b}}{n}) - \frac{n\log(\frac{\delta b}{\sigma^2}+1)}{b}\delta} \\
&= e^{\frac{n\sigma^2}{b^2}(\frac{\delta b}{\sigma^2} + 1 - 1 - \log(\frac{\delta b}{\sigma^2}+1) - \frac{\delta b}{\sigma^2}\log(\frac{\delta b}{\sigma^2}+1))} \\
&= e^{-\frac{n\sigma^2}{b^2}((1+\frac{\delta b}{\sigma^2})\log(1+\frac{\delta b}{\sigma^2}) - \frac{\delta b}{\sigma^2})} \\
&= e^{-\frac{n\sigma^2}{b^2}h(\frac{b\delta}{\sigma^2})}
\end{aligned}
$$

(c) (Bonus)

3. (a)

$$
\begin{aligned}
\mathbb{E}\max_{i=1}^{n} X_i &= \frac{1}{\lambda}\mathbb{E}\max_{i=1}^{n} \lambda X_i \text{ for positive } \lambda \\
&= \frac{1}{\lambda}\mathbb{E}\log e^{\max_{i=1}^{n} \lambda X_i} \\
&= \frac{1}{\lambda}\mathbb{E}\log \max_{i=1}^{n} e^{\lambda X_i} \text{ because exp is monotonic} \\
&\leq \frac{1}{\lambda}\mathbb{E}\log \sum_{i=1}^{n} e^{\lambda X_i} \\
&\leq \frac{1}{\lambda}\log \mathbb{E} \sum_{i=1}^{n} e^{\lambda X_i} \text{ because log is concave and using Jensen's inequality} \\
&= \frac{1}{\lambda}\log \sum_{i=1}^{n} \mathbb{E} e^{\lambda X_i} \text{ linearity of expectation} \\
&\leq \frac{1}{\lambda}\log \sum_{i=1}^{n} e^{\frac{\lambda^2\sigma^2}{2}} \text{ subgaussianity} \\
&= \frac{1}{\lambda}\log n e^{\frac{\lambda^2\sigma^2}{2}} \\
&= \frac{1}{\lambda}\log n + \frac{\lambda\sigma^2}{2}
\end{aligned}
$$

Minimising this wrt to λ using differentiating, setting to zero, solving, and checking the positivity condition on it, we get the best bound by using

$$\lambda = \frac{\sqrt{2\log n}}{\sigma}$$

Substituting it back, we get

$$
\begin{aligned}
\mathbb{E}\max_{i=1}^{n} X_i &\leq \frac{1}{\frac{\sqrt{2\log n}}{\sigma}}\log n + \frac{\frac{\sqrt{2\log n}}{\sigma}\sigma^2}{2} \\
&= \sqrt{2\sigma^2\log n}
\end{aligned}
$$

(b) Let's define $Y_i = X_i$ and $Y_{i+n} = -X_i$ for $i \in \{1, 2, ..., n\}$.

$$
\begin{aligned}
\mathbb{E}\max_{i=1}^{n} |X_i| &= \mathbb{E}\max_{i=1}^{2n} Y_i \\
&\leq \sqrt{2\sigma^2\log 2n} \text{ using part a) on } Y \qquad (1) \\
&= \sqrt{2\sigma^2(\log 2 + \log n)} \\
&\leq \sqrt{2\sigma^2 2\log n} \text{ for } n \geq 2 \\
&= 2\sqrt{\sigma^2\log n}
\end{aligned}
$$

where in line (1) we used the fact that Y satisfies all the preconditions of part (a):

- zero mean: $\mathbb{E}Y_i = \mathbb{E}X_i = 0$ and $\mathbb{E}Y_{i+n} = \mathbb{E}(-X_i) = -\mathbb{E}X_i = 0$
- subgaussian: $\mathbb{E}e^{\lambda Y_i} = \mathbb{E}e^{\lambda X_i} \leq e^{\frac{\lambda^2\sigma^2}{2}}$ and $\mathbb{E}e^{\lambda Y_{i+n}} = \mathbb{E}e^{\lambda(-X_i)} = \mathbb{E}e^{(-\lambda)X_i} \leq e^{\frac{(-\lambda)^2\sigma^2}{2}} = e^{\frac{\lambda^2\sigma^2}{2}}$

4. My strategy is to take the median of the sample means of $-d\log(\delta)$ batches of size $\frac{c\sigma^2}{\epsilon^2}$, where c and d are constants to be chosen later. Let Y_i denote the random variable for the sample mean of batch i. Using Chebyshev's inequality on Y, we have

$$\begin{aligned}
P(|Y_i - \mathbb{E}Y_i| > \epsilon) &= P(|Y_i - \mu| > \epsilon) \text{ by linearity of expectation} \\
&\leq \frac{V(Y_i)}{\epsilon^2} \\
&= \frac{\frac{\sigma^2}{(\frac{c\sigma^2}{\epsilon^2})}}{\epsilon^2} \\
&= \frac{1}{c}
\end{aligned} \tag{2}$$

Let Z_i denote the indicator random variable for the ith sample mean being outside the required interval. It has probability $\frac{1}{c}$, from the previous calculation. I will use the default version of the Chernoff bound (https://en.wikipedia.org/wiki/Chernoff_bound#Multiplicative_form_(relative_error)) to show that the probability that at least half of the sample means are outside of the required interval is $\leq \delta$. This probability upper bounds the probability that the median is outside the required interval.

$$\begin{aligned}
P(\sum_{i=1}^{-d\log\delta} Z_i \geq \frac{-d\log\delta}{2}) &\leq e^{-\mathbb{E}\sum_{i=1}^{-d\log\delta} Z_i}(\frac{e\mathbb{E}\sum_{i=1}^{-d\log\delta} Z_i}{\frac{-d\log\delta}{2}})^{\frac{-d\log\delta}{2}} \text{ by the Chernoff bound} \\
&= e^{-\mathbb{E}\sum_{i=1}^{-d\log\delta} Z_i}(\frac{e\mathbb{E}\sum_{i=1}^{-d\log\delta} Z_i}{\frac{-d\log\delta}{2}})^{\frac{-d\log\delta}{2}} \text{ by linearity of expectation} \\
&= e^{\frac{d\log\delta}{c}}(\frac{-ed\log\delta}{\frac{-cd\log\delta}{2}})^{\frac{-d\log\delta}{2}} \\
&= e^{\frac{d\log\delta}{c}}(\frac{2e}{c})^{\frac{-d\log\delta}{2}} \\
&= \delta^{\frac{d}{c}} e^{\frac{-d\log\delta}{2}}(\frac{2}{c})^{\frac{-d\log\delta}{2}} \\
&= \delta^{\frac{d}{c}} \delta^{\frac{-d}{2}}(\frac{2}{c})^{\frac{-d\log\delta}{2}} \\
&= \delta^{\frac{d}{c}} \delta^{\frac{-d}{2}} e^{\log\frac{2}{c}\frac{-d\log\delta}{2}} \\
&= \delta^{\frac{d}{c}} \delta^{\frac{-d}{2}} \delta^{\frac{-d\log\frac{2}{c}}{2}} \\
&= \delta^{d(\frac{1}{c}-\frac{1}{2}-\frac{\log\frac{2}{c}}{2})} \\
&= \delta^{d(\frac{1}{c}+\frac{\log c}{2}-\frac{\log 2}{2}-\frac{1}{2})}
\end{aligned} \tag{3}$$

there are various ways to finish from here, and since we don't care about the best constants (i.e. optimising $c \cdot d$), we can just set $d = 1$, and do some loose bounds on c:

$$\begin{aligned}
\delta^{\frac{1}{c}+\frac{\log c}{2}-\frac{\log 2}{2}-\frac{1}{2}} &\leq \delta^{\frac{\log c}{2}-\frac{\log 2}{2}-\frac{1}{2}} \text{ since } \delta < 1 \\
&\leq \delta^{\frac{\log c}{2}-1}
\end{aligned} \tag{4}$$

now setting $c = e^4$, we get

$$\begin{aligned}
&\leq \delta^{\frac{\log e^4}{2}-1} \\
&= \delta
\end{aligned} \tag{5}$$

Note that there are various other bounds one could use for the second part of the proof, probably some of them being much easier than using the Chernoff-bound.

5. (a)

$$\begin{aligned}
P(\epsilon) &\leq \sum_{k=1}^{K} P(\bigcup_{t=1}^{\infty}\{|\hat{\mu}_{k,t}-\mu_k| > U(t,\delta/K)\}) \text{ by union bound} \\
&\leq \sum_{k=1}^{K} \delta/K \text{ by the property of an anytime confidence interval} \\
&= \delta
\end{aligned}$$

(b) The only way the algorithm can fail, is if at some iteration i, there is an arm k such that

$$\hat{\mu}_{k,t} - U(t,\delta/K) > \hat{\mu}_{k^*,t} + U(t,\delta/K)$$

However,

$$\begin{aligned}
P(\bigcup_{k=1,k\neq k^*}^{K}\bigcup_{t=1}^{\infty}\hat{\mu}_{k,t} - U(t,\delta/K) > \hat{\mu}_{k^*,t} + U(t,\delta/K)) &\leq P(\bigcup_{k=1,k\neq k^*}^{K}\bigcup_{t=1}^{\infty}\hat{\mu}_{k,t} - U(t,\delta/K) > \hat{\mu}_{k^*,t} + U(t,\delta/K)) \\
&\leq P(\bigcup_{k=1,k\neq k^*}^{K}\bigcup_{t=1}^{\infty}(\hat{\mu}_{k,t} - \mu_k > U(t,\delta/K))\cup \\
&(\mu_k^* - \hat{\mu}_{k^*,t} > U(t,\delta/K))) \\
&\leq P(\bigcup_{k=1,k\neq k^*}^{K}\bigcup_{t=1}^{\infty}(|\hat{\mu}_{k,t} - \mu_k| > U(t,\delta/K))\cup \\
&(|\hat{\mu}_{k,t} - \mu_k^*| > U(t,\delta/K))) \\
&\leq P(\bigcup_{k=1}^{K}\bigcup_{t=1}^{\infty}(\{|\hat{\mu}_{k,t} - \mu_k| > U(t,\delta/K)\}) \\
&\leq \delta \text{ by part a)}
\end{aligned}$$

(c)

$$\begin{aligned}
P(\bigcup_{t=1}^{\infty}|\hat{\mu}_t - \mu| \geq U(t,\delta)) &\leq \sum_{t=1}^{\infty} P(|\hat{\mu}_t - \mu| \geq U(t,\delta)) \text{ by union bound} \\
&\leq \sum_{t=1}^{\infty} P(|\sum_{i=1}^{t}\frac{1}{t}Z_i - \mu| \geq U(t,\delta)) \\
&\leq \sum_{t=1}^{\infty} 2e^{\frac{-2U(t,\delta)^2}{t(\frac{b}{t}-\frac{a}{t})^2}} \text{ by two-sided Hoeffding-inequality} \\
&= \sum_{t=1}^{\infty} 2e^{\frac{-2\frac{(b-a)^2\log(\frac{4t^2}{\delta})}{2t}}{t(\frac{b}{t}-\frac{a}{t})^2}} \\
&= \sum_{t=1}^{\infty} 2e^{\log(\frac{\delta}{4t^2})} \\
&= \sum_{t=1}^{\infty} 2\frac{\delta}{4t^2} \\
&= \frac{\delta}{2}\sum_{t=1}^{\infty}\frac{1}{t^2} \\
&= \frac{\delta}{2}\frac{\pi^2}{6} \text{ famous Basel problem} \\
&\leq \delta
\end{aligned}$$

(d) (Bonus)