Spectral properties of large dimensional random matrices

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Introduction

- Random matrices turn out to be accurate models for a large number of mathematical and physical problems.

Example (Quantum many-body problem and highly correlated systems)

$\psi = U\psi_0$, U is the evolution (unitary) operator. Large number of degrees of freedom, interactions $\rightarrow$ hopless to find analytical expression for U. Idea: replace U (or the Hamiltonian) by a random matrix of finite, but large, dimension. What are the matrix space and probability distribution that best model our system?

- In physics, the symmetries of a problem are often known a priori, even if the details of the dynamics remain obscure. $\rightarrow$ The corresponding symmetries are the only constraints we can impose to these random matrices.

Classical ensembles

- **Gaussian orthogonal ensemble** (GOE(n), $\beta = 1$) is a space of $n \times n$ real symmetric matrices equipped with a measure invariant under orthogonal conjugation $M \to OMO^T \; \forall O \in O(\mathbb{R}, n)$. Models Hamiltonians with time-reversal and/or rotational symmetry.
- **Gaussian unitary ensemble** (GUE(n), $\beta = 2$) is a space of $n \times n$ hermitian matrices equipped with a measure invariant under unitary conjugation $M \to U^{-1}MU \; \forall U \in U(\mathbb{C})$. Models Hamiltonians lacking time-reversal symmetry.
- **Gaussian symplectic ensemble** (GSE(n), $\beta = 4$) is a space of $n \times n$ symetric quaternionic matrices equipped with a measure invariant under unitary conjugation $M \to W^{\dagger}MW \; \forall W \in Sp(n)$. Models Hamiltonians with time-reversal symmetry but no rotational symmetry.

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Fundamental belief of universality of random matrices

- We study statistical properties of the eigenvalue spectrum of such random matrices, typically in the limit of large matrix dimension.
- To compare different systems in the same symmetry class, the eigenvalues of the system need to be rescaled ("unfolded").
- Spectral properties can be studied at different scales (local/global limit). Is the obtained limit universal (independent on the chosen potential)?

General phenomenon:

The macroscopic statistics (spectral density) depend on the models. Microscopic statistics (spectral fluctuations) are universal i.e. independent on the details of the systems except the symmetries.

Level spacing

- Let $P(s)$ be the probability density to find two adjacent eigenvalues at a distance s. P(s) probes the strength of the eigenvalues repulsion due to interactions.
- Can be computed analytically for the classical ensembles. Rather complicated expressions (prolate spheroidal functions). For most practical purpose it is sufficient to use Wigner surmise (1956):

Idea: guess the probability law $P(s)$ from the eigenvalues of 2x2 matrices in the same symmetry class. It is given by

$$P_\beta(s) = a_\beta s^\beta e^{-b_\beta s^2}$$

with $\beta = 1, 2, 4$ (GOE, GUE, GSE). The quantities a_β and b_β are chosen such that

$$\int_0^{+\infty} ds P_\beta(s) = 1 \quad \text{and} \quad \langle s \rangle = \int_0^{+\infty} ds P_\beta(s) s = 1$$

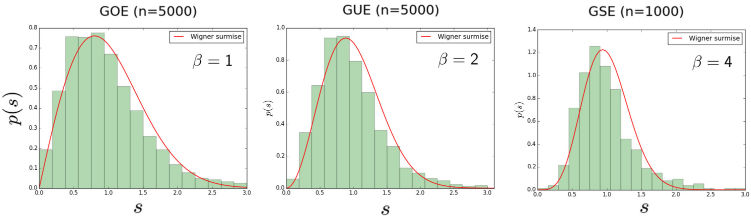
- In contrast, $P(s)$ of a quantum system whose classical counterpart is integrable is given by a Poisson process, $P_0(s) = e^{-s}$, corresponding to uncorrelated eigenvalues.

Level spacing

- Explicit formulas:

$$
\begin{aligned}
P_1(s) &= \frac{\pi}{2} s \exp\left(-\frac{\pi}{4} s^2\right) \\
P_2(s) &= \frac{32}{\pi^2} s^2 \exp\left(-\frac{4}{\pi} s^2\right) \\
P_4(s) &= \frac{2^{18}}{3^6 \pi^3} s^4 \exp\left(-\frac{64}{9\pi} s^2\right)
\end{aligned}
$$

- Numerical test, with rescaling $s \leftarrow \frac{s}{\langle s \rangle}$:



Wigner Semi-cercle Law

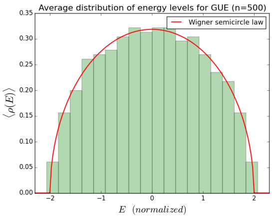
Theorem

Let $X \in$ GUE(n) such that $\mathbb{E}(X_{ij}) = 0$ and $\mathbb{E}(X_{ij}^2) = \sigma^2$. For $k \in \mathbb{N}^*$, we assume that $\sup_{i,j} \mathbb{E}\left(|X_{ij}|^k\right) < \infty$. Then the empirical distribution of eigenvalues of the matrix $\frac{X_{i,j}}{\sqrt{n}}$ converges almost surely to the semi-circle law:

$$\lim_{n\to\infty} \frac{1}{n}\mathrm{tr}\mathbb{E}\left(\frac{X}{\sqrt{n}}\right)^k = \frac{1}{2\pi\sigma^2}\int_{-2\sigma}^{2\sigma} x^k\sqrt{4\sigma^2 - x^2}dx$$

In particular, upon a suitable normalization ($E \leftarrow \frac{E}{\sqrt{n}}$), the spectral density $\rho(E)$ converges to $\frac{1}{2\pi}\sqrt{4 - E^2}$ in the interval $[-2, 2]$ as $n \to +\infty$.

Result is true as well for GOE and GSE.



- Let X be a random *nxn* matrix from the Gaussian ensemble of index $\beta = 1, 2, 4$, $F_{\beta,n}(s) = \mathbb{P}(\lambda_{max} \leq s)$, the partition function of its largest eigenvalue $\lambda_{max}(\sim 2\sigma\sqrt{n})$ and σ, the standard deviation of the Gaussian distribution of its off-diagonal elements.

- Eigenvalues rescaling (for GUE): $\lambda \leftarrow \left(\lambda - 2\sqrt{n}\right) n^{1/6}$

- Tracy and Widom: the limit

$$F_\beta(s) := \lim_{n\to+\infty} F_{\beta,n}\left(2\sigma\sqrt{n} + \frac{\sigma s}{n^{1/6}}\right) \quad \text{exists.}$$

$f_\beta(s) := \frac{dF_\beta(s)}{ds}$ is called Tracy-Widom distribution.

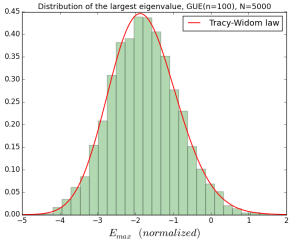
Extremal eigenvalues and Tracy-Widom Law

- For $\beta = 2$, $\sigma = 1/\sqrt{2}$ and

$$F_2(s) = \exp\left(-\int_s^{+\infty} (x-s) q^2(x) dx\right)$$

where $q(x)$ is a solution of the Painlevé II differential equation:

$$q'' = xq + 2q^3$$



with the asymptotic condition:

$$\lim_{x\to+\infty} q(x) = \mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{t^3}{3} + xt\right) dt \quad \text{(Airy function)}$$

- For other Gaussian ensembles

$$\begin{aligned} F_1(s) &= \exp\left(-\frac{1}{2}\int_s^{+\infty} q(x)dx\right)\sqrt{F_2(s)} \\ F_4(s) &= \cosh\left(\frac{1}{2}\int_s^{+\infty} q(x)dx\right)\sqrt{F_2(s)} \end{aligned}$$

Conclusion and perspectives

- Random matrix theory describes universal quantities determined by global symmetries that are shared by all systems in a given symmetry class.
- The study of their spectral properties finds a lot of applications: Mathematics (gap distribution of zeros of Riemann ζ-function is given by GUE, graph theory, analytic number theory, combinatorics), quantum gravity, string theory, quantum chaos, quantum information, nuclear physics, statistical mechanics, wireless telecommunications, biology, finance, traffic, etc.
- Often physical systems consist of parts with different symmetries, or of a classically integrable and a chaotic part. Changing a parameter of the system may then result in transitions between different symmetry classes. The symmetry transition in a given physical system can be described by a transition between ensembles (or Poisson).
  Example: transition between GOE and GUE behavior takes place in the spectrum of a kicked rotor when time-reversal symmetry is gradually broken.