31 T-1

$\xi \sim R(0, \theta)$ вер. $\theta > 0$ $\rightarrow 100$

$\vec{X}_n = (X_1, \dots, X_n)$ $X_i \sim \xi$

$\tilde{\theta}_1 = \frac{2}{n} \sum_{i=1}^{n} X_i$ $\tilde{\theta}_4 = X_{min} + X_{max}$

$\tilde{\theta}_2 = X_{min}$ $\tilde{\theta}_5 = X_1 + \frac{1}{n-1} \sum_{i=2}^{n} X_i$

$\tilde{\theta}_3 = X_{max}$

Исслед. на несмещ. и состоят., выбрать эффект.

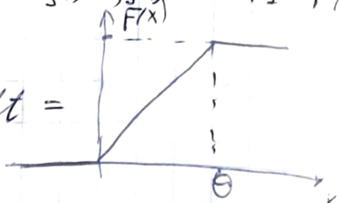
$M[\xi] = \frac{\theta}{2}$

$M[\xi^2] = \int_0^{\theta} x \frac{1}{\theta} dx = \frac{\theta^2}{3}$ $D[\xi] = \frac{\theta^2}{12}$

1) $M[\tilde{\theta}_1] = \frac{2}{n} \sum_{i=1}^{n} MX_i = 2 M\xi = \theta$ $\Rightarrow$ несмещ.

$D[\tilde{\theta}_1] = \frac{4}{n^2} \sum DX_i = \frac{4}{n} D\xi = \frac{\theta^2}{3n} \xrightarrow[n \to \infty]{} 0$ $\Rightarrow$ сост. по дост. усл.

$\xi \sim F(x)$ незав.
$\min(\xi_1, \dots, \xi_n) \sim 1-(1-F(y))^n$



2) $M[\tilde{\Theta}_2] = M[X_{min}] = \int_0^\theta y\,\varphi(y)\,dy =$

$= \int_0^\theta y n \left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\,dy \underset{t=\frac{y}{\theta}}{=} \int_0^1 t n (1-t)^{n-1} \theta\,dt =$

$= n\theta B(2,n) = n\theta \frac{\Gamma(2)\Gamma(n)}{\Gamma(n+2)} = \frac{n\theta}{n(n+1)} = \frac{\theta}{n+1}$

$\Gamma(n+2) = (n+1)n\Gamma(n)$

$\varphi(y) = \Phi'(y) = n(1-F(y))^{n-1} p(y)$

$p(y) = \frac{1}{\theta}\ (0, \theta)$

$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$

$\tilde{\Theta}_2' = (n+1)\tilde{\Theta}_2$ — несмещ.

$M[\tilde{\Theta}_2^2] = \int_0^\theta y^2 n \left(1-\frac{y}{\theta}\right)^{n-1} \frac{1}{\theta}\,dy \underset{t=\frac{y}{\theta}}{=}$

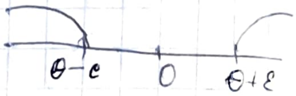
$= \int_0^1 t^2 n (1-t)^{n-1} \theta^2\,dt = n\theta^2 B(3,n) =$

$= n \frac{\theta^2\,\Gamma(3)\Gamma(n)}{\Gamma(n+3)} = \frac{2\theta^2}{(n+2)(n+1)}$

$\Gamma(n+3) = (n+2)(n+1)n\Gamma(n)$

$D[\tilde{\Theta}_2] = \frac{2\theta^2}{(n+2)(n+1)} - \frac{\theta^2}{(n+1)^2} = \theta^2 \frac{n}{(n+2)(n+1)^2}$

$D[\tilde{\Theta}_2'] = (n+1)^2 D[\tilde{\Theta}_2] = \theta^2 \frac{n}{n+2} \xrightarrow[n\to\infty]{} \theta^2 \neq 0$



По опр-ю: $\exists\ \tilde{\Theta}_2' \xrightarrow{P} \theta \Rightarrow P(|\tilde{\Theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

$P(|\tilde{\Theta}_2' - \theta| \geq \varepsilon) \geq \underset{\text{один хвост}}{P}(\tilde{\Theta}_2' \geq \theta+\varepsilon) = P((n+1)\tilde{\Theta}_2 \geq \theta+\varepsilon) = P\left(\tilde{\Theta}_2 \geq \frac{\theta+\varepsilon}{n+1}\right) =$

$= P\left(X_1 \geq \frac{\theta+\varepsilon}{n+1}, \dots, X_n \geq \frac{\theta+\varepsilon}{n+1}\right) = \prod_i P\left(X_i \geq \frac{\theta+\varepsilon}{n+1}\right) = \left(P\left(\xi \geq \frac{\theta+\varepsilon}{n+1}\right)\right)^n =$ (X_{min})

$= \left(1 - F\left(\frac{\theta+\varepsilon}{n+1}\right)\right)^n = \left(1 - \frac{\theta+\varepsilon}{(n+1)\theta}\right)^n \longrightarrow e^{-\frac{\theta+\varepsilon}{\theta}} > 0$ (?!) $\Rightarrow$ не явл. сост.

3) $M[\tilde{\Theta}_3] = M[X_{max}]$

$\xi \sim F(x)$

$\max(\xi_1, \dots, \xi_n) \sim (F(z))^n = \Psi(z)$

$\psi(z) = nF^{n-1}(z)\,p(z) = n\left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta}\ (0, \theta)$

$M[\tilde{\Theta}_3] = \int_0^\theta z n \left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta}\,dz = n\theta \int_0^1 t^n\,dt = \frac{n}{n+1}\theta$

$\tilde{\Theta}_3' = \frac{n+1}{n} X_{max}$ несмещ.

$M[\tilde{\Theta}_3^2] = \int_0^\theta z^2 n \left(\frac{z}{\theta}\right)^{n-1} \frac{1}{\theta} = \int_0^1 n\theta^2 t^{n+1}\,dt = \frac{n}{n+2}\theta^2$

$D[\tilde{\Theta}_3] = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2\left[\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right] = \theta^2 \frac{n^3+2n^2+n-n^3-2n^2}{(n+2)(n+1)^2} = \frac{n\theta^2}{(n+2)(n+1)^2}$

$D[\tilde{\Theta}_3'] = D\left[\frac{n+1}{n}\tilde{\Theta}_3\right] = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$ по дост. усл. сост.

4) $M[\tilde{\Theta}_4] = M[\tilde{\Theta}_2] + M[\tilde{\Theta}_3] = \frac{\theta}{n+1} + \frac{n\theta}{n+1} = \theta \Rightarrow$ несмещ.

$D[\tilde{\Theta}_4] = D[\tilde{\Theta}_2] + D[\tilde{\Theta}_3] + 2\,\mathrm{cov}(\tilde{\Theta}_2, \tilde{\Theta}_3) =$

$\mathrm{cov}(\tilde{\Theta}_2, \tilde{\Theta}_3) = M(\tilde{\Theta}_2\tilde{\Theta}_3) - M(\tilde{\Theta}_3)M(\tilde{\Theta}_2)$

$K(y,z) = \begin{cases} F^n(z) - (F(z)-F(y))^n, & z \geq y \\ F^n(z), & z < y \end{cases}$

$$\varkappa(y,z)=\frac{\partial^2 K}{\partial y\,\partial z}=\frac{\partial}{\partial z}\left(n\left(F(z)-F(y)\right)^{n-1}F'(y)\right)=n(n-1)\left(F(z)-F(y)\right)^{n-2}\underset{\frac{1}{\theta}}{F'(y)}\,\underset{\frac{1}{\theta}}{F'(z)}$$

$$M(\tilde\theta_2\tilde\theta_3)=\iint_{-\infty} yz\,\varkappa(y,z)\,dy\,dz=\int_0^\theta z\,dz\int_0^z y\cdot n(n-1)\left(\frac{z}{\theta}-\frac{y}{\theta}\right)^{n-2}\frac{1}{\theta^2}\,dy\ \underset{t=\frac{y}{z}}{=}$$

$$=\int_0^\theta dz\int_0^1 tz^2\,n(n-1)\left(\frac{z}{\theta}-\frac{tz}{\theta}\right)^{n-2}\frac{1}{\theta^2}\,z\,dt=\int_0^\theta dz\int_0^1 tz^2\,\frac{z^{n-2}}{\theta^{n-2}}(1-t)^{n-2}\frac{1}{\theta^2}\,z\,dt=$$

$$=\int_0^\theta dz\ n(n-1)\frac{z^{n+1}}{\theta^n}\underbrace{\int_0^1 t(1-t)^{n-2}dt}_{B(2,n-1)=\frac{\Gamma(n-1)}{\Gamma(n+1)}=\frac{\Gamma(n-1)}{n(n-1)\Gamma(n-1)}}=\int_0^\theta\frac{z^{n+1}}{\theta^n}\,dz=\frac{1}{n+2}\theta^2$$

$$\operatorname{cov}(\tilde\theta_2,\tilde\theta_3)=\frac{1}{n+2}\theta^2-\frac{n}{n+1}\theta\cdot\frac{1}{n+1}\theta=\theta^2\left(\frac{1}{n+2}-\frac{n}{(n+1)^2}\right)=\theta^2\frac{1}{(n+2)(n+1)^2}$$

$$D[\tilde\theta_4]=\theta^2\left(\frac{n}{(n+1)^2(n+2)}+\frac{n}{(n+1)^2(n+2)}+\frac{2}{(n+2)(n+1)^2}\right)=\frac{2\theta^2}{(n+2)(n+1)}\xrightarrow[n\to\infty]{}0\Rightarrow\tilde\theta_4\text{ сост.}$$

5) $M[\tilde\theta_5]=M[X_1]+\frac{1}{n-1}(n-1)M[X_2]=\frac{\theta}{2}+\frac{\theta}{2}=\theta\Rightarrow$ несмещ.

$$D[\tilde\theta_5]=D[X_1]+\frac{1}{(n-1)^2}(n-1)D[\xi]=\frac{\theta^2}{12}\left(1+\frac{1}{n-1}\right)\xrightarrow[n\to\infty]{\not\to}0$$

$$\tilde\theta_5=\underset{\xrightarrow{p}X_1}{X_1}+\underbrace{\frac{1}{n-1}\sum_{k=2}^n X_k}_{\xrightarrow{p}M\tilde\theta_5=\frac{\theta}{2}\ (\text{ЗБЧ Хинчина})}\xrightarrow{p}X_1+\frac{\theta}{2}\Rightarrow\text{ не явл. сост.}$$

6) Выберем самую эффект. (из 1, 3, 4)

$$\frac{\theta^2}{3n},\quad\frac{\theta^2}{n(n+2)},\quad\frac{2\theta^2}{(n+2)(n+1)}$$

(3) и (4)

$\frac{1}{n},\ \frac{2n}{n+1}\qquad n+1<2n,\ n>1\ \Rightarrow(3)<(4)$

(1) и (3)

$\frac{1}{3}\quad\frac{1}{n+2}\qquad n+2>3,\ n>1\Rightarrow(3)<(1)$