Задача №4 $\xi \sim R(\theta, 2\theta)$

а) Оценки:

ММП:

$$\alpha_1 = \int_\theta^{2\theta} x \cdot \frac{1}{\theta}\, dx = \frac{3}{2}\theta = M\xi$$

$$\frac{3}{2}\theta = \bar{x} \Rightarrow \tilde{\theta}_1 = \frac{2}{3}\bar{x}$$

ОММП:

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \left(\frac{1}{\theta}\right)^n \quad (\theta; 2\theta) \Rightarrow \begin{cases} \theta \le x_{min} \\ 2\theta \ge x_{max} \end{cases}$$

$$\tilde{\theta}_2 = \frac{1}{2} x_{max}$$

б) 1) Несмещённость

$$M\tilde{\theta}_1 = \frac{2}{3} M\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{3} M\xi = \theta \Rightarrow \text{несмещ.}$$

$$M\tilde{\theta}_2 = \frac{1}{2} M[x_{max}] = \frac{1}{2}\int_\theta^{2\theta} y \cdot n(\varphi(y))^{n-1} q(y)\, dy =$$

$$= \frac{n}{2\theta}\int_\theta^{2\theta} y \left(\frac{y-\theta}{\theta}\right)^{n-1} dy = \frac{n}{2\theta}\left[\frac{\theta}{n}\left(y\left(\frac{y-\theta}{\theta}\right)^n\Big|_\theta^{2\theta} - \int_\theta^{2\theta}\left(\frac{y-\theta}{\theta}\right)^n dy\right)\right] = \frac{2n+1}{2(n+1)}\theta$$

$$\Rightarrow \text{смещ.}, \quad \tilde{\theta}_2' = \frac{2(n+1)}{2n+1}\tilde{\theta}_2 = \frac{n+1}{2n+1} x_{max}$$

$$M\tilde{\theta}_3 = \frac{1}{5} M[x_{min}] + \frac{2}{5} M[x_{max}]$$

$$M[x_{min}] = \int_\theta^{2\theta} y \cdot \left(1 - \frac{y-\theta}{\theta}\right)^{n-1} \frac{n}{\theta}\, dy = -y\left(\frac{2\theta - y}{\theta}\right)^2 + \int_\theta^{2\theta}\left(\frac{2\theta - y}{\theta}\right)^2 dy =$$

$$= \frac{\theta(n+2)}{n+1}$$

$$M\tilde{\theta}_3 = \frac{1}{5}\theta\frac{n+2}{n+1} + \frac{2}{5}\theta\frac{2n+1}{n+1} = \frac{\theta}{5}\cdot\frac{5n+4}{n+1} \Rightarrow \text{смещ.}$$

$$\tilde{\theta}_3' = \frac{5(n+1)}{5n+4}\tilde{\theta}_3 = \frac{n+1}{5n+4}(x_{min} + 2x_{max})$$

2) Состоятельность

$$D\tilde{\theta}_1 = \frac{4}{9n} D\xi = \frac{4}{9n}\left(\frac{7}{3}\theta^2 - \frac{9}{4}\theta^2\right) = \frac{1}{9n}\cdot\frac{1}{12}\theta^2 = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ сост. по достат. усл-ю

$$D\tilde{\theta}_2 = \frac{1}{4} D[x_{max}] = \frac{1}{4}\left[\frac{4n^2+8n+2}{(n+1)(n+2)}\theta^2 - \frac{(2n+1)^2}{(n+1)^2}\theta^2\right] = \frac{1}{4(n+1)}\left[\frac{4n^2+8n+2}{n+2} - \frac{(2n+1)^2}{n+1}\right]\theta^2 =$$

$$= \frac{n\theta^2}{4(n+1)^2(n+2)}$$

$$D\tilde{\theta}_2' = \frac{4(n+1)^2}{(2n+1)^2}\cdot\frac{n\theta^2}{4(n+1)^2(n+2)} = \frac{n\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост. по достат. усл-ю}$$

$$D\tilde{\theta}_3 = \frac{1}{25}\left(D[x_{min}] + 4D[x_{max}] + 4\,\underline{cov(x_{min}, x_{max})}\right)$$

$$M[x_{min}x_{max}] - M[x_{min}]M[x_{max}]$$

$$g(y,z) = n(n-1)(\varphi(z) - \varphi(y))^{n-2} q(y) q(z)$$

$$M[x_{min}x_{max}] = \iint_{-\infty}^{+\infty} yz\, g(y,z)\, dy\, dz = \frac{2n^2+7n+5}{(n+1)(n+2)}$$

$$D[X_{min}] = \frac{1}{\theta}\int_{\theta}^{2\theta} y^2 \cdot n\left(1-\frac{y-\theta}{\theta}\right)^{n-1} dy - \frac{1}{4}\left(\int_{\theta}^{2\theta} yn\left(1-\frac{y-\theta}{\theta}\right)^{n-1} dy\right)^2 =$$

$$= \frac{\theta^2(n^2+5n+8)}{(n+1)(n+2)} - \frac{\theta^2(n+2)^2}{(n+1)^2} = \frac{n\theta^2}{(n+1)^2(n+2)^2}$$

$$D\tilde{\theta}_3 = \frac{1}{25}\left[\frac{n\theta^2}{(n+1)^2(n+2)} + \frac{4n\theta^2}{(n+1)^2(n+2)} + \frac{4\theta^2(2n^2+7n+5)}{(n+1)(n+2)} - \frac{4\theta^2(2n+1)(n+2)}{(n+1)^2}\right] =$$

$$= \frac{\theta^2(5n+4)}{25(n+1)^2(n+2)}$$

$$D\tilde{\theta}_3' = \frac{25(n+1)^2}{(5n+4)^2} \cdot \frac{(5n+4)\theta^2}{25(n+1)^2(n+2)} = \frac{\theta^2}{(5n+4)(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$ сост. по дост. усл.

(c) $D\tilde{\theta}_2' = \frac{n\theta^2}{4(n+1)^2(n+2)}$ $\quad D\tilde{\theta}_3' = \frac{\theta^2}{(5n+4)(n+2)}$

$$\frac{n\theta^2}{4(n+1)^2(n+2)} \qquad \frac{\theta^2}{(5n+4)(n+2)} \qquad \frac{n}{5n^2+4n}$$

$$\frac{1}{4n^2+8n+4} \underset{n\to\infty}{>} \frac{1}{5n^2+4n}$$

$\Rightarrow \tilde{\theta}_3'$ более эфф. при больших n

$$\frac{D\tilde{\theta}_1}{D\tilde{\theta}_3} = \frac{(n+1)(5n+4)}{27n} \xrightarrow[n\to\infty]{} \infty \Rightarrow$$ $\tilde{\theta}_3$ – наиболее ас. эфф.

(e) $\tilde{\theta} = \frac{2}{3}\bar{X}$ (из ОММ)

ЦПТ: $\frac{\bar{X} - M\xi}{\sqrt{D\xi}}\sqrt{n} \rightsquigarrow N(0,1)$

$$\frac{\frac{3}{2}(\tilde{\theta}-\theta)\sqrt{12n}}{\theta} \rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < \frac{3(\tilde{\theta}-\theta)\sqrt{12n}}{2\theta} < u_{\frac{1+\beta}{2}} \Rightarrow u_{\frac{1-\beta}{2}} + 3\sqrt{3n} < \frac{\tilde{\theta}}{\theta}\cdot 3\sqrt{3n} < u_{\frac{1+\beta}{2}} + 3\sqrt{3n}$$

$$\frac{3\sqrt{3n}\,\tilde{\theta}}{(u_{\frac{1+\beta}{2}} + 3\sqrt{3n})} < \theta < \frac{3\sqrt{3n}\,\tilde{\theta}}{(u_{\frac{1-\beta}{2}} + 3\sqrt{3n})}, \quad \tilde{\theta} = \frac{2}{3}\bar{x}$$

$\tilde{\theta} = \frac{n+1}{2n+1} X_{max}$ (из ОМП)

$$\frac{\tilde{\theta}-\theta}{\sigma}\sqrt{n} \rightsquigarrow N(0,1)$$

$$\sigma = \sqrt{\nabla^T\theta \cdot I^{-1}(\theta)\,\nabla\theta} = \sqrt{I^{-1}(\theta)}$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = \int_{\theta}^{2\theta}\frac{1}{\theta}\cdot\frac{1}{\theta^2}dx = \frac{1}{\theta^2} \Rightarrow \sigma = \left(\frac{1}{\theta}\right)^{-1} = \theta$$

$$\left(\frac{\tilde{\theta}}{\theta} - 1\right)\sqrt{n} \rightsquigarrow N(0,1)$$

$\frac{1}{\sqrt{n}}u_{\frac{1+\beta}{2}} < \frac{\tilde{\theta}}{\theta} - 1 < \frac{1}{\sqrt{n}}u_{\frac{1+\beta}{2}} \Rightarrow \frac{\tilde{\theta}}{u_{\frac{1-\beta}{2}}\frac{1}{\sqrt{n}} + 1} > \theta > \frac{\tilde{\theta}}{u_{\frac{1+\beta}{2}}\frac{1}{\sqrt{n}} + 1}$, $\tilde{\theta} = \frac{n+1}{2n+1} x_{max}$

① $y_i = \frac{x_i}{\theta} - 1 \sim R(0,1)$

$y_{max} = \frac{x_{max}}{\theta} - 1 = G(x,\theta)$ → ф-ция распределения:

$F(y) = \begin{cases} 0, & y \le 0 \\ y^n, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases}$ $\quad p(y) = n y^{n-1} \quad [0,1]$

$P(g_1 < G(x,\theta) < g_2) = P\left(\frac{x_{max}}{g_2+1} < \theta < \frac{x_{max}}{g_1+1}\right)$ $\quad max \Rightarrow g_2 = 1$

$\beta = P(g_1 < y_{max} < 1) = F(1) - F(g_1) = 1 - g_1^n = \beta \to g_1 = \sqrt[n]{1-\beta}$

$\frac{x_{max}}{1+\sqrt[n]{\beta}} < \theta < \frac{x_{max}}{1+\sqrt[n]{1-\beta}}$

Задача №5

$p(x) = \begin{cases} \frac{\theta-1}{x^\theta}, & x \ge 1 \\ 0, & x < 1 \end{cases}$ $\quad \theta > 1$

② $L(\theta) = \prod_{i=1}^{n} p(x_i,\theta) = (\theta-1)^n \cdot \prod_{i=1}^{n} \frac{1}{x_i^\theta}$, $\quad \min x_i \ge 1$

$\max L(\theta) = (\theta-1)^n$

$\ln L(\theta) = n \ln(\theta-1) - \theta \sum_{i=1}^{n} \ln x_i$

$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta-1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$

$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{n}{(\theta-1)^2} \Big|_{\theta=\tilde{\theta}} = -\sum_{i=1}^{n} \ln x_i \cdot \frac{1}{n} < 0 \Rightarrow max$

③ $\frac{f(\tilde{\theta}) - f(\theta)}{\sigma}\sqrt{n} \rightsquigarrow N(0,1)$

$f(\theta) = x_{med}$: $\quad F(x_{med}) = \int_{-\infty}^{x_{med}} \frac{\theta-1}{t^\theta}\,dt = \frac{(\theta-1)(x_{med}^{\theta} - x_{med})}{x_{med}^{\theta}(\theta-1)} = \frac{1}{2}$

$1 - x_{med}^{1-\theta} = \frac{1}{2} \to x_{med}^{1-\theta} = \frac{1}{2} \to x_{med} = \left(\frac{1}{2}\right)^{\frac{1}{1-\theta}} = 2$

$f'(\theta) = 2^{\frac{1}{\theta-1}} \cdot \ln 2 \cdot \frac{(-1)}{(1-\theta)^2} = -\ln 2 \cdot 2^{\frac{1}{\theta-1}} \cdot \frac{1}{(\theta-1)^2}$

$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial \theta}\right)^2\right] = \int_{1}^{+\infty} \frac{\theta-1}{x^\theta} \cdot \left(\frac{1}{\theta-1} - \ln x\right)^2 dx = \frac{1}{(\theta-1)^2}$

$$\frac{2^{\frac{1}{\Theta-1}} - 2^{\frac{1}{\Theta-1}}}{+\ln 2 \cdot 2^{\frac{1}{\Theta-1}} \left(\frac{1}{\Theta-1}\right)^2 \cdot (\Theta-1)} = \frac{(\Theta-1)}{\ln 2}\left(1 - \underbrace{2^{\frac{1}{\Theta-1}}}_{x_{med}} - \frac{1}{\Theta-1}\right)$$

$$2^{\frac{1}{\Theta-1}}\left(1 - \frac{u_{\frac{1+\beta}{2}} \cdot \ln 2}{\sqrt{n}(\Theta-1)}\right) < x_{med} < 2^{\frac{1}{\Theta-1}}\left(1 - \frac{u_{\frac{1-\beta}{2}} \cdot \ln 2}{(\Theta-1)\sqrt{n}}\right)$$

d) $(\tilde{\Theta} - \Theta)\sqrt{I(\Theta)} \cdot \sqrt{n} \rightsquigarrow N(0,1)$

$(\tilde{\Theta} - \Theta) \cdot \frac{\sqrt{n}}{\Theta-1} \rightsquigarrow N(0,1)$

$u_{\frac{1-\beta}{2}} < \frac{\sqrt{n}}{\Theta-1}(\tilde{\Theta} - \Theta) < u_{\frac{1+\beta}{2}}$

$\tilde{\Theta} - \frac{(\tilde{\Theta}-1)u_{\frac{1+\beta}{2}}}{\sqrt{n}} < \Theta < \tilde{\Theta} - \frac{(\tilde{\Theta}-1)u_{\frac{1-\beta}{2}}}{\sqrt{n}}$, $\tilde{\Theta} = 1 + \frac{n}{\sum_{i=1}^{n} \ln x_i}$