Задача №4 $\quad \xi \sim R(\theta, 2\theta)$

ⓐ ОММ:

$$\alpha_1 = \int_{\theta}^{2\theta} x \cdot \frac{1}{\theta}\, dx = \frac{3}{2}\theta = M\xi$$

$$\frac{3}{2}\theta = \overline{X} \Rightarrow \tilde{\theta}_1 = \frac{2}{3}\overline{X}$$

ОМП:

$$L(\theta) = \prod_{i=1}^{n} p(x_i, \theta) = \left(\frac{1}{\theta}\right)^n \quad (\theta; 2\theta) \Rightarrow \begin{cases} \theta \le X_{min} \\ 2\theta \ge X_{max} \end{cases}$$

$$\tilde{\theta}_2 = \frac{1}{2} X_{max}$$

ⓑ 1) Несмещённость

$$M\tilde{\theta}_1 = \frac{2}{3} M\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{2}{3} M\xi = \theta \Rightarrow \text{несмещ.}$$

$$M\tilde{\theta}_2 = \frac{1}{2} M[X_{max}] = \frac{1}{2}\int_{\theta}^{2\theta} y \cdot n(\Phi(y))^{n-1} q(y)\, dy =$$

$$= \frac{n}{2\theta}\int_{\theta}^{2\theta} y\left(\frac{y-\theta}{\theta}\right)^{n-1} dy = \frac{n}{2\theta}\left[\frac{\theta}{n}\left(y\left(\frac{y-\theta}{\theta}\right)^n\Big|_{\theta}^{2\theta} - \int_{\theta}^{2\theta}\left(\frac{y-\theta}{\theta}\right)^n dy\right)\right] = \frac{2n+1}{2(n+1)}\theta$$

$$\Rightarrow \text{смещ.} \quad \tilde{\theta}_2' = \frac{2(n+1)}{2n+1}\tilde{\theta}_2 = \frac{n+1}{2n+1} X_{max}$$

$$M\tilde{\theta}_3 = \frac{1}{5} M[X_{min}] + \frac{2}{5} M[X_{max}]$$

$$M[X_{min}] = \int_{\theta}^{2\theta} y \cdot \left(1 - \frac{y-\theta}{\theta}\right)^{n-1}\frac{n}{\theta}\, dy = -y\left(\frac{2\theta-y}{\theta}\right)^2 + \int_{\theta}\left(\frac{2\theta-y}{\theta}\right)^2 dy =$$

$$= \frac{\theta(n+2)}{n+1}$$

$$M\tilde{\theta}_3 = \frac{1}{5}\theta\frac{n+2}{n+1} + \frac{2}{5}\theta\frac{2n+1}{n+1} = \frac{\theta}{5}\cdot\frac{5n+4}{n+1} \Rightarrow \text{смещ.}$$

$$\tilde{\theta}_3' = \frac{5(n+1)}{5n+4}\tilde{\theta}_3 = \frac{n+1}{5n+4}(X_{min} + 2X_{max})$$

2) Состоятельность

$$D\tilde{\theta}_1 = \frac{4}{9n} D\xi = \frac{4}{9n}\left(\frac{7}{3}\theta^2 - \frac{9}{4}\theta^2\right) = \frac{4}{9n}\cdot\frac{1}{12}\theta^2 = \frac{\theta^2}{27n} \xrightarrow[n\to\infty]{} 0$$

$\Rightarrow$ сост. по дост. усл-ю

$$D\tilde{\theta}_2 = \frac{1}{4}D[X_{max}] = \frac{1}{4}\left[\frac{4n^2+8n+2}{(n+1)(n+2)}\theta^2 - \frac{(2n+1)^2}{(n+1)^2}\theta^2\right] = \frac{1}{4(n+1)}\left[\frac{4n^2+8n+2}{n+2} - \frac{(2n+1)^2}{n+1}\right]$$

$$= \frac{n\theta^2}{4(n+1)^2(n+2)}$$

$$D\tilde{\theta}_2' = \frac{\cancel{4}\cancel{(n+1)^2}}{(2n+1)^2}\cdot\frac{n\theta^2}{\cancel{4}\cancel{(n+1)^2}(n+2)} = \frac{n\theta^2}{(2n+1)^2(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow \text{сост. по дост. усл-ю}$$

$$D\tilde{\theta}_3 = \frac{1}{25}\left(D[X_{min}] + 4D[X_{max}] + 4\underbrace{cov(X_{min}, X_{max})}_{M[X_{min}X_{max}] - M[X_{min}]M[X_{max}]}\right)$$

$$q(y,z) = n(n-1)(\Phi(z) - \Phi(y))\cdot q(y)q(z)$$

$$M[X_{min}X_{max}] = \iint_{-\infty}^{+\infty} yz\, q(y,z)\, dy dz = \frac{2n^2+7n+5}{(n+1)(n+2)}$$

$$D[X_{min}] = \frac{1}{\theta}\int_\theta^{2\theta} y^2 \cdot n\left(1-\frac{y-\theta}{\theta}\right)^{n-1} dy - \frac{1}{4}\left(\int_\theta^{2\theta} yn\left(1-\frac{y-\theta}{\theta}\right)^{n-1} dy\right)^2 =$$

$$= \frac{\theta^2(n^2+5n+8)}{(n+1)(n+2)} - \frac{\theta^2(n+2)^2}{(n+1)^2} = \frac{n\theta^2}{(n+1)^2(n+2)^2}$$

$$D\tilde{\theta}_3 = \frac{1}{25}\left[\frac{n\theta^2}{(n+1)^2(n+2)} + \frac{4n\theta^2}{(n+1)^2(n+2)} + \frac{4\theta^2(2n^2+7n+5)}{(n+1)(n+2)} - \frac{4\theta^2(2n+1)(n+2)}{(n+1)^2}\right] =$$

$$= \frac{\theta^2(5n+4)}{25(n+1)^2(n+2)}$$

$$D\tilde{\theta}_3' = \frac{\cancel{25(n+1)^2}}{(5n+4)^2} \cdot \frac{\cancel{(5n+4)}\theta^2}{\cancel{25(n+1)^2}(n+2)} = \frac{\theta^2}{(5n+4)(n+2)} \xrightarrow[n\to\infty]{} 0 \Rightarrow$$ сост. по дост. усл.

(c) $D\tilde{\theta}_2' = \dfrac{n\theta^2}{4(n+1)^2(n+2)}$ $\qquad D\tilde{\theta}_3' = \dfrac{\theta^2}{(5n+4)(n+2)}$

$$\frac{n\cancel{\theta^2}}{4(n+1)^2\cancel{(n+2)}} \qquad \frac{\cancel{\theta^2}}{(5n+4)\cancel{(n+2)}} \qquad \frac{n}{5n^2+4n}$$

$$\frac{1}{4n^2+8n+4} \underset{n\to\infty}{>} \frac{1}{5n^2+4n}$$

$\Rightarrow \tilde{\theta}_3'$ более эфф. при больших n

$$\frac{D\tilde{\theta}_1}{D\tilde{\theta}_3} = \frac{(n+1)(5n+4)}{27n} \xrightarrow[n\to\infty]{} \infty \Rightarrow \tilde{\theta}_3$$ – наиболее ас. эфф.

(e) $\tilde{\theta} = \frac{2}{3}\bar{X}$ (из ОММ)

ЦПТ: $\dfrac{\bar{X} - M\xi}{\sqrt{D\xi}}\sqrt{n} \rightsquigarrow N(0,1)$

$$\frac{\frac{3}{2}(\tilde{\theta}-\theta)\sqrt{12n}}{\theta} \rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < \frac{3(\tilde{\theta}-\theta)\sqrt{12n}}{2\theta} < u_{\frac{1+\beta}{2}} \Rightarrow u_{\frac{1-\beta}{2}} + 3\sqrt{3n} < \frac{\tilde{\theta}}{\theta}\cdot 3\sqrt{3n} < u_{\frac{1+\beta}{2}} + 3\sqrt{3n}$$

$$\frac{3\sqrt{3n}\,\tilde{\theta}}{(u_{\frac{1+\beta}{2}} + 3\sqrt{3n})} < \theta < \frac{3\sqrt{3n}\,\tilde{\theta}}{(u_{\frac{1-\beta}{2}} + 3\sqrt{3n})}, \quad \tilde{\theta} = \frac{2}{3}\bar{x}$$

$\tilde{\theta} = \dfrac{n+1}{2n+1} X_{max}$ (из ОМП)

$$\frac{\tilde{\theta}-\theta}{\sigma}\sqrt{n} \rightsquigarrow N(0,1)$$

$$\sigma = \sqrt{\nabla^T\theta \cdot I^{-1}(\theta)\,\nabla\theta} = \sqrt{I^{-1}(\theta)}$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p}{\partial\theta}\right)^2\right] = \int_\theta^{2\theta} \frac{1}{\theta}\cdot\frac{1}{\theta^2}\,dx = \frac{1}{\theta^2} \Rightarrow \sigma = \left(\frac{1}{\theta}\right)^{-1} = \theta$$

$$\left(\frac{\tilde{\theta}}{\theta} - 1\right)\sqrt{n} \rightsquigarrow N(0,1)$$

$$\frac{1}{\sqrt{n}}u_{\frac{1-\beta}{2}} < \frac{\tilde{\Theta}}{\Theta} - 1 < \frac{1}{\sqrt{n}}u_{\frac{1+\beta}{2}} \Rightarrow \frac{\tilde{\Theta}}{\frac{u_{\frac{1-\beta}{2}}}{\sqrt{n}}+1} > \Theta > \frac{\tilde{\Theta}}{\frac{u_{\frac{1+\beta}{2}}}{\sqrt{n}}+1}, \quad \tilde{\Theta} = \frac{n+1}{2n+1} X_{max}$$

б) $y_i = \frac{X_i}{\Theta} - 1 \sim R(0,1)$

$y_{max} = \frac{X_{max}}{\Theta} - 1 = G(x, \Theta) \to$ ф.-ция распределения:

$$F(y) = \begin{cases} 0, & y < 0 \\ y^n, & 0 \le y \le 1 \\ 1, & y > 1 \end{cases} \qquad p(y) = ny^{n-1} \quad [0,1]$$

$$P(g_1 < G(x,\Theta) < g_2) = P\left(\frac{X_{max}}{g_2+1} < \Theta < \frac{X_{max}}{g_1+1}\right) \quad max \Rightarrow g_2 = 1$$

$$\beta = P(g_1 < y_{max} < 1) = F(1) - F(g_1) = 1 - g_1^n = \beta \to g_1 = \sqrt[n]{1-\beta}$$

$$\frac{X_{max}}{1+\sqrt[n]{\beta}} < \Theta < \frac{X_{max}}{1+\sqrt[n]{1-\beta}}$$

Задача №5

$$p(x) = \begin{cases} \frac{\Theta-1}{x^\Theta}, & x \ge 1 \\ 0, & x < 1 \end{cases} \qquad \Theta > 1$$

а) $L(\Theta) = \prod_{i=1}^{n} p(x_i, \Theta) = (\Theta-1)^n \cdot \prod_{i=1}^{n} \frac{1}{x_i^\Theta}$, $\quad \min x_i \ge 1$

$\max L(\Theta) = (\Theta - 1)^n$

$$\ln L(\Theta) = n \ln(\Theta-1) - \Theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \Theta} = \frac{n}{\Theta-1} - \sum_{i=1}^{n} \ln x_i = 0 \Rightarrow \tilde{\Theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

$$\frac{\partial^2 \ln L}{\partial \Theta^2} = -\frac{n}{(\Theta-1)^2} \underset{\Theta=\tilde{\Theta}}{=} -\sum_{i=1}^{n} \ln x_i \cdot \frac{1}{n} < 0 \Rightarrow max$$

б) $\frac{f(\tilde{\Theta}) - f(\Theta)}{\sigma}\sqrt{n} \rightsquigarrow N(0,1)$

$$f(\Theta) = X_{med}: \quad F(X_{med}) = \int_{-\infty}^{X_{med}} \frac{\Theta-1}{t^\Theta}\,dt = \frac{\cancel{(\Theta-1)}(X_{med}^{\Theta} - X_{med})}{X_{med}^{\Theta}\cancel{(\Theta-1)}} = \frac{1}{2}$$

$$1 - X_{med}^{1-\Theta} = \frac{1}{2} \to X_{med}^{1-\Theta} = \frac{1}{2} \to X_{med} = \left(\frac{1}{2}\right)^{\frac{1}{1-\Theta}} = 2^{\frac{1}{\Theta-1}}$$

$$f'(\Theta) = 2^{\frac{1}{\Theta-1}} \cdot \ln 2 \cdot \frac{(-1)}{(1-\Theta)^2} = -\ln 2 \cdot 2^{\frac{1}{\Theta-1}} \cdot \frac{1}{(\Theta-1)^2}$$

$$I(\Theta) = M\left[\left(\frac{\partial \ln p}{\partial \Theta}\right)^2\right] = \int_{1}^{+\infty} \frac{\Theta-1}{x^\Theta} \cdot \left(\frac{1}{\Theta-1} - \ln x\right)^2 dx = \frac{1}{(\Theta-1)^2}$$

$$\frac{2^{\frac{1}{\tilde\theta-1}} - 2^{\frac{1}{\theta-1}}}{+\ln 2 \cdot 2^{\frac{1}{\tilde\theta-1}} \left(\frac{1}{\tilde\theta-1}\right)^2 \cdot (\tilde\theta-1)} = \frac{(\tilde\theta-1)}{\ln 2}\left(1 - \underbrace{2^{\frac{1}{\theta-1}}}_{x_{med}}{}^{-\frac{1}{\tilde\theta-1}}\right)$$

$$2^{\frac{1}{\tilde\theta-1}}\left(1 - \frac{u_{\frac{1+\beta}{2}} \cdot \ln 2}{\sqrt{n}\,(\tilde\theta-1)}\right) < x_{med} < 2^{\frac{1}{\tilde\theta-1}}\left(1 - \frac{u_{\frac{1-\beta}{2}} \cdot \ln 2}{(\tilde\theta-1)\sqrt{n}}\right)$$

d) $(\tilde\theta-\theta)\sqrt{I(\theta)} \cdot \sqrt{n} \rightsquigarrow N(0,1)$

$$(\tilde\theta-\theta) \cdot \frac{\sqrt{n}}{\theta-1} \rightsquigarrow N(0,1)$$

$$u_{\frac{1-\beta}{2}} < \frac{\sqrt{n}}{\theta-1}(\tilde\theta-\theta) < u_{\frac{1+\beta}{2}}$$

$$\tilde\theta - \frac{(\tilde\theta-1)\,u_{\frac{1+\beta}{2}}}{\sqrt{n}} < \theta < \tilde\theta - \frac{(\tilde\theta-1)\,u_{\frac{1-\beta}{2}}}{\sqrt{n}}, \quad \tilde\theta = 1 + \frac{n}{\sum_{i=1} \ln X_i}$$

© Априорная: $p(y) = \begin{cases} e^{1-y}, & y \geq 1 \\ 0, & y < 1 \end{cases}$

$\theta \sim \begin{cases} e^{1-\theta}, & \theta \geq 1 \\ 0, & \theta < 1 \end{cases}$

$\ln p(\theta / \overline{X}_n) \to \max$

$\ln C + \ln L + \ln p(\theta) \to \max$

$L = \left(\frac{\theta - 1}{x_i^{\theta}}\right)^n, \quad x \geq 1 \qquad L = \frac{(\theta-1)^n}{\prod x_i^{\theta}} \; (\min x \geq 1)$

$\ln p(\theta / \overline{X}_n) = \ln C + n \ln(\theta - 1) - \sum \theta \ln x_i + 1 - \theta \to \max$

$\frac{\partial \ln p}{\partial \theta} = \frac{n}{\theta - 1} - 1 - \sum \ln x_i \Rightarrow \frac{n}{\theta - 1} = 1 + \ln x_i \Rightarrow \theta = \frac{n}{1 + \sum \ln x_i} + 1$

Дов. интервал

$p(\theta / \overline{X}_n) = \frac{p(\theta) L(\theta)}{p(\overline{X}_n)} = e^{1-\theta} \cdot \frac{c(\theta-1)^n}{\prod x_i^{\theta}}$

$\int_1^{+\infty} e^{1-\theta} \cdot \frac{c(\theta-1)^n}{\prod x_i^{\theta}} \, d\theta = 1$

$\int_1^{q_1} p(\theta / \overline{X}_n) \, d\theta = 0.025 \to q_1 \approx 5,75$

$\int_{q_2}^{+\infty} p(\theta / \overline{X}_n) \, d\theta = 0,025 \to q_2 \approx 8,05$