CHEME 132 Module 3: Multiple Asset Geometric Brownian Motion

Jeffrey D. Varner
Smith School of Chemical and Biomolecular Engineering
Cornell University, Ithaca NY 14853

Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm, is described by a stochastic differential equation. Geometric Brownian motion was popularized as a financial model by Samuelson in the 1950s and 1960s [1], but is arguably most commonly associated with the Black–Scholes options pricing model, which we will describe later [2]. Previously, we considered the single asset GBM model, where the share price of a firm was modeled as a deterministic drift term that is corrupted by a Wiener noise process. We showed that the share price of a firm follows a log-normal distribution, and derived the analytical solution for the share price of a firm at a future time point. However, in practice, investors often hold portfolios of assets, and the share price of a firm is correlated with the share price of other firms. Thus, let's consider the GBM for multiple simulataneous assets, where the share price of a firm is modeled as a deterministic drift term that is corrupted by a Wiener noise process (same as before), but now we consider how the noise is correlated with other firms in a portfolio.

Consider an asset portfolio $\mathcal{P}$, e.g., a collection of equities with a logarithmic return covariance matrix Σ and drift vector μ. The multi-dimensional geometric Brownian motion model describing the share price $S_i(t)$ for asset $i \in \mathcal{P}$ at time t is given by:

$$\frac{dS_i\left(t\right)}{S_i(t)} = \mu_i\,dt + \sum_{j=1}^{\mathcal{P}} a_{ij}\cdot dW_j(t) \qquad \text{for} \quad i = 1, 2, \dots, \mathcal{P}$$

where $a_{ij} \in \mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\top} = \Sigma$ are noise coefficients describing the connection between firms i and firms j in the portfolio $\mathcal{P}$, and μ_i denotes the drift parameter for asset i in the portfolio $\mathcal{P}$. The multi-dimensional GBM model has the analytical solution:

$$S_i(t_k) = S_i(t_{k-1})\cdot\exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sqrt{\Delta t}\cdot\sum_{j\in\mathcal{P}} a_{ij}\cdot Z_j(0,1)\right] \quad i \in \mathcal{P}$$

where $S_i(t_{k-1})$ is the share price for firm $i \in \mathcal{P}$ at time t_{k-1}, the term $\Delta t = t_k - t_{k-1}$ denotes the time difference (step-size) for each time step (fixed), and $Z_j(0,1)$ is a standard normal random variable for firm $j \in \mathcal{P}$.

Drift and Covariance Estimation

The drift vector μ and covariance matrix Σ are key parameters in the multi-dimensional GBM model. We can estimate the drift vector μ, which will be a $\dim\mathcal{P}\times 1$ vector, from historical data by computing the logarithmic excess growth rate of the assets in the portfolio $\mathcal{P}$, as we have shown previosly; we simply repeat the process for each asset in the portfolio $\mathcal{P}$. On the other hand, the covariance matrix, a $\dim\mathcal{P}\times\dim\mathcal{P}$ symmetric matrix, describes the relationship between the logarithmic return series of firms i and j. The (i,j) element of the covariance matrix $\sigma_{ij}\in\Sigma$ is given by:

$$\sigma_{ij} = \text{cov}\left(r_i, r_j\right) = \sigma_i\sigma_j\rho_{ij} \qquad \text{for} \quad i,j\in\mathcal{P}$$

where $\sigma_{\star}$ denote the standard deviation of the logarithmic return of asset $\star$, i.e., the volatility parameter, and ρ_{ij} denotes the correlation between the returns of asset i and j in the portfolio $\mathcal{P}$.

The noise coefficients a_{ij} modify the noise terms in the multi-dimensional GBM model, and describe the connection between firms i and firms j in the portfolio $\mathcal{P}$. The noise coefficients are given by the Cholesky decomposition of the covariance matrix Σ:

$$\Sigma = \mathbf{A}\mathbf{A}^{\top}$$

where $\mathbf{A}$ is a lower triangular matrix. The Cholesky decomposition is a matrix factorization that decomposes the covariance matrix Σ into the product of a lower triangular matrix $\mathbf{A}$ and its complex conjugate transpose $\mathbf{A}^{\top}$; given that Σ is a positive-definite matrix, the Cholesky decomposition is unique.

Multiasset simulation

Now that we have estimated the drift vector μ, the covariance matrix Σ, and the noise coefficients a_{ij}, we can simulate the multi-dimensional GBM model to predict the share price of a firm at a future time point. Much like the single asset GBM model, we can simulate the multi-dimensional GBM model using the analytical solution, except in this case we will simulate the share price of each firm in the portfolio $\mathcal{P}$ at each time point. Thus, we'll need a vector of initial share prices $\mathbf{S}(t_0)$, the drift vector μ, the covariance matrix Σ, along with user-defined time points $t_0, t_1, \dots, t_N$, and the number of samples to simulate N_{samples}. Given this data, we can simulate the multi-dimensional GBM model using Algorithm 1, and then fit a distribution to the simulated share price of each firm in the portfolio $\mathcal{P}$ at each time point.

Computing the Net Present Value (NPV) of the portfolio $\mathcal{P}$

To compute the Net Present Value (NPV) of a portfolio $\mathcal{P}$, we can use Algorithm 1 to simulate the share price of each firm in $\mathcal{P}$ over a time-horizon, e.g., from now to until next year. Given the share price of each firm over that horizon, we can compute what combinations of shares of each firm in the portfolio $\mathcal{P}$ are the best in some way. For example, we can compute which share combinations lead to the largest projected NPV of the portfolio $\mathcal{P}$ at time t by computing the two cash flow events that occur, first the purchase of the shares of each firm in the portfolio $\mathcal{P}$ at time t_0, and then the liquidation of the portfolio $\mathcal{P}$ at time T, where the future share price value is discounted back to the present value using an appropriate discount rate, such as the average risk-free rate over the time period $\bar{r}_f$:

$$\text{NPV}(T,\bar{r}_f) = \sum_{i\in\mathcal{P}} n_i\cdot\Big(\mathcal{D}_{T,0}^{-1}(\bar{r}_f)\cdot S_i(T) - S_i(0)\Big) \tag{1}$$

where n_i denotes the number of shares of firm $i \in \mathcal{P}$, $S_i(T)$ denotes the share price of firm $i \in \mathcal{P}$ at time T, $S_i(0)$ denotes the share price of firm $i \in \mathcal{P}$ at time 0, and $\mathcal{D}_{T,0}(\bar{r}_f)$ denotes the multiperiod discount factor between time $0 \rightarrow T$ with the discount rate equal to the risk-free rate $\bar{r}_f$.

To calculate the net present value at $t = T$, we need the share price of each firm in the portfolio $\mathcal{P}$ at time T, and the number of shares of each firm in the portfolio $\mathcal{P}$ at time t_0, where we assume the investor does not buy or sell any shares of any firm in the portfolio $\mathcal{P}$ over the time period $t_0 \rightarrow T$. The number of shares of each firm in the portfolio $\mathcal{P}$ at time t is given by the fraction of the total wealth invested in each firm in the portfolio $\mathcal{P}$. Let's imagine a scenario where an investor has a total wealth of W_{total} at time t, and the fraction of the total wealth invested in each firm $i \in \mathcal{P}$ is given by $0 \leq \omega_i \leq 1$:

$$\omega_i(t) = \frac{W_i(t)}{W_{\text{total}}} \qquad \text{for} \quad i \in \mathcal{P}$$

where $W_i(t)$ denotes the wealth invested in firm $i \in \mathcal{P}$ at time t. However, we know the total wealth of portfolio $\mathcal{P}$, is just the liqidation value of the portfolio:

$$W(t) = \sum_{i \in \mathcal{P}} n_i(t) \cdot S_i(t)$$

where $n_i(t)$ denotes the number of shares of firm $i \in \mathcal{P}$ at time t, and $S_i(t)$ denotes the share price of firm $i \in \mathcal{P}$ at time t. We can rewrite the number of shares of each firm in the portfolio $\mathcal{P}$ at time t as:

$$n_i(t) = \frac{\omega_i(t) \cdot W_{\text{total}}}{S_i(t)} \qquad \text{for} \quad i \in \mathcal{P}$$

Further, we can rewrite the fraction of the total wealth invested in each firm $i \in \mathcal{P}$ as:

$$\omega_i(t) = \frac{n_i(t) \cdot S_i(t)}{\displaystyle\sum_{k \in \mathcal{P}} n_k(t) \cdot S_k(t)} \qquad \text{for} \quad i \in \mathcal{P} \tag{2}$$

The fractions $\omega_i(t)$, i.e., the portfolio weights at time t, have a few interesting properties (Concept 1).

Concept 1: Properties of allocation ω

The fraction of the total wealth invested in each firm $i \in \mathcal{P}$ at time t, in the absence of borrowing shares, sums to one for all time points t:

$$\sum_{i \in \mathcal{P}} \omega_i(t) = 1 \quad \forall t \geq t_0$$

Further, because the share price of each firm in the portfolio $\mathcal{P}$ is positive, the fraction of the total wealth invested in each firm $i \in \mathcal{P}$ at time t is non-negative $\omega_i(t) \geq 0$ for all $t \geq t_0$. Finally, ω_i, even with constant shares, is a function of time as the share price of each firm in the portfolio $\mathcal{P}$ changes over time.

The investor can choose the fraction of the total wealth ω_i invested in each firm $i \in \mathcal{P}$ by some approach, e.g., by formulating the choice as an optimization problem, or by sampling from probability distribution such as a Dirchlet distribution $\omega_i \sim$ Dirchlet$(\alpha_1, \alpha_2, \dots, \alpha_{\dim \mathcal{P}})$, where α_i denotes the concentration parameter for firm $i \in \mathcal{P}$, and then compute the NPV of the portfolio $\mathcal{P}$ at time T by sampling the final share price of each firm in the portfolio $\mathcal{P}$.

Summary

In this module, we have considered the multi-dimensional GBM model, which describes the share price of a firm in a portfolio $\mathcal{P}$. We have shown how to estimate the drift vector μ, the covariance matrix Σ, and the noise coefficients a_{ij}, and provided an algorithm to simulate the multi-dimensional GBM model to predict the share price of a firm at a future time point. Finally, we have shown how to compute the wealth of a portfolio $\mathcal{P}$ at time t, and its relationship to the fraction of the total wealth invested in each firm in the portfolio $\mathcal{P}$. Later, we'll develop tools to compute these fractions, i.e., the portfolio weights, but for we'll consider the case where the portfolio weights are equal, or are chosen by the investor by some approach (perhaps even randomly).

References

1. Merton RC. Paul Samuelson And Financial Economics. The American Economist. 2006;50(2):9–31.
2. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973;81(3):637–654.

Algorithm 1 Multiasset Geometric Brownian Motion

Require: The initial share price vector $\mathbf{S}(t_0) = \{S_i(t_0)\}_{i=1}^{\dim\mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: The drift vector $\mu = \{\mu_i\}_{i=1}^{\dim\mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: The covariance matrix Σ for the portfolio $\mathcal{P}$.
Require: The time step Δt (units: years), the initial time $t_\circ$, the final time t_f, and the number of samples N_s.
Require: A Cholesky decomposition function $\texttt{cholesky}(\Sigma)$ that returns the Cholesky factor $\mathbf{A}$.

procedure MULTIASSET GBM(μ, Σ, $\mathbf{S}_\circ$, $(t_\circ, t_f, \Delta t)$, N_{samples})

  $N_a \leftarrow$ length(μ) ▷ Number of assets in the portfolio $\mathcal{P}$
  $N_t \leftarrow (t_f - t_\circ)/\Delta t$ ▷ Number of time steps
  $T \leftarrow \texttt{arange}(t_\circ, t_f, \Delta t)$ ▷ Time points
  $X \leftarrow \texttt{zeros}(N_s, N_t, N_a + 1)$ ▷ Pre-allocate the share price array
  $\mathcal{Z} \leftarrow \mathcal{N}(0, 1)$ ▷ Instantiate a standard normal distribution
  $\mathbf{A} \leftarrow \texttt{cholesky}(\Sigma)$ ▷ Cholesky decomposition of the covariance matrix Σ

  for $i \in 1$ to N_s **do** ▷ Loop over the number of samples
    for $j \in \texttt{eachindex}(T)$ **do** ▷ Loop over the time points
      $X[i, j, 1] \leftarrow T[j]$ ▷ Set the time points in first column
    end for

    for $j \in 1$ to N_a **do** ▷ Loop over the number of assets in the portfolio $\mathcal{P}$
      $X[i, 1, j+1] \leftarrow \mathbf{S}_\circ[j]$ ▷ Set the initial share price
    end for

    for $j \in 2$ to N_t **do** ▷ Loop over the number of time steps
      for $k \in 1$ to N_a **do** ▷ Loop over the number of assets in the portfolio $\mathcal{P}$

        noise $\leftarrow 0.0$
        for $l \in 1$ to N_a **do** ▷ Loop over the number of assets in the portfolio $\mathcal{P}$
          noise $\leftarrow$ noise $+ \mathbf{A}[k, l] \cdot \texttt{rand}(\mathcal{Z})$ ▷ Compute the noise term for asset k
        end for

$$X[i, j, k+1] \leftarrow X[i, j-1, k+1] \cdot \exp\left[\left(\mu[k] - a_{kk}/2\right)\Delta t + \sqrt{\Delta t} \cdot \text{noise}\right]$$

      end for
    end for
  end for
end procedure