CHEME 132 Module 2: Single Asset Geometric Brownian Motion

Jeffrey D. Varner
Smith School of Chemical and Biomolecular Engineering
Cornell University, Ithaca NY 14853

Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm. Geometric Brownian motion was popularized as a financial model by Samuelson in the 1950s and 1960s [1], but is arguably most commonly associated with the Black–Scholes options pricing model, which we will describe later [2]. Let's start with the single asset case (in the absence of dividends) and then consider the multiple asset case in the next module.

Geometric Brownian motion (GBM) assumes that the share price $S(t)$ of a firm can be modeled as a deterministic drift term (which is proportional to the share price) that is corrupted by a Wiener noise process, also proportional to the share price:

$$\frac{dS}{S} = \mu\, dt + \sigma\, dW \tag{1}$$

The constant μ denotes a drift parameter, i.e., the growth rate of the share price return, σ is a volatility parameter, i.e., the dispersion of the return, dt denotes an infinitesimal time step, and dW is the output of a Wiener noise process. A Wiener Process (also often referred to as a standard Brownian motion) is a real-valued continuous-time stochastic process named after Norbert Wiener for the study of one-dimensional Brownian motion (Defn. 1):

Definition 1: Wiener Process

A Wiener process is a continuous one-dimensional stochastic process $\{W\left(t\right), 0 \leq t \leq T\}$ with the following properties:

- W(0) = 0 with probability 1
- The increments $\{W(t_1) - W(t_o), \dots, W(t_k) - W(t_{k-1})\}$ are independent for any k and $0 \leq t_o < t_1 < \dots < t_k \leq T$
- The increment W(t) - W(s) $\sim\ N\left(0, t-s\right)$ for any $0 \leq s < t \leq T$, where $N\left(0, t-s\right)$ denotes a normally distributed random variable with mean 0 and variance $t-s$.

Equation 1 is a continuous-time analog of the discrete-time binomial lattice model we developed previously. It has several exciting properties; for example, it has an analytical solution and analytical expressions for the expectation and variance of the share price. In this module, we will develop analytical solutions to Eqn. 1, and tools to estimate the parameters μ and σ from historical data. We'll then use these tools to simulate the share price of firms and analyze properties of the return distributions in the example and project for this module.

Analytical solution

Using Ito's lemma, we can formulate an analytical solution to the GBM equation for a single asset. K. Ito's Lemma, developed in 1951, is an analog of the Taylor series for stochastic systems. Let the random variable $X(t)$ be governed by the general stochastic differential equation:

$$dX = a\left(X(t), t\right) dt + b\left(X(t), t\right) dW(t)$$

where $dW(t)$ is a one-dimensional Wiener process and a and b are functions of $X(t)$ and t. Let $Y(t) = \phi\left(t, X(t)\right)$ be twice differentiable with respect to $X(t)$, and singly differentiable with respect to t. Then, $Y(t)$ is governed by the equation:

$$dY = \left(\frac{\partial Y}{\partial t} + a\frac{\partial Y}{\partial X} + \frac{b^2}{2}\frac{\partial^2 Y}{\partial X^2}\right) dt + b\left(\frac{\partial Y}{\partial X}\right) dW(t)$$

Let $Y = \ln(S)$, $a = \mu \cdot S$, and $b = \sigma \cdot S$. Then, Y is governed by the stochastic differential equation (using Ito's Lemma):

$$d\ln(S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \cdot dW(t)$$

We integrate both sides of the equation to obtain from $t_\circ$ to t:

$$\int_{t_\circ}^{t} d\ln(S) = \int_{t_\circ}^{t} \left(\mu - \frac{\sigma^2}{2}\right) dt + \int_{t_\circ}^{t} \sigma \cdot dW(t)$$

which gives:

$$\ln\left(\frac{S_t}{S_\circ}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(t - t_\circ) + \sigma \cdot \sqrt{t - t_\circ} \cdot Z(0,1)$$

where the noise term makes use of the definition of the integral of a Wiener process. Finally, we exponentiate both sides of the equation to obtain the analytical solution to the GBM model:

$$S(t) = S_\circ \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t - t_\circ) + (\sigma\sqrt{t - t_\circ}) \cdot Z_t(0,1)\right] \tag{2}$$

where $S_\circ$ denotes the share price at $t_\circ$, and $Z_t(0,1)$ denotes a standard normal random variable at time t. The expectation and variance of the GBM model is given by:

$$\begin{aligned}
\mathbb{E}\left(S_t\right) &= S_o \cdot \exp\left(\mu \cdot \Delta t\right) \\
\text{Var}\left(S_t\right) &= S_\circ^2 e^{2\mu\cdot\Delta t}\left[e^{\sigma^2\Delta t} - 1\right]
\end{aligned}$$

where $\Delta t = t - t_\circ$.

Model parameters

Estimating the drift parameter μ

Let's assume that we have a time series of share price values $S(t_1), S(t_2), \dots, S(t_k)$ and we want to estimate the detereministic growth of the share price, i.e., the drift parameter μ. There are several ways to do this, but we will use a detereministic linear model of the natural log of the share price values. To estimate the deterministic component of the share price, we first set the volatility parameter $\sigma = 0$ in Eqn. 2. Then, at some future time t, the share price (after some algebra) is given by:

$$\ln\,S_i = \ln\,S_\circ + \mu\cdot(t_i - t_\circ) \tag{3}$$

where $\ln\,S_\star$ denotes the natural log of the share price at time $t_\star$. Equation 3 is a linear model of the form $y = mx + b$, where $y = \ln\,S_i$, $x = t_i - t_\circ$, $m = \mu$, and $b = \ln\,S_\circ$. Thus, we can estimate the growth parameter μ by fitting a linear model to the log of the share price values by solving an overdetermined system of linear equations.

Let $\mathbf{A}$ denote a $\mathcal{S}\times 2$ matrix, where each row corresponds to a time value $t > 0$. The first column of $\mathbf{A}$ is all 1's while the second column holds the $(t_k - t_\circ)$ values. Further, let $\mathbf{Y}$ denote the natural log of the share price values (in the same temperal order as the $\mathbf{A}$ matrix). Then, the y-intercept and slope (drift parameter) can be estimated by solving the overdetermined system of equations:

$$\mathbf{A}\theta + \epsilon = \mathbf{Y}$$

where θ denotes the vector of unknown parameters (the y-intercept and slope), and ϵ denotes an error model, e.g., a normal distribution with mean zero and variance σ_ϵ^2. This system can be solved as:

$$\theta = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} - (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\epsilon$$

where $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$, and $(\mathbf{A}^T\mathbf{A})^{-1}$ denotes the inverse of the square matrix product $\mathbf{A}^T\mathbf{A}$. Finally, we can estimate the error term ϵ by calculating the residuals:

$$\epsilon = \mathbf{Y} - \mathbf{A}\theta$$

and then fitting a normal distribution to the residuals, using some technique such as maximum likelihood estimation, to compute the uncertainty in the estimate of the drift parameter $\hat{\mu}$ (where the $\hat{\star}$ denotes an estimate of the parameter).

Estimating the volatility parameter σ

Generally, methods to estimate the volatility parameter can be classified into two categories - historical volatility estimates based on past return data and future volatility estimates based on the Implied Volatility (IV) of options contracts. Here, we estimate the volatility σ from historical data, i.e., from the past returns of the share price.

A return refers to the increase or decrease in the price of an asset, e.g., shares of a stock, over a specific period, e.g., minutes, days, weeks, or even years. Assume the share price of a stock at time t is given by $S(t)$. Then, if we assume continuous compounding, the price of the stock at time

$t + \Delta t$ is given by:

$$S(t+\Delta t) = S(t)\cdot\exp\left(\mu\cdot\Delta t\right) \tag{4}$$

where μ denotes the instantenous growth rate (units: inverse years) of the share price over the time horizon $t \rightarrow t+\Delta t$. The continuously compounded share price gives rise to the logarithmic return model (Defn. 2):

Definition 2: Logarithmic Return

Let $S_{i,t-1}$ denote the continuously compounding share price of ticker i at time $t-1$. Then, at time t, the share price for ticker i, denoted as $S_{i,t}$, is given by:

$$S_{i,t} = S_{i,t-1}\cdot\exp\left(\mu^{(i)}_{t,t-1}\cdot\Delta t\right) \tag{5}$$

where $\mu^{(i)}_{t,t-1}$ denotes the instantenous growth rate (units: inverse years) of ticker i over time horizon $(t-1)\rightarrow t$, and Δt is the time interval between $t-1$ and t (units: years). The logarithmic return on asset i over time horizon $(t-1)\rightarrow t$, denoted by $r^{(i)}_{t,t-1}$, is defined as:

$$r^{(i)}_{t,t-1} \equiv \ln\left(\frac{S_{i,t}}{S_{i,t-1}}\right) = \mu^{(i)}_{t,t-1}\cdot\Delta t \tag{6}$$

Given Defn. 2, we compute the logarithmic return distribution using Algorithm 1. For each stock i, we compute the logarithmic return $\mu^{(i)}_{t,t-1}$ for each time interval $(t-1)\rightarrow t$, and then compute the mean and variance of the logarithmic return distribution. Once we estimated the logarith-

Algorithm 1 Logarithmic Excess Growth Rate

Require: data set $\mathcal{D}_i = \{S_{i,t}\}_{t=1}^{N} \in \mathcal{D}$ where $S_{i,t}$ denotes the price of stock i at time t, all stocks have the same time horizon $N \gg 2$, and $\mathcal{D}$ denotes the data set of all stocks.
Require: The time interval Δt between t and $t-1$ (units: years), and a list of stocks $\mathcal{L} = \{i\}_{i=1}^{M}$ where $M = \dim\mathcal{L}$.
Require: The risk-free rate r_f (units: inverse years).

procedure LOG GROWTH RATE($\mathcal{D}$, $\mathcal{L}$, Δt, r_f)
  $N \leftarrow$ length($\mathcal{D}$) ▷ Number of trading days for each stock $i \in \mathcal{L}$
  for $i \in \mathcal{L}$ **do**
    $\mathcal{D}_i \leftarrow \mathcal{D}[i]$ ▷ Select the data for stock i from the dataset collection $\mathcal{D}$
    for $t = 2 \rightarrow N$ **do**
      $S_{i,t-1} \leftarrow \mathcal{D}_i[t-1]$ ▷ Select the price of stock i at time $t-1$
      $S_{i,t} \leftarrow \mathcal{D}_i[t]$ ▷ Select the price of stock i at time t
      $\mu^{(i)}_{t,t-1} \leftarrow (1/\Delta t)\cdot\ln\left(S_{i,t}/S_{i,t-1}\right) - r_f$ ▷ Set $r_f = 0$ for regular growth rate
    end for
  end for
  return $\mu^{(1)},\dots,\mu^{(\dim\mathcal{L})}$ ▷ Return the logarithmic growth rate array for each stock $i \in \mathcal{L}$
end procedure

mic growth distribution for each stock, we can estimate the volatility parameter σ by computing the standard deviation of the return distribution. Typically, we use the daily data, but we want the annualized standard deviation of the logarithmic return distribution, thus we multiply the daily standard deviation by $\sqrt{252}$, where 252 is the average number of trading days in a year, i.e, $\sigma \leftarrow \sqrt{252} \cdot \sigma$.

Stylized facts

Geometric Brownian motion is a widely used as a pricing model. However, whether geometric Brownian motion replicates many of the statistical properties of actual pricing and return data is unclear. These properties, referred to as *stylized facts* have been observed for decades, dating back to early work of Mandelbrot [3, 4] and later by Cont [5] and more recently by Ratliff-Crain et al. [6] who reviewed the original stylized facts proposed by Cont using newer data.

Computing and understanding the stylized facts of financial data is important for several reasons. However, here we'll focus on only two, namely, the shape of the return distribution and the autocorrelation of the returns. The shape of the return distribution is important because it is used to compute the probability of extreme events. For example, the probability of a 10% drop in the share price of a firm in a single day. Typiclly, we assume that the return distribution is normal, but this is not always the case. Instead, retruns are often leptokurtic, i.e., they have fat tails, for example following a Laplace distribution. The autocorrelation of the returns is important because it is used to compute the probability of a run of consecutive up or down days. If the returns are uncorrelated in time, then consecutive up or down days are independent events, i.e., the return (up or down) on day t is independent of the return on day $t-1$. Thus, the market is behaving like a random walk. However, if the returns are correlated in time, then consecutive up or down days are not independent events, i.e., past returns influence future returns. This is a very active area of research, and we will not be able to cover all of the stylized facts here. However, the the lack of autcorrelation goes to the heart of the efficient market hypothesis, which is at the core of modern finance theory, and one of the foundation principeles of quantitative finance over the last 50 years.

In the example and project for this module, we will compute the stylized facts of the share price of firms in the S&P 500 index. In particular, we will compute the shape of the return distribution and the autocorrelation of the returns. We'll show that the returns are leptokurtic, for the majority of the firms in the S&P 500 index, and that the returns are uncorrelated in time.

Summary

In this module we introduced the single asset geometric Brownian motion model, and developed analytical solutions to the model. This model is a continuous-time analog of the discrete-time binomial lattice model we developed previously. The GBM model requires two parameters, the drift parameter μ and the volatility parameter σ. We developed methods to estimate these parameters from historical data, and then used these parameters to simulate the share price of firms in the example and project for this module. Finally, we discussed the stylized facts of financial data, and particularly the shape of the return distribution and the autocorrelation of the returns. The shape of the return distribution is important because it is used to compute the probability of extreme events, while the autcorrelation of the returns is important because it is used to compute the correlation in time of the returns.

References

1. Merton RC. Paul Samuelson And Financial Economics. The American Economist. 2006;50(2):9–31.
2. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973;81(3):637–654.
3. Mandelbrot B. The Variation of Certain Speculative Prices. The Journal of Business. 1963;36(4):394–419.
4. Mandelbrot B. The Variation of Some Other Speculative Prices. The Journal of Business. 1967;40(4):393–413.
5. Cont R. Empirical properties of asset returns: stylized facts and statistical issues. Quantitative Finance. 2001;1(2):223–236.
6. Ratliff-Crain E, Oort CMV, Bagrow J, Koehler MTK, Tivnan BF. Revisiting Stylized Facts for Modern Stock Markets; 2023.