CHEME 132 Module 3: Multiple Asset Geometric Brownian Motion

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Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm. Geometric Brownian motion was popularized as a financial model by Samuelson in the 1950s and 1960s [1], but is arguably most commonly associated with the Black–Scholes options pricing model, which we will describe later [2]. Previously, we considered the single asset GBM model, where the share price of a firm is modeled as a deterministic drift term that is corrupted by a Wiener noise process. Now, we will consider the multi-asset GBM model, where the share price of a firm is modeled as a deterministic drift term that is corrupted by a Wiener noise process, and the share price of a firm is correlated with the share price of other firms through the noise process.

Consider an asset portfolio $\mathcal{P}$ with a return covariance matrix Σ and drift vector μ. The multi-dimensional geometric Brownian motion model describing the share price $S_i(t)$ for asset $i \in \mathcal{P}$ is given by:

$$\frac{dS_i\left(t\right)}{S_i(t)} = \mu_i\,dt + \sum_{j=1}^{\mathcal{P}} a_{ij}\cdot dW_j(t) \qquad \text{for} \quad i = 1, 2, \dots, \mathcal{P}$$

where $a_{ij} \in \mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\top} = \Sigma$, and μ_i denotes the drift parameter for asset i. The multi-dimensional GBM model has the analytical solution:

$$S_i(t_k) = S_i(t_{k-1})\cdot\exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sqrt{\Delta t}\cdot\sum_{j\in\mathcal{P}} a_{ij}\cdot Z_j(0,1)\right] \quad i \in \mathcal{P}$$

where $S_i(t_{k-1})$ is the share price at time t_{k-1} for asset $i \in \mathcal{P}$, $\Delta t = t_k - t_{k-1}$ is the time difference (step-size) for each time step (fixed), and $Z_j(0,1)$ is a standard normal random variable.

Estimating the covariance matrix Σ

The covariance between the logarithmic return $r_\star$ on assets i and j, denoted as cov (r_i, r_j), quantifies the relationship between assets i and j in the portfolio $\mathcal{P}$.

$$\Sigma_{ij} = \text{cov}\left(r_i, r_j\right) = \sigma_i\sigma_j\rho_{ij} \qquad \text{for} \quad i,j \in \mathcal{P}$$

where $\sigma_\star$ denote the standard deviation of the return of asset $\star$, and ρ_{ij} denotes the correlation between assets i and j in the portfolio $\mathcal{P}$. The correlation is given by:

$$\rho_{ij} = \frac{\mathbb{E}(r_i - \mu_i)\cdot\mathbb{E}(r_j - \mu_j)}{\sigma_i\cdot\sigma_j} \qquad \text{for} \quad i,j \in \mathcal{P}$$

The diagonal elements of the covariance matrix Σ are the variances, while the off-diagonal measure the relationship between the assets in the portfolio $\mathcal{P}$.

Summary

Fill me in.

References

1. Merton RC. Paul Samuelson And Financial Economics. The American Economist. 2006;50(2):9–31.
2. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973;81(3):637–654.