CHEME 132 Module 3: Multiple Asset Geometric Brownian Motion

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Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm, is described by a stochastic differential equation. Geometric Brownian motion was popularized as a financial model by Samuelson in the 1950s and 1960s [1], but is arguably most commonly associated with the Black–Scholes options pricing model, which we will describe later [2]. Previously, we considered the single asset GBM model, where the share price of a firm was modeled as a deterministic drift term that is corrupted by a Wiener noise process. We showed that the share price of a firm follows a log-normal distribution, and derived the analytical solution for the share price of a firm at a future time point. However, in practice, investors often hold portfolios of assets, and the share price of a firm is correlated with the share price of other firms. Thus, let's consider the GBM for multiple simulataneous assets, where the share price of a firm is modeled as a deterministic drift term that is corrupted by a Wiener noise process (same as before), but now we consider how the noise is correlated with other firms in a portfolio.

Consider an asset portfolio $\mathcal{P}$, e.g., a collection of equities with a logarithmic return covariance matrix Σ and drift vector μ. The multi-dimensional geometric Brownian motion model describing the share price $S_i(t)$ for asset $i \in \mathcal{P}$ at time t is given by:

$$\frac{dS_i\left(t\right)}{S_i(t)} = \mu_i\,dt + \sum_{j=1}^{\mathcal{P}} a_{ij}\cdot dW_j(t) \qquad \text{for} \quad i = 1,2,\dots,\mathcal{P}$$

where $a_{ij} \in \mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\top} = \Sigma$ are noise coefficients describing the connection between firms i and firms j in the portfolio $\mathcal{P}$, and μ_i denotes the drift parameter for asset i in the portfolio $\mathcal{P}$. The multi-dimensional GBM model has the analytical solution:

$$S_i(t_k) = S_i(t_{k-1})\cdot\exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right)\Delta t + \sqrt{\Delta t}\cdot\sum_{j\in\mathcal{P}} a_{ij}\cdot Z_j(0,1)\right] \quad i \in \mathcal{P}$$

where $S_i(t_{k-1})$ is the share price for firm $i \in \mathcal{P}$ at time t_{k-1}, the term $\Delta t = t_k - t_{k-1}$ denotes the time difference (step-size) for each time step (fixed), and $Z_j(0,1)$ is a standard normal random variable for firm $j \in \mathcal{P}$.

Estimating the drift vector μ and covariance matrix Σ

The drift vector μ and covariance matrix Σ are key parameters in the multi-dimensional GBM model. We can estimate the drift vector μ, which will be a $\dim\mathcal{P}\times 1$ vector, from historical data by computing the logarithmic excess growth rate of the assets in the portfolio $\mathcal{P}$, as we have shown previosly; we simply repeat the process for each asset in the portfolio $\mathcal{P}$. On the other hand, the covariance matrix, a $\dim\mathcal{P}\times\dim\mathcal{P}$ symmetric matrix, describes the relationship between the logarithmic return series of firms i and j. The (i,j) element of the covariance matrix $\sigma_{ij}\in\Sigma$ is given by:

$$\sigma_{ij} = \text{cov}\left(r_{i},r_{j}\right) = \sigma_{i}\sigma_{j}\rho_{ij}\qquad\text{for}\quad i,j\in\mathcal{P}$$

where $\sigma_{\star}$ denote the standard deviation of the logarithmic return of asset $\star$, i.e., the volatility parameter, and ρ_{ij} denotes the correlation between the returns of asset i and j in the portfolio $\mathcal{P}$. The correlation is given by:

$$\rho_{ij} = \frac{\mathbb{E}(r_{i}-\mu_{i})\cdot\mathbb{E}(r_{j}-\mu_{j})}{\sigma_{i}\cdot\sigma_{j}}\qquad\text{for}\quad i,j\in\mathcal{P}$$

where $\mathbb{E}(r_{i}-\mu_{i})$ is the expected value of the difference between the logarithmic return of asset i and its drift parameter μ_{i}, i.e., mean of the logorithmic return. The diagonal elements of the covariance matrix $\sigma_{ii}\in\Sigma$ are the variances, while the off-diagonal measure the relationship between assets i and j in the portfolio $\mathcal{P}$.

Algorithm 1 Logarithmic Excess Growth Rate

Require: data set $\mathcal{D}_{i} = \left\{S_{i,t}\right\}_{t=1}^{N}\in\mathcal{D}$ where $S_{i,t}$ denotes the price of stock i at time t, all stocks have the same time horizon $N\gg 2$, and $\mathcal{D}$ denotes the data set of all stocks.
Require: The time interval Δ{t} between t and $t-1$ (units: years), and a list of stocks $\mathcal{L} = \left\{i\right\}_{i=1}^{M}$ where $M=\dim\mathcal{L}$.
Require: The risk-free rate r_{f} (units: inverse years).

procedure LOG GROWTH RATE($\mathcal{D}$, $\mathcal{L}$, Δ{t}, r_{f})
  $N\leftarrow$ length($\mathcal{D}$) ▷ Number of trading days for each stock $i\in\mathcal{L}$
  for $i\in\mathcal{L}$ **do**
    $\mathcal{D}_{i}\leftarrow\mathcal{D}[i]$ ▷ Select the data for stock i from the dataset collection $\mathcal{D}$
    for $t=2\rightarrow N$ **do**
      $S_{i,t-1}\leftarrow\mathcal{D}_{i}[t-1]$ ▷ Select the price of stock i at time $t-1$
      $S_{i,t}\leftarrow\mathcal{D}_{i}[t]$ ▷ Select the price of stock i at time t
      $\mu_{t,t-1}^{(i)}\leftarrow(1/\Delta{t})\cdot\ln\left(S_{i,t}/S_{i,t-1}\right)-r_{f}$ ▷ Set $r_{f}=0$ for regular growth rate
    end for
  end for
  return $\mu^{(1)},\dots,\mu^{(\dim\mathcal{L})}$ ▷ Return the logarithmic growth rate array for each stock $i\in\mathcal{L}$
end procedure

Estimating the noise coefficients a_{ij}

The noise coefficients a_{ij} modify the noise terms in the multi-dimensional GBM model, and describe the connection between firms i and firms j in the portfolio $\mathcal{P}$. The noise coefficients are given by the Cholesky decomposition of the covariance matrix Σ:

$$\Sigma = \mathbf{A}\mathbf{A}^{\top}$$

where $\mathbf{A}$ is a lower triangular matrix. The Cholesky decomposition is a matrix factorization that decomposes the covariance matrix Σ into the product of a lower triangular matrix $\mathbf{A}$ and its complex conjugate transpose $\mathbf{A}^{\top}$; given that Σ is a positive-definite matrix, the Cholesky decomposition is unique.

Multiasset GBM simulation algorithm

Now that we have estimated the drift vector μ, the covariance matrix Σ, and the noise coefficients a_{ij}, we can simulate the multi-dimensional GBM model to predict the share price of a firm at a future time point. Much like the single asset GBM model, we can simulate the multi-dimensional GBM model using the analytical solution, except in this case we will simulate the share price of each firm in the portfolio $\mathcal{P}$ at each time point. Thus, we'll need a vector of initial share prices $\mathbf{S}(t_0)$, the drift vector μ, the covariance matrix Σ, along with user-defined time points $t_0, t_1, \dots, t_N$, and the number of samples to simulate N_{samples}. Given this data, we can use the following algorithm to simulate the multi-dimensional GBM model (2).

Algorithm 2 Logarithmic Excess Growth Rate

Require: The initial share price vector $\mathbf{S}(t_0) = \{S_i(t_0)\}_{i=1}^{\dim\mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: The drift vector $\mu = \{\mu_i\}_{i=1}^{\dim\mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: The covariance matrix Σ for the portfolio $\mathcal{P}$.
Require: The time step Δt (units: years), the initial time $t_\circ$, the final time t_f, and the number of samples N_{samples}.
Require: A Cholesky decomposition function $\texttt{cholesky}(\Sigma)$ that returns the Cholesky factor $\mathbf{A}$.

procedure MULTIASSET GBM(μ, Σ, $\mathbf{S}_\circ$, $(t_\circ, t_f, \Delta t)$, N_{samples})
  $N_a \leftarrow \text{length}(\mu)$ ▷ Number of assets in the portfolio $\mathcal{P}$
  $N_t \leftarrow (t_f - t_\circ)/\Delta t$ ▷ Number of time steps
  $X \leftarrow \texttt{zeros}(N_{\text{samples},N_t,N_a+1})$ ▷ Pre-allocate the share price array
  $\mathbf{A} \leftarrow \texttt{cholesky}(\Sigma)$ ▷ Cholesky decomposition of the covariance matrix Σ
end procedure

Summary

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References

1. Merton RC. Paul Samuelson And Financial Economics. The American Economist. 2006;50(2):9–31.

2. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973;81(3):637–654.