CHEME 132 Module 3: Multiple Asset Geometric Brownian Motion

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Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm is described by a stochastic differential equation. Geometric Brownian motion was popularized as a financial model by Samuelson in the 1950s and 1960s [1], but is arguably most commonly associated with the Black–Scholes options pricing model, which we will describe later [2]. Previously, we considered the single asset GBM model, where a firm's share price was modeled as a deterministic drift term corrupted by a Wiener noise process. We showed that a firm's share price follows a log-normal distribution and derived the analytical solution for a firm's share price at a future point in time. However, in practice, investors often hold portfolios of assets, and a firm's share price is correlated with other firms' share price. Thus, let's consider the GBM for multiple simultaneous assets, where the share price of a firm is modeled as a deterministic drift term that is corrupted by a Wiener noise process (same as before), but now we consider how the noise is correlated with other firms in a portfolio.

Consider an asset portfolio $\mathcal{P}$, e.g., a collection of equities with a logarithmic return covariance matrix Σ and drift vector μ. The multi-dimensional geometric Brownian motion model describing the share price $S_i(t)$ for asset $i \in \mathcal{P}$ at time t is given by:

$$\frac{dS_i(t)}{S_i(t)} = \mu_i \, dt + \sum_{j=1}^{\mathcal{P}} a_{ij} \cdot dW_j(t) \qquad \text{for} \quad i = 1, 2, \ldots, \mathcal{P}$$

where $a_{ij} \in \mathbf{A}$ and $\mathbf{A}\mathbf{A}^{\top} = \Sigma$ are noise coefficients describing the connection between firms i and firms j in the portfolio $\mathcal{P}$, and μ_i denotes the drift parameter for asset i in the portfolio $\mathcal{P}$. The multi-dimensional GBM model has the analytical solution:

$$S_i(t_k) = S_i(t_{k-1}) \cdot \exp\left[\left(\mu_i - \frac{\sigma_i^2}{2} \right) \Delta t + \sqrt{\Delta t} \cdot \sum_{j \in \mathcal{P}} a_{ij} \cdot Z_j(0, 1) \right] \quad i \in \mathcal{P}$$

where $S_i(t_{k-1})$ is the share price for firm $i \in \mathcal{P}$ at time t_{k-1}, the term $\Delta t = t_k - t_{k-1}$ denotes the time difference (step-size) for each time step (fixed), and $Z_j(0, 1)$ is a standard normal random variable for firm $j \in \mathcal{P}$. For more information on the derivation of the multi-dimensional GBM model, see Glasserman [3].

Drift and Covariance Estimation

The drift vector μ and covariance matrix Σ are critical parameters in the multi-dimensional GBM model. We can estimate the drift vector μ, which will be a $\dim \mathcal{P} \times 1$ vector, from historical data

by computing the logarithmic excess growth rate of the assets in the portfolio $\mathcal{P}$, as we have shown previously; we repeat the process for each asset in the portfolio $\mathcal{P}$. On the other hand, the covariance matrix, a $\dim\mathcal{P} \times \dim\mathcal{P}$ symmetric matrix, describes the relationship between the logarithmic return series of firms i and j. The (i, j) element of the covariance matrix $\sigma_{ij} \in \Sigma$ is given by:

$$\sigma_{ij} = \mathsf{cov}\left(r_i, r_j\right) = \sigma_i \sigma_j \rho_{ij} \qquad \text{for} \quad i, j \in \mathcal{P}$$

where $\sigma_\star$ denote the standard deviation of the logarithmic return of asset $\star$, i.e., the volatility parameter, and ρ_{ij} denotes the correlation between the returns of asset i and j in the portfolio $\mathcal{P}$.

The noise coefficients a_{ij} modify the noise terms in the multi-dimensional GBM model, and describe the connection between firms i and firms j in the portfolio $\mathcal{P}$. The noise coefficients are given by the Cholesky decomposition of the covariance matrix Σ:

$$\Sigma = \mathbf{A}\mathbf{A}^\top$$

where $\mathbf{A}$ is a lower triangular matrix, and $\mathbf{A}^\top$ denotes the complex conjugate transpose of $\mathbf{A}$. The diagonal elements of the Cholesky factor $\mathbf{A}$ are the standard deviations of the logarithmic return series of the assets in the portfolio $\mathcal{P}$, i.e., the volatility of firm i. On the other hand, the off-diagonal elements of the Cholesky factor $\mathbf{A}$ can be used to compute σ_{ij}, i.e., the covariance between the logarithmic return series of the assets in the portfolio $\mathcal{P}$. The Cholesky decomposition of the covariance matrix Σ is unique and can be computed using the Cholesky decomposition algorithms provided by many common numerical linear algebra libraries.

Multiasset Portfolio Simulations

We simulate the multi-dimensional GBM model using the analytical solution for the share price of each firm in the portfolio $\mathcal{P}$, given the initial share price $\mathbf{S}(t_0)$ vector, the drift vector μ, the covariance matrix Σ, along with user-defined time points $t_0, t_1, \ldots, t_N$, and the number of samples to simulate N_{samples}.

Definition 1: Distribution of Share Price

The share price of each firm in the portfolio $\mathcal{P}$ at time t follows a log-normal distribution:

$$S(t) \sim \mathsf{LogNormal}\left(\mu_t, \Sigma_t\right)$$

where μ_t denotes the mean log-transformed share price vector, and Σ_t denotes the covariance matrix of the log-transformed share prices at time t. A log-normal distribution is a right-skewed distribution. It is often used to model the share price because of its positive support and its ability to model the exponential growth of the share price.

We simulate the multi-dimensional GBM model using Algorithm 1, and then fit a distribution to the simulated share price of each firm in the portfolio $\mathcal{P}$ (Defn. 1).

Algorithm 1 Multiasset Geometric Brownian Motion

Require: The initial share price vector $\mathbf{S}(t_0) = \{S_i(t_0)\}_{i=1}^{\dim \mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: The drift vector $\mu = \{\mu_i\}_{i=1}^{\dim \mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: The covariance matrix Σ for the portfolio $\mathcal{P}$.
Require: The time step Δt (units: years), the initial time $t_\circ$, the final time t_f, and the number of samples N_s.
Require: A Cholesky decomposition function $\texttt{cholesky}(\Sigma)$ that returns the Cholesky factor $\mathbf{A}$.

1: **procedure** MULTIASSET GEOMETRIC BROWNIAN MOTION

2:      $N_a \leftarrow \text{length}(\mu)$ ▷ Number of assets in the portfolio $\mathcal{P}$
3:      $N_t \leftarrow (t_f - t_\circ)/\Delta t$ ▷ Number of time steps
4:      $T \leftarrow \texttt{arange}(t_\circ, t_f, \Delta t)$ ▷ Time points
5:      $X \leftarrow \texttt{zeros}(N_s, N_t, N_a + 1)$ ▷ Pre-allocate the share price array
6:      $\mathcal{Z} \leftarrow \mathcal{N}(0, 1)$ ▷ Instantiate a standard normal distribution
7:      $\mathbf{A} \leftarrow \texttt{cholesky}(\Sigma)$ ▷ Cholesky decomposition of the covariance matrix Σ

8:      **for** $i \in 1$ to N_s **do**   ▷ Loop over the number of samples
9:          **for** $j \in \texttt{eachindex}(T)$ **do**   ▷ Loop over the time points
10:             $X_{i,j,1} \leftarrow T_j$ ▷ Set the time points in first column
11:          **end for**

12:          **for** $j \in 1$ to N_a **do**   ▷ Loop over the number of assets in the portfolio $\mathcal{P}$
13:             $X_{i,1,j+1} \leftarrow S_{\circ,j}$ ▷ Set the initial share price for asset j
14:          **end for**

15:          **for** $j \in 2$ to N_t **do**   ▷ Loop over the number of time steps
16:             **for** $k \in 1$ to N_a **do**   ▷ Loop over the number of assets in the portfolio $\mathcal{P}$

17:                 noise $\leftarrow 0.0$
18:                 **for** $l \in 1$ to N_a **do**   ▷ Loop over the number of assets in the portfolio $\mathcal{P}$
19:                     noise $\leftarrow$ noise $+ \mathbf{A}_{k,l} \cdot \texttt{rand}(\mathcal{Z})$ ▷ Compute the noise term for asset k
20:                 **end for**

21:             $X_{i,j,k+1} \leftarrow X_{i,j-1,k+1} \cdot \exp\left[\left(\mu_k - a_{kk}^2/2\right) \cdot \Delta t + \sqrt{\Delta t} \cdot \text{noise}\right]$
22:             **end for**
23:          **end for**
24:      **end for**
25: **end procedure**

Computing the Net Present Value (NPV) of the portfolio $\mathcal{P}$

To compute the Net Present Value (NPV) of a portfolio $\mathcal{P}$, we can use Algorithm 1 to simulate the share price of each firm in $\mathcal{P}$ over a time-horizon, e.g., from now to until next year. Given the share price of each firm over that horizon, we can compute what combinations of shares of each firm in

the portfolio $\mathcal{P}$ are the best in some way. For example, we can compute which share combinations lead to the largest projected NPV of the portfolio $\mathcal{P}$ at time t by computing the two cash flow events that occur, first the purchase of the shares of each firm in the portfolio $\mathcal{P}$ at time t_0, and then the liquidation of the portfolio $\mathcal{P}$ at time T, where the future share price value is discounted back to the present value using an appropriate discount rate, such as the average risk-free rate over the period $\bar{r}_f$:

$$\text{NPV}(T, \bar{r}_f) = \sum_{i \in \mathcal{P}} n_i \cdot \left(\mathcal{D}_{T,0}^{-1}(\bar{r}_f) \cdot S_i(T) - S_i(0) \right) \tag{1}$$

where n_i denotes the number of shares of firm $i \in \mathcal{P}$, $S_i(T)$ denotes the share price of firm $i \in \mathcal{P}$ at time T, $S_i(0)$ denotes the share price of firm $i \in \mathcal{P}$ at time 0, and $\mathcal{D}_{T,0}(\bar{r}_f)$ denotes the multiperiod discount factor between time $0 \to T$ with the discount rate equal to the risk-free rate $\bar{r}_f$.

To calculate the net present value at $t = T$, we need the share price of each firm in the portfolio $\mathcal{P}$ at time T and the number of shares of each firm in the portfolio $\mathcal{P}$ at time t_0, where we assume the investor does not buy or sell any shares of any firm in the portfolio $\mathcal{P}$ over the period $t_0 \to T$. The number of shares of each firm in the portfolio $\mathcal{P}$ at time t is given by the fraction of the total wealth invested in each firm in the portfolio $\mathcal{P}$. Let's imagine a scenario where an investor has a total wealth of W_{total} at time t, and the fraction of the total wealth invested in each firm $i \in \mathcal{P}$ is given by $0 \le \omega_i \le 1$:

$$\omega_i(t) = \frac{W_i(t)}{W_{\text{total}}} \qquad \text{for} \quad i \in \mathcal{P}$$

where $W_i(t)$ denotes the wealth invested in firm $i \in \mathcal{P}$ at time t. However, we know the total wealth of portfolio $\mathcal{P}$, is just the liqidation value of the portfolio:

$$W(t) = \sum_{i \in \mathcal{P}} n_i(t) \cdot S_i(t)$$

where $n_i(t)$ denotes the number of shares of firm $i \in \mathcal{P}$ at time t, and $S_i(t)$ denotes the share price of firm $i \in \mathcal{P}$ at time t. We can rewrite the number of shares of each firm in the portfolio $\mathcal{P}$ at time t as:

$$n_i(t) = \frac{\omega_i(t) \cdot W_{\text{total}}}{S_i(t)} \qquad \text{for} \quad i \in \mathcal{P}$$

Further, we can rewrite the fraction of the total wealth invested in each firm $i \in \mathcal{P}$ as:

$$\omega_i(t) = \frac{n_i(t) \cdot S_i(t)}{\sum_{k \in \mathcal{P}} n_k(t) \cdot S_k(t)} \qquad \text{for} \quad i \in \mathcal{P} \tag{2}$$

The fractions $\omega_i(t)$, i.e., the portfolio weights at time t, have a few interesting properties (Concept 1).

Concept 1: Properties of allocation ω

The fraction of the total wealth invested in each firm $i \in \mathcal{P}$ at time t, in the absence of

borrowing shares, sums to one for all time points t:

$$\sum_{i \in \mathcal{P}} \omega_i(t) = 1 \quad \forall t \geq t_0$$

Further, because the share price of each firm in the portfolio $\mathcal{P}$ is positive, the fraction of the total wealth invested in each firm $i \in \mathcal{P}$ at time t is non-negative $\omega_i(t) \geq 0$ for all $t \geq t_0$. Finally, ω_i, even with constant shares, is a function of time as the share price of each firm in the portfolio $\mathcal{P}$ changes over time.

The investor can choose the fraction of the total wealth ω_i invested in each firm $i \in \mathcal{P}$ by some approach, e.g., by formulating the choice as an optimization problem or by sampling from a probability distribution such as a Dirchlet distribution $\omega_i \sim \text{Dirchlet}(\alpha_1, \alpha_2, \ldots, \alpha_{\dim \mathcal{P}})$, where α_i denotes the concentration parameter for firm $i \in \mathcal{P}$. Then, when combined with the multi-dimensional GBM model, the investor can simulate the share price of each firm in the portfolio $\mathcal{P}$ over a time period and compute the NPV of the portfolio. This process can be repeated many times, and the expected NPV of the portfolio $\mathcal{P}$ at time T can be computed by averaging the NPV over all the samples (Algorithm 2).

Algorithm 2 Monte-Carlo Portfolio Net Present Value

Require: The initial share price vector $\mathbf{S}(t_0) = \{S_i(t_0)\}_{i=1}^{\dim \mathcal{P}}$ for each firm $i \in \mathcal{P}$.
Require: Share price distribution $S(T) \sim \text{LogNormal}(\mu_T, \Sigma_T)$
Require: An allocation distribution function $\omega \sim \text{Dir}(\alpha_1, \alpha_2, \ldots, \alpha_{\dim \mathcal{P}})$, where we assume equal concentration parameters $\alpha_i = 1/\dim \mathcal{P}$ firm $i \in \mathcal{P}$.
Require: The average risk-free rate $\bar{r}_f$, the number of samples N_s and the total wealth W_{total}.

1: **procedure** PORTFOLIO SIMULATION

2:      $\text{NPV} \leftarrow \texttt{zeros}(N_s, \dim \mathcal{P} + 1)$ ▷ Pre-allocate the NPV array
3:      **for** $i \in 1$ to N_s **do**   ▷ Loop over the number of samples
4:          $\mathbf{S} \leftarrow S(T)$ ▷ Sample the share price of each firm in the portfolio $\mathcal{P}$ at time T
5:          $\omega \leftarrow \text{Dir}(\alpha)$ ▷ Sample allocation ω at $t = t_0$

6:          $n \leftarrow \texttt{zeros}(\dim \mathcal{P})$ ▷ Pre-allocate the number of shares array
7:          **for** $j \in 1$ to $\dim \mathcal{P}$ **do**   ▷ Loop over the number of assets in the portfolio $\mathcal{P}$
8:              $n_j \leftarrow \omega_j \cdot W_{\text{total}} \cdot \left(S_j(t_0)^{-1}\right)$ ▷ Compute the number of shares of firm j at time $t = t_0$
9:          **end for**

10:          $\text{NPV}_{i,1} \leftarrow \sum_{k \in \mathcal{P}} n_k \cdot \left(\mathcal{D}_{T,0}^{-1}(\bar{r}_f) \cdot S_k(T) - S_k(0)\right)$ ▷ NPV of $\mathcal{P}$ at T, store column 1
11:          **for** $j \in 1$ to $\dim \mathcal{P}$ **do**   ▷ Loop over the number of assets in the portfolio $\mathcal{P}$
12:              $\text{NPV}_{i,j+1} \leftarrow \omega_j$ ▷ Store the portfolio weights in columns $2, \ldots, \dim \mathcal{P} + 1$
13:          **end for**
14:      **end for**

15: **end procedure**

Summary

In this module, we have considered the multi-dimensional GBM model, which describes the share price of firms in a portfolio $\mathcal{P}$. We demonstrated how to estimate the drift vector μ, the covariance matrix Σ, and the noise coefficients a_{ij}, and provided an algorithm to simulate the multi-dimensional GBM model. Finally, we introduced the concept of the Net Present Value (NPV) of a portfolio $\mathcal{P}$. We presented an algorithm to compute the NPV of a portfolio $\mathcal{P}$ at time T, where we sampled the fraction of the total wealth ω_i invested in each firm $i \in \mathcal{P}$ from a Dirichlet distribution and the share price at $t = T$ from a LogNormal distribution. Later, we'll develop tools to compute optimal values of these fractions, i.e., the portfolio weights, but for now, we considered the case where the portfolio weights are chosen by the investor by some approach (perhaps even randomly).

References

1. Merton RC. Paul Samuelson And Financial Economics. The American Economist. 2006;50(2):9–31.
2. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973;81(3):637–654.
3. Glasserman P. Monte Carlo methods in financial engineering. vol. 53. New York: Springer; 2004. Available from: `http://www.loc.gov/catdir/enhancements/fy0813/2003050499-d.html`.