CHEME 132 Module 1: Lattice Models of Equity Share Price

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Introduction

A lattice model discretizes the potential future states of the world into a finite number of options. For instance, a binomial lattice model has two future states: `up` and `down`, while a ternary lattice model has three: `up`, `down`, and `flat`. To make predictions, we must assign values and probabilities to each of these future states and then calculate the expected value and variance of future values. Thus, we do not precisely know quantities such as share price because we are projecting into the future. Instead, we have only a probabilistic model of the possible future values. We'll begin with the simplest possible lattice model, a binomial lattice (Fig. 1).

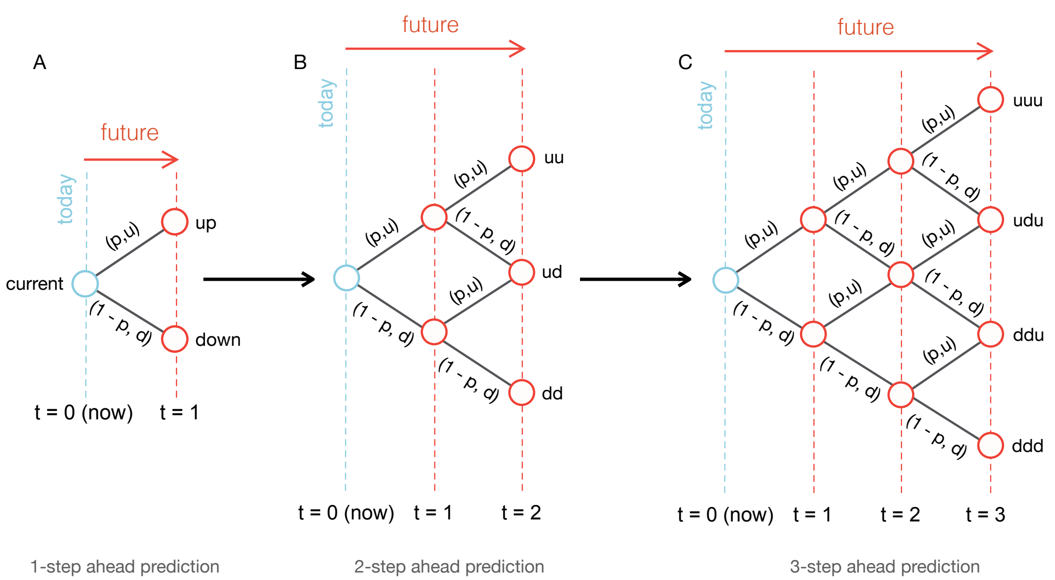

Fig. 1: Binomial lattice model schematic. At each node, the share price can either go `up` by u or `down` by d. The probability of going `up` is p, and the likelihood of going `down` is $1-p$. **A**: Single time-step lookahead. **B**: Two time-step lookahead. **C**: Three time-step lookahead. At the tree l level, the potential share price can take on $l+1$ values.

Let's start with a single time-step lookahead, with two possible future states (Fig. 1A). Let the initial share price at time `0` be $S_{\circ}$ and the share price at future time `1` be S_1. During the transition from time `0`$\rightarrow$`1`, the world transitions from the current state to one of two possible future states: `up` or `down`. We move to the `up` state with probability p or the `down` state with probability $(1-p)$. Thus, at the time `1`, the share price S_1 can take on one of two possible values: $S^{u} = u\cdot S_{\circ}$ if the world moves to the `up` state, or $S^{d} = d\cdot S_{\circ}$ if the world moves to the `down` state. As we move to the future, we can continue to build out the lattice model by adding additional time steps; for example, consider a two-step ahead prediction (Fig. 1B). At time `2`, the share price can take on one of three possible values: $S^{uu} = u^2 \cdot S_\circ$ if the world moves to the up-up state, $S^{ud} = ud \cdot S_\circ$ if the world moves to the up-down state, or $S^{dd} = d^2 \cdot S_\circ$ if the world moves to the down-down state. We can continue to build out the lattice model by adding additional time steps; for example, consider a three-step ahead prediction (Fig. 1C).

Analytical solution

Let's consider a binomial lattice model with n time-steps. At each time step, the share price can either go up by a factor of u or down by a factor of d. Then, at time n, the share price can take on $n+1$ possible values:

$$S_n = S_\circ \times D_1 \times D_2 \times D_3 \times \cdots \times D_n \tag{1}$$

where D_i is a random variable that can take on one of two values: u or d, with probabilities p and $(1-p)$ respectively. Thus, at each time-step, the world flips a coin and lands in either the up state with probability p or the down state with probability $(1-p)$. For a single time step, we model this random process as a Bernoulli trial, where the probability of success is p and the likelihood of failure is $(1-p)$. As the number of time steps increases, we have a series of Bernoulli trials, which is a binomial distribution (Defn: 1):

Definition 1: Binomial Share Price and Probability

Let $S_\circ$ denote the current share price at t = 0, u and d denote the up and down factors, and p denote the probability of going up. At time t, the binomial lattice model predicts the share price S_t is given by:

$$S_t = S_\circ \cdot u^{t-k} \cdot d^k \qquad \text{for} \quad k = 0, 1, \dots, t$$

The probability that the share price takes on a particular value at time t is given by:

$$P(S_t = S_\circ \cdot u^{t-k} \cdot d^k) = \binom{t}{k} \cdot (1-p)^k \cdot p^{t-k} \qquad \text{for} \quad k = 0, 1, \dots, t$$

where $\binom{t}{k}$ denotes the binomial coefficient.

Models of u, d and p

The up and down factors u and d, and the probability p can be computed in various ways. For example, we can estimate them from historical data or propose models for their values. Let's start by looking at approaches to estimate these parameters from historical data. Then, we'll introduce an important model developed by Cox, Ross, and Rubinstein (CRR) to compute u, d, and p in the context of options pricing that we'll see later [1].

Real-world probability p

Suppose we have a historical share price dataset from time $1, \dots, T$ for some ticker j denoted as $\mathcal{D}_j = \left\{S_1^{(j)}, S_2^{(j)}, \dots, S_T^{(j)}\right\}$, where $S_i^{(j)}$ denotes the share price of ticker j at time i. We can use different values for the share price, e.g., the opening price, closing price, high price, low price, etc. When dealing with historical data, we will typically use the volume-weighted average price (VWAP) for the period, e.g., the VWAP for the day, week, month, etc. Over the time range of the dataset $\mathcal{D}_j$, we can calculate the number of `up` and `down` moves occurring between time $i-1$ and i, and the magnitude of these moves. Then, the fraction of `up` moves is an estimate of the probability p, while some measured of the magnitude of the `up` and `down` moves, e.g., the average value are estimates of u and d respectively.

Suppose we assume the share price of ticker j is continuously compounded with an instantaneous discount (interest) rate of $r^{(j)}_{i,i-1} \equiv \mu^{(j)}_{i,i-1} \cdot \Delta t$, i.e., we split the return into a growth rate $\mu^{(j)}_{i,i-1}$ and a time step size Δt. Then, the share price at time an expression of the form governs i:

$$S^{(j)}_{i} = \exp\left(\mu_{i,i-1}\cdot\Delta t\right)\cdot S^{(i)}_{i-1} \tag{2}$$

where $\mu^{(j)}_{i,i-1}$ denotes the *growth rate* (units: 1/time) for ticker j, and Δt (units: time) is the time step size during the time period $(i-1)\rightarrow i$. Solving for the growth rate (and dropping the ticker j superscript for simplicity) gives:

$$\mu_{i,i-1} = \left(\frac{1}{\Delta t}\right)\cdot\ln\left(\frac{S_i}{S_{i-1}}\right) \tag{3}$$

We often use daily price data; thus, a single trading day is a natural time frame between S_{i-1} and S_i. However, subsequently, it will be easier to use an annualized value for the μ parameter; thus, we let $\Delta t = 1/252$, i.e., the fraction of an average trading year that occurs in a single trading day; thus, our base time will be years. We compute the growth rate for each trading day i in a collection of datasets $\mathcal{D}$ using Algorithm 1. Then we can estimate the `up` and `down` factors u and d from

Algorithm 1 Logarithmic excess growth rate

Require: Collection of price datasets $\mathcal{D}$, where $\mathcal{D}_j \in \mathcal{D}$. All datasets have the same length N.
Require: list of stocks $\mathcal{L}$, where $\dim\mathcal{L} = M$.
Require: time step size Δt between t and $t-1$ (units: years),
Require: risk-free rate r_f (units: inverse years).

$M \leftarrow$ length$(\mathcal{D}_1)$ ▷ Number of trading days in the dataset $\mathcal{D}_j$
$N \leftarrow$ length$(\mathcal{L})$ ▷ Number of stocks in the dataset $\mathcal{D}$
$\mu \leftarrow$ Array$(M-1, N)$ ▷ Initialize empty array of growth rates

for $i \in \mathcal{L}$ **do**
    $\mathcal{D}_i \leftarrow \mathcal{D}[i]$ ▷ Get dataset for stock i
    for $t = 2 \rightarrow N$ **do**
        $S_1 \leftarrow$ VWAP$(\mathcal{D}_i[t-1])$ ▷ Get volume weighted average price for stock i at time $t-1$
        $S_2 \leftarrow$ VWAP$(\mathcal{D}_i[t])$ ▷ Get volume weighted average price for stock i at time t
        $\mu[t-1, i] \leftarrow \left(\frac{1}{\Delta t}\right)\cdot\ln\left(\frac{S_2}{S_1}\right) - r_f$ ▷ Set $r_f = 0$ for regular growth rate
    end for
end for

the growth rate array generated using something like Algorithm 2. While this strategy is simple,

Algorithm 2 Estimating u, d and p from the μ-array

Require: Growth-rate μ-array from Algorithm 1.
Require: Effective risk-free rate $\bar{r}$ (units: inverse years).
Require: time step size Δt between t and $t-1$ (units: years) of the price data
Require: `mean` function, `find` function, `length` function, and `push` function.

procedure REAL WORLD PROBABILITY(μ, $\bar{r}$, Δt)

Initialize:
$N \leftarrow$ length(μ) ▷ Number of growth rates
up $\leftarrow$ Array() ▷ Initialize empty array of `up` factors
down $\leftarrow$ Array() ▷ Initialize empty array of `down` factors

Compute u:
$i_{+} \leftarrow$ find($\mu > 0$) ▷ Find the indices of all `positive` growth rates
for $i \in i_{+}$ **do**
push(up, $\exp(\mu[i] \cdot \Delta t)$)) ▷ Push the `positive` return $\mu \cdot \Delta t$ onto up-array
end for
$u \leftarrow$ `mean`(`up`) ▷ mean is our estimate of the `up` factor u

Compute d:
$i_{-} \leftarrow$ find($\mu < 0$) ▷ Find the indices of all `negative` growth rates
for $i \in i_{-}$ **do**
push(down, $\exp(\mu[i] \cdot \Delta t)$) ▷ Push the `negative` return $\mu \cdot \Delta t$ onto down-array
end for
$d \leftarrow$ `mean`(`down`) ▷ mean is our estimate of the `down` factor d

$N_{+} \leftarrow$ length(i_{+}) ▷ Number of `positive` growth rates
$p \leftarrow N_{+}/N$ ▷ Estimate of the probability p

return u, d and p

end procedure

it may not be robust. For example, if the dataset $\mathcal{D}_j$ is short, i.e., only a few trading days, then the number of `up` and `down` moves will be small, and the estimates of u, d and p will be poor. If a precise estimate of u, d and p is required, the number of trading days in the dataset $\mathcal{D}_j$ should be large. Furthermore, the estimates of u, d and p are not robust to outliers in the dataset $\mathcal{D}_j$. Thus, we may want to consider other models for computing u, d and p.

Risk-neutral probability q

Another approach to computing the parameters in the lattrice is to use the risk-neutral probability q. Let's consider a single-step binomial lattice model (Fig. 1A). The expected value of the share price at time 1 for a single step is given by:

$$\mathbb{E}_{\mathbb{Q}}(S_1|S_\circ) = q\cdot S^u + (1-q)\cdot S^d$$

where $\mathbb{E}_{\mathbb{Q}}(\dots)$ denotes the expectation operator written with respect to the risk-neutral probability measure $\mathbb{Q}$, the term q denotes the risk-neutral probability of the `up` state, and S^u and S^d are the share prices in the `up` and `down` states respectively. The hypothetical probability q is used to price derivatives (as we shall see later), however, we could also think of it as a tool to compute a *fair* price for a share of stock. Suppose we rewrite Eqn. (2) as:

$$\mathcal{D}_{1,0}(\bar{r})\cdot S_\circ = \mathbb{E}_{\mathbb{Q}}\left(S_1|S_\circ\right) \tag{4}$$

where $\mathcal{D}_{1,0}(\bar{r})$ is the continuous discount factor between period $0\rightarrow 1$, and $\bar{r}$ is the effective (constant) risk-free rate. Thus, unlike the previous case, where the share price S_i was discounted by the return $\mu_{i,i-1}\cdot\Delta t$ (which could vary in time), here we discount the share price S_i by an effective (constant) risk-free rate $\bar{r}$. The expectation operator $\mathbb{E}_{\mathbb{Q}}(\dots)$ can be matched with the expansion:

$$\mathcal{D}_{1,0}(\bar{r})\cdot S_\circ = q\cdot S^u + (1-q)\cdot S^d \tag{5}$$

The share prices in the `up` and `down` states are the product of the `up` factor u (or a `down` factor d) and the initial share price, i.e., $S^u = u\cdot S_\circ$ and $S^d = d\cdot S_\circ$. Substiuting these price values into Eqn. (5) and solving for q gives:

$$q = \frac{\mathcal{D}_{1,0}(\bar{r}) - d}{u-d} \tag{6}$$

Thus, we can compute the risk-neutral probability q from the `up` and `down` factors u and d and the effective risk-free rate $\bar{r}$. The risk neutral probability q, and the `up` and `down` factors u and d can be estimated using Algorithm 3.

Cox-Ross-Rubinstein (CRR) model

The Cox, Ross, and Rubinstein (CRR) lattice model assumes a risk-neutral probability q to describe the probability of an `up` move. However, unlike a purely data-driven risk-neutral probability where we estimnate values for u and d from data, the CRR model proposes a model for u and d that is consistent with the variance of the return of the underlying asset. The `up` factor u is constructed so that the CRR model matches the variance of the log return Var$(S_j/S_{j-1}) = \sigma^2\Delta t$:

$$\text{Var}(S_j/S_{j-1}) = \mathbb{E}\left[(S_j/S_{j-1})^2\right] - \mathbb{E}\left[S_j/S_{j-1}\right]^2 = \sigma^2\Delta t$$

which after subtitution of the price and probability expressions from Defn. 1 gives:

$$q(u-1)^2 + (1-q)(d-1)^2 + [p(u-1) + (1-q)(d-1)]^2 = \sigma^2\Delta t \tag{7}$$

where σ is the volatility (standard deviation) of the return, and Δt is the time step. Equation (7) is a quadratic equation in u, which can be solved for u (assuming Δt^2 terms and above are ignored):

$$u = \exp(\sigma\cdot\sqrt{\Delta t})$$

Algorithm 3 Risk-neutral estimate of u, d and q

Require: Growth-rate μ-array from Algorithm 1.
Require: Effective risk-free rate $\bar{r}$ (units: inverse years).
Require: time step size Δt between t and $t-1$ (units: years) of the price data
Require: `mean` function, `find` function, `length` function, and `push` function.

procedure RISK NEUTRAL PROBABILITY(μ, $\bar{r}$, Δt)

  Initialize:
  $N \leftarrow$ length(μ) ▷ Number of growth rates
  up $\leftarrow$ Array$()$ ▷ Initialize empty array of `up` factors
  down $\leftarrow$ Array$()$ ▷ Initialize empty array of `down` factors

  Compute u:
  $i_{+} \leftarrow$ find$(\mu > 0)$ ▷ Find the indices of all `positive` growth rates
  for $i \in i_{+}$ **do**
    push(up, $\exp(\mu[i] \cdot \Delta t)))$ ▷ Push the `positive` return $\mu \cdot \Delta t$ onto up-array
  end for
  $u \leftarrow \texttt{mean}(\texttt{up})$ ▷ mean is our estimate of the `up` factor u

  Compute d:
  $i_{-} \leftarrow$ find$(\mu < 0)$ ▷ Find the indices of all `negative` growth rates
  for $i \in i_{-}$ **do**
    push(down, $\exp(\mu[i] \cdot \Delta t))$ ▷ Push the `negative` return $\mu \cdot \Delta t$ onto down-array
  end for
  $d \leftarrow \texttt{mean}(\texttt{down})$ ▷ mean is our estimate of the `down` factor d

  $q \leftarrow \frac{\exp(\bar{r} \cdot \Delta t) - d}{u - d}$ ▷ Compute the risk-neutral probability q

  return u, d and q

end procedure

In the CRR approach, the `up` and `down` factors are related by $ud = 1$, i.e., the lattice is symmetric.

To use the CRR model, we must first specify the volatility σ of the return of the underlying asset, e.g., shares of stock. We can estimate this from data by first computing the return (or the growth) and the estimatinf the volatility of the return, which is a backward-looking approach, or we can use the implied volatility from the options market, which is a forward-looking approach (the market's estimate of future price movements).

Binomial lattice trade strategy

Suppose we purchase n_o share of ticker `XYZ` at time $t = 0$ for S_o USD/share. Then, at some later date $t = T$, we sell all n_o shares for S_T USD/share. The net present value (NPV) of this trade has two cashflow events, the initial purchase of n_o shares for S_o USD/share, and the sale of n_o shares for S_T USD/share at some later date $t = T$:

$$\text{NPV}(\bar{r}, T) = -n_o \cdot S_o + n_o \cdot S_T \cdot \mathcal{D}_{T,0}^{-1}(\bar{r}) \tag{8}$$

where $\mathcal{D}_{T,0}^{-1}(\bar{r})$ is the inverse of the continuous discount factor for period $0 \rightarrow T$, assuming an effective discount rate $\bar{r}$. Factoring the initial number of shares n_o from the NPV expression gives:

$$\text{NPV}(\bar{r}, T) = n_o \cdot \left(S_T \cdot \mathcal{D}_{T,0}^{-1}(\bar{r}) - S_o\right) \tag{9}$$

The term in the parenthesis is the net change in the share price, i.e., the change in the share price S_T discounted by the effective discount rate $\bar{r}$. However, if we hold the shares for only a short period, e.g., on order days or months, then $\mathcal{D}_{T,0}^{-1}(\bar{r}) \approx 1$, which simplifies the NPV expression to:

$$\text{NPV}(\bar{r}, T) \simeq n_o \cdot (S_T - S_o) \tag{10}$$

Dividing by the initial investment $n_o \cdot S_o$ gives the fraction return on investment, or just the fractional return:

$$\frac{\text{NPV}(\bar{r}, T)}{n_o \cdot S_o} \simeq \frac{S_T - S_o}{S_o} = \frac{S_T}{S_o} - 1 \tag{11}$$

In a binomial world, the share price S_T is a random variable which takes on $T + 1$ possible values:

$$S_T \in \left\{S_\circ \cdot u^{T-k} \cdot d^k\right\}_{k=0}^{T} \tag{12}$$

where $S_\circ$ is the initial share price, u and d are the `up` and `down` factors, and p is the probability of going `up`. Substiuting the S_T expression into Eqn. (11) gives:

$$\frac{\text{NPV}(\bar{r}, T)}{n_o \cdot S_o} \in \left\{u^{T-k} \cdot d^k - 1\right\}_{k=0}^{T} \tag{13}$$

Equation (13) describes the distribution of the fractional return on investment predicted by the binomial lattice model. Because we also have the probability of each future share price, we can compute a number of interesting properties of the distribution of the fractional returns, e.g., the mean, variance, etc. Furthermore, we can compute the probability that the fractional return is greater than some desired threshold, e.g., 10%, by reconstructing the cumulative distribution function (CDF) of the fractional return. We'll illustrate this idea with the project for this module.

Summary

In this module we introduced the binomial lattice model for equity share price. The binomial lattice model is a discrete-time model, where the furture share price can take on one of two possible values at each time step: `up` or `down`. In the `up` state, the share price increases by a factor of u, while in the `down` state, the share price decreases by a factor of d. The binomial lattice model is a probabilistic model, where the probability of going `up` is p and the probability of going `down` is $(1 - p)$. The `up` and `down` factors u and d, and the probability p can be computed in various ways, but are assumed to be constant over time. For example, we can compute the `up` and `down` factors u

and d from historical data, or we can propose a model for u and d. Finally, we developed a simple trade strategy which relied on the binomial lattice model to compute a future price distribution, where the net present value was used to value the trade strategy.

References

1. Cox JC, Ross SA, Rubinstein M. Option pricing: A simplified approach. Journal of Financial Economics. 1979;7(3):229–263. doi:https://doi.org/10.1016/0304-405X(79)90015-1.