CHEME 132 Module 2: Single Asset Geometric Brownian Motion

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Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm. Geometric Brownian motion was popularized as a financial model by Samuelson in the 1950s and 1960s [1], but is arguably most commonly associated with the Black–Scholes options pricing model, which we will describe later [2]. Let's start with the single asset case (in the absence of dividends), and then consider the multiple asset case in the next module.

Geometric Brownian motion (GBM) assumes that the share price $S(t)$ of a firm can be modeled as a detereministic drift term (which is proportional to the share price) that is corrupted by a Wiener noise process, also proportional to the share price:

$$\frac{dS}{S} = \mu\,dt + \sigma\,dW \tag{1}$$

The constant μ denotes a drift parameter, i.e., the growth rate of the share price return, σ is a volatility parameter, i.e., the dispersion of the return, dt denotes an infinitesimal time step, and dW is the output of a Wiener noise process. A Wiener Process (also often referred to as a standard Brownian motion) is a real-valued continuous-time stochastic process named after Norbert Wiener for the study of one-dimensional Brownian motion (Defn. 1):

Definition 1: Wiener Process

A Wiener process is a continuous one-dimensional stochastic process $\{W\left(t\right), 0 \leq t \leq T\}$ with the following properties:

- W(0) = 0 with probability 1
- The increments $\{W(t_1) - W(t_o), \dots, W(t_k) - W(t_{k-1})\}$ are independent for any k and $0 \leq t_o < t_1 < \cdots < t_k \leq T$
- The increment W(t) - W(s) $\sim\ N\left(0, t-s\right)$ for any $0 \leq s < t \leq T$, where $N\left(0, t-s\right)$ denotes a normally distributed random variable with mean 0 and variance $t-s$.

Equation 1 is a continuous-time analog of the discrete-time binomial lattice model we developed previously. It has several interesting properties, for example it has an analytical solution, and that solution is a lognormal distribution. Futher, it has analytical expressions for the expectation and variance of the share price. In this module, we will develop analytical solutions to Eqn. 1, and tools to estimate the parameters μ and σ from historical data. We'll then use these tools to simulate the share price of firms.

Analytical solution

Using Ito's lemma, we can formulate an analytical solution to the GBM equation for a single asset. Ito's Lemma, developed by K. Ito in 1951, is an analog of the Taylor series for stochastic systems. Let the random variable $X(t)$ be governed by the general stochastic differential equation:

$$dX = a\left(X(t), t\right) dt + b\left(X(t), t\right) dW(t)$$

where $dW(t)$ is a one-dimensional Wiener process and a and b are functions of $X(t)$ and t. Let $Y(t) = \phi\left(t, X(t)\right)$ be twice differentiable with respect to $X(t)$, and singly differentiable with respect to t. Then, $Y(t)$ is governed by the equation:

$$dY = \left(\frac{\partial Y}{\partial t} + a\frac{\partial Y}{\partial X} + \frac{b^2}{2}\frac{\partial^2 Y}{\partial X^2}\right) dt + b\left(\frac{\partial Y}{\partial X}\right) dW(t)$$

Let $Y = \ln(S)$, $a = \mu \cdot S$, and $b = \sigma \cdot S$. Then, Y is governed by the stochastic differential equation (using Ito's Lemma):

$$d\ln(S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \cdot dW(t)$$

We integrate both sides of the equation to obtain from $t_\circ$ to t:

$$\int_{t_\circ}^{t} d\ln(S) = \int_{t_\circ}^{t} \left(\mu - \frac{\sigma^2}{2}\right) dt + \int_{t_\circ}^{t} \sigma \cdot dW(t)$$

which gives:

$$\ln\left(\frac{S_t}{S_\circ}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(t - t_\circ) + \sigma \cdot \sqrt{t - t_\circ} \cdot Z(0,1)$$

where the noise term makes use of the definition of the integral of a Wiener process. Finally, we exponentiate both sides of the equation to obtain the analytical solution to the GBM model:

$$S(t) = S_\circ \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t - t_\circ) + (\sigma\sqrt{t - t_\circ}) \cdot Z_t(0,1)\right] \qquad (2)$$

where $S_\circ$ denotes the share price at $t_\circ$, and $Z_t(0,1)$ denotes a standard normal random variable at time t. The expectation and variance of the GBM model is given by:

$$\begin{aligned}
\mathbb{E}\left(S_t\right) &= S_o \cdot \exp\left(\mu \cdot \Delta t\right) \\
\text{Var}\left(S_t\right) &= S_\circ^2 e^{2\mu\cdot\Delta t}\left[e^{\sigma^2\Delta t} - 1\right]
\end{aligned}$$

where $\Delta t = t - t_\circ$.

Model parameters

Estimating the drift parameter μ

Let's assume that we have a time series of share price values $S(t_1), S(t_2), \dots, S(t_k)$ and we want to estimate the detereministic growth of the share price, i.e., the drift parameter μ. There are several ways to do this, but we will use a detereministic linear model of the natural log of the share price values. To estimate the deterministic component of the share price, we first set the volatility parameter $\sigma = 0$ in Eqn. 2. Then, at some future time t, the share price (after some algebra) is given by:

$$\ln\,S_i = \ln\,S_\circ + \mu\cdot(t_i - t_\circ) \tag{3}$$

where $\ln\,S_\star$ denotes the natural log of the share price at time $t_\star$. Equation 3 is a linear model of the form $y = mx + b$, where $y = \ln\,S_i$, $x = t_i - t_\circ$, $m = \mu$, and $b = \ln\,S_\circ$. Thus, we can estimate the growth parameter μ by fitting a linear model to the log of the share price values by solving an overdetermined system of linear equations.

Let $\mathbf{A}$ denote a $\mathcal{S}\times 2$ matrix, where each row corresponds to a time value $t > 0$. The first column of $\mathbf{A}$ is all 1's while the second column holds the $(t_k - t_\circ)$ values. Further, let $\mathbf{Y}$ denote the natural log of the share price values (in the same temperal order as the $\mathbf{A}$ matrix). Then, the y-intercept and slope (drift parameter) can be estimated by solving the overdetermined system of equations:

$$\mathbf{A}\theta + \epsilon = \mathbf{Y}$$

where θ denotes the vector of unknown parameters (the y-intercept and slope), and ϵ denotes an error model, e.g., a normal distribution with mean zero and variance σ_ϵ^2. This system can be solved as:

$$\theta = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} - (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\epsilon$$

where $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$, and $(\mathbf{A}^T\mathbf{A})^{-1}$ denotes the inverse of the square matrix product $\mathbf{A}^T\mathbf{A}$. Finally, we can estimate the error term ϵ by calculating the residuals:

$$\epsilon = \mathbf{Y} - \mathbf{A}\theta$$

and then fitting a normal distribution to the residuals, using some technique such as maximum likelihood estimation, to compute the uncertainty in the estimate of the drift parameter $\hat{\mu}$ (where the $\hat{\star}$ denotes an estimate of the parameter).

Estimating the volatility parameter σ

Generally, methods to estimate the volatility parameter can be classified into two categories - historical volatility estimates based on past return data and future volatility estimates based on the Implied Volatility (IV) of options contracts. Here, we estimate the volatility σ from historical data, i.e., from the past returns of the share price.

A return refers to the increase or decrease in the price of an asset, e.g., shares of a stock, over a specific period, e.g., minutes, days, weeks, or even years. Assume the share price of a stock at time t is given by $S(t)$. Then, if we assume continuous compounding, the price of the stock at time

$t + \Delta t$ is given by:

$$S(t + \Delta t) = S(t) \cdot \exp\left(\mu \cdot \Delta t\right) \tag{4}$$

where μ denotes the instantenous growth rate (units: inverse years) of the share price over the time horizon $t \rightarrow t + \Delta t$. The continuously compounded share price gives rise to the logarithmic return model (Defn. 2):

Definition 2: Logarithmic Return

Let $S_{i,t-1}$ denote the continuously compounding share price of ticker i at time $t-1$. Then, at time t, the share price for ticker i, denoted as $S_{i,t}$, is given by:

$$S_{i,t} = S_{i,t-1} \cdot \exp\left(\mu_{t,t-1}^{(i)} \cdot \Delta t\right) \tag{5}$$

where $\mu_{t,t-1}^{(i)}$ denotes the instantenous growth rate (units: inverse years) of ticker i over time horizon $(t-1) \rightarrow t$, and Δt is the time interval between $t-1$ and t (units: years). The logarithmic return on asset i over time horizon $(t-1) \rightarrow t$, denoted by $r_{t,t-1}^{(i)}$, is defined as:

$$r_{t,t-1}^{(i)} \equiv \ln\left(\frac{S_{i,t}}{S_{i,t-1}}\right) = \mu_{t,t-1}^{(i)} \cdot \Delta t \tag{6}$$

Given Defn. 2, we compute the logarithmic return distribution using Algorithm 1. For each stock i, we compute the logarithmic return $\mu_{t,t-1}^{(i)}$ for each time interval $(t-1) \rightarrow t$, and then compute the mean and variance of the logarithmic return distribution.

Algorithm 1 Logarithmic Excess Growth Rate

Require: data set $\mathcal{D}_i = \{S_{i,t}\}_{t=1}^{N} \in \mathcal{D}$ where $S_{i,t}$ denotes the price of stock i at time t, all stocks have the same time horizon $N \gg 2$, and $\mathcal{D}$ denotes the data set of all stocks.
Require: The time interval Δt between t and $t-1$ (units: years), and a list of stocks $\mathcal{L} = \{i\}_{i=1}^{M}$ that we are interested in, where $M = \dim \mathcal{D}$.
Require: The risk-free rate r_f (units: inverse years).

procedure LOG RETURN($\mathcal{D}$, $\mathcal{L}$, Δt, r_f)
  $N \leftarrow$ length(μ) ▷ Number of growth rates
  for $i \in \mathcal{L}$ **do**
    $\mathcal{D}_i \leftarrow \mathcal{D}[i]$ ▷ Select the data for stock i from the dataset collection $\mathcal{D}$
    for $t = 2 \rightarrow N$ **do**
      $S_{i,t-1} \leftarrow \mathcal{D}_i[t-1]$ ▷ Select the price of stock i at time $t-1$
      $S_{i,t} \leftarrow \mathcal{D}_i[t]$ ▷ Select the price of stock i at time t
      $\mu_{t,t-1}^{(i)} \leftarrow (1/\Delta t) \cdot \ln\left(S_{i,t}/S_{i,t-1}\right) - r_f$ ▷ Set $r_f = 0$ for regular growth rate
    end for
  end for
end procedure

Summary

Fill me in.

References

1. Merton RC. Paul Samuelson And Financial Economics. The American Economist. 2006;50(2):9–31.
2. Black F, Scholes M. The Pricing of Options and Corporate Liabilities. Journal of Political Economy. 1973;81(3):637–654.