CHEME 132 Module 2: Single Asset Geometric Brownian Motion Simulations

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Introduction

Geometric Brownian motion (GBM) is a continuous-time stochastic model in which the random variable $S(t)$, e.g., the share price of a firm, is described by a deterministic trajectory corrupted by a Wiener stochastic noise process:

$$\frac{dS}{S} = \mu \, dt + \sigma \, dW \tag{1}$$

The constant μ denotes a drift parameter, i.e., the growth rate of the share price return, σ is a volatility parameter, i.e., the dispersion of the return, dt denotes an infinitesimal time step, and dW represents the output of a Wiener noise process. Thus, Eqn. 1 is the continuous-time analog of the discrete-time binomial lattice model we developed previously. In this module, we will develop analytical solutions to Eqn. 1, and tools to estimate the parameters μ and σ from historical data.

Analytical solution

Using Ito's lemma, we can formulate an analytical solution to the GBM equation for a single asset. Ito's Lemma, developed by K. Ito in 1951, is an analog of the Taylor series for stochastic systems. Let the random variable $X(t)$ be governed by the general stochastic differential equation:

$$dX = a\left(X(t), t\right) dt + b\left(X(t), t\right) dW(t)$$

where $dW(t)$ is a one-dimensional Wiener process and a and b are functions of $X(t)$ and t. Let $Y(t) = \phi\left(t, X(t)\right)$ be twice differentiable with respect to $X(t)$, and singly differentiable with respect to t. Then, $Y(t)$ is governed by the equation:

$$dY = \left(\frac{\partial Y}{\partial t} + a\frac{\partial Y}{\partial X} + \frac{b^2}{2}\frac{\partial^2 Y}{\partial X^2}\right) dt + b\left(\frac{\partial Y}{\partial X}\right) dW(t)$$

Let $Y = \ln(S)$, $a = \mu \cdot S$, and $b = \sigma \cdot S$. Then, Y is governed by the stochastic differential equation (using Ito's Lemma):

$$d\ln(S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma \cdot dW(t)$$

We integrate both sides of the equation to obtain from $t_\circ$ to t:

$$\int_{t_\circ}^{t} d\ln(S) = \int_{t_\circ}^{t} \left(\mu - \frac{\sigma^2}{2}\right) dt + \int_{t_\circ}^{t} \sigma \cdot dW(t)$$

which gives:

$$\ln\left(\frac{S_t}{S_\circ}\right) = \left(\mu - \frac{\sigma^2}{2}\right)(t - t_\circ) + \sigma \cdot \sqrt{t - t_\circ} \cdot Z(0,1)$$

where the noise term makes use of the definition of the integral of a Wiener process. Finally, we exponentiate both sides of the equation to obtain the analytical solution to the GBM model:

$$S(t) = S_\circ \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)(t - t_\circ) + (\sigma\sqrt{t - t_\circ}) \cdot Z_t(0,1)\right] \quad (2)$$

where $S_\circ$ denotes the share price at $t_\circ$, and $Z_t(0,1)$ denotes a standard normal random variable at time t. The expectation and variance of a GBM model is:

$$\begin{aligned}
\mathbb{E}\left(S_t\right) &= S_o \cdot \exp\left(\mu \cdot \Delta t\right) \\
\text{Var}\left(S_t\right) &= S_\circ^2 e^{2\mu \cdot \Delta t}\left[e^{\sigma^2 \Delta t} - 1\right]
\end{aligned}$$

Model parameters

Estimating the growth parameter μ

Let $\mathbf{A}$ denote a $\mathcal{S} \times 2$ matrix, where each row corresponds to a time value. The first column of $\mathbf{A}$ is all 1's while the second column holds the $(t_k - t_\circ)$ values. Further, let $\mathbf{Y}$ denote the $\ln$ of the share price values (in the same order as the $\mathbf{A}$ matrix). Then, the y-intercept and slope (drift parameter) can be estimated by solving the overdetermined system of equations:

$$\mathbf{A}\theta + \epsilon = \mathbf{Y}$$

where θ denotes the vector of unknown parameters. This system can be solved as:

$$\theta = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Y} - (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\epsilon$$

where $\mathbf{A}^T$ denotes the transpose of the matrix $\mathbf{A}$, and $(\mathbf{A}^T\mathbf{A})^{-1}$ denotes the inverse of the square matrix product $\mathbf{A}^T\mathbf{A}$. Finally, we can estimate the error term ϵ by calculating the residuals:

$$\epsilon = \mathbf{Y} - \mathbf{A}\theta$$

and then fitting a normal distribution to the residuals to compute the uncertainty in the estimate of the mean of the drift parameter $\hat{\mu}$.

Summary

Fill me in.

References