Multiobjective Evolution Strategy with Linear and Nonlinear Constraints

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ABSTRACT

This paper introduces a new evolution strategy for solving multiobjective optimization problems subject to linear and nonlinear constraints. Here the handling constraints and (in)feasible individuals are successfully performed using the concept of $\mathcal{F}$- and $\mathcal{C}$-fitness. Therefore the infeasible population quickly evolves towards the feasibility. A new selection algorithm for searching, maintaining a set of feasible pareto-optimal solutions in every generation is proposed. Finally, some selected test cases are shown to illustrate a remarkable efficiency of the proposed evolution strategy.

MOTIVATION

The general multiobjective optimization problem with linear and nonlinear constraints can be formulated as below:

$$\min_{\boldsymbol{x}} \ \boldsymbol{f}(\boldsymbol{x}) = \min_{\boldsymbol{x}}(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \cdots, f_{\mathrm{N}}(\boldsymbol{x}))$$

where

- $\boldsymbol{x} = (x_1, x_2, \cdots, x_{\mathrm{n}})^T \in \mathcal{F} \subseteq \mathcal{S} \subseteq \mathbb{R}^n$.
- $\mathcal{S}$ denotes so-called *search space*, for example, a n-dimensional rectangle defined by lower and upper bounds of variables:

$$x_{\mathrm{i}}^{(\mathrm{lower})} \leq x_{\mathrm{i}} \leq x_{\mathrm{i}}^{(\mathrm{upper})}, \ \forall i = \overline{1, n}$$

- $\mathcal{F} \subseteq \mathcal{S}$ (so-called *feasible region*) is often defined by a set of m additional linear and nonlinear constraints ($m \geq 0$):

$$G_{\mathrm{j}}(\boldsymbol{x}) \leq 0, \ \forall j = \overline{1, q}$$
$$H_{\mathrm{j}}(\boldsymbol{x}) = 0, \ \forall j = \overline{q+1, m}.$$

To solve this problem is to find all feasible trade-offs among the multiple, conflicting objectives, known as a set $\mathcal{P}_{\mathrm{x}}$ of *feasible pareto-optimal solutions* in the variable space:

A vector $\boldsymbol{x}^* \in \mathcal{F}$ is said to be feasible pareto-optimal if and only if there exists no other vector $\boldsymbol{x} \in \mathcal{F}$ such that: $f_{\mathrm{i}}(\boldsymbol{x}) \leq f_{\mathrm{i}}(\boldsymbol{x}^*)$, $\forall i = \overline{1, N}$ and $f_{\mathrm{j}}(\boldsymbol{x}) < f_{\mathrm{j}}(\boldsymbol{x}^*)$ for at least one j.

$\mathcal{P}_{\mathrm{x}}$ corresponds to a set $\mathcal{P}_{\mathrm{f}}$ of nondominated or noninferior vectors (solutions), lying on a surface known as a *pareto-optimal surface*, in the objective function space.
Recently, some evolutionary algorithms (EAs) for solving multiobjective optimization problems are proposed [9,5,4,7,6,1]. The goal of EAs is to approximate a set of pareto-optimal solutions by a population of individuals (namely, to find and to maintain a representative sampling of solutions on the pareto-optimal surface). Traditional multiobjective evolution strategies (ESs)

provided a well approximation of pareto-optimal solutions, but they required a feasible starting point for optimization. The finding a feasible point is itself a difficult problem especially in cases the ratio between the feasible and the search region is too small.
The proposed evolution strategy MOBES (*MultiOBjective Evolution Strategy*) will be intended to overcome drawbacks of traditional ones.

HANDLING CONSTRAINTS

Without loss of generality only constraints in terms of inequalities should be taken into account because constraints in terms of linear and nonlinear equations:

$$H_{\rm j}(\boldsymbol{x})=0,\ \forall j=\overline{q+1,m}$$

can be replaced by a pair of inequalities:

$$-\epsilon \le H_{\rm j}(\boldsymbol{x}) \le \epsilon,\ \forall j=\overline{q+1,m}$$

where ϵ is an additional parameter to define the precision of the system (for example, $\epsilon = 10^{-16}$). Therefore, the feasible region can be redefined as

$$\mathcal{F} = \{\boldsymbol{x} \in \mathcal{S} : G_{\rm i}(\boldsymbol{x}) \le 0, \forall i = \overline{1,M}\},$$

where $M = q + 2(m-q) = 2m - q$. Let $c_{\rm i}(\boldsymbol{x}) = \max\{G_{\rm i}(\boldsymbol{x}), 0\}$, a measure of violation of constraints by a vector $\boldsymbol{x}$ can be determined by

$$\mathcal{C}(\boldsymbol{x}) = (c_1(\boldsymbol{x}), c_2(\boldsymbol{x}), \cdots, c_{\rm M}(\boldsymbol{x}))^T.$$

Clearly, $\mathcal{C} \equiv \mathbf{0}$ for $\forall \boldsymbol{x} \in \mathcal{F}$ and $\mathcal{C} > \mathbf{0}$ for $\forall \boldsymbol{x} \in \mathcal{S}\backslash\mathcal{F}$. Then $\mathcal{C}$ is an operator mapping the search space into the constraint space so that the original point of the constraint space corresponds to the feasible region. For optimization with many constraints a large amount of memory is necessary to save $\mathcal{C}$ for every individual. Moreover, it takes much time for ranking and comparing infeasible individuals using the vectorial measure. To overcome these problems the following scalar measure based on the "distance" between one point and the original point in the constraint space, is used [8,2]:

$$C(\boldsymbol{x}) = \Big(\sum_{i=1}^{M} [c_{\rm i}(\boldsymbol{x})]^p \Big)^{\frac{1}{p}}, \quad (p > 0).$$

Our first experiments have shown that the scalar measure of violation of constraints is good and acceptable. $C(\boldsymbol{x})$ is called C-fitness of an individual in the constraint space. Using the concept of C-fitness, the population can be divided into classes. Individuals of the same class have the same value of C-fitness (i. e. the "same distance" to the feasible region) and they are said to be individuals of the C-class. Of course, all feasible individuals belong to the 0-class.

From this consideration an individual can be characterized by the four following properties: an objective variable vector $\boldsymbol{x} = (x_1, \cdots, x_{\rm n})^T$, a strategy parameter vector $\boldsymbol{s} = (s_1, \cdots, s_{\rm n})^T$, an objective function vector $\boldsymbol{f}(\boldsymbol{x})$ (so called $\mathcal{F}$-fitness) and C-fitness $C(\boldsymbol{x})$. Unlike other ESs the fourth property is specially used to handle constraints.

SEARCHING FOR FEASIBLE PARETO-OPTIMAL SOLUTIONS

In opposition to traditional ESs mutation and reproduction operators can generate both feasible and infeasible offspring. For highly constrained problems so that no feasible solution can be found through first several generations, it is necessary to give the first priority for making an infeasible population to evolve towards the feasible region. To do it the C-fitness is used to rank infeasible individuals; better infeasible individuals belong to lower classes ("nearer" to the feasible region).

Criterion 1. An individual of the $\mathcal{C}_1$-class ($\mathcal{C}_1 > 0$) is said to be better than the other of the $\mathcal{C}_2$-class iff $\mathcal{C}_1 < \mathcal{C}_2$.

In other words this criterion enables to reject "father" infeasible individuals from a population and after some stage of the evolution process the population can consist of only feasible individuals. In optimization cases by which the feasible region is non-convex or the ratio between the feasible and the search region is small, it is also difficult to generate feasible offspring even from feasible individuals in many generations. Moreover many experiments showed any feasible individual is *not always better* than any infeasible one. In *Fig. 1*, for example, an infeasible individual (denoted by b) can generate feasible offspring which are better than offspring of a feasible individual (denoted by a). It is reasonable to maintain a

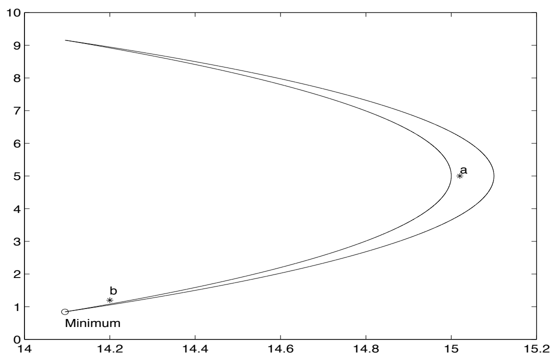


Fig. 1. An example about an infeasible individual which can be "better" than feasible one

mixed population through all generations (containing feasible and infeasible individuals lying in a neigbourhood of the feasible region). For a mixed population infeasible individuals should belong to a so-called extension class.

Criterion 2. Infeasible individuals up to the $\mathcal{C}_{\text{extra}}$-class (i. e. individuals of $\mathcal{C}$-classes so that $0 < \mathcal{C} \leq \mathcal{C}_{\text{extra}}$) are said to be in the same class, so-called the *extra class.*

Criterion 3. An infeasible offspring of a feasible individual is said to be "viable" iff it belongs to the extension $\mathcal{C}_{\text{extension}}$-class defined by $\mathcal{C}_{\text{extension}} = \max\{\mathcal{C}_{\text{extra}}, \mathcal{C}_{\text{pop}}\}$, where $\mathcal{C}_{\text{pop}}$ is the highest class in the current population (corresponding to the "farthest" infeasible individual of the current population).

For comparing individuals of the same class and updating a current pareto-optimal frontier the following criterion based on the $\mathcal{F}$-fitness is used:

Criterion 4. An individual is said to be viable iff its $\mathcal{F}$-fitness is not dominated by $\mathcal{F}$-fitness of the others.

MAINTAINING A FEASIBLE PARETO-OPTIMAL SET

An algorithm for maintaining a representative sampling of solutions on the pareto-optimal surface was proposed in the old versions of the MOBES [1]. The weakness of this algorithm is that the density of individuals in a neigbourhood of the selfish minima (an own minimum of each objective function) is still higher than in other regions of the pareto-optimal surface. For this reason it should be slighly modified as below:

Criterion 5. Let

$$
\begin{aligned}
\boldsymbol{f}^{(\min)} &= (\min f_1, \min f_2, \cdots, \min f_{\mathrm{N}}) \\
&= (f_1^{(\min)}, f_2^{(\min)}, \cdots, f_{\mathrm{N}}^{(\min)}) \\
\boldsymbol{f}^{(\max)} &= (\max f_1, \max f_2, \cdots, \max f_{\mathrm{N}}) \\
&= (f_1^{(\max)}, f_2^{(\max)}, \cdots, f_{\mathrm{N}}^{(\max)}),
\end{aligned}
$$

where the minimum and maximum operators are performed along each coordinate axes of the objective function space for all feasible noninferior individuals of the population. Then, the current feasible trade-offs surface is bound in a hyperparallelogram defined by vectors $\boldsymbol{f}^{(\min)}$ and $\boldsymbol{f}^{(\max)}$.

- Dividing each interval $[f_{\mathrm{i}}^{(\min)}, f_{\mathrm{i}}^{(\max)}]$ into N_{pop} (the desired number of feasible individuals per population) small sections (denoted by $\mathcal{H}_{\mathrm{j}}^{\mathrm{i}},\ j = \overline{1, N_{\mathrm{pop}}}$) with the length δ_{i}, i. e.:

$$
\delta_{\mathrm{i}} = \frac{f_{\mathrm{i}}^{(\max)} - f_{\mathrm{i}}^{(\min)}}{N_{\mathrm{pop}}}.
$$

- Evaluating the density of the population per section (the number of noninferior individuals on a section).
- Along the *i-th* coordinate axes of the objective function space ($i = \overline{1, N}$) the best individual in each of the $N_{\mathrm{best}} = \dfrac{N_{\mathrm{pop}}}{N+k}$ first sections (i. e. $\mathcal{H}_{\mathrm{j}}^{\mathrm{i}},\ j = \overline{1, N_{\mathrm{best}}}$) is selected, where k is an integer number.
- Other individuals can be selected from remaining sections with the lowest density.

TEST CASES

In this section some test problems are selected to illustrate that the new multiobjective evolution strategy is successful and powerful in practice. For all test cases the following important parameters of the MOBES were used:

- Population size = 100
- $\mathcal{C}_{extra} = 0.1$
- Number of parents for mutation and reproduction = 10
- Number of offspring per mutation = 5.

Test Case 1

The problem [7] is min $(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$:

$$
\begin{aligned}
f_1(\boldsymbol{x}) &= (x_1 - 2)^2 + (x_2 - 1)^2 + 2 \\
f_2(\boldsymbol{x}) &= 9x_1 - (x_2 - 1)^2
\end{aligned}
$$

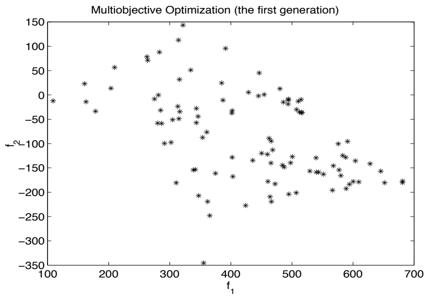
subject to non-linear constraints:

$$
\begin{aligned}
x_1^2 + x_2^2 - 225 &\leq 0 \\
x_1 - 3x_2 + 10 &\leq 0
\end{aligned}
$$

and bounds: $-20 \leq x_{\mathrm{i}} \leq 20,\ \forall i = 1, 2$.
A set of pareto-optimal solutions is found after running 5 generations. In opposition to [7] the current population more quickly runs to the pareto-optimal frontier (29 generations in [7]) and is more uniformly distributed on it (see *Fig. 2*).

Test Case 2

The problem is min $(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x}))$:



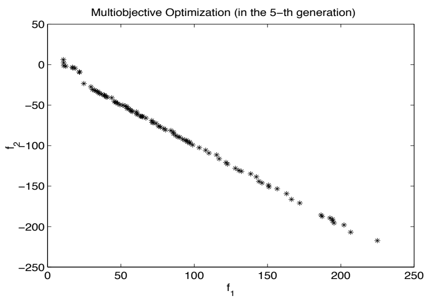


Fig. 2. Pareto optimal solutions for Test Case 1

$$
\begin{aligned}
f_1(\boldsymbol{x}) &= 1.5 - x_1(1 - x_2)\\
f_2(\boldsymbol{x}) &= 2.25 - x_1(1 - x_2^2)\\
f_3(\boldsymbol{x}) &= 2.625 - x_1(1 - x_2^3)
\end{aligned}
$$

subject to non-linear constraints:

$$
\begin{aligned}
-x_1^2 - (x_2 - 0.5)^2 + 9 &\leq 0\\
(x_1 - 1)^2 + (x_2 - 0.5)^2 - 6.25 &\leq 0
\end{aligned}
$$

and bounds: $-10 \leq x_i \leq 10,\ \forall i = 1, 2.$
This optimization problem has an unique solution at $\boldsymbol{x}^* = (3, 0.5)^T$ lying on the boundary of the non-convex feasible region. The objective function vector at $\boldsymbol{x}^*$ is $\mathbf{0}$. Starting from an feasible point $\boldsymbol{x} = (10, 10)^T$ the current population gets the feasible region after 10 generations and concentrates on the global feasible minimum after 60 generations.

Test Case 3

The optimization problem is min $(f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$:

$$
\begin{aligned}
f_1(\boldsymbol{x}) &= 4x_1^2 + 4x_2^2\\
f_2(\boldsymbol{x}) &= (x_1 - 5)^2 + (x_2 - 5)^2
\end{aligned}
$$

subject to non-linear constraints:

$$
\begin{aligned}
(x_1 - 5)^2 + x_2^2 - 25 &\leq 0\\
-(x_1 - 8)^2 - (x_2 + 3)^2 + 7.7 &\leq 0
\end{aligned}
$$

and bounds: $-15 \leq x_i \leq 30,\ \forall i = 1, 2.$
This optimization problem has the following properties:

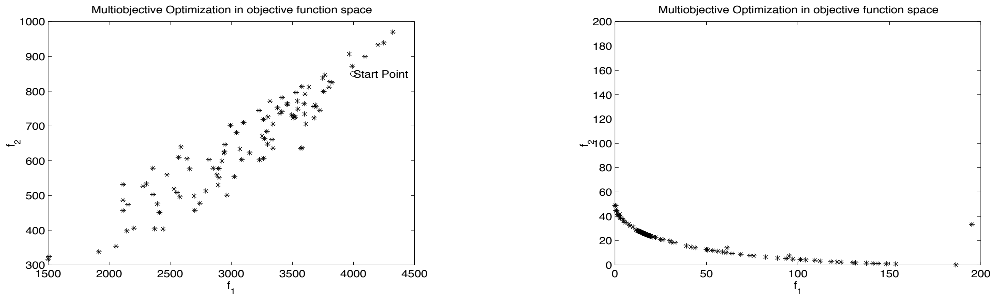


Fig. 3. Optimization process for Test Case 3

- The feasible region is non-convex.
- Some feasible pareto-optimal solutions lie on the boundaries of the feasible region.
- Both objectives are in conflict so that a little reduction of the second objective f_2 leads to a big increase in the first one.

The optimization proccess with an infeasible starting point at $\boldsymbol{x}_0 = (-10, 30)^T$ is illustrated in *Fig. 3*. The experimental results with different feasible regions have shown that MOBES is very robust and a set of pareto-optimal solutions can be always found in each cases.

CONCLUSION

In this paper the new evolution strategy for multiobjective constrained optimization problems based on the concept of $\mathcal{C}$- and $\mathcal{F}$-fitness is proposed. The existence of infeasible individuals in a $\mathcal{C}_{\text{extension}}$-class allows MOBES to explore new feasible areas and then to find new feasible pareto-optimal solutions. MOBES is a promising, powerful tool and it is implemented in a MATLAB–based environment [3].

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