MIT ES.S20 Lecture 2: Classical Combinatorial Game Theory

Lecture Outline

1. Nim
2. Nimbers
3. Graphical Game Representation
4. Sprouts
5. Hex
6. Demonstrations

Introduction

- Combinatorial Games

  We have two players with a finite set of positions, which are fully specified by rule-based moves. We alternate turns and we eventually reach a win or lose.

- Impartial Games

  The moves don't depend on which player is moving (i.e. the game is *unbiased*, or non-*partisan*). Games like Chess and Go are *not* impartial.

Nim

- Game Rules

  There are three *nim-heaps* of stones and each player alternates taking any number of stones off of each heap until no stones remain.

  In *normal* scoring, the player to make the last move wins; and in *misere* scoring, the player forced to make the last move loses.

```
A B C     Moves

3 4 5
          I take 2 from A
1 4 5
          You take 3 from C
1 4 2
          I take 1 from B
1 3 2
          You take 1 from B
1 2 2
          I take the last in A
0 2 2
          You take 1 from B
0 1 2
          I take 1 from C
0 1 1
          You take 1 from B
0 0 1
          I take the last C and win
0 0 0
```

- Game Positions

  Now, let's say that we are in a *P-position* if it guarantees a win for the previous player; and we say the game is in an *N-position* if it guarantees a win for the next player. These are relative to the position, as above.

  If we call every terminal position P and then every position from which we can only reach a P-position an N position, then induct forward, we can label every position.

- Winning Strategy

  We can make a winning strategy if we can make an *invariant* such that every move we make transitions the game into a P-position.

  To do this, note that every time we transition from P to N in backward induction, we change the sign of at least one value in the XOR of the values in all three piles (demonstrate XOR). Since our goal state is (0,0,0), whose XOR is clearly 0, maintaining an XOR of 0 is our invariant!

- Nimbers

Now, say that each nimber s of a Nim-state is the net XOR of the 3-tuple of the numbers in each pile. Our invariant is that s=0, and each move can be represented as some t not equal to 0 which will update the state.

- It is always possible to make the next state's nimber 0 if the current nimber is nonzero
- Poker Nim

  Suppose that we can also add chips to a heap.

- Nimble

  Put any number of coins on a row of squares, which can be moved left. The game ends when all coins are on the far left square.

- 2D Nim

  The same as Nimble in two dimensions, in which case you can move down or left.

Graphical Games

- Graph representation of Games

  A game is a graph G = (S, M), where S is the set of legal game states and M is the set of move arrows S -> S. The start position is some s0 and each player alternately moves along some arrow in M. A player who is unable to make a move loses. Assume all paths in G have finite length (*progressively bounded*).

- Sprague-Grundy Functions

  Define a function g on S (the *Sprague-Grundy function*) that takes each game state and assigns it the smallest number that is different from the number assigned to the states it can move to.

```
g(s) = min({n >= 0 : n != g(b) for each (m : s -> z) in M})
```

This looks like the min-coloring of the game graph! Note also that in the context of N- and P- positions, it obeys the same rules – i.e. a winning strategy is one that moves to the same "color" or Sprague-Grundy number as the winning state)

- Sums on Games

  Using this representation, we can "add" games together by making a graph C from graphs A and B such that the states are in Sa x Sb and the moves are in Ma x Mb! For example, a game of 3-pile Nim is the sum of 3 1-pile games of Nim.

  Note that this means that on each turn we can move in one game or both and the maximum number of possible moves is the sum of the maximum number of possible numbers in each game.

- The Sprague-Grundy Theorem

  The SG function of games on a graph is just the Nim-sum of the SG functions of its components.

Other Impartial Games

- Northcott's Game

  Take a checkerboard with one black and one white checker on each row, which are moved by their respective players. Players can move any number of spaces along each row (without jumping) until one of them can't any more.

  This game is not finite and not impartial.

- 21 Takeaway

  Start with a pile of 21 coins. Each player alternately takes 1-3 coins away from the pile. The player that takes the last coin wins.

- Turning Turtles

  Lay out a row of n coins, randomly on H or T. Each move, you change one coin from H to T, and optionally change another on the left of it from H to T or T to H.

- Silver Dollar Game

  Starting with a long row of squares, with coins on some of them, a legal move is to move one coin any number of squares left without jumping. The game ends when there are no more legal moves. Silver dollars are optional.

- Green Hackenbush

  Draw one or more connected rooted graphs. On each turn, cut a branch; this deletes the branch and any branches that are not connected anymore. The player who cuts the last branch wins.

- Sprouts

  Start with any number of dots on the page. A move connects two dots and makes a new dot in the middle of them. The rules are that there cannot be any intersections; no edge can have degree more than three; and a curve may connect to its origin. The person who draws the last curve wins.

Hex

- Game Rules

  Diamond-shaped board of hexagonal cells. Alternate turns moving to unoccupied spaces and the first to find a path from one side to the other wins.

- Lemma: Having extra pieces of your own color cannot hurt you

  Take the additional piece at position x on the board: if x is part of your optimal strategy, then you just gain a turn; if x is not, then it should not influence your strategy.

- Lemma: The first player can always win Hex

  Suppose the second player has an optimal strategy; then if he takes the first position and simply makes a random move on his first turn, he still wins. If hex cannot end in a draw (next theorem), we can therefore always construct optimal play for the first player.

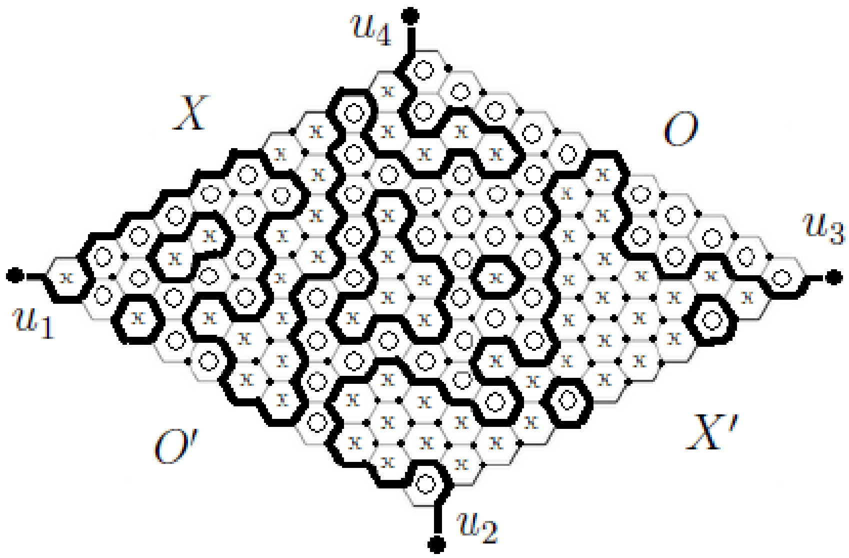

Figure 1: Hex Game Example

- The Hex Theorem: The game cannot end in a draw

  Assume the board is filled; then mark all edges at an interface between opposing colors. This will form some set of closed surfaces and a draw implies that there is some marked path from the top of the board to the bottom.

  However, if this path exists, then there cannot be a path between the other ends. However, this means that the path that starts from them cannot terminate because cycles do not make sense in this context. This gives a contradiction.

- The Hex Theorem is equivalent to Brouwer's Fixed Point Theorem

  There is a proof in `simple_games/hex.pdf`, but I didn't think we'd have enough time to cover it (it's a bit complex). The primary insight comes from changing representations to a lattice (see Figure 2).

Demonstration

1. Try to find optimal strategies for the other similar games, such as Northcott, Silver-Dollar, or higher-dimensional Nim

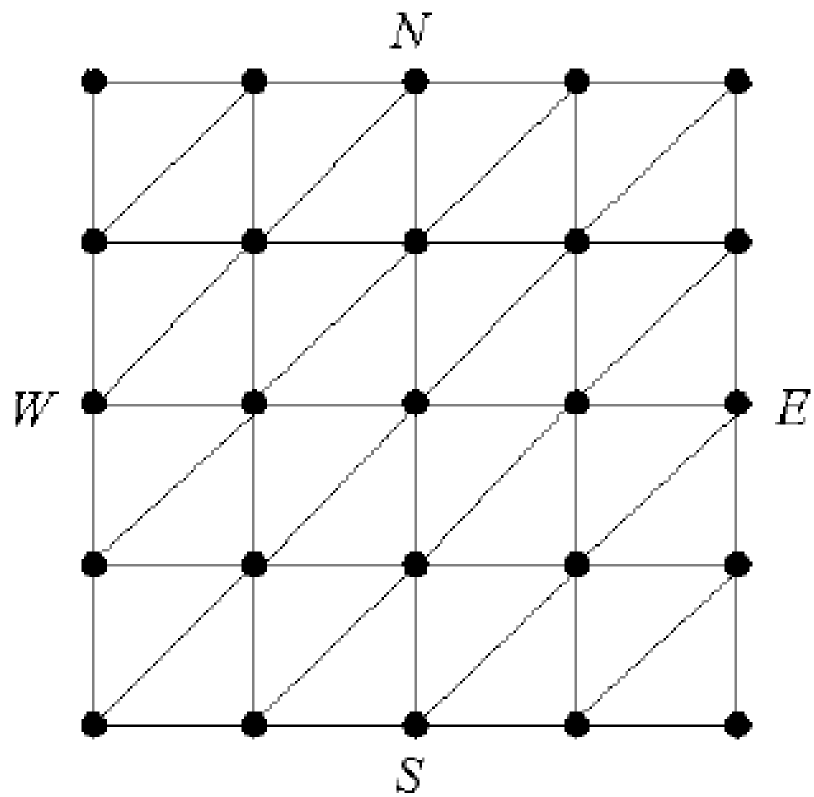

Figure 2: Hex Game Adjacency Lattice

2. Write a sprouts position and try to convince your partner to play it.
3. Play Hex!