$$\frac{\partial}{\partial x}\tan^{-1}\left(\frac{y_j - y}{x - x_j}\right) = \frac{x\frac{\partial}{\partial x}(y) - x_j\frac{\partial}{\partial x}(y) - y + y_j}{(y-y_j)^2 + (x-x_j)^2}$$

$$\cos(\beta_i) = \frac{\partial x}{\partial y} \qquad \sin(\beta_i) = \frac{\partial y}{\partial x}$$

$$f(x) = y - y_j \qquad \frac{\partial f(x)}{\partial x_i} = \frac{\partial y_i}{\partial x_i} - \frac{\partial y_j}{\partial x_i}$$

$$g(x) = x - x_j \qquad \frac{\partial g(x)}{\partial x_i} = \frac{\partial x_i}{\partial x_i} - \frac{\partial x_j}{\partial x_i}$$

$$\boxed{\begin{array}{l}\cos(B_i) = -\sin\theta_j \\ \sin(B_i) = \cos(\theta_j)\end{array}}$$

$$\left(\frac{\partial}{\partial x_i}\right)_{(\hat{x}_i, \hat{y}_i)} \tan^{-1}\left(\frac{y - y_j}{x - x_j}\right) ds_j$$

(Quotient + chain rule)

$$= \frac{1}{\left(1 + \left(\frac{(y_i - y_j)}{(x_i - x_j)}\right)^2\right)}\left(\frac{(x_i - x_j)\left(\frac{\partial y_i}{\partial x_i} - \frac{\partial y_j}{\partial x_i}\right) - (y_i - y_j)\left(\frac{\partial x_i}{\partial x_i} - \frac{\partial x_j}{\partial x_i}\right)}{(x - x_j)^2}\right)$$

$$= \frac{(\hat{x}_i - x_j)\sin(B_i) - (\hat{y}_i - y_j)\cos(B_i)}{(\hat{x}_i - x_j)^2 + (\hat{y}_i - y_j)^2}$$

$$= \frac{\hat{x}_i\sin(B_i) - x_j\cos(\theta_j) - \hat{y}_i\cos(B_i) - y_j\sin(\theta_j)}{(\hat{x}_i - x_j)^2 + (\hat{y}_i - y_j)^2}$$

RELATIONS:

$$\hat{x}_j = x_j + \Delta s_j\cos\theta_j \qquad \hat{y}_j = y_j + \Delta s_j\sin\theta_j$$

$$\hat{x}_i = x_i + \Delta s_i\cos\theta_i \qquad \hat{y}_i = y_i + \Delta s_i\sin\theta_i$$

NUMERATOR

$$\hat{x}_i\sin(\beta_i) = \hat{x}_i(\cos\theta_j)$$

$$\hat{x}_j\cos(\theta_j) = x_j\cos(\theta_j) + \Delta s_j\cos(\theta_j)\cos(\theta_j)$$

$$\hat{y}_i \cos(\theta_i) = -\hat{y}_i \sin\theta_j$$

$$-\hat{y}_j \sin(\theta_j) = -\left(y_j + \frac{\Delta s_j \sin\theta_j)}{2}\right) \sin(\theta_j)$$

$$= -y_j \sin\theta_j \quad - \Delta s_j \sin(\theta_j) \sin(\theta_j)$$

$$\hat{x}_i \cos\theta_j - x_j \cos(\theta_j) - \Delta s_j \cos(\theta_j)\cos(\theta_j) + \hat{y}_i \sin\theta_j$$
$$- y_j \sin\theta_j - \Delta s_j \sin\theta_j \sin\theta_j$$

$$= (\hat{x}_i - x_j)\cos(\theta_j) + \sin(\theta_j)(\hat{y}_i - y_j) - \Delta s_j (\cos\theta_i - \cos\theta_j)$$

DENOMINATOR:

$$(\hat{x}_i - \hat{x}_j)^2 + (\hat{y}_i - \hat{y}_j)^2$$

$$\hat{x}_j = x_j + \Delta s_j \cos\theta_j \qquad \hat{y}_j = y_j + \Delta s_j \sin\theta_j$$

$$\hat{x}_i = x_i + \Delta s_i \cos\theta_i \qquad \hat{y}_i = y_i + \Delta s_i \sin\theta_i$$

$$\hat{x}_i^2 + (x_j + \Delta s_j \cos\theta_j)^2 - 2\hat{x}_i(x_j + \Delta s_j \cos\theta_j)$$

$$\hat{x}_i + x_j^2 + \Delta s_j^2 \cos^2\theta_j + 2 x_j \Delta s_j \cos\theta_j - 2\hat{x}_i x_j - 2\hat{x}_i \Delta s_j \cos\theta_j$$

$$(\hat{x}_i - x_j)^2 + \Delta s_j^2 \cos^2\theta_j + 2x_j \Delta s_j \cos\theta_j - 2\hat{x}_i \Delta s_j \cos\theta_j$$

$$(\hat{x}_i - x_j)^2 + \Delta s_j^2 \cos^2\theta_j + 2x_j \Delta s_j \cos\theta_j - 2\hat{x}_i \Delta s_j \cos\theta_j$$

$$(\hat{y}_i - y_j)^2 + \Delta s_j^2 \sin^2\theta + 2y_j \Delta s_j \sin\theta_j - 2\hat{y}_i \Delta s_j \sin\theta_j$$

$$\rightarrow (\hat{x}_i - x_j)^2 + \Delta s_j^2 + \Delta s_j (2x_j \cos\theta_j - 2\hat{x}_i \cos\theta_j + 2y_j \sin\theta_j - 2\hat{y}_i \sin\theta_j)$$

$$\boxed{\begin{aligned} D &= (\hat{x}_i - x_j)^2 + (\hat{y}_i - y_j)^2 \\ C &= (2x_j\cos\theta_j - 2\hat{x}_i\cos\theta_j + 2y_j\sin\theta_j - 2\hat{y}_i\sin\theta_j) \\ A &= -\cos(\theta_i - \theta_j) \\ B &= (\hat{x}_i - x_j)\cos(\theta_j) + \sin(\theta_j)(\hat{y}_i - y_j) \end{aligned}}$$

$$\frac{\gamma(s_j)}{2\pi} = \frac{1}{2\pi}\left(\gamma_j + (\gamma_{j+1} - \gamma_j)\frac{s_j}{\Delta s_j}\right)$$

$$= \frac{1}{2\pi}\left(\gamma_j + \frac{s_j(\gamma_{j+1})}{\Delta s_j} - \frac{\gamma_j s_j}{\Delta s_j}\right)$$

$$= \frac{\gamma_j}{2\pi} + \frac{(\gamma_{j+1} - \gamma_j)s_j}{2\pi\,\Delta s_j}$$

($\frac{1}{2\pi}$ can be pulled out of integral)

$$\boxed{\begin{aligned} b &= \gamma_j \\ a &= \frac{(\gamma_{j+1} - \gamma_j)}{\Delta s_j} \end{aligned}}$$

INTEGRAL CULMINATION

$$\int_0^{\Delta s_j} (a s_j + b) \frac{A s_j + B}{s_j^2 + C s_j + D} \, ds_j$$

(Hessian):

$$\begin{bmatrix} \partial\phi/\partial x \\ \partial\phi/\partial y \end{bmatrix} = \begin{bmatrix} U\cos\alpha - \sum_{j=1}^{n} \int_0^{\Delta(s_j)} \frac{\gamma(s_j)}{2\pi} \cdot \frac{y_j - y}{(x-x_j)^2 + (y-y_j)^2} \, ds_j \\ U\sin\alpha - \sum_{j=1}^{n} \int_0^{\Delta(s_j)} \frac{\gamma(s_j)}{2\pi} \cdot \frac{x - x_j}{(x-x_j)^2 + (y-y_j)^2} \, ds_j \end{bmatrix} = 0$$

$$\left(U_\infty - \frac{1}{2\pi} \sum_{j=1}^{n} \int_0^{(\Delta s_j)} (a s_j + b) \frac{A s_j + B}{s_j^2 + C s_j + D} \, ds_j \right) \cdot n_i = 0$$

$$\text{where } n_i = \begin{bmatrix} \sin(\theta_1 - \alpha) \\ \sin(\theta_2 - \alpha) \\ \vdots \\ \sin(\theta_m - \alpha) \end{bmatrix} = \sin(\theta_i - \alpha)$$

$$\Rightarrow U_\infty 2\pi \sin(\theta_i - \alpha) = \sum_{j=1}^{n} \int_0^{(\Delta s_j)} (a s_j + b) \frac{A s_j + B}{s_j^2 + C s_j + D} \, ds_j$$

Assuming $[C_{1,ij}, C_{2,ij}]$ for $\sum_{j=1}^{n} \int_0^{(\Delta s_j)} (a s_j + b) \frac{A s_j + B}{s_j^2 + C s_j + D} \, ds_j$

$$\begin{bmatrix} C_{1_{11}} & C_{2_{11}} + C_{1_{12}} & \cdots & C_{2_{1(m-1)}} + C_{1_{1m}} & C_{2_{1m}} \\ C_{1_{21}} & C_{2_{21}} + C_{1_{22}} & \cdots & C_{2_{2(m-1)}} + C_{1_{2m}} & C_{2_{2m}} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_{1_{m1}} & C_{2_{m1}} + C_{1_{m2}} & \cdots & C_{2_{m(m-1)}} + C_{1_{mm}} & C_{2_{mm}} \\ 1 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \cdots \\ \gamma_m \\ \gamma_{m+1} \end{bmatrix} = 2\pi U_\infty \begin{bmatrix} \sin(\theta_1 - \alpha) \\ \sin(\theta_2 - \alpha) \\ \cdots \\ \sin(\theta_m - \alpha) \\ 0 \end{bmatrix} :$$

where $[\gamma_1, \ldots, \gamma_{m+1}]$ are the unknowns in the linear system.