Bifurcation Theory: A Brief Examination

Source: https://github.com/varshakrishnakumar/Bifurcation-Theory-Examination

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I. An Introduction to Bifurcation Theory

In a *dynamical system*, a **bifurcation** is a qualitative change in behavior produced by varying parameters (known as *bifurcation parameters* in the system's dynamics. Within a family of Ordinary Differential Equations (ODEs), **Bifurcation Theory** imparts a strategy for investigating bifurcations that can occur. In a general case, the local stability properties of equilibria (or other invariants) at a bifurcation enforces changes in the dynamical system's behavior.[1]

Considering the autonomous system of Ordinary Differential Equations:

$$\dot{x} = f(x, \lambda), x \in \mathbb{R}^n, \lambda \in \mathbb{R}^p$$

A **bifurcation** happens at parameter $\lambda = \lambda_0$ if there are other parameter values close to λ_0 with different behavior of dynamics from those at λ_0. For example, there may be a nuance in the number of stability of equilibrium points. When a *bifurcation diagram* of a system is produced, the λ parameter space can be divided into regions of *topologically equivalent systems* (that is, systems that have similar dynamic behavior).

Conventionally, bifurcations are divided into two types: Local bifurcations and Global bifurcations. Since the objective of this examination is a simple overview, local bifurcations will be examined further in the upcoming sections.

I.I Basic Case of Bifurcation: Local Bifurcation

Suppose a differential equation of this form is considered

$$\frac{du}{dt} = f(u, \mu)$$

[1] Guckenheimer, J., "Bifurcation," Scholarpedia: http://www.scholarpedia.org/article/Bifurcation.

Here, u is a function dependent on the parameter t (signifying time), and f is a vector field dependent on u and μ. From the structure of this ODE form, μ represents the bifurcation parameter.

For each initial condition of the ODE, there must exist a unique solution for the equation. It must also be assumed that C^k represents the class of the vector field, with $k >= 2$, and satisfies:

$$f(0,0) = 0, \frac{\partial f}{\partial u}(0,0) = 0$$

The first condition indicated that $u = 0$ is an equilibrium equation at $\mu = 0$. Varying parameter μ can provide an interesting set of bifurcation diagrams to study, as local bifurcations can occur in close vicinity of this equilibrium condition.

I.I.A Local Bifurcation: Saddle-Node

Consider the following autonomous Ordinary Differential Equation[2]:

$$\frac{dx}{dt} = f(x) = x^2 + r$$

The behavior of $f(x)$ is dependent on the parameter r.

When $r < 0$, there are two equilibrium solutions: a stable equilibrium at $x(t) = -\sqrt{-r}$, and an unstable equilibrium $x(t) = \sqrt{-r}$. The stable equilibrium solution is always less than the unstable solution.

When $r > 0$, there are no equilibrium solutions. However, at $r = 0$, there is one unique semi-stable equilibrium solution called the *Saddle-Node Bifurcation*.

I.I.B Local Bifurcation: Transcritical

Consider the following Ordinary Differential Equation:

$$\frac{dx}{dt} = f(x) = rx - x^2 = x(r - x)$$

In this particular case, $x = 0$ and $x = r$ is always an equilibrium solution. As the value of r changes, the behaviour of the solution curves change as well.

When $r < 0$, $x = 0$ is a stable equilibrium solution, while $x = r$ is an unstable solution. When $r > 0$, $x = 0$ is an unstable equilibrium solution, while $x = r$ is a stable solution.

At $r = 0$, a semi-stable equilibrium solution is found.

[2] "Intro to bifurcation theory": https://www.math.ksu.edu/~albin/teaching/math340_fall2018/slides/03_bifurcations.html.

I.I.C Local Bifurcation: Pitchfork

Consider the following Ordinary Differential Equation:

$$\frac{dx}{dt} = f(x) = rx - x^3 = x(r - x^2)$$

In this particular case, $x = 0$ and $x = \pm\sqrt{r}$ is always an equilibrium solution. When $r < 0$, $x = 0$ is a stable equilibrium solution, and the only equilibrium solution present. When $r >= 0$, $x = 0$ is an unstable equilibrium solution, while $x = \pm\sqrt{r}$ are stable solutions.

II. A Brief Examination of Bifurcation Theory's Applications

Do something here