Fourth International Olympiad, 1962

Name: Varshini G N
ID:COMETFWC031
Date: August 1, 2025

1962/1.

Find the smallest natural number n which has the following properties:

(a) Its decimal representation has 6 as the last digit.

(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

1962/2.

Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

1962/3.

Consider the cube $ABCDA'A''B''C''D''$ (where $ABCD$ and $A'B'C'D'$ are the upper and lower bases, and edges AA', BB', CC', DD' are vertical). The point X moves at constant speed along the perimeter of square $ABCD$ in the direction $A \to B \to C \to D \to A$, and the point Y moves at the same speed along the perimeter of square $B'C'CBB'$ in the direction $B' \to C' \to C \to B \to B'$. Points X and Y start simultaneously from positions A and B', respectively.

Determine and draw the locus of the midpoints of the segments XY.

1962/4.

Solve the equation:

$$\cos^2 x + \cos^2 2x + \cos^2 3x = 1.$$

1962/5.

On the circle K, three distinct points A, B, and C are given. Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the quadrilateral $ABCD$.

1962/6.

Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is:

$$d = \sqrt{r(r - 2\rho)}.$$

1962/7.

The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges SA, SB, SC, BC, CA, AB, or to their extensions.

(a) Prove that the tetrahedron $SABC$ is regular.

(b) Prove conversely that for every regular tetrahedron, five such spheres exist.