STAT 110: Probability Lecture 1 Notes

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Fundamental Concepts

Sample Space and Events

- **Sample Space (S)**: The complete set of all possible outcomes of a random experiment or process.
- It represents every possible result that could occur when the experiment is conducted.
    - *Example*: When flipping a coin twice, the sample space is S = {HH, HT, TH, TT}, consisting of all possible arrangements of heads and tails.
- **Event (E)**: A subset of the sample space, representing a collection of outcomes that share a common characteristic or property of interest.
    - *Example*: The event of "getting at least one head" when flipping a coin twice would be E = {HH, HT, TH}, which is a subset of the sample space.

Probability Definition

- **Naive (Classical) Definition of Probability**: For equally likely outcomes, the probability of an event A is:

  P(A) = Number of favorable outcomes / Number of possible outcomes

  = Number of outcomes in event A / Size of the Sample Space

- This definition assumes all outcomes are equally likely to occur (equiprobable).

Counting Principles

The Multiplication Rule

The multiplication rule is a fundamental principle for counting the number of ways to perform a sequence of tasks.

When to Use the Multiplication Rule:

1. When choices are made in sequence (one after another)

2. When each choice is independent of previous choices (the number of options at each step does not depend on the choices made in previous steps)

Formal Statement:

- If an experiment consists of a sequence of r tasks
- And task 1 can be done in n_1 ways
- And task 2 can be done in n_2 ways for each way of doing task 1
- And task 3 can be done in n_3 ways for each way of doing tasks 1 and 2
- ...
- And task r can be done in n_r ways for each way of doing all previous tasks
- Then the total number of ways to perform the entire sequence is: $n_1 \times n_2 \times n_3 \times ... \times n_r$

Example - Three-Course Meal:

1. First course (Appetizer): 2 options (Soup or Salad) → $n_1 = 2$
2. Second course (Main dish): 3 options (Chicken, Fish, Pasta) → $n_2 = 3$
3. Third course (Dessert): 2 options (Cake or Ice Cream) → $n_3 = 2$

Total possible meal combinations = 2 × 3 × 2 = 12 different possible meals

Visual Interpretation: The multiplication rule counts all possible paths through a decision tree, where each level represents a decision point, and branches represent available options.

Special Cases of the Multiplication Rule

1. Dependent Events (Restricted Choices)

When choices are dependent (i.e., previous choices affect subsequent options), we still use the multiplication rule, but the number of options changes at each step.

Example - Password with No Repeating Digits:

- First digit: 10 options (0-9) → $n_1 = 10$
- Second digit: 9 options (all digits except the first) → $n_2 = 9$
- Third digit: 8 options (all digits except the first two) → $n_3 = 8$

Total possible passwords = 10 × 9 × 8 = 720

2. Repeated Independent Trials (Exponents)

When the same experiment with k possible outcomes is repeated n times independently, the total number of possible outcome sequences is k^n.

Example - Flipping a Coin 5 Times:

- Each flip has 2 possible outcomes (H or T)
- Total number of possible sequences = $2^5 = 32$

Example - Full House in Poker: A full house consists of a three of a kind and a pair.

- Ways to choose the value for three of a kind: 13 (A through K)
- Ways to choose 3 cards of that value: C(4,3) = 4 (combinations of 4 cards taken 3 at a time)
- Ways to choose the value for the pair: 12 (any value except the three of a kind)
- Ways to choose 2 cards of that value: C(4,2) = 6

Total ways to get a full house = 13 × 4 × 12 × 6 = 3,744

Probability = 3,744 / C(52,5) ≈ 0.00144

Combinatorial Mathematics

Permutations and Combinations

Binomial Coefficient: The number of ways to select k objects from a set of n distinct objects, regardless of order:

(n choose k) = C(n,k) = n! / [(n-k)! × k!]

where $0 \le k \le n$

Permutation: The number of ways to arrange k objects from a set of n distinct objects, where order matters:

P(n,k) = n! / (n-k)!

Sampling Methods

The following table summarizes the formulas for different sampling scenarios:

Sampling Method	Order Matters	Order Doesn't Matter
With Replacement	n^k	(n+k-1 choose k)
Without Replacement	n!/(n-k)!	(n choose k)

Sampling With Replacement and Order Matters

- Each selection has n options, regardless of previous selections
- Total number of possibilities: n^k

Sampling Without Replacement and Order Matters

- Also called a permutation of k items from n
- Formula: $P(n,k) = n!/(n-k)!$

Sampling Without Replacement and Order Doesn't Matter

- Also called a combination of k items from n
- Formula: $C(n,k) = n!/[(n-k)!k!]$

Sampling With Replacement and Order Doesn't Matter

- Also known as "stars and bars" or multisets
- Formula: $C(n+k-1,k) = (n+k-1)!/[k!(n-1)!]$

Example - Ice Cream Shop:

- 3 flavors (vanilla, chocolate, strawberry)
- Selecting 2 scoops with replacement (can choose the same flavor twice)
- Order doesn't matter (vanilla+chocolate is the same as chocolate+vanilla)

Possible combinations:

1. Vanilla + Vanilla
2. Vanilla + Chocolate
3. Vanilla + Strawberry
4. Chocolate + Chocolate
5. Chocolate + Strawberry
6. Strawberry + Strawberry

Total number of combinations = $C(3+2-1,2) = C(4,2) = 4!/(2!\times2!) = 6$

Applied Examples

Example 1: Password Combinations

For a 3-digit password where repetition is allowed:

- First digit: 10 possibilities (0-9)
- Second digit: 10 possibilities (0-9)
- Third digit: 10 possibilities (0-9)

Total possible passwords = $10 \times 10 \times 10 = 1{,}000$

Example 2: Three-Course Meal

A restaurant offers:

- 2 appetizer choices (soup or salad)

- 3 main course options (chicken, fish, pasta)
- 2 dessert options (cake or ice cream)

Total meal combinations = 2 × 3 × 2 = 12

Example 3: Ice Cream Selection

At an ice cream shop with n flavors, selecting k scoops:

- For n = 3 (vanilla, chocolate, strawberry) and k = 2 scoops
- When the same flavor can be chosen multiple times and order doesn't matter

Number of possible selections = C(n+k-1,k) = C(3+2-1,2) = C(4,2) = 6