Lecture 2: Story proofs, Axioms of probability.

§ Label people, objects e.t.c.
If you have n people then label them 1, 2 ..., n

→ 10 people, Split into team of 6, team of 4 → $\binom{10}{4} = \binom{10}{6}$

choosing 4 people from a group of 10 = choosing 6 people from group 7.

Since the total numbers of people is fixed (10), selecting a team of 6 automatically determines the remaining 4 people.

Ex: 10 people labeled A, B, C, D, E, F, G, H, I, J. The no of ways to divide them into a team of 6 and team of 4 :

1) choosing 4 people from the 10 (for small team)

$$\binom{10}{4} = \frac{10!}{4!\,(10-4)!} = 210 \text{ ways}$$

2) choosing 6 people from 10 (for large team)

$$\binom{10}{6} = \frac{10!}{6!\,(10-6)!} = \underline{\underline{210}} \text{ ways}$$

Pick k times from a set of n objects, where order doesn't matter, with replacement

→ $\binom{n+k-1}{k}$ ways

<u>Extreme cases</u>: $k=0 \rightarrow \binom{n-1}{0} = 1$

$k=1 \rightarrow \binom{n}{1} = n$

Simplest non trivial example $n=2 \rightarrow \binom{k+1}{k} = \binom{k+1}{1}$

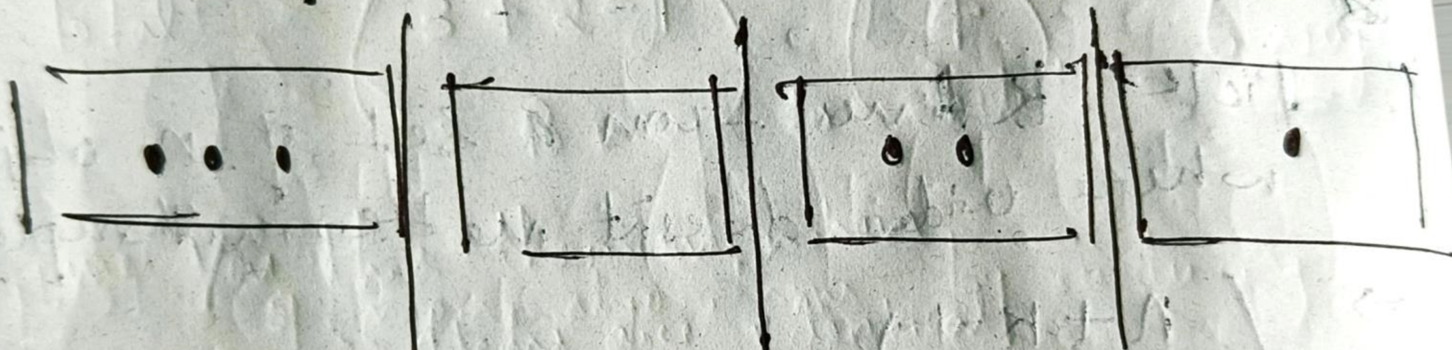
$= \underline{k+1}$

1 2

dots in in

$\{0, 1, \ldots, k\}$

Equivalent: How many ways are there to put k indisting particles into n distinguishable boxes?



$n = 4 \to$ no of boxes

$k = 6 \to$ no of particles

$\bullet\bullet\bullet \,|\,|\, \bullet\bullet \,|\, \bullet$

k $\bullet$'s, $n-1$ $|$'s

dots separators

$$\Rightarrow \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$$

This theorem is actually Stars and bars theorem.

→ Represent Selections as Stars(*) and Category dividers as bars (|)

→ Ex: for $n = 2$ (A, B) and $K = 3$, the combination "AAB" is **|*

→ Total arrangements $\rightarrow \binom{n+K-1}{K}$

→ To seperate n categories we need <u>n-1</u> bars

→ stars and bars in a line of length <u>K+n-1</u>

→ we choose K position for stars (a n-1 bars)

placing K stars among n-1 bars

<u>Ex</u>: How many ways can you distribute 3 identical candies to 2 children?

Sol) Stars and bars representation.

→ ***| = All 3 candies to child 1

→ **|* = 2 to child 1, 1 to child 2

→ *|** = 1 to child 1, 2 to child 2

→ |*** = All 3 to child 2

Total ways = 4, which matches

$$\binom{3+2-1}{3} = \binom{4}{3} = 4$$

for $n = 3$ (Categories: A, B & C)
and $K = 2$ (items)

$**||$ = AA

$*|*|$ = AB

$*||*$ = AC

$|**|$ = BB

$|*|*$ = BC

$||**$ = CC

Total → $\binom{3+2-1}{2} = \binom{4}{2} = 6$ ways

→ use You this theorem when
→ Items are identical
→ Categories are distinct
→ order doesnt matter

formula → $\binom{n+k-1}{k}$

Ex:) Lower Bounds (minimum items per Category)

Problem: How many ways can You distribute 10 identical candles to 4 children if Each child must get at least 2 Candles?

Sol: → First give 2 candies to each child

(Total distribute : $4 \times 2 = 8$)

→ Remaining candies : $10 - 8 = 2$

left to distribute freely.

→ Use stars and bars : Now distribute the remaining 2 candies to 4 children with no restrictions.

$$\binom{4+2-1}{2} = \binom{5}{2} = \underline{\underline{10 \text{ ways}}}$$

$n = 4$

$k = 2$

__Story Proof__ : proof by interpretation

Ex: 1) $\binom{n}{k} = \binom{n}{n-k}$

2) $n\binom{n-1}{k-1} = k\binom{n}{k}$

(count same thing in 2 ways)

pick k people out of n, with designates as president

3) $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$

(Vandermonde identity)

1) Identity $\binom{n}{k} = \binom{n}{n-k}$

choose k people from n to form a team

choose $n-k$ people to exclude from the team (leaving k members)

→ Both count the same thing (team compensatiton), so they're equal.

ex: $n = 5$, $k = 2$

$\binom{5}{2} = 10$ ways to pick 2 numbers

$\binom{5}{3} = 10$ ways to exclude 3 ~~no~~

Application: optimizing algorithms by reducing computations (ex $\binom{100}{98} = \binom{100}{2}$)

2) Identity: $n\binom{n-1}{k-1} = k\binom{n}{k}$

pick a captain (from n people) then choose $k-1$ more members from the remaining $n-1$

first pick k team members from n, then choose a captain among those k

ex: $n = 4$, $k = 2$

left $4 \cdot \binom{3}{1} = 12$

right $2 \cdot \binom{4}{2} = 12$

3) Vandermonde's identity

$$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$$

Scenario: choose k people from a group of m men and n women.

Sum over cases:

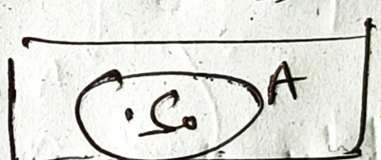
j men and k-j women

$\forall$ possible j (from 0 to k)

ex: $m=2$, $n=2$, $k=2$

$$\binom{4}{2} = \binom{2}{0}\binom{2}{2} + \binom{2}{1}\binom{2}{1} + \binom{2}{2}\binom{2}{0}$$

$$= 1 + 4 + 1 = 6$$

non-naive definition : 

A probability sample consists of S and P,

where S → sample space, and

P a function which takes an event $A \subseteq S$ as input, returns $P(A) \in [0,1]$ as outputs. Such that

1) $P(\phi) = 0$, $P(S) = 1$

(empty set (impossible event))

~~2) P(~~

2) $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$ if

Rolling a fair six-sided:

$S = \{1, 2, 3, 4, 5, 6\}$

$P(\{1\}) = P(\{2\}) = \cdots = P(\{6\}) = \frac{1}{6}$

Even

$A = \{2, 4, 6\} \rightarrow$ rolling an even number

$P(A) = P(\{2\}) + P(\{4\}) + P(\{6\})$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \underline{0.5}$

$P(\phi) = 0$

$P(S) = \frac{1}{6} \times 6 = \underline{1}$

Probability of a union of events A_n, possibly referencing the Boole's inequality (or the union bound).

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n)$$

→ the probability of any one of multiple events happening is at most the sum of their individual probabilities.

Ex: Suppose A_1 = "Rain tomorrow"

$$(P(A_1) = 0.3)$$

A_2 = "Snow tomorrow"

$$(P(A_2) = 0.2)$$

$$P(\text{Rain} \cap \text{Snow}) \leq P(\text{Rain}) + P(\text{Snow})$$
$$= 0.3 + 0.2 = 0.5$$

1) finite case: for two events A_1 & A_2

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

Since $P(A_1 \cap A_2) \geq 0$

$$P(A_1 \cup A_2) \leq P(A_1) + P(A_2)$$

2) Induction: Extend to n Events

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$

3) Infinite case: the inequality holds for $n \to \infty$

Disjoint events: If A_n are mutually exclusive (no overlap) then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$$

Ex: A_n: "Roll a n", $P\left(\bigcup_{n=1}^{6} A_n\right) = \sum_{n=1}^{6} \frac{1}{6} = 1$