Lecture 4: Conditional probability

Independent Events:

Defn: Events A, B are independent if

$$P(A \cap B) = P(A) P(B)$$

Note: completely different from disjointness

A, B, C are independent if

$$P(A, B) = P(A) P(B)$$
$$P(A, C) = P(A) P(C)$$
$$P(B, C) = P(B) P(C)$$
$$P(A, B, C) = P(A) P(B) P(C)$$

Similarly for Events $A_1, \dots A_n$. "independence means multiply"

Newton - Pepys problem:

Have fair dice, which is most likely?

(A) at least one 6 with 6 dice ← truth

(B) at least two 6's with 12 dice

(C) at least three 6's with 18 dice ← Pepys

i) $P(A) = 1 - \left(\dfrac{5}{6}\right)^6$

$= 1 - 0.3349 \approx \boxed{0.6651}$

ii) $P(\text{At least two } 6's) = 1 - P(\text{zero } 6s \cap \text{ one } 6)$

$P(0) = \left(\dfrac{5}{6}\right)^{12} \approx 0.1122$

$P(1) = \left(\dfrac{12}{1}\right) \cdot \left(\dfrac{1}{6}\right)^1 \left(\dfrac{5}{6}\right)^{11}$

Binominal probability

$$\boxed{P(K \text{ sives}) = \left(\begin{array}{c} n \\ k \end{array}\right) \cdot p^k \cdot (1-p)^{n-k}}$$

$= 12 \cdot \dfrac{1}{6} \cdot \left(\dfrac{5}{6}\right)^{11}$

$= \left(\dfrac{5}{6}\right)^{11} \approx 0.1615$

$P(1 \text{ six}) \approx 12 \cdot \dfrac{1}{6} \approx 0.1615 \approx \underline{0.2702}$

$P(1 \text{ six}) = 0.2702$

$P(0 \cap 1 \text{ six}) = 0.1122 + 0.2702$

$\qquad\qquad = 0.3824$

$P(\text{at least 2 sixes}) = 1 - 0.3824$

$\qquad\qquad \approx \underline{0.6176}$

$P(C) = 1 - \sum\limits_{K=0}^{3} \left(\begin{array}{c} 18 \\ k \end{array}\right) \left(\dfrac{1}{6}\right)^{k} \left(\dfrac{5}{6}\right)^{18-k}$
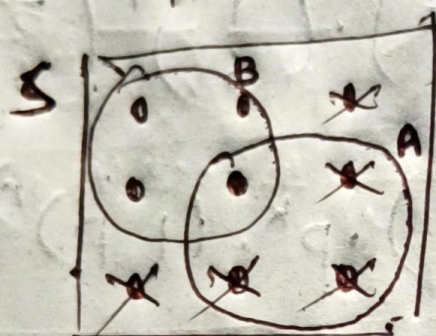
$\qquad\qquad \approx \underline{0.597}$

Conditional probability!

"Conditioning is the soul of statistics"

Defn: $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, if $P(B) > 0$

Intuition1: pebble world

9 Pebbles, total mass 1

$P(A|B)$: get rid of pebbles in B^c.

renormalize to make mass again.

Intuition 2: frequentist world

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1 0 0 1 0 1 1 0 1
0 0 1 0 0 1 0 1 1
1 1 1 1 1 1
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then!

$$P(A \cap B) = P(B) \, P(A|B) = P(A) \, P(B|A)$$

Thm 2 $P(A_1 \dots A_n) = P(A_1) P(A_2|A_1) P(A_3|A_1 A_2)$

$$\dots P(A_n | A_1 \dots A_{n-1})$$

Bayes rule!

Theorem $P(A|B) = \dfrac{P(B|A) P(A)}{}$