① Verify CMVT for $f(x) = \log x$ and $g(x) = \frac{1}{x}$ in $[1, e]$

Sol We have $f'(x) = \frac{1}{x}$, $g'(x) = -\frac{1}{x^2}$ $\Rightarrow$ f, g are continuous on $[1, e]$ are derivable on $(1, e)$. Also $g'(x) \neq 0$ for every $x$ in $(1, e)$.

All the conditions of CMVT are satisfied. Then there exists a real no. 'c' in $(1, e)$ such that

$$\frac{\frac{1}{c}}{-\frac{1}{c}} = \frac{\log e - \log 1}{\frac{1}{e} - 1} \Rightarrow c = e/e-1$$
$$= 1.6$$

② If $0 \leq x < 1$, prove that $\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$

Sol Let $f(x) = \log(1+x)$; $g(x) = \sin^{-1}x$

Also $f'(x) = \frac{1}{1+x}$; $g'(x) = \frac{1}{\sqrt{1-x^2}}$ & $g'(x) \neq 0$ in $0 < x < 1$

All the conditions of CMVT as satisfied in $(0, x)$. Then there exist a real number

c in $(0, x)$ such that $\frac{1}{1+c}\sqrt{1-c^2} = \frac{\log(1+x) - \log 1}{\sin^{-1}x - \sin^{-1}0}$

Now $0 < c < x < 1$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\sqrt{1-c}}{\sqrt{1+c}} < 1$$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} < \frac{\log(1+x)}{\sin^{-1}x} < 1$$

③ Prove $\sin\beta - \sin\alpha < \beta - \alpha$ if $0 < \alpha < \beta < \pi/2$

Sol Let $f(x) = \sin x$ in $[\alpha, \beta]$

$f'(x) = \cos x$

$\therefore$ $f'(x)$ is cont. on $[\alpha, \beta]$ & $\therefore$ $f'(x)$ is diff. on $(\alpha, \beta)$

All cond$^n$ LMVT are satisfied

Thus; $f'(\alpha) = \frac{f(\beta) - f(\alpha)}{\beta - \alpha}$

$$\cos c = \frac{\sin\beta - \sin\alpha}{\beta - \alpha} \quad (\alpha < c < \beta)$$

$$|\cos c| = \frac{|\sin\beta - \sin\alpha|}{|\beta - \alpha|} \quad [\because |\cos c| \leq 1].$$

$$= \frac{\sin\beta - \sin\alpha}{\beta - \alpha} < 1 \quad = \sin\beta - \sin\alpha < \beta - \alpha$$

Hence proved.

(4) Using LMVT, PT $1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1$ & decide that $\frac{1}{6} < \log(1.2) < \frac{1}{5}$

Sol $f(x)$ is cont. on $[a,b]$ & $f(x)$ is diff. on $(a,b)$

$\therefore$ All cond$^{ns}$ on LMVT verify

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad ; \quad c \in (a,b)$$

$$\therefore f'(c) = \frac{\log b - \log a}{b - a} \Rightarrow \frac{1}{c} = \frac{\log b - \log a}{b - a} \longrightarrow (1)$$

$(a < c < b)$

also; $\frac{1}{b} < \frac{1}{c} < \frac{1}{a}$

Thus from (1);

$$\Rightarrow \frac{1}{b} < \frac{\log b - \log a}{b - a} < \frac{1}{a}$$

$$\Rightarrow \frac{1}{b} < \frac{\log(b/a)}{b - a} < \frac{1}{a} \Rightarrow \frac{b - a}{b} < \log\left(\frac{b}{a}\right) < \frac{b - a}{a}$$

$$\Rightarrow 1 - \frac{a}{b} < \log\left(\frac{b}{a}\right) < \frac{b}{a} - 1$$

Hence Proved.

(5) Using CMTV prove that

$$\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c \quad (a < c < b)$$

$(a < c < b)$

Sol $f(x) = \sin x; \quad g(x) = \cos x$

$f'(x) = \cos x \quad ; \quad g'(x) - \sin x$

$$\frac{\cos x}{\sin x} = \frac{\sin b - \sin a}{(-\cos b) - (-\cos a)}$$

$$\Rightarrow \cot c = \frac{\sin b - \sin a}{\cos a - \cos b}$$

Hence Proved.