REAL ANALYSIS NOTES

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s.t → such that

- A is a subset of set X $A \subset X$ or $X \supset A$ if every element of A belongs to X if $x \in A \Rightarrow x \in X$

- The powerset $P(X)$ to the set of all subsets of X.

Some Notations

$\mathbb{N}$: $[1, \infty)$
$\mathbb{Z}$: $(-\infty, \infty)$ ⟹ unbounded/ on $-\infty, \infty$'s. open bounded
$\mathbb{Q}$: $(-\infty, \infty)$ all rational numbers.

Ordered set: set on which an order is defined

Let S be a set. An order on S is a relation denoted by '<' with the following prop's.

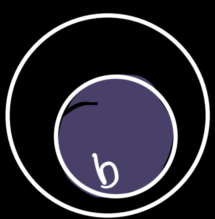
eg. If $x, y \in S$, then $x < y$ / $x = y$ / $x > y$ only one of them is true.

$x, y, z \in S \longrightarrow x < y,\ y < z \Longrightarrow x < z$

$\mathbb{Q}$ is an O·S IF $r<s$ implies $s-r \in \mathbb{Q}$ (Positive or sign irrespective)

Suppose S is an O·S & $E \subset S$. if $\exists$ a $\beta \in S$ s.t $x \leq \beta \ \forall \ x \in E \longrightarrow$
we say 'E is bounded above'
and 'β' $\rightarrow$ upper bound of E.

$A = \mathbb{Q}$ The set $X = \{x \in \mathbb{Q} \mid 2 < x < 4\}$



If A be an ordered set $X \subseteq A$ an element $b \in A$ is called the least upper bound/ supremum for X IF b' is an upper bound & $b \leq b'$ $\forall$ upper bound b for X.

DOUBT: Is x bounded?, If not, how can $b \leq b'$ = True)

Consider set X or $\mathbb{Q}$ · no's between 2 & 4.

Infimum: A point $l \in A$ is a lower bound for $X \subset A$
IF $l \leq x \ \forall \ x \in A$.

$\downarrow$

Or greatest lower bound $\rightarrow$ (inf X).
and, lowest upper bound $\rightarrow$ (sup).

If i chose the intervals at which i take numbers, then, there will be a defined upper bound, right?

may or may not belong to the set depending $\leq \geq$ or $< >$ for constraints.

- Assume $A, B \in \mathbb{R}$, $B > A$. B is an upper bound for all the sets → $[a,b]$, (a,b), $[a,b)$, $(a,b]$
  (and, all numbers greater than b)

- $\mathbb{Z}, \mathbb{Q}, \mathbb{N}$ → X bounded above.
- $\mathbb{N}$ → Bounded below. 1 → infimum.
- $r \in \mathbb{Q} : (0 \leq r < \sqrt{2})$
  ↓
  Here, the upper doesn't have to belong in the constraints.

- $\boxed{\{n^{(-1)^n}, n \in \mathbb{N}\}}$ → for $n = 2k$, upper bound $= +\infty$
  → lowerbound $= 0$.
  [Occurs for large $n = 2k+1$]
  ↓
  called subsequences

$\boxed{A = \frac{1}{n^2} \quad n \in \mathbb{N} \ (\& \ N \geq 3).}$ → G.L.B $= \to 0$
→ L.U.B $= 1/9$

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* Remember: lower bound ⇒ greatest of lowest numbers.

Maximum: Every element of the set should be $<$ than it.

- $\{0,1\} \rightarrow$ Max, supremum exists.
  $\downarrow$? $\downarrow$ 1 ?
- $\{m+n\sqrt{2} \mid m,n \in \mathbb{Z}\}$
- Consider $D = \{x \in R : x^2 < 10\}$

The set R of real no's satisfies the archimedian property:

let $a, b \in \mathbb{R}$, then $\exists\, ma > b$, where $m \in \mathbb{I}$.
Except, when $a < b$, $n \leq 1$.
"Spoonfuls of water can fill a bathtub"

- Supremum: Satisfies the number type of set too or not reqdire to fit in?
- Does Maximum HAVE to be part of the considered set (satisfying its number type constraint)

For a finite set, supremum & maximum coincide.

$\sup A = \max A \qquad A = \{1, 2, \dots\}$

$\inf A = \min A \qquad A = \{1, 2, \dots\}$

$\mathbb{Q}$ satisfies the density property.

AXIOM

If $c < d \in \mathbb{R}$, then $\exists$ a real no $c < q < d$.

↳ Infact, it can also be a rational number

- Between any 2 irrational numbers, $\exists$ a rational no.
- vice-versa

Axiom - Just accept 'em

<u>Field & it's properties</u>

A1. S is closed under addition: if x, y ∈ S then x + y ∈

A2. Addition is commutative: if x, y ∈ S then x + y = y
x.

A3. Addition is associative: if x, y, z ∈ S then x + (y + z
= (x + y) + z.

A4. There exists an element, called 0, in S which is an
additive identity: if x ∈ S then 0 + x = x.

A5. Each element of S has an additive inverse: if x ∈ S
then there is an element -x ∈ S such that x + (-x) = 0.

Integers don't have multiplicative inverses, hence, don't form fields

M1. S is closed under multiplication: if x, y ∈ S then x · y ∈ S.
M2. Multiplication is commutative: if x, y ∈ S then x · y = y · x.
M3. Multiplication is associative:
if x, y, z ∈ S then x · (y · z) = (x · y) · z.
M4. There exists an element, called 1, which is a multiplicative identity: if x ∈ S then 1 · x = x.
M5. Each nonzero element of S has a multiplicative inverse: if 0 6= x ∈ S then there is an element x_{-1} ∈ S such that (x_{-1}) · x = 1. The element x_{-1} is sometimes denoted 1/x.
D1. Multiplication distributes over addition:
if x, y, z ∈ S then x · (y + z) = x · y + x · z.

A field is an ordered field if there is an *relation* $<$ defined on it such that
1. *Transitive*
If $a<b$, and $b<c$, then $a<c$.

2. For every two elements a, b, exactly one of the following holds.
$$a<b,\ a=b,\ a>b$$

3. For any c, if $a<b$, then $a+c<b+c$.

4. For any $c>0$, if $a<b$, then $ac<bc$.

The set R of Real Numbers is an ordered field
C is a field but not an ordered field.

<u>Real no' line</u>

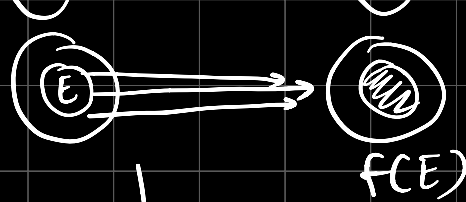
$$\boxed{-\infty < x < \infty}$$

↓ extended real number line.

Finite countable & uncountable sets.

Mapping:-

$$\begin{array}{ccc} A & \mathbb{R} & B \\ x & \longrightarrow & f(x) \end{array}$$

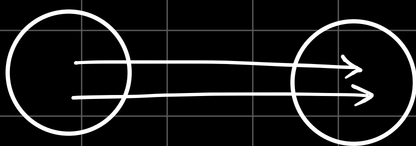


(Onto mapping

↓
can be one-one or
many-one.

[one-many is not a valid mapping]

For inverse to exist, one-one & onto mapping has to exist.



→ Bijection, Countably finite?

George Cantor → "There are different kinds of infinities → Countable & Uncountable ∞.

Cardinality of sets.

$|A| = |B|$

$|x| \Rightarrow$ Number of elements in x.

eg. set of $\mathbb{N} = \{1, 2, 3 \ldots\}$

$f(1) = 1, f(2) = 2 \ldots f(n) = n$

↓ Countably infinite.

set of even no. $= \{2, 6, 8 \ldots\}$

$f(1) = 2, f(2) = 4 \ldots$

Such a set where meaning can be

attached to each element is called. countably finite. Bijection matters because uniqueness is guaranteed

Any $\mathbb{I}$ 'n'

$J_n = \{1, 2, 3 \ldots n\}$.

$J = \{1, 2, 3 \ldots\}$ all integers

for any set 'A',

$A \sim J_n$ for any val. of n.

↓ one-one correspondence

A is countable if $A \sim J$ (Discrete)

??? A is X countable if its neither finite nor contable.

insert! Eg. of uncountable set:

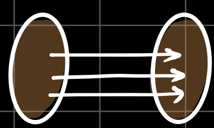
J_n : be the set where elements are Integers $1, 2, \dots n$.

J : " " " of all the integers.

A is finite if $A \sim J_n$

$\sim$ 1-1 correspondence IF $\exists$ a 1-1 mapping of

$A \underset{\text{(onto)}}{\Longrightarrow} B$,



then, $|A| = |B|$.

① $A \sim A$

② $A \sim B$ then $B \sim A$

③ $A \sim B$, $B \sim C$ then $B \sim C$.

Equivalence

- A is said to be infinite if its not finite
- A is countable if $A \sim J$ ($J \to$ All natural no's)
- A is uncountable if $A \neq$ finite, $\neq$ countable.

A or A

Finite Infinite

Countably ∞ uncountably ∞

$\downarrow$

enumerable / denumerable.

(Each number can be given a number to count it w/)

LINGO

$\boxed{\text{Countable} = \text{Countably } \infty \text{ set}}$

- Set of even natural number

$2, 4, 6, 8 \ldots$
$\updownarrow \; \updownarrow \; \updownarrow \; \updownarrow$
$1 \;\; 2 \;\; 3 \;\; 4$

$f(2n) = n$

$f(n) = n/2$

if $f(x_1) = f(x_2)$.
$x_1 = x_2$
$f(2m) \neq f(2n)$
$m \neq n$

$f(2m) = f(2m)$
$m = m$ → what?! should be n?

The beauty: (Behind coun.ty & uncoun.ty.

What is an uncountable set ?
A set where

$x_1 \; x_2 \; x_3 \; x_{3.4} \; x_4.$
$\searrow 1 \; \searrow 2 \; \searrow 3 \; \searrow ? \; \searrow$

The real interval $(0,1)$
If it's countable, every no' b/w 0, 1 should be put in 1-1 correspondence w/ J (nat. no's)

WRONG!

Set of rational no.
Algebraic no's are countable
PROOF (of Cantor)

① Proof that Real no's b/w 1 & 0 ain't countable.

1	$\leftrightarrow$	$x_1 = 0.256173\ldots$	
2	$\leftrightarrow$	$x_2 = 0.654321\ldots$	
3	$\leftrightarrow$	$x_3 = 0.876241\ldots$	
4	$\leftrightarrow$	$x_4 = 0.60000\ldots$	
5	$\leftrightarrow$	$x_5 = 0.67678\ldots$	
6	$\leftrightarrow$	$x_6 = 0.38751\ldots$	
n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5\ldots a_n\ldots$	

Construct the number

$b = 0.b_1b_2b_3b_4b_5\ldots$

Choose

b_1 not equal to 2 say 4

b_2 not equal to 5 say 7

b_3 not equal to 6 say 8

b_4 not equal to 0 say 3

b_5 not equal to 8 say 7

b_n not equal to a_n

You can assign numbers like this. Then add another no. at the end of x. y2ab...

This can be done ∞.

Same cannot be said for rationals. ∵

→ Blocks of decimal repeat

→ Decimal terminates

Countability & uncountability

- uncountability of real line.

→ Proof by contradiction.

- (0,1) is countable.

① $\mathbb{Q}_i \rightarrow \frac{p_i}{q_i}$

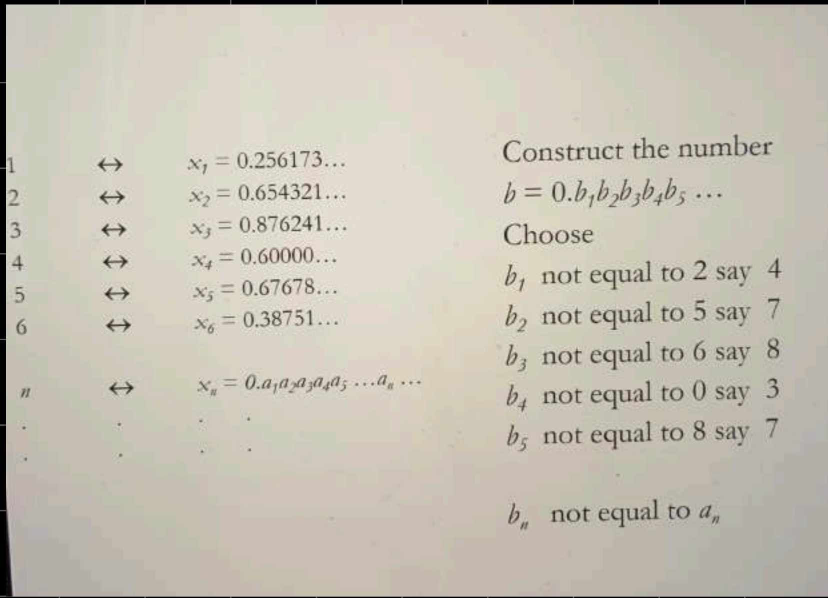
- Express a rational no. in decimal.
- Decimal point → Terminates / Doesn't terminate, but has a repeating pattern.

Irrational no ① it does not terminate.
② Doesn't repeat sequentially

A real number is Algebraic if it is the solⁿ of polynomial eqⁿ, where coeff of the polynomial

$$a_n x^n + a_{n-1} x^{n-1} \ldots + a_0 = 0$$

Transcendental numbers - No's which are ✗ algebr-aic. eg. e, π are uncountably infinite
(BTW, the field of complex numbers is not ordered)

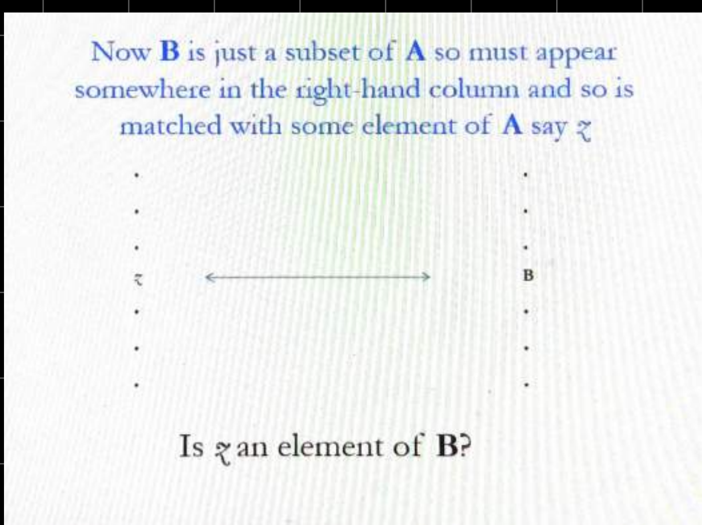


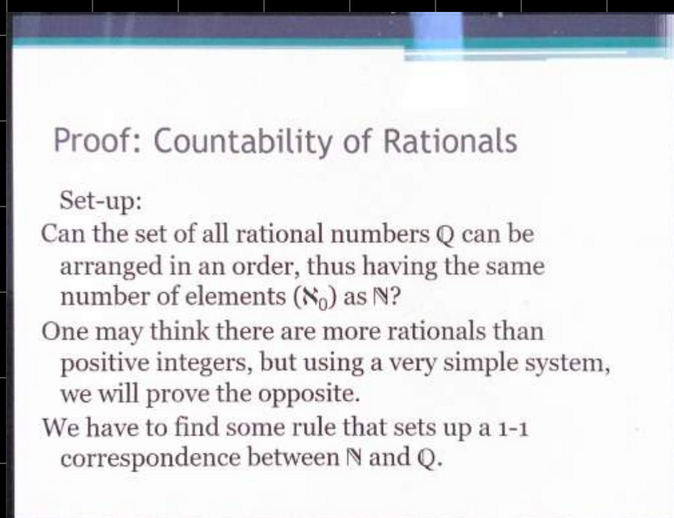
1	$\leftrightarrow$	$x_1 = 0.256173\ldots$	Construct the number
2	$\leftrightarrow$	$x_2 = 0.654321\ldots$	$b = 0.b_1b_2b_3b_4b_5\ldots$
3	$\leftrightarrow$	$x_3 = 0.876241\ldots$	Choose
4	$\leftrightarrow$	$x_4 = 0.60000\ldots$	b_1 not equal to 2 say 4
5	$\leftrightarrow$	$x_5 = 0.67678\ldots$	b_2 not equal to 5 say 7
6	$\leftrightarrow$	$x_6 = 0.38751\ldots$	b_3 not equal to 6 say 8
n	$\leftrightarrow$	$x_n = 0.a_1a_2a_3a_4a_5\ldots a_n\ldots$	b_4 not equal to 0 say 3
.	.	. .	b_5 not equal to 8 say 7
.	.	. .	
			b_n not equal to a_n

→ Used to prove that the subset is a proper subset.

Positive rationals are countable:
(using the diagonal proof)
- If every number can be in a correspondence w/ an $\mathbb{N}$, then it is countable.

→ Finite irrationals are also countable.





Proof of Countability of rationals

$$
\begin{array}{ccc}
1 & -1 & 2 \\
1/2 & -1/2 & 2/2 \\
1/3 & -1/3 & 2/3 \\
1/4 & -1/4 & 2/4 \\
1/5 & -1/5 & \vdots \\
\vdots & \vdots & \vdots
\end{array}
$$

- All of the numbers are accountable.
- It doesn't have to be a 'function'
- Just attach $\mathbb{N}$ w/ each $\mathbb{Q}$

A countable union of countable sets is countable

$\{E_n\}$ is a sequence of countable sets

$$S = \bigcup_{n=1}^{\infty} E_n.$$

Is S countable?

$\because$ each $E_n \rightarrow$ countable.

Each E_n can be arranged in a sequence.

$E_1 = x_{11}, x_{12}, x_{13}, \ldots$

$E_2 = x_{21}, x_{22}, x_{23} \ldots$

$\vdots$

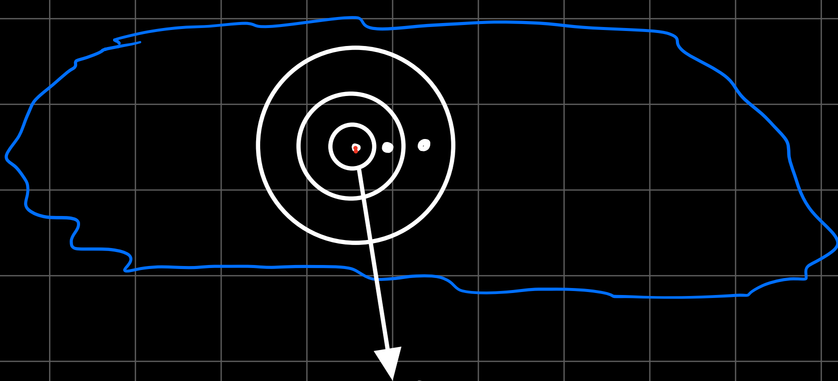
infinitely many sets.

$x_{11} \quad x_{21} \quad \cancel{x_{12}} \quad x_{13} \quad x_{41} \quad \cancel{x}$

Neighbourhood in $\mathbb{R}^3$ → All points within a sphere

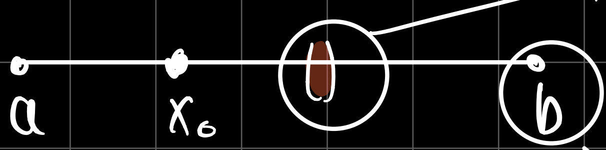
<u>Limit point of a set</u> v/s <u>limit of a function.</u>

A point $p \in E$ is a limit point of it, if every nbd. of p contains pt. $q \neq p$, $q \in E$ or nbd. of p contains a pt other than itself in it



p.

Consider the open interval (a, b



→ Neighbourhood (however small, has points).

E: set of all points in the interval (a, b).

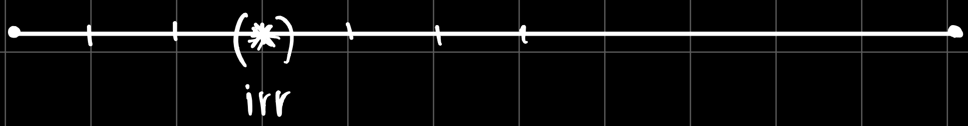
→ limit point. (Every point is a

x_0 → Point.

limit point)

What is the set of limit pt. of set.

(a, b) $[a, b]$

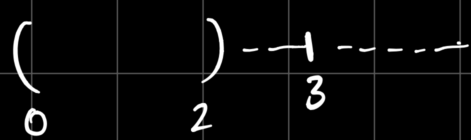


An example of set $A \subseteq \mathbb{R}$

x is a limit pt. of A

an $x \in A$.

$A = (0, 2) \cup \{3\}$



- Here, 3 is not a limit point but belongs to set.
  $\because$ 3 doesn't have any neighbourhood w/ an element in A.
- Now, for eg. $-1 \notin A$.

"EVERY neighourhood of a point should contain atleast 1 element of set."

if $p \in E$ & p is not a limit point of E, then p is called an isolated point

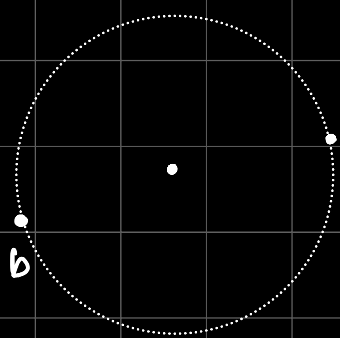
eg. $(0, 2) \cup \{3\}$

A set A is closed if every limit point of E is A point of E.

eg. $[0,2]$

eg. $(0,2)$; $0, 2$ & the set is not closed

Point p is an interior point of E if there is A nbd. such that $N \subset E$



a → not interior

$[a, b]$

↓ ↓

Border elements

A set E is open if every point is an interior point

closed

Definition (i) A subset G of $\mathbb{R}$ is **open** in $\mathbb{R}$ if for each $x \in G$ there exists a neighborhood V of x such that $V \subseteq G$.

∞ union of ordered sets $\bigcup_{i,j} (a_i, a_j)$ is countable

Set of all functions $f: \mathbb{N} \to \{0,1\}$.

$$f_1(1) = 0 \text{ or } 1$$
$$f_1(2) = 0 \text{ or } 1$$
$$\vdots$$
$$f_1(n) = 0 \text{ or } 1$$

Countable :
$$\begin{matrix} (a_1, a_2), & (a_1, a_3) & \cdots & (a_1, a_n) \\ (a_2, a_1), & (a_2, a_2) & \cdots & (a_2, a_n) \\ \vdots & & & \\ (a_n, a_1) & \cdots & & (a_n, a_n) \end{matrix}$$

Since each element can be accountable for, this is a countable union of countable sets.

Binary sequence

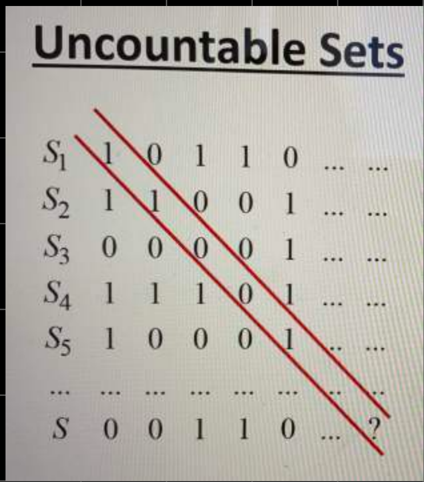
$f_1(1) = 0$ or 1
$f_1(2) = 0$ or 1
$f_1(3) = 0$ or 1
$\vdots$
$f_1(n) = 0$ or 1

$f_1\{1, 2, 3, \ldots n, \ldots\} = 01000 \ldots$
$f_2\{1, 2, 3, n \ldots\} = 10010 \ldots$
$f_n\{1, 2, 3, n \ldots\} = 000 \ldots 1 \ldots$

Such a set of 0's and 1's of all such seq

↓

Such Sets are uncountable.



Uncountable Sets

S_1	1	0	1	1	0	...	...
S_2	1	1	0	0	1	...	...
S_3	0	0	0	0	1	...	...
S_4	1	1	1	0	1	...	...
S_5	1	0	0	0	1	...	...
...	...	...	...	...	...	...	...
S	0	0	1	1	0	...	?

(Similar proof as real no's)

$S_j[J] = S[J]$

But, by definit^n

$S_j[J] \neq S[J]$

Metric space

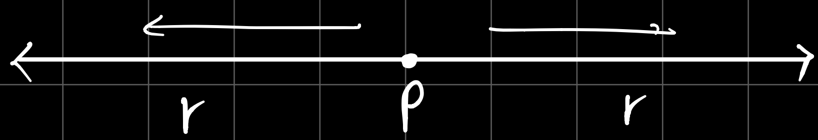
In analysis, we also consider spaces in which there is no distance (Open sets) ⟶ studied under topology

① $d(p,q) > 0$ if $p \neq q$

$d(p,q) = 0$ if $p = q$

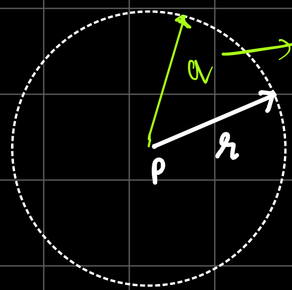
$d(p,q) = d(q,p)$

Neighbourhood of pt P ($N_r(P)$) is a set consisting of all q, $d(p,q) < r$ $r > 0$. r is called radius of the neighbourhood.



in $\mathbb{R}$

$$N_r(p) = \{ q \in N_r(p) \mid d(p,q) < r \}$$



generic distance
↓
reaches but, never reaches border

$$\boxed{\text{The } \infty \text{ seq of integers is uncountable}}$$

(Not proved)

↓

Not just sequences in order, random number

What is ξ ?

$2^{\aleph_0}$

$\aleph_0 \rightarrow$ Cardinality of continuum.

Cantor posed: Is there a cardinality in b/w the cardinality of a countable set of . → continuum.

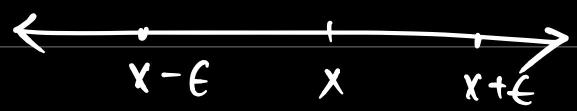
1900 → Hilbert → " Is there a cardinality b/w countability & uncountable ∞?

① NBD of pt p $\{q \in N_r(p) \mid d(p,q) < r\}$ $r>0$

② Interior pt. of E is a point p such that $\exists$ a NBD which contains ∞ points $\in$ the set, which has atleast neighbourhood to be fully contained in the set

③ Open set. If every pt. is an interior point, then the set is an open set. $\exists$ an ϵ, the interval $(x-\epsilon, x+\epsilon)$.
(Let U be the set) $U \subset R$

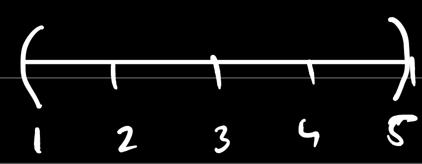
$x \in U$



This interval wrt. R $(x-\epsilon, x+\epsilon)$ is called a NBD of x.

Eg. Set $U = \{x \in R : |x-3| < 2\}$

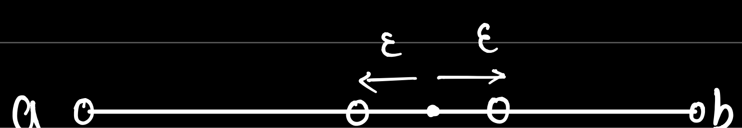
$(x-\epsilon, x+\epsilon) \subset U$



$\epsilon = 2 - |x-3|$

$x = 2, \quad 2 - |2-3| = \underline{\underline{1}}$

$(x-1, x+1) \subset U$



Every NBD is an open set.

A open set is a set where EVERY point in it is an interior point. Interior point is a pt. s.t $\exists$ a NBD of the pt completely contained in the set.

Interior of a set : Set of all interior points of a set

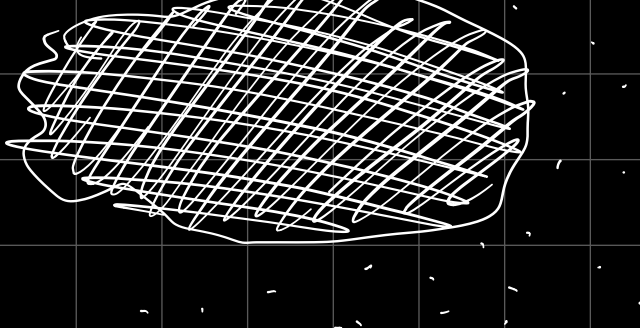
eg.

Interior $\rightarrow$ $(-1, 2)$

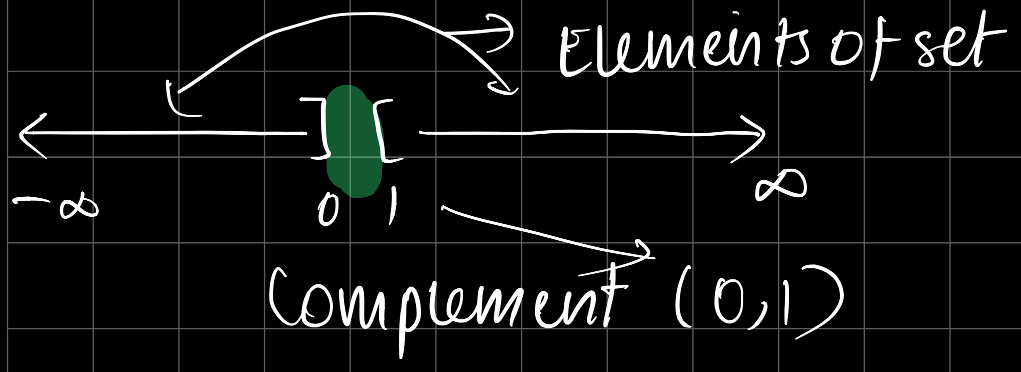
Boundary set :
- Let $b \in R$, let b be a boundary point of S, then every non-empty NBD of b $(b-\epsilon, b+\epsilon)$ contains both the points of S & of R/S.
- A boundary of b may be in S or in the S^c.

$\rightarrow$ 'mesh'

$(0,1)$ $\quad$ Q :



All of R is the boundary of Q.



$\boxed{\mathbb{Q} \text{ is neither open nor closed}}$

$\downarrow$

$r \in \mathbb{Q}$, an N_r of r contains irrational points.

$N_r \not\subset \mathbb{Q} \Rightarrow \mathbb{Q}$ is NOT open.

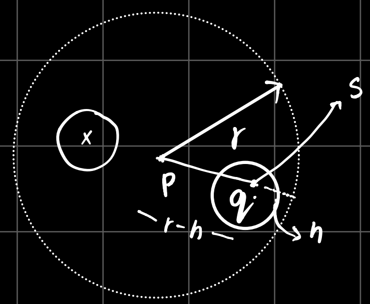
- Let s any Irrational pt.
- Let B_s be an NBD of s
- $B_s \cap \mathbb{Q} \neq \emptyset$ IF s is a limit point of $\mathbb{Q}$ BUT it's not a rational no. $\mathbb{Q}$ doesn't contain all its limit points.

$\Rightarrow \mathbb{Q}$ is not.

$\mathbb{R}$ numberline's members has ALL memb-ers, and hence it can be called the boundar-y of $\mathbb{Q}$.

We take a NBD & s.t. every point in the NBD is an interior point.

(we show that the boundary is not included)



NBD E

$N_r(p) = \{q \mid d(p,q) < r\}$

$q \in E$

$d(p,q) < r$

$d(p,q) = r-h$

Construct a NBD around q so that it is $\subset E$,

s, $d(R,S) < h$. $d(p,s) \leq d(p,q) + d(q,s)$ $\llcorner (\Delta \neq)$

(is it p, s, r??)

$\leq r-h+h$

$d(p,s) \leq r$

$d(p,s) < r \Rightarrow$ we can always pick any point in E & exhibit a NBD $\subset E$.

If p is a limit point of a set E, then every NBD of p contains ∞ points of E.

Prove by contradiction (contrapositive works too?)

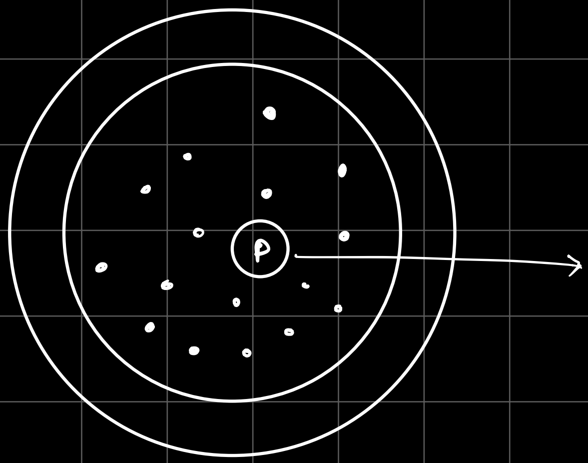
$\boxed{\neg B \rightarrow \neg A.}$

$\downarrow$

(eh?! this is contrapositive, right?)

$\neg B \rightarrow$ Assume that $\exists$ one NBD N of p which contains only a finite number of points of E.

$\neg A \rightarrow$ A is not a limit point.



smallest NBD with only P.

$\downarrow$

But this is not possible if P is an actual limit point.

$\downarrow$

A.K.A. cluster/accumulation point.

Q. How do we know that this is indeed the smallest distance?

→ Take distances b/w P & $q_1, q_2, q_3, q_4 \ldots$

→ Take min of them

→ Divide by a constant to ensure no point is contained

Bolzano - Weierstrass's theorem: Every bounded ∞ subset of R has a limit point.

Def: ① A closed set is a set which contains all its limit points

② A closed set is the complement of an open set.

Proof By Contradiction

- C is a closed set if its complement is open $\Rightarrow$ C contains all its limit points.
- Let R be the limit pt. of C to S.T $x \in C$ using ①.
- Suppose $x \notin C \Rightarrow x \in C^c$. But, C^c is open. $\Rightarrow \exists$ a NBD V of x. $x \in V \subset C^c$
$\Rightarrow$ There is a NBD V of x which doesn't have a single point of C $\rightarrow$ But this is a CONTRADICTION that x is a limit pt.

(word w/C)
- Conv. C as a closed set contains all its limit points, we prove that $C^c \rightarrow$ open.
Let $x \notin C \Rightarrow x \in C^c$

- x is, hence, not a limit point $\Longrightarrow$ we need to prove that C^c is open.

- Let $x \notin C \Rightarrow x \in C^c$ (assume)
x is not a limit pt $\Longrightarrow$ has atleast one NBD w/o a single point of C in it.

$\Rightarrow$ This NBD $\subset C^c$ {QED}

- we have exhibited any pt in C^c with a NBD $\subset C^c \Rightarrow C^c \rightarrow$ Open

(we basically contradict our initial statement that x is a limit point.

(UNOFFICIAL COROLLARY)

Also, proves that limit point MUST be in the set

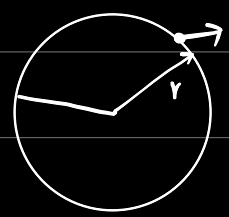
x

$\boxed{\text{every open interval} \downarrow \text{open set}}$

④ <u>Closed Sets</u> : $[a, b) \rightarrow$ not open

↓

not opp of open



Set E is a closed set if every limit point of E is a point of E.

⑤ <u>Limit point</u> : p is a limit point of the set E if every NBD of point p contains a pt (q) of set E other than itself $(p \neq q)$

$a \notin R$ is a limit point, b is a limit point.

$(a, b) \rightarrow$ except a, b This is not a closed set. This set is closed if its complement is open.

eg. $[a, b] \longrightarrow$ The complement $\rightarrow$

$(-\infty, a) \cup (b, \infty)$

eg. $\{1, 1/2, 1/3, 1/4 \ldots\} \cup 0$

complement

$\downarrow$

$(-\infty, 0) \cup (1, 1/2, 1/3 \ldots)$

$\downarrow$

Union of open intervals is open

$\neg\{\ \}$

$=$ $\cup$ of open sets, so, another open set.

$0 \ \frac{1}{4} \ \frac{1}{3} \ \frac{1}{2} \qquad 1$

- Consider the union $\bigcup \frac{1}{n+1}$ $n = 1 \ldots$

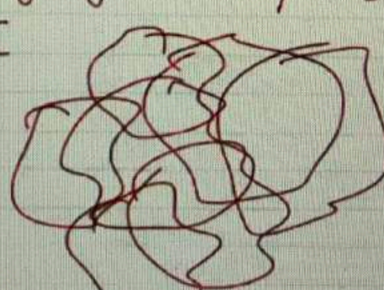
$\left(0, \frac{1}{100}\right) \left(\frac{1}{100}, \frac{1}{99}\right) \left(\frac{1}{99}, \frac{1}{98}\right) \ldots$ Is a union of open intervals is open $\therefore$ This is an open

<u>Result</u>

An Arbitrary union of open sets is open
and a Finite Intersection of open sets is open.

$\}$ Starting pt of the def of a "Topology"

$A_i \subset \mathbb{R} : i \in I$



$x \in \bigcup_{i \in I} A_i$

$\Rightarrow x \in A_i$ for some I

A_i is an open set

$(x-\delta, x+\delta) \subset A_i \subset \bigcup_{i \in I} A_i$

$\Rightarrow$ An arb union of open sets is open.

Supp

Finite intersection is open. WHY?

$\Rightarrow$ An arb union of open sets is open

let i

$\Rightarrow x \in$ every A_i

every A_i is open

Supp

$(x-\delta_i, x+\delta_i) \subset A_i$

Choose the min $\{\delta_1, \delta_2 - - \delta_n\} > 0$

δ_m

$(x-\delta_m, x+\delta_m) \subset \bigcap_{i=1}^{n} A_i \Rightarrow$ This open

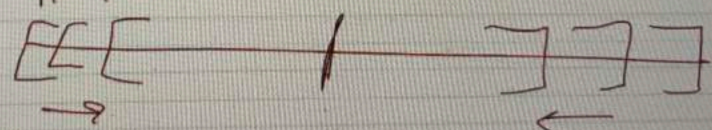
$[\frac{1}{n}, 1]$

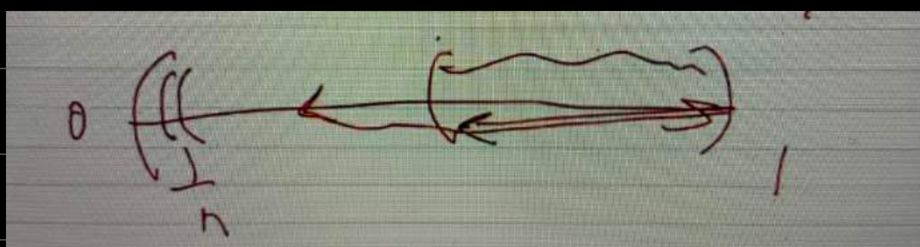
$I_n = (-\frac{1}{n}, \frac{1}{n})$ ∞ $\quad n \to \infty$

$\bigcap_{n=1}^{\infty} I_n = \{0\}$

$I_n = \bigcap_{n=1}^{\infty} (\frac{1}{n}, 1)$

empty





Housedorf → sets where metric cannot be defined

A sequence of Real no's or a sequence in $\mathbb{R}$ is a function numbers $\subset \mathbb{R}$ $(\{1, 2, 3 \ldots\} \subset \mathbb{R})$

Sequence Representation.

1. First, list out the members $\rightarrow a_1, a_2, a_3 \ldots$

   $2, 4, 6 \ldots 2n$

2. Representing sequences by a rule.

   $a_n = \sqrt{n}$ $\quad 1, \sqrt{2}, \sqrt{3}.$

   $b_n = (-1)^{n+1} \cdot \frac{1}{n}$ $\quad b_1 = +1$

   $b_2 = -1/2$

   $b_3 = 1/3$

3. Recursion: Recurrence Relation

   eg. Fibonacci Sequences

   (The same operation carried out repeatedly)

   * Insert eg*

Best representation – Visual

Types of sequences

- Constant sequences
- Oscillating sequences. eg. $1, -1, 1, -1 \ldots$

- $\cos \frac{n\pi}{3}$ $\quad n \in \mathbb{N}$

  $(1/2, -1/2, -1, -1/2, 1/2, 1 \ldots)$

- $a_n = \sqrt[n]{n}$ $\quad 1, \sqrt{2}, \sqrt[3]{3} \ldots\ldots$

  (interestingly, the value is generally ≈ 1.4)

SEQUENCES AND SERIES

Section 3

#2

Limit of a sequence (a_n) is a real no. s.t. the values of S_n are close to each other for large values of n.

Def. The seq $\{a_n\}$ converges to the number L if $\forall$ the no ϵ. There corresponds to an integer N such that for all $n > N$,

$$\boxed{|a_n - L| < \epsilon \quad \boxed{\text{for all } n > N}}$$

If $\{a_n\}$ converges to L, we say, $\lim\limits_{n\to\infty} a_n = L$. or $a_n \to L$.

Sequence $\left(\frac{1}{2^n}\right) = 1/2, 1/2^2, 1/2^3, 1/2^4, 1/2^5 \ldots 1/2^\infty \ldots$

$L \to$ limit

$a_n \to$ term

$\epsilon \to$ arbitrary number

You can always find a distance greater than the distance b/w 2 numbers at large N's in a converging sequences.

$\boxed{\text{In any metric space } X \text{, a seq } \{P_n\} \text{ converges if } \exists \text{ a pt. } \in X \ \forall \epsilon > 0 \ \exists \text{ an } n > N,\ d(P_n, P) < \epsilon}$ → Alternative, topological definition

If a series is convergent: then ↓ ~~⇐~~ (converse may not be true)
- Limit of the set is unique
- The sequence is bounded

eg. $1, -1, 1, -1 \to$ Bounded but, no unique limits & not convergent

eg. $S_n = \dfrac{3n+1}{7n-4} \quad \dfrac{3 + 1/n}{7 - 4/n}$ (The general rules of limits)

What are the theorems?

Sandwich theorem: 1

$\{a_n\}, \{b_n\}, \{c_n\} \ \forall$ n beyond N. → (tail of sequence)

$a_n < b_n < c_n$

2

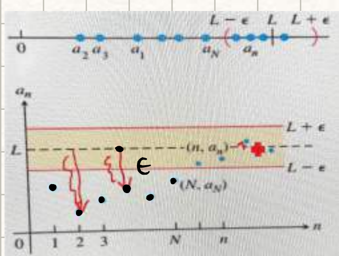
If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

$\Rightarrow \lim\limits_{n\to\infty} b_n = L$

3

Cauchy criterion

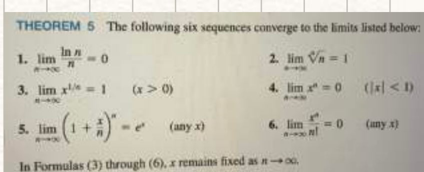
→ Let $\{a_j\}$ be a sequence of no's (r/c), satisfies the criterion if $\forall \epsilon > 0, \exists$ an $N > 0$, s.t if $j, k > N \Rightarrow$



L is called the horizontal asymptote.

Divergence (calculated at $n\to\infty$)
→ Approaches to ∞
or
→ Oscillates

→ TOOL: Sandwich theorem: Useful for trigonometric expression



THEOREM 5 The following six sequences converge to the limits listed below:

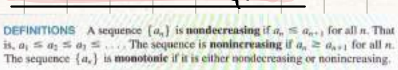
1. $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$
2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$
3. $\lim\limits_{n\to\infty} x^{1/n} = 1 \quad (x > 0)$
4. $\lim\limits_{n\to\infty} x^n = 0 \quad (|x| < 1)$
5. $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x \quad (\text{any } x)$
6. $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0 \quad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

Newton's formula

$x_0 = 1$
$x_{n+1} = x_n - [(\sin x_n - x_n^2)/(\cos x_n - 2x_n)]$
for $n > 0$.
and gives soln to eqn $\sin x - x^2 = 0$.

Bounded Monotonic Sequences



DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le \ldots$. The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

Converging sequences are both bounded & monotonic at the nth element.

For checking convergence,

only check tail

$|a_j - a_k| < \epsilon$

$a_1, a_2, a_3 \ldots a_N \mid a_{N+1}, a_{N+2} \ldots a_{N\to\infty}$

Head of the seq

$\underbrace{\text{Tail of the sequence}}$

$\Downarrow$

Determines the convergence of the sequence

SEQUENCES AND SERIES

Section 3

If a sequence converges to L,

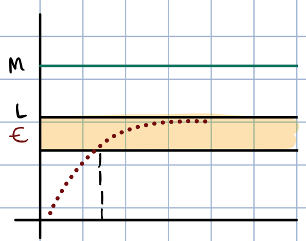
$|(a_n - L)| < 1$

But not all bounded sequences converge. e1

$\boxed{\text{Monotonic \& Bounded} \to \text{Congruent}}$

$\overset{\times}{\longleftarrow}$ (not converse)

Proof.



A sequence (a_j) is said bounded if there is a no. $M>0$ such that $|a_j| \le M \; \forall j$.

Result: let $\{a_j\}$ be a convergent seq. Then the sequence is bounded.

Convergent $\to$ Bounded.

Let (x_n) be a convergent sequence with limit x. There is exists a $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon \quad \forall \; n > N$.

In particular $\exists$ an n, $|x_n - x| < 1$ for $n \in \mathbb{N}$.

We can use the Δ inequality $|x_n| = |x_n + x - x| \le |x_n - x| + |x|$

Define $M = \max\{|x_1|, |x_2|, \ldots\ldots 1 + |x|\}$ $\quad \le 1 + |x|$ for $n > N$

$|x_n| \le M$ for all $n \in \mathbb{N}$.

This sequence can be called bounded (x_n)

If a sequence is unbounded it diverges?

$p \to q$, then $\neg q \to \neg p$

We'll show that for $n > N$, the tail of the sequence.

COMPLETE THE SENTENCE!!

$x_1, x_2, x_3 \ldots.$

$1, 8, 27, 64, \ldots \to n^3$

as $n^3 \to \infty$ as $n \to \infty$ this is not a convergent seq.

$1, -1, 1, -1$

The boundaries or convergence of a sequence of a seq depends on only the behavior of 'infinite point' tail $(x_n)_{n=N}^{\infty}$

"Series"   sequences

$\{a_n\}$ if $a_n \to L$, f is a continous function at L defined. at all a_n. Then $f(a_n) \to f(L)$

$\lim_{n\to\infty} a_n$, $\lim_{x\to\infty} f(a)$. L'hopital's rule.

$\lim_{n\to\infty} \frac{\ln n}{n} = \frac{\ln 1}{1} = \frac{\ln 2}{2} \ldots$

- $\dfrac{\ln x}{x} = \lim\limits_{x\to\infty} \dfrac{1/x}{1} = \dfrac{0}{1} = \underline{\underline{0}}$

- $a_n = 2 + (0.1)^n$ converges to 2.

- $\dfrac{n+(-1)^n}{n} \quad \lim\left(1 + \dfrac{(-1)^n}{n}\right) = 1.$

A monotone seq. of $\mathbb{R}$ converges iff it is bounded.
- If (x_n) is monotone increasing & bounded then $\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$

- Furthermore if (x_n) is monotone increasing and unbounded, then $\lim\limits_{n\to\infty} x_n \to \infty$.

Axiom: Dedekind completeness. (1872)

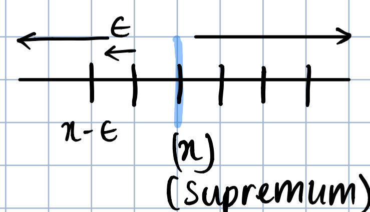
" $\mathbb{R}$ are complete ordered fields"

formally

↓

Every non empty set of $\mathbb{R}$ that is bounded from Above has a supremum.

<u>Supremum Property</u>



It is the least upper bound.
$x-\epsilon$ is lower than the L.U.B.

$x-\epsilon$ has atleast one greater element than it as it is not an upper bound. Infact, it's lesser than the least upper bound.

$$x-\epsilon < x_N \leq x.$$

seq: Convergent → Bounded.

Conversely, let (x_n) be bounded, monotone. increasing, $x = \sup\{x_n : n \in \mathbb{N}\}$

From the def of a sup, $\exists$ an $N \in \mathbb{N}$

$x_n > x-\epsilon$, $x_n > x_N \;\forall\; n > N$.

$x-\epsilon < x_n \leq x$. $\quad |x_n - x| < \epsilon$, $x_n > x_N$ for $n > N$.

$x_n \to x$ as $n \to \infty$

$x_n > x_N$, $x_n \geq x_N$

eg.

$1-\frac{1}{n}$	$1-1$	$1-\frac{1}{2}$	$1-\frac{1}{3}$	$1-\frac{1}{4}$.

Sequences & series

The cauchy criterion for sequences.
A sequence is said to satisfy the cauchy criterion if $\forall \epsilon > 0$. $\exists$ an integer $N > 0$ such that if $j, k > N$ then $|a_j - a_k| < \epsilon$.

This says that the elements of the sequence get closer and closer together for large N.

$$(a_j) = \left(\frac{1}{j}\right) \quad n > N \quad j, k > N \ (j > k > N)$$

$$|a_j - a_k| = \left|\frac{1}{j} - \frac{1}{k}\right| < \left|\frac{j-k}{jk}\right| < \frac{j}{jk} = \frac{1}{k} < \frac{1}{N} < \epsilon$$

This seq. satisfies the cauchy criterion. $\forall$ series that satisfy this $\rightarrow$ Convergent. Converse is also true.
$\downarrow$ In $\mathbb{R}$

Bolzano weirstrass'

Every bounded sequence has a convergent subsequence

let $\{a_j\}$ be a given sequence. If $0 < j_1 < j_2 < j_3 \ldots$ are the integers the $f: k \mapsto a_{j_k}$ called a subsequence of a given seq.

$$\{a_j\}_{k=1}^{\infty}$$

* Even if I remove ∞ no. of elements from this, ∞ numbers remain.

1, 2, 3, 4, 5, 6, 7, ...

$a_{k_1}, a_{k_2}, a \ldots$
$\rightarrow$ Double subscript $\rightarrow$ shows that it's a subsequence

- I can remove these elements, and a subsequence remains.
- RULE: An order must exist. a_3 cannot come after a_{10}.

Limit superior & limit inf:

If we take diff subsequences, FORM their respective

limits. Their set could be finite or infinite (Again, it depends on the superset)
The inf of these is lim inf, sup → lim sup.

Consider : $-1, 0, 1, -1, 0, 1 \ldots$

$a_b \quad a_c \quad a_k$

subsequences.
(They all have their own limits)

$a_b = (-1, -1, -1, \ldots\ldots) \longrightarrow -1$
$a_c = (0, 0, 0, 0 \ldots) \longrightarrow 0.$
....

- If all the subsequences as above converge to a SINGLE point, o a single limit point, the SEQUENCE is converging.

- If a seq. converges, its limit point is unique.
  $0, 1, 0, 1, \ldots$
  ↳ X converging

Infinite series

$$S = \sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 \ldots$$

A rudimentary view - Addition of all terms → series
Sophisticated RA definition - series is a limit of sequence of partial sums.

$a_1, a_2, a_3, a_4, a_5, \ldots$

Partial sums: $S_1 = a_1$
$S_2 = a_1 + a_2$
$S_3 = a_1 + a_2 + a_3$
$\vdots$
$S_n = a_1 + a_2 + a_3 \ldots\ldots a_n$

The sum is supposed to be ∞, right? but nope.

Xeno's paradox

Each time you're adding an ∞ numbers as. $n \to \infty$. a_n : $\frac{1}{2^n}$ or $\frac{1}{n}$ etc. $\longrightarrow 0$. so the impact of a new term tends almost to nil.

" A series will converge if for $n > N$ the change in the sum is negligible $< \epsilon$

A series is said to converge if the sequence of partial sums converge.

$$\lim_{n \to \infty} \{S_n\} \to L$$

Tests: · P series
· Geometric series.
· Ratio test
★ · Comparision test
· Integral tests.

If a sequence converges, it is a cauchy sequence. These sequences are not bounded. $\boxed{-1 \ 1 \ -1 \ 1 \ \ldots \quad n>N}$ – How is this converging?

eg. $1/2^n$

"The terms of the sequences can be made arbitrarily close". $\epsilon > 0$.
(As close as one wishes

Mathematically: for a given $\epsilon > 0$, $\exists$ an N, m such that $m > N$
$|S_n - S_m| < \epsilon$
(This thing has been said about sums but the same thing can be said about a_n (terms) too)

Geometric sequences

$$a + ar + ar^2 \ldots = \sum_{n=1}^{\infty} ar^n = \frac{a(1-r^n)}{1-r}$$

Consider
- $|r| < 1$: $r^n \to 0$ as $n \to \infty$, the series will converge.
- $|r| > 1$: $r^n \to \infty$ as $n \to \infty$, the series will diverge

Geometric sequences' limit can be found out. But, we shall focus on whether it IS convergent or not.

$S_n = a + ar + \cdots ar^{n-1}$
$rS_n = ar + ar^2 + ar$

$S_n - rS_n = a^r - 1$
$S_n = \boxed{\frac{a^r - 1}{1-r}}$

If you have a telescoping series

eg. $a_1 + a_2 + a_3 + \ldots\ldots$
such that the terms in the middle cancel out.

eg. $\Rightarrow$ $\boxed{\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}}$

$$S_K = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) \ldots\ldots \left(\frac{1}{K} - \frac{1}{K+1}\right)$$

$S_K = 1 - \frac{1}{K+1}$. If $K \to \infty$, the series converges to $\underline{\underline{1}}$

The nth term test for divergence

→ Some series don't converge because the n^{th} terms don't become small

$a_n = S_n - S_{n-1} = 0$ then the series converges.

$a_n \to 0$ only then a series (sum) is converging.

$a_n \to 0$ can also mean diverging but not vice versa.

(For series the criteria is different. a_n doesn't have to be 0.)
↓
For series → convergent if area under the limit is finite

Harmonic series: Each term is a reciprocal of A.P.

P series: $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} \ldots\ldots}$

If $p > 1$, the series will converge
If $p < 1$, the series will diverge.

TOOL:
For proof q's, $\left(\frac{1}{2}\right)^n$, $\left(\frac{1}{4}\right)^n \ldots$
G.P works

In the integral test, $f(1)$ & $f(2)$ are areas of the rectangles. The $\int$ is less than $\sum f(1) \ldots f(n)$.

The integral test: Let $\{a_n\}$ be a sequence of the terms. Suppose $a_n = f(n)$ where f is a continous, +ve, decreasing function of x of x for all $x \geq N$ ($N \to$ +ve I)

$\underbrace{a_1 + a_2 + a_3 + a_4 \ldots\ldots a_N}_{\text{Finite}}$ | $a_{N+1} \ldots\ldots \infty$ Infinite

↓
Does not matter to us.

Basically, just integrate, apply limits, if $\int$ is finite → convergent.

The answer you get is only a rough approximation, not the actual sum.

The series) $\sum_{n=N}^{\infty}$ and the the integral $\int_N^{\infty} f(x) \cdot dx$ and $\sum_{n=0}^{N} \left(\int_0^n f(x) \cdot dx\right)$

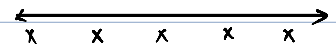
↳ They BOTH should be either convergent / Divergent

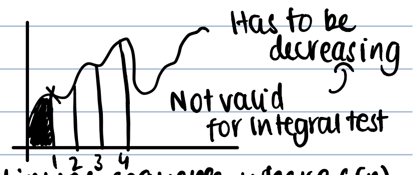
The most imp observatn

Why should the term be +ve? If a series has negative no's, the series

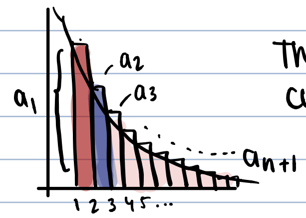
MAY not converge at ∞. when $n>0$ then the chance for divergence. → >>

$a_1 = f(1)$
$a_2 = f(2)$

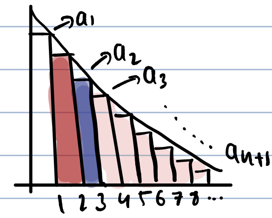




Assume that f is a decreasing, NON-continuos sequence where $f(n) = a_n \ \forall\, n$.



The area under this curve is less than the area under the curve on the right.



wrong?

It is of quientessence to note that if $\int^{\infty} f(x) \cdot dx \to \infty$, $a_1 + a_2 \cdots$ can be infinite, but having the boundary ($\int^{n+1} f(x)$ is also infinite → (why? finiteness depends on the tail)) ensures that $a_1 + a_2 + a_3 \cdots$ is indeed ∞

if $\int f(x) \cdot dx$. is finite, then $\sum a_n$ is finite.

So the $\sum$ and the $\int$ should BOTH be either ∞ or finite

$$\int_1^{n+1} f(x) < a_1 + a_2 + \cdots a_n \rightarrow \text{①}$$

Now, we can infer that this series is sandwiched between $\int_1^{n+1} f(x)$ & $\int_1^{n} f(x) + a_1$

$$\boxed{\int_1^{n+1} f(x) < a_1 + a_2 + \cdots a_n < \int_1^{n} f(x) + a_1}$$

(To be completed)

n^{th} Term test

If a series converges then the term $a_n \to 0$ as $n \to \infty$. The converse, however, isn't true.
Counter eg. $1/n$. $n \to \infty$, $a_n \to 0$, But not converging.

Assume that the series converges. It satisfies the cauchy criterion.

$$\boxed{\left|\sum_{j=m}^{n} a_j\right| < \epsilon \quad n, m > N}$$

(consider the sequence of partial sums)

$|S_n - S_m|$ $\lim_{n\to\infty} S_n = S$

Suppose the cauchy criterion holds.

$|S_n - S_m| = \left|\sum_{j=m} a_j\right| < \epsilon \quad n, m > N$

The sequence of partial sums converges. $\{S_n\}$ is a cauchy sequence
$\Rightarrow$ $\{S_n\}$ converges.
$\Rightarrow$ series converges

If the series converges, then the cauchy criterion $\to$ satisfied.

$|S_n - S_m| < \epsilon$, $\left|\sum_{j=m}^{n} a_j\right| < \epsilon$

If the series converges $\to$ the n^{th} lim $\to 0$.

$a_1 + a_2 + a_3 \dots \; a_n \underset{n\to\infty}{\to} \underline{\underline{0}}$.

$a_m, a_n \to$ Terms at the tail.
let's take $n = m$, $m > N$. (where $n, m > N$)
i for large m (or bounds the tail of the series). $|a_m| < \epsilon$ i.e $|a_m|$ is arbitrari-ly small.

$S_1, S_2, S_3, S_4 \dots S_\infty$

If a series converges $\Rightarrow a_n \to 0$
But not vice versa.
(insert more contrapositive stuff)

The comparison test let $\sum a_n$, $\sum c_n$, & $\sum d_n$ be series of non negative terms, $\forall$ that for some $n > N$, $d_n \le a_n \le c_n$ $\forall$ $n > N$

(a) If $\sum c_n$ converges, then $\sum a_n$ converges.
(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges

If the partial sums of $\sum a_n$ are bounded above by .

$a_1 + a_2 + \dots \; a_N + a_{N+1} + \dots$
$\le a_1 + a_2 + a_3 \dots a_N + \sum_{n=N+1}^{\infty} c_n$.

consider $a_N \to$ onwards.
$\sum_{n=N+1}^{m} a_n$. ... $S_{N+1}, S_{N+2}, \dots$ C_n's.

↓ These are bounded by the S_{N+1}, S_{N+2}

we have an increasing seq $\{S_n\}$ bounded above $\Rightarrow$ seq of partial sum converges. $\sum d_n \le \sum a_n \to$ diverges.

Real Analysis

Convergence of ∞ series.

The limit comparison test.

Suppose $a_n > 0, b_n > 0 \ \forall \ n \geq N$ (N is an integer)

① If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ the Σa_n and b_n both converge or both diverge.

② If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. If Σb_n converges $\Rightarrow \Sigma a_n$ conve-rges.

③ If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and Σb_n diverges $\Rightarrow \Sigma a_n$ diverges.

$c > 0, \frac{c}{2} > 0$, $\exists$ an $n > N$ s.t. $\left| \frac{a_n}{b_n} - c \right| < \frac{c}{n}$

$\Rightarrow n > N \quad \frac{c}{2} < \frac{a_n}{b_n} < \frac{3c}{2} \Rightarrow \frac{b_n \cdot c}{2} < a_n < \frac{3 \cdot c \cdot b_n}{2}$

Bounded by $3c/2 \cdot b_n$ for $n > N$

If Σb_n converges then $\Sigma \frac{3c}{2} b_c$ also converges.

If Σb_n converges, then $\Sigma \frac{3c}{2}$ also converges. By the direct comparison, Σa_n converges, if Σb_n diverges. $\Rightarrow \Sigma a_n$ also diverges.

ⓒ If $\lim \frac{a_n}{b_n} (> 1) = \infty$
$\Rightarrow n > N$ for all $n > N$
$\Rightarrow a_n > b_n$
$\Rightarrow$ if Σb_n diverges then, by direct comparison, Σa_n will diverge

Supporting examples

① $\frac{3}{4}, \frac{5}{9}, \frac{7}{16} + \frac{9}{25} \Rightarrow \frac{2n+1}{n^2+2n+1} \times \frac{1}{1/n} = \frac{2+1/n}{1+1+1} = 2$

$1+\frac{1}{2n}+\frac{1}{n^2}$

Here, $(n+1)^2 \to$ diverging,

$\lim \sum \frac{a_n}{b_n} = C$, hence $2n+1 \to$ Diverging

General way to find convergence with limit comparison tes-:

1. take sequence a_n. (To test convergence of)
2. Divide by a term for which convergence / lack thereof is known.
   Or - take a polynomial with the same order (Big O and small o)
3) Do the steps done in eg.

Absolute Convergence

A series converges absolutely if the corresponding series of absolute values $\sum |a_n|$ converges.

The absolute convergence test

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges "Mathematical rigour"

$\forall n \quad -|a_n| \leq a_n \leq |a_n| \Rightarrow$ add $|a_n|$

$0 = |a_n| - |a_n| \leq a_n + |a_n| \leq 2|a_n|$

if $\sum |a_n|$ converges $\to \sum 2|a_n|$ converges

$\Rightarrow \sum_{n=1} a_n + |a_n|$ converges.

$a_n = (a_n + |a_n|) - |a_n|$

express $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|$

(converges) conver.

It's the difference of 2 convergent series

$\Rightarrow \sum a_n$ converges.

$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} \ldots$ converges

absolutely too, so this series in its original form converges.

(Provided by Imami)

Ex : $\sum_{n=1}^{\infty} \sqrt{\frac{n+1}{n^2+2}}$ order = $1/\sqrt{n}$

as $\sum 1/\sqrt{n}$ diverges by the p series with $p<1$

so let $b_n = 1/\sqrt{n}$

$$\lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{\sqrt{\frac{n+1}{n^2+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \sqrt{\frac{n^2+n}{n^2+2}}$$

$$= 1 = c > 0$$

so Both $\sum a_n$ & $\sum b_n$ diverges as b_n diverge

Ex : $\sum_{n=1}^{\infty} \frac{2^n}{3+4^n}$

$O(Nr) = 2^n$

$O(Dr) = (2^2)^n = 2^{2n}$

$Ord = 1/2^n$

so let $b_n = 1/2^n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{3+4^n} \cdot 2^n = \lim_{n\to\infty} \frac{4^n}{3+4^n} = 1$$

so by comparision test

$\sum a_n$ converges as $\sum b_n$ converges

Statement:

A series $\sum a_n$ converges absolutely if the corresponding series of absolute value $\sum |a_n|$ converges

Alternating Series

$a_n = (-1)^{n+1} u_n$ or $(-1)^n u_n$ where $u_n \to$ positive number

Alternating harmonic series -(1/geometric)

The series is an alternating geometric series. $r = \left(\frac{1}{2}\right)$

n^{th} term test $\lim_{n \to 0} a_n \not\to 0$ then the series diverges.

The alternating series test

The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 \cdots\cdots$

converges if All of below 3 conditions [free] are satisfied

1. The u_n's are all +ve, decreasing.
2. if $\lim_{n \to \infty} u_n \to 0$, then the alt. series converges
3. $u_n \to 0$.

$\underline{N=1}$ if n is $= 2m$, then Σ of the 1st n terms is: (taking the partial sums)

$eq^n_A \leftarrow$ $S_n = S_{2m} = (u_1 - u_2) + (u_3 - u_4) + \cdots (u_{2m-1} - u_{2m})$

$\downarrow u_2 \geq u_1$ $\quad \downarrow u_3 > u_4$ $\quad \downarrow u_{2m-1} > u_{2m}$

$eq^n_B \leftarrow$ Observe that S_{2m} - sum of non-negative terms

$u_1 - (u_2 - u_3) - (u_4 - u_5) \cdots\cdots - u_{2m}$

eq. a is telling us, S_{2m} is the sum of m non negative terms, hence, $S_{2m+2} \geq S_{2m}$ and the seq. of partial sums.

$\{S_{2m}\}$ is non decreasing.

The second equality shows that

$$S_{2m} \leq u_1$$

$\{S_{2m}\}$ is a non decreasing and bounded series from above by a_1.

↳ It has a limit by Bolzano Weierstass' Theorem

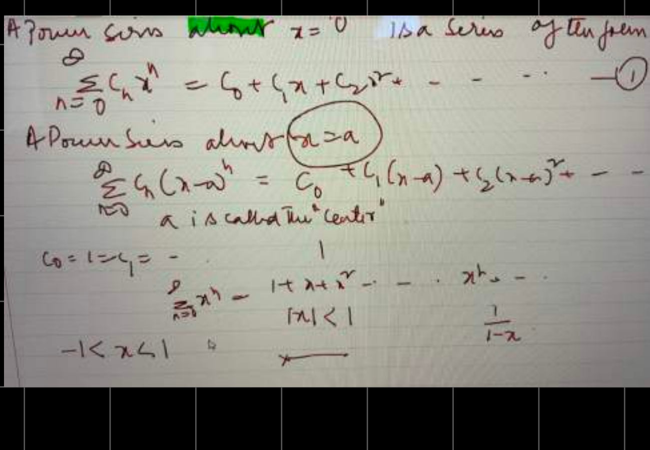
↓

$$\lim_{m \to \infty} S_{2m} = L.$$

$$S_n = S_{2m} = \quad S_{2m+1} \quad = S_{2m} + u_{2n+1} \quad (\because a_n \to 0 \ \ n \to \infty)$$

$$\underbrace{S_{m+1} = S_{2m} + u_{2n+1}}_{(\text{all} \to \infty)} = L + 0.$$

Power series & Convergence.


A Power series about $x = 0$ is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots \quad — (1)$$

A Power Series about $x = a$

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

a is called the "center"

$c_0 = 1 = c_1 = \cdots$

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 \cdots x^n \cdots$$

$|x| < 1$ $\quad \frac{1}{1-x}$

$-1 < x < 1$

$3 \quad 4 \quad 5$ By the Alternating series test

$|x| < 1 \quad \left.\begin{array}{l} -1 < x < 1 \\ \text{at } x = 1 \end{array}\right\}$ The series converges

1. $\exists$ a +ve no R such that the series diverges for $|x-a| > R$

But converges absolutely $|x-a| < R$

The series may or may not converge

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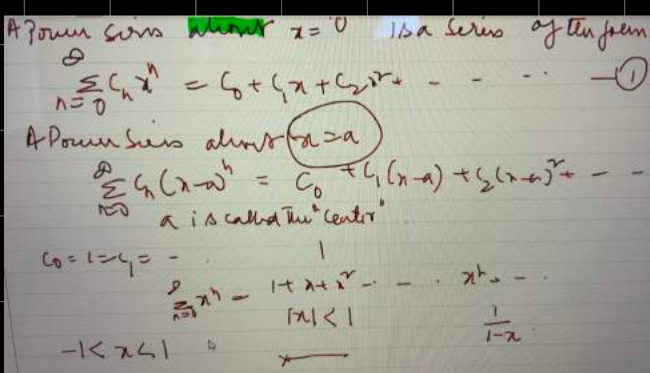
Power series & Convergence.



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$|x| < 1$

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$3 \quad 4 \quad 5$ By the Alternating series test

$$\left.\begin{array}{l} |x| < 1 \quad -1 < x < 1 \\ \text{at } x = 1 \end{array}\right\} \text{The series converges}$$

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But converges absolutely $|x-a| < R$

The series may or may not converge

3 4 5 By the Alternating series test

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