Classical Mechanics

Lagrangian Mechanics

, Lagrangian mechanics is a formulation of classical mechanics that is based on the principle of stationary action and in which energies are used to describe motion. The equations of motion are then obtained by the Euler-Lagrange equation, which is the condition for the action being stationary.

Energy → Motion
↓
Fundamental

- We obtain Euler-Lagrangian equation

WHY? Defines conditions for stationary state

2 Fundamental Concepts of Lagrangian mech

Lagrangian → ≈ Function that describes a particle through KE & PE

Action →
- Used to describe a route through space & time
- Required to be stationary to get correct eqn's
- WHY?

This is called the principle of stationary action.

WHAT?
- Very fundamental in physical
- Called 'least action principle'
- Has to be minimised.

- We shall use a very abstract approach. And, call the 'to be minimised' quantity as 'S'.
- Upon more calculations, we arrive at the Euler-Lagrangian eqn
- We finally arrive at a quantity familiar to us, in terms of KE and PE.

Saddle point: zero derivative point

Lagrangian mechanics is fundamentally an optimization process of the kinetic and potential energies of objects and systems; this is how we predict their motion.

$T = KE$
$V = PE$

This would also make intuitive sense. If we know the kinetic and potential energy (i.e. the value of this Lagrangian function) at each point, we can determine the entire trajectory by simply adding all of them up.

Now, in classical mechanics, energy is a continuous variable, which means that the action wouldn't be a discrete sum, but rather an integral over time (the time it takes from the starting point to the end point of the trajectory).

$$S = \int_{t_1}^{t_2} L(T, V) \cdot dt$$

Principle of stationary action

Based on a lot of evidence, we've seen that physical objects and even fields, will always behave and move in such a way that the action is minimized (or more accurately, stationary).

In the context of Lagrangian mechanics, this means that the trajectory of an object will always be the one in which the action is stationary. This is called the principle of stationary action.

LAGRANGIAN → NO GENERALISED FORM

↓ in classical mechanics: $L = T - V$

WHY NOT $T+V$

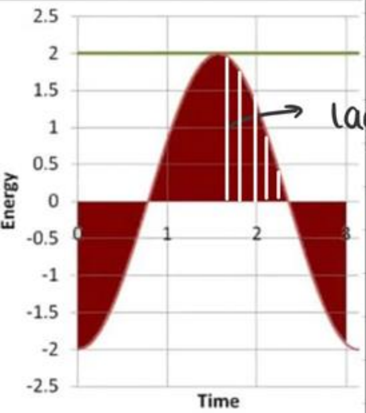
↳ Hamiltonian mechanism

BUT WHY $T-V$

- First, $L \neq$ Total energy
  - $L \rightarrow$ state of motion based on KE & PE

PE ⇄ KE motion: Described by lagrangian

Kind of analogous to $F=ma$



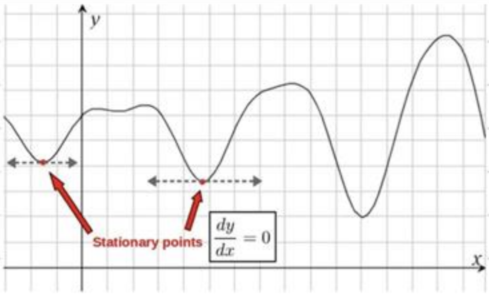
Hamiltonian

Lagrangian

- $T+V \rightarrow$ constant with time
  - Not useful in describing motion
- $T-V \rightarrow$ Not constant
  - Variation thereof can help describe motion

SO... WHY? Because it gives us desirable results

A stationary point for a function is simply a point at which the tangent line is vertical (i.e. the derivative at this point is zero):



$$\delta \int_{t_1}^{t_2} L \cdot dt = 0$$

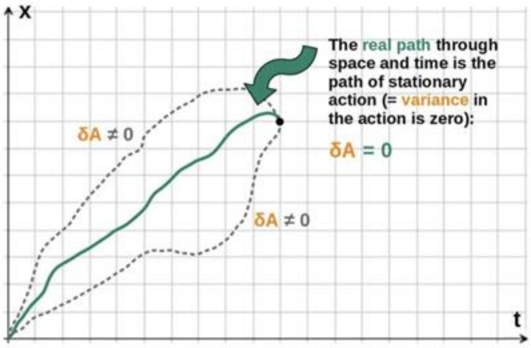
(Mathematical rep. of L.A.P)

$$\approx \delta S = 0$$

↓

Means that a slight variation / Differential in the action should be 0.

A functional differential essentially means varying the value of the action a little (infinitesimal) bit. The stationary points are then those at which a slight variance doesn't actually affect the value of the action.
That is essentially the principle of stationary action explained as simply as possible.



WHY LEAST ACTION PRINCIPLE?

- Physical systems tend to evolve towards equilibrium.
- Trajectory that makes least action stationary ≈ equilibrium state of act^n

↳ In short, everything is governed by it.

The most important eq^n:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x}$$

— END OF THE BEGINNING —

But we really only need q and $\dot{q}$ to describe motion w/ time.

↓

Or just KE & PE

Degree of freedom :

No. of independent quantities we need to define its position.
For an event moving particle across n places, we need to specify Deg $\times$ n values.

The quantities needn't be cartesian coordinates.

Generalised Quantities

- In a system of s d.o.f's, any s quantities that fully determine the position are called the generalised coordinates. (q_i)
- Time derivatives thereof : generalised velocity : $\dot{q}_i$
- If we know $\dot{q}_i$, q_i, we can know $\ddot{q}_i$ (generalised accelerat^n)

The motion can be known if states of the system is known.

→ The relationship b/w q_i, $\dot{q}_i$, $\ddot{q}_i$ are called equations of motion

Only position $\not\rightarrow$ velocity
Only velocity $\not\rightarrow$ acceleration.

In principle, q_i, $\dot{q}_i$, $\ddot{q}$ ... are fully independent.

Hamilton's principle

Nature is usually the most "efficient/lazy"
How do we represent this mathematically?
Lazy: Minimise energy/work.
Let there be a function of q_i's, $\dot{q}_i$'s, t : $L(q, \dot{q}, t)$
This needs to be 'minimised' by the laziness. In general, Magnitude of $L(q, \dot{q}, t)$ can shift wrt. time.

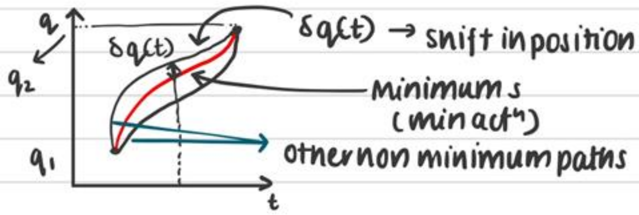
Feynmann's note.

Time integral:

$$\underset{\text{Action}}{S} = \int_{t_1}^{t_2} \underset{\text{Lagrangian}}{L}(q, \dot{q}, t)\, dt \quad \Big\} \; ①$$

Action should be mininum (generality : extremum)

?



(Discussed in the beginning)

Let $q = q(t)$ be the function [path] for which S is min.

at a time 't', if $q(t) \rightarrow q(t) + \delta q(t) \Rightarrow S$ increases.

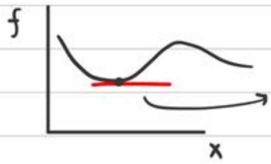
- $\delta q(t)$ is a variation of $q(t)$.
- $\delta q(t_1) = \delta q(t_2)$ [end and starting points of path must be fixed]

$$\Delta S = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + d\dot{q} + t) \cdot dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) \cdot dt \rightarrow ②$$

If S is min for $q = q(t) \Rightarrow \delta S = 0$ → From least action principle.

$$\delta S = \delta \left(\int_{t_1}^{t_2} L(q, \dot{q}, t) \right) = 0$$

Means: If we deviate infinitesimally from this path, change in energy is minimum.
('This' path is an extremum)



$\frac{df}{dx} = 0$ for minimum path.

!! S is a function of L. It isn't really path independent/ work yet

Approximating value of function
[from neighbouring values] } Taylor expansion.

Chain rule of total change

$$\boxed{\delta f(x,y) = \frac{\partial f(x,y)}{\partial x}\cdot \delta x + \frac{\partial f(x,y)}{\partial y}\cdot \delta y}$$

Euler Lagrangian Equation Derivation

$$L(q+\delta q, \dot{q}+\delta\dot{q}, t) = L(q,\dot{q},t) + \boxed{\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}}$$

if q and/or $\dot{q}$ changes infinitesimally, the S is previous S + small correction

$$\int_{t_1}^{t_2} \delta L(q,\dot{q},t)\cdot dt$$

$$\downarrow$$

$$\boxed{\delta S}$$

$$\therefore \delta S = \int_{t_1}^{t_2}\left\{\overbrace{\frac{\partial L}{\partial q}\delta q(t)}^{①} + \overbrace{\frac{\partial L}{\partial \dot{q}}\delta\dot{q}(t)}^{②}\right\} dt = 0$$

We only evaluate the second term, keeping the first as is.

$$\Rightarrow \delta\dot{q}(t) = \frac{\partial L}{\partial \dot{q}}\cdot\frac{d}{dt}\delta x$$

$$\Rightarrow \frac{\partial L}{\partial \dot{q}}\delta\dot{q}(t) = \frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

$$\left(\frac{d}{dx}(fg) = g\frac{d}{dx}f + f\frac{d}{dx}g\right)$$

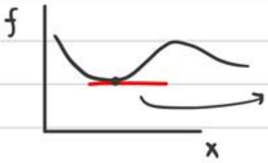
$$\Rightarrow \int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}}\delta\dot{q}(t)\cdot dt = \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q\right)dt - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\, dt$$

(integrating previous exp^n)

second term

$q = q(t)$ ⟵

$$= \left(\cancel{\frac{\partial L}{\partial \dot{q}}\,\delta q(t)}\right)\Bigg|_{t_1}^{t_2} \;\; \boxed{-\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt}$$



This term evaluates to 0 since the values $\delta q(t_1)$ and $\delta q(t_2)$ both tend to 0.
This equals 0 because the variance in position at start, end = 0
(t_1) (t_2)

$$\therefore \; \delta S = \int_{t_1}^{t_2} \left\{ \frac{\partial L}{\partial q}\,\delta q(t) + \frac{\partial L}{\partial \dot{q}}\,\delta \dot{q}(t) \right\} dt = 0 \quad — (4)$$

(Our derivation till now)

$$= \int_{t_1}^{t_2} \delta q(t) \underbrace{\left\{ \frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \right\}} dt = 0$$

Must vanish, = 0
(To make it work for all cases)

This is true for any variation $\delta q(t)$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q}$$

(Arbitrary over multiples of L, all αL work)

Lagrangian eq^n

If the system has more than one d.o.f's

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q_i} = 0 \quad \text{for all } i = 1, 2 \ldots s$$

s ⟶ d.o.f

⟶ Lagrange's equations

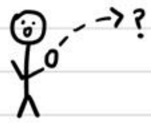
⟶ Give eq^n of motion.

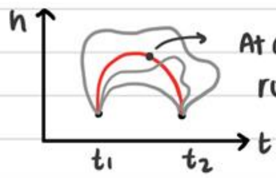
Ordinarily, one can use newton's second law to to figure path of particle.

$\vec{F} = m\vec{a}$

↳ If we know F, we can find $\vec{a}$, using which we calculate position in time

But we can use principle of least action too.





At any point, rock has a specific K.E and P.E.

At time t:

$T(t)$ ↳ KE $\frac{1}{2}mv^2$

$U(t)$ ↳ PE mgh

Action for any point:
$$\sum_{t=i}^{t=T} KE_i - PE_i$$

"Action"
$$S = \int_{t_1}^{t_2} T - U \, dt$$

It turns out, the path chosen by the particle is the one that minimises this. On such a path:

T ↓ small

U ↓ large

Well behaved function for physics - smooth, unfluctuating graph

Observations:

① For 2 non interacting systems (or if negligible), Lagrangian for joint system =

$$L_{A+B} = L_A + L_B \quad \text{(linearly additive)} \longrightarrow ⑥$$

In general, all lagrangians can be determined upto a multiplicative const.
BUT
Additivity indicates: ALL lagrangian parts must be scaled w/ the same constant (A choice of unit for Lagrangians)

- We can added a total time derivative quantity which is differentiable.
- It must vary only on q and atmost t!
- $L \to L' = L + \frac{dF(q,t)}{dt}$

② A lagrangian for a system remains valid upon addition of a complete derivative of a term in the eq^n. This changes the act^n, but doesn't affect the path, may only affect the ends. EVERYTHING shifts by constant, no effect.
- You can add ANY Constant and equation remains the same.
- We get same eq^n's of motion.

Lagrangian must be scaled by same const. ∀ t.

(L) + C ⟹ Another valid (L)
[Lagrangian]
Can be written as absolute derivative of a term × constant

Mechanics in an Inertial Frame

⇓

Any q, t dependent term, like the time der. quantity added, is removed. Hence has no effect on the final equation.

Space in itself is uniform, not warped, symmetric

Frame of reference can be chosen ALWAYS such that space → homogenous, isotopic, time → homogenious.

Example: Spaceship where you feel no pseudo force is an inertial frame.

Homogenity of space: Space looks same from no matter which direction.

WHY we can add a total time derivative q, proof

of the system in question. Moreover, L and L' differ by $\dot{F} \equiv \frac{dF(q,t)}{dt}$. This allows us to calculate:

$$S[q(t)] = \int_{t_i}^{t_f} dt\, L\Big(q(t), \dot{q}(t), t\Big)$$

$$\begin{aligned}
S'[q(t)] &= \int_{t_i}^{t_f} dt\, L'\Big(q(t), \dot{q}(t), t\Big) \\
&= \int_{t_i}^{t_f} dt\, \Big(L\Big(q(t), \dot{q}(t), t\Big) + \dot{F}\Big(q(t), t\Big)\Big) \\
&= \Big(\int_{t_i}^{t_f} dt\, L\Big) + F\Big(q(t_f), t_f\Big) - F\Big(q(t_i), t_i\Big) \\
&= S + F\Big(q(t_f), t_f\Big) - F\Big(q(t_i), t_i\Big).
\end{aligned} \quad (7.40)$$

(More proof in 'No-nonsense Classical Mech - Jakob .... pg. 143 of pdf)

$$\int_a^b dx \frac{df(x)}{dx} = f(b) - f(a)$$

A lagrangian should not be affected explicitly by space (position vector) and neither time.

eg ⇒ same lagrangian in US & India, 1890 and 2021

If object is not in any external force field.

⇓ Definition of a 'Free Particle'

We can also use $x_1\hat{x} + x_2\hat{x} + x_3\hat{x}$ instead of $x\hat{i} + y\hat{j} + z\hat{k}$

- If we want to find the lagrangian of a miniscule particle (This mostly applies to miniscule particles)

⇒ The homogenity of space & time imply L cannot be a function of $\vec{r}$ or $\vec{t}$.

∴ It ONLY depends on velocity.

ALSO, the isotopy of space ⟹ it can only depend on magnitude, $\vec{v}^2 = v^2$ (v^2, ∵ magnitude, observed dependence). even $|v|$ works.

WHY only MAGNITUDE? If direction is needed, isotropy is broken.

$L \cancel{\propto} v^2 \implies \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$ would depend on something other than $\ddot{q}$.

[v^2 is initially only used for convenience

NOTE: For n dimensional spaces: $v^2 = \Sigma v_i^2$

$L = L(v^2) \cdot \Sigma L(v_i^2)$

$\implies \frac{\partial L}{\partial x_i} = 0$ [lagrangian doesn't vary over position in any axis]

- we use $\vec{r} = x_1\hat{x}_1 + \cdots\cdots x_n\hat{x}_n$ instead of $\vec{x}, \vec{y}, \vec{z}$.

$\therefore \frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0 \Rightarrow \frac{\partial L}{\partial v_i}$ = constant value.

indicates inertial frame

we don't even know about L, but we see the analogue b/w this abstract notion & newtonian units s.t. this quantity only varies w/ v, and this variation is constant.

we consider another frame of reference which has a SMALL relative velocity wrt. the earlier frame.

$$\vec{V} = \vec{v} + \vec{\epsilon} = \vec{v}'$$

$\because$ eq^n of motion will have the same form in every frame, L & L' must differ (if at all) by ONLY a time derivative (total not partial $\Rightarrow \frac{d}{dt}$ not $\frac{\partial}{\partial t}$)

?

$L' = L(v'^2) = L(v^2 + 2\vec{v}\cdot\vec{\epsilon} + \epsilon^2)$ $\quad [v' = \vec{v} + \vec{\epsilon}]$

if $\vec{\epsilon}$ is very small, $L' = L(v^2) + \frac{\partial L}{\partial v^2}(2\vec{v}\cdot\vec{\epsilon}) + \underbrace{\cdots\cdots}$

Must be a total time derivative

Taylor expansion.

$f(x+\Delta x) = f(x) + \frac{\partial f}{\partial x}\Delta x \cdots$

$L(v^2) \propto v^2$. The constant = $1/2\, m$ (could be anything really)

WHY 1/2? Because then $L = 1/2 mv^2$. We know of this quantity. we can now say this L for a free particle = KE, and it tries to minimise this quantity.

Another case: If frame of references differ by a non infinitesimal quantity ?

→ Same calc as before, don't ignore ϵ term.

$$L' = 1/2\, m v'^2 = 1/2\, m(\vec{v} + \vec{V})^2 = \frac{1}{2} m v^2 + m\vec{v}\cdot\vec{V} + \frac{1}{2} m V^2$$

$$L = \frac{d}{dt}\left(m\vec{r}\vec{V} + \frac{1}{2} m V^2 t\right)$$

↳ Total time derivative

For SOP that do not interact:

$$L = \Sigma\, 1/2\, m_a v_a^2$$

(implies)

m cannot be (−)ve.

WHY? ↳ ∵ $S = \int_{t_1}^{t_2} 1/2\, m v^2 \cdot dt$

cannot have a defined finite negative

?: Any expression f(x) can be written as a time derivative of an expression q(x), which is a time integral of q(x).

→ So, can we add any such expression that can depend on time, not v, but L cannot depend on t. (You would violate addition of a constant quantity)

Q Represent Lagrangian in other coordinate systems (cylindrical and spherical)?

Cartesian: $1/2\, m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

Lagrangian for a closed system of particles

Consider a SOP interacting w/ one other BUT w/ other external matter.

How will L change from $\sum \frac{1}{2} m_a v_a^2$?

- Add a term (f^n) that would depend on the nature of interaction.
- We could also represent the other quantity

$$L = \frac{1}{2}\sum_a m_a(\underbrace{\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2}_{KE}) - \underbrace{U(\vec{r})}_{PE} \longrightarrow (13)$$

In general, $L = \frac{1}{2}\sum_{i,j} a_{ik}(q)\dot{q}_i\dot{q}_k - U(q) \longrightarrow (14)$

$$(1) = \frac{d}{dt}\frac{1}{2}m_a v_a^2$$

$$= m_a \cdot v \cdot a$$

$$= m_a a$$

(nothing to do w/ position)

No dependence on position, hence derivate only second term wrt. X.

$\Downarrow$

From here, we get from lagrangian $\Rightarrow$

$$\boxed{m_a \frac{dq_i^a}{dt} = -\frac{\partial U}{\partial x_i^a}}$$

Conservation Laws

Integrals of motion: In a system with s d.o.f., 2s quantities vary with time: $q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s$

There are functions of these 2s quantities which remain constant

[General solution of s e.o.m. will have 2s arbitrary constants. One can ~~all~~ always be eliminated using the arbitrary starting time t_0.]

Some of these integrals have profound significance.

Let us first investigate $\boxed{\text{homogeneity of time}}$. The Lagrangian of a closed system does not have an explicit dependence on time.

$$L = L(q_i, \dot{q}_i)$$

$$\frac{dL}{dt} = \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i$$

$$= 0$$

[Chain rule: $f = f(x, y, z)$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Just like: $g = g(x)$

$$\frac{dg}{dt} = \frac{dg}{dx}\frac{dx}{dt}$$

From Lagrange eqns: $\frac{\partial L}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)$

$$\therefore \frac{dL}{dt} = \sum_i \left[\dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i\right]$$

$$= \sum_i \frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right)$$

$$\Rightarrow \frac{d}{dt}\left[\sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L\right] = 0$$

-Doubt

Remember

$\dot{q}_i = \frac{dq_i}{dt}$

$\ddot{q}_i = \frac{d^2 q_i}{dt^2} = \frac{d}{dt}\dot{q}_i$

$$\therefore \sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L = \text{Constant}. \quad \text{— (15)}$$

To recognise the constant consider a system of ~~free~~ particles (in one dimension) in a constant potential U

$$L = \frac{1}{2}m\left(\dot{x}_1^2 + \dot{x}_2^2 + \cdots\right) - U$$

$$\therefore \sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L = \sum_i \left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) - \frac{1}{2}m\sum_i \dot{x}_i^2 + U$$

$$= \frac{1}{2}m\sum_i \dot{x}_i^2 + U = \text{Total energy}.$$

mv^2
(as in pg. 14 of this pdf)

The lagrangian of a single particle moving in an external field:

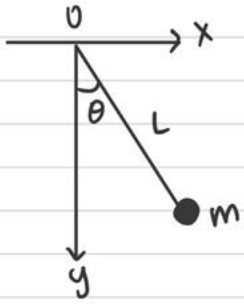
$$L = \tfrac{1}{2} m v^2 - U(\vec{r}, t)$$

$$\text{e.o.m} = m\dot{\vec{v}} = -\frac{\partial U}{\partial \vec{r}} = -\nabla U \iff U = -\vec{F}\cdot\vec{r}$$

(iff F = uniform)

Grad operator

Example



let generalised coord $= \theta$

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m L^2 \dot{\theta}^2$$

$$U = -mgL\cos\theta$$

$$L = \frac{1}{2} m L^2 \dot{\theta}^2 + mgL\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(mL^2\dot{\theta}\right), \quad \frac{\partial L}{\partial \theta} =$$

(PE) (KE)

(The above is an equation we have for SHM)

Some quantities, like abs. position of things, we can consider that a constant.
For eg. in class eg. $U(q)$ is a constant w.r.t. $\dot{q}$.
eg. if we keep x constant, U will not change ∵ of it, perhaps other forces can affect it.

One needs to determine region of 0 potential, rest is a constant additive.

Remaining notes in slides

Eq^n 15:

$L = L(q, \dot{q})$

$\frac{dL}{dt} = \sum \frac{\partial L}{\partial q_i} \dot{q}_i + \sum \frac{\partial L}{\partial \dot{q}_i} \ddot{q}$

$\hookrightarrow = 0$ (Homogenity)

(15) $\rightarrow$

$\sum_i \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i}\right) - L = \text{Const.}$

(17)

So we can rewrite eq^n (15) as

Homogeneity of time

$$\boxed{E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = \text{constant}} \quad - (16)$$

Wait! We derived the equation for closed systems but considered a potential term that is constant!

This is okay as the derivation remains valid as long as the potential does not time explicitly.

In general: $L = \underbrace{T(q, \dot{q})}_{\text{quadratic function of the velocities}} - U(q)$

Euler's theorem of homogeneous functions gives us

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T$$

$$\therefore E = 2T - T + U = T + U$$

(18)

What about homogeneity of space?

To avoid unnecessary complictions we restrict ourselves to 1D.
[It is easy to generalize for 3D if you have done your homework on vector calculus. ☺]

$$L(x, \dot{x}) \equiv L(x, v)$$

If we move the entire system by ϵ i.e., $x_a \rightarrow x_a + \epsilon$

$$\delta L = \sum_{a=1}^{N} \frac{\partial L}{\partial x_a} \delta x_a = \epsilon \sum_{a=1}^{N} \frac{\partial L}{\partial x_a} = 0$$ [a denotes the particle index, $a = 1, 2, \ldots, N$]

$$\Rightarrow \boxed{\sum \frac{\partial L}{\partial x_a} = 0}$$ This we already know

$\rightarrow$ shows homogenity of space

From Lagrange's eq^{n}s we get,

$$\sum_{a=1}^{N} \frac{d}{dt}\left(\frac{\partial L}{\partial v_a}\right) = \frac{d}{dt}\left(\sum_{a=1}^{N} \frac{\partial L}{\partial v_a}\right) = 0$$

Central force motion

$E = \frac{1}{2}m\dot{x}^2 - U(x)$

$E = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - U(x)$

$\frac{2(E+U(x))}{m} = \left(\frac{dx}{dt}\right)^2$

$\sqrt{\frac{2}{m}(E+U(x))} = \frac{dx}{dt}.$

$\int dt = \int\sqrt{\frac{m}{2(E+U(x))}} + C$

Lagrangian for a motion with d.o.f. = 1

$$L = \frac{1}{2}a(q)\dot{q}^2 - U(q) \quad \text{(General)}$$

$$= \frac{1}{2}m\dot{x}^2 - U(x) \quad \text{(Cartesian)}$$

The first integral of motion $E = \frac{1}{2}m\dot{x}^2 + U(x)$

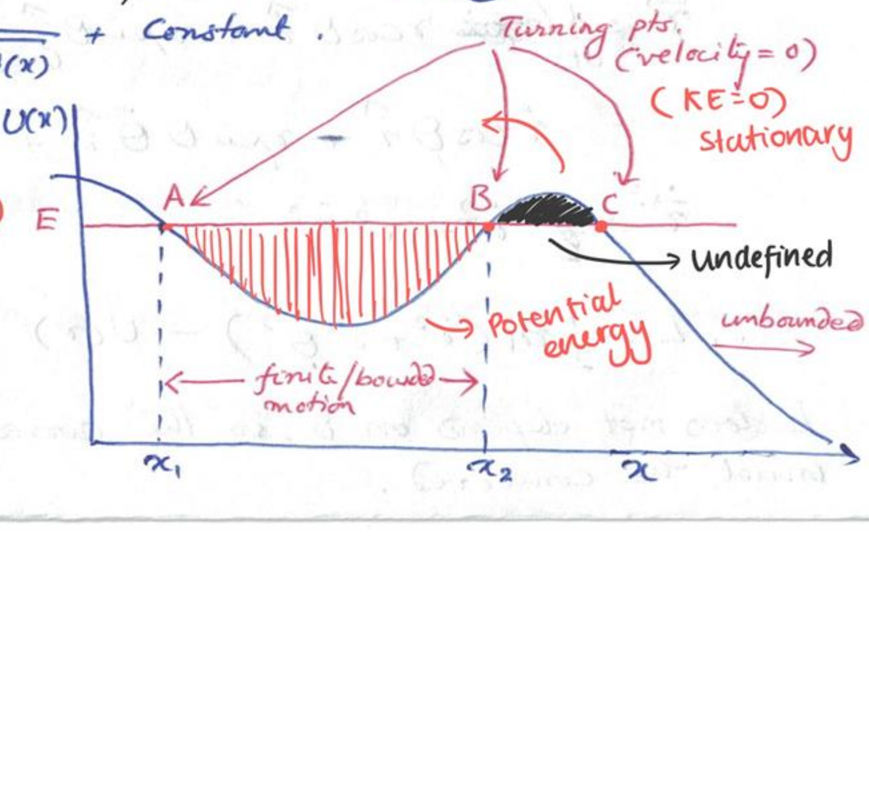
If we know E & $U(x)$, we can figure out how much time is needed for the system (particle) to cover certain distance

$$t = \sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{E-U(x)}} + \text{Constant}. \qquad (25)$$

For motion $U(x) < E$

If A finite motion in 1D is oscillatory WHY?

$$T(E) = \sqrt{2m}\int_{x_1(E)}^{x_2(E)}\frac{dx}{\sqrt{E-U(x)}} \qquad (26)$$

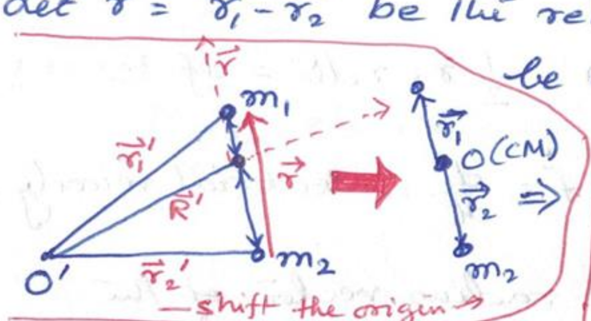


Two-body Problem

$$\left[\vec{r}^{\,2} = \vec{r}\cdot\vec{r} = r^2\right]$$

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U\left(|\vec{r}_1 - \vec{r}_2|\right)$$

Let $\vec{r} = \vec{r}_1 - \vec{r}_2$ be the relative position vector and let the origin be at the centre of mass,



$$\boxed{\begin{aligned} &\Rightarrow m_1 \vec{r}_1 + m_2 \vec{r}_2 = 0 \quad [R = 0] \\ &\Rightarrow \vec{r}_1 = \frac{m_2 \vec{r}}{m_1 + m_2}, \quad \vec{r}_2 = \frac{-m_1 \vec{r}}{m_1 + m_2} \quad \text{— (27)} \end{aligned}}$$

We can rewrite the Lagrangian in terms of $\vec{r}$, i.e. $\{\vec{r}_1, \vec{r}_2\} \to \{\vec{r}, \underset{\substack{\| \\ 0}}{\vec{R}}\}$

$$L = \frac{1}{2} m_1 \frac{m_2^2 \dot{\vec{r}}^{\,2}}{(m_1+m_2)^2} + \frac{1}{2} m_2 \frac{m_1^2 \dot{\vec{r}}^{\,2}}{(m_1+m_2)^2} - U(r) \quad \text{— (28)}$$

$$= \frac{1}{2} m \dot{r}^2 - U(r)$$ where $m = \frac{m_1 m_2}{m_1 + m_2}$ is called the reduced mass.

So we have replaced the two-body problem by a formally identical problem of a particle of mass m moving in a central field $U(r)$.

Motion in a central field

A particle moves under: $\vec{F} = -\bar{\nabla} U = -\frac{\partial U}{\partial \vec{r}} = -\frac{dU}{dr}\hat{r}$ — (29)

Rotational symmetry about the force centre (origin)

$\Rightarrow \vec{M} = \vec{r} \times \vec{p}$ is conserved.

Since $\vec{r}$ is $\perp$ to $\vec{M}$ $\Rightarrow$ the motion happens in a plane $\perp$ to $\vec{M}$.

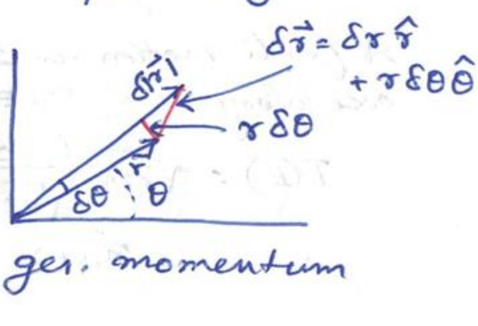
So, a priori, we need two co-ordinates to describe the motion.

$$\vec{r}^{\,2} = x^2 + y^2 = r^2 \sin^2\theta + r^2\cos^2\theta$$

$$\dot{\vec{r}} = \frac{d}{dt}\left(r\cos\theta\,\hat{i} + r\sin\theta\,\hat{j} \right)$$

$$= \dot{r}\cos\theta\,\hat{i} - r\sin\theta\,\dot{\theta}\,\hat{i} + \dot{r}\sin\theta\,\hat{j} + r\cos\theta\,\dot{\theta}\,\hat{j}$$

$$\dot{\vec{r}}^{\,2} = \dot{r}^2 + r^2\dot{\theta}^2$$

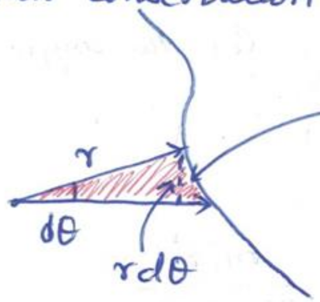


$$\therefore L = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - U(r) \quad \text{— (30)}$$

L does not depend on θ, so the corresponding gen. momentum must be conserved.

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} = M = \text{constant}. \quad — (31)$$

This conservation law has a simple geometric interpretation. $\sin d\theta = d\theta$.



→ area of the red region $= \frac{1}{2} r . r d\theta = df$

$\therefore \; M = 2m\dot{f} \qquad \dot{f} = \frac{df}{dt} =$ sectorial velocity. — (32)

⇒ In equal times the radius vector of the particle sweeps out equal area (Kepler's 2nd law)

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + U(r) \qquad \frac{M}{mr^2} = \dot{\theta}$$

$$= \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\frac{M^2}{mr^2} + U(r) \leftarrow \text{One dof problem}$$

$$\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{m}\left(E - U(r)\right) - \frac{M^2}{m^2r^2}} \Rightarrow t = \int \frac{dr}{\sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}}} + \text{Const}.$$

Generally in this type of problems one does not need to solve e.o.m. (29)

But we can get them.

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow \frac{d}{dt}(m\dot{r}) = m r\dot{\theta}^2 - \frac{dU}{dr}$$

$U = U(r)$

$\vec{F}(r) = -\frac{dU}{dr}\hat{r}$

$$\Rightarrow m\left(\ddot{r} - r\dot{\theta}^2\right) = F(r) \quad \text{— (33)}$$

Change of variable: $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{1}{r^2}\dot{r}/\dot{\theta}$

$\dot{\theta} = \frac{M}{mr^2}$

$$= \frac{-1}{r^2}\frac{\dot{r}}{M/mr^2} = -\frac{\dot{r}m}{M}$$

$$\frac{d^2u}{d\theta^2} = \frac{d}{d\theta}\left(-\frac{m}{M}\dot{r}\right) = \frac{1}{\dot{\theta}}\frac{d}{dt}\left(-\frac{m}{M}\dot{r}\right)$$

$$= \frac{1}{M/mr^2}\cdot\left(-\frac{m}{M}\right)\ddot{r} = -\frac{m^2}{M^2}r^2\ddot{r}$$

$$\text{or } \ddot{r} = -\frac{M^2}{m^2}u^2\frac{d^2u}{d\theta^2}$$

We can rewrite eqn (33) as

$$m\left(-\frac{M^2}{m^2}u^2\frac{d^2u}{d\theta^2} - r\frac{M^2}{m^2r^4}\right) = F\left(\frac{1}{u}\right)$$

or $$\frac{M^2}{m}\left(u^2\frac{d^2u}{d\theta^2} + u^3\right) + F\left(\frac{1}{u}\right) = 0$$

or $$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{m}{M^2}\frac{1}{u^2}F\left(\frac{1}{u}\right)}$$ — (34)

or $$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = -\frac{mr^2}{M^2}F(r)}$$

The θ eqn will give us the conservation of M.

The e.o.m. lets us find out the nature of the force from the orbit.

For example, a particle is moving in a logarithmic spiral orbit $r = ke^{\alpha\theta}$ where k and α are constants (Example 8.1 of M&T)

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{\alpha}{k}e^{-\alpha\theta}, \quad \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = \frac{\alpha^2 e^{-\alpha\theta}}{k} = \frac{\alpha^2}{r}$$

$$\therefore F(r) = \frac{-M^2}{mr^2}\left(\frac{\alpha^2}{r} + \frac{1}{r}\right) = -\frac{M}{mr^3}(\alpha^2+1)$$

$$\boxed{F \propto \frac{1}{r^3} \text{ \& attractive}}$$

Coming back to our original problem of finding the trajectories we recall

$$\left.\begin{aligned} \dot{r} &= \sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}} \\ t &= \int \frac{dr}{\sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}}} \end{aligned}\right\} \quad \text{——— } (35)$$

Since $r^2\dot{\theta}$ is constant we can trade t with θ

$$\frac{dr}{dt} = \frac{dr}{d\theta}\dot{\theta} = \frac{dr}{d\theta}\frac{M}{mr^2}$$

$$\therefore \frac{dr}{d\theta} = \frac{mr^2}{M}\sqrt{\frac{2}{m}(E-U) - \frac{M^2}{m^2r^2}}$$

or
$$\boxed{\theta(r) = \int \frac{\frac{M}{r^2}\,dr}{\sqrt{2m\left(E-U-\frac{M^2}{2mr^2}\right)}} + \text{Const.}} \quad -(36)$$

Since $\dot{\theta}$ cannot change sign, $\theta(r)$ increases or decreases monotonically.

centrifugal energy.

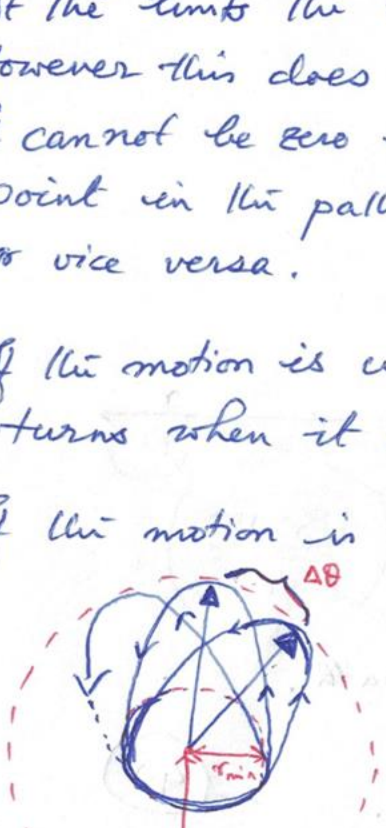
$$\boxed{U_{eff}(r) = U(r) + \frac{M^2}{2mr^2}} \quad -(37)$$

The values of r for which $U_{eff}(r) = E$, determines the limits of the motion.

At the limits the radial velocity $\dot{r}$ becomes zero (see eq. 35). However this does not mean the particle stops at those points as $\dot{\theta}$ cannot be zero if M is not. Hence, $\dot{r}$ indicates a turning point in the path where $r(t)$ begins to increase instead decreasing or vice versa.

If the motion is unbound, $r \geqslant$ [redacted] r_{min}, then the particle turns when it reaches r_{min}.

If the motion is bound, $r_{min} \leq r \leq r_{max}$, the particle moves in a region bounded by two circles [redacted] [redacted] $r = r_{min}$ & $r = r_{max}$. The path may or may not be a closed one.



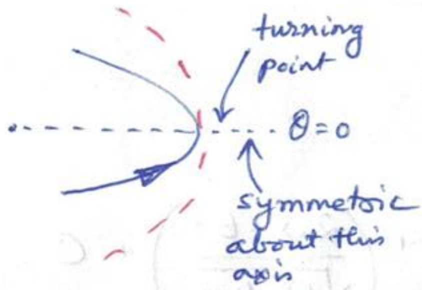
$$\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{M\,dr/r^2}{\sqrt{2m(E - U_{eff})}} \quad \text{— (38)}$$

If $\Delta\theta$ is a rational fraction of 2π, i.e, of the form $\Delta\theta = 2\pi\frac{m}{n}$ then the path is closed

→ rational ∴ the same path will come again

There are only two types of central fields in which all finite (bound) motions take closed paths: $U \propto \frac{1}{r}$ or r^2.

$\frac{1}{r^2}$



At the turning points the factor $\pm\sqrt{2m(E-U_{eff})}$ just changes sign.

> Homework: What are the conditions for which a particle fall into the force centre?

Kepler's problem

$$U(r) = -\frac{\alpha}{r}$$

$$\therefore\; U_{eff} = -\frac{\alpha}{r} + \frac{M^2}{2mr^2}$$

39

$$\frac{dU_{eff}}{dr} = +\frac{\alpha}{r_m^2} - \frac{M^2}{mr_m^3} = 0 \Rightarrow r_{min} = \frac{M^2}{m\alpha}$$

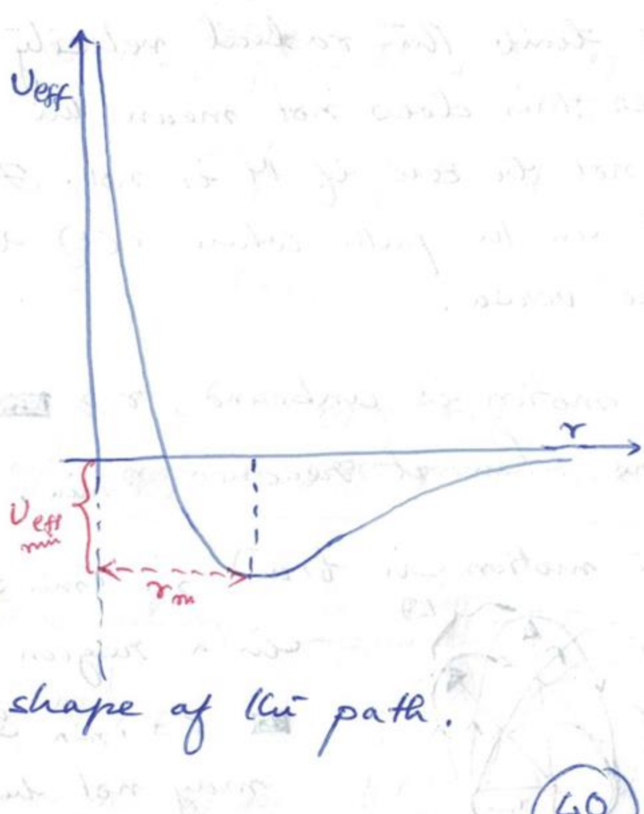
$$U_{eff}(r_m) = -\frac{m\alpha^2}{2M^2}$$

As $r \to 0$, $U_{eff} \to \infty$

As $r \to \infty$, $U_{eff} \to 0$ from the -ve side.

Motion is finite for $E < 0$

Motion is infinite for $E > 0$.



Eq. 36 would give us the shape of the path.

$$\theta(r) = \int \frac{\frac{M}{r^2}\,dr}{\sqrt{2m\left(E + \frac{\alpha}{r} - \frac{M^2}{2mr^2}\right)}} + \text{Const.} \qquad (40)$$

We can now perform the integration.

$$\theta(r) = -\int \frac{M\,du}{\sqrt{2mE + 2m\alpha u - M^2u^2}} + \text{Const.}$$

$$= -\int \frac{du}{\sqrt{\frac{2mE}{M^2} + \frac{2m\alpha}{M^2}u - u^2}} + \text{Const.}$$

Standard formula : $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}} = \frac{-1}{\sqrt{-a}} \sin^{-1}\left(\frac{2ax+b}{\sqrt{b^2-4ac}}\right) + C$

when $a < 0$.

$$\therefore\ \theta(r) = \sin^{-1}\left[\frac{-2u + \frac{2m\alpha}{M^2}}{\sqrt{\frac{4m^2\alpha^2}{M^4} + \frac{8mE}{M^2}}}\right] + \text{Const.}$$

$$\sin(\theta + \theta_0) = \frac{\frac{M^2}{m\alpha}\frac{1}{r} - 1}{\sqrt{1 + \frac{2EM^2}{m\alpha^2}}}$$

Let $\theta_0 = \pi/2$. We get

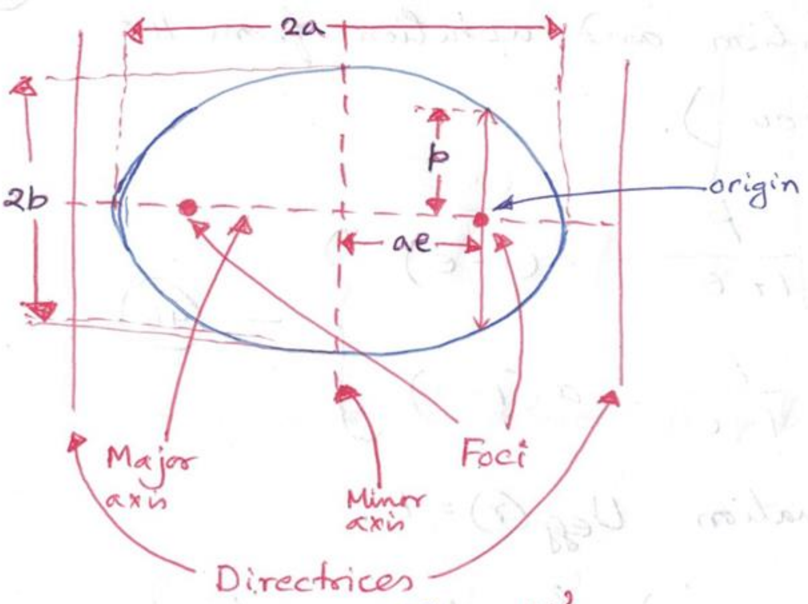
$$\boxed{\cos\theta = \frac{\frac{M^2}{m\alpha}\frac{1}{r} - 1}{\sqrt{1 + \frac{2EM^2}{m\alpha^2}}}} \quad \text{— (42)}$$

We can recognise the relation by putting

$$\boxed{p = \frac{M^2}{m\alpha}} \quad \& \quad \boxed{e = \sqrt{1 + \frac{2EM^2}{m\alpha^2}}} \quad \text{— (43)}$$

$$\therefore \quad \cos\theta = \frac{\frac{p}{r} - 1}{e} \quad \Rightarrow \quad \boxed{\frac{p}{r} = 1 + e\cos\theta} \quad \text{— (44)}$$

This is the equation of a conic section with one focus at the origin. $2p$ is called the latus rectum of the orbit and e is the eccentricity.



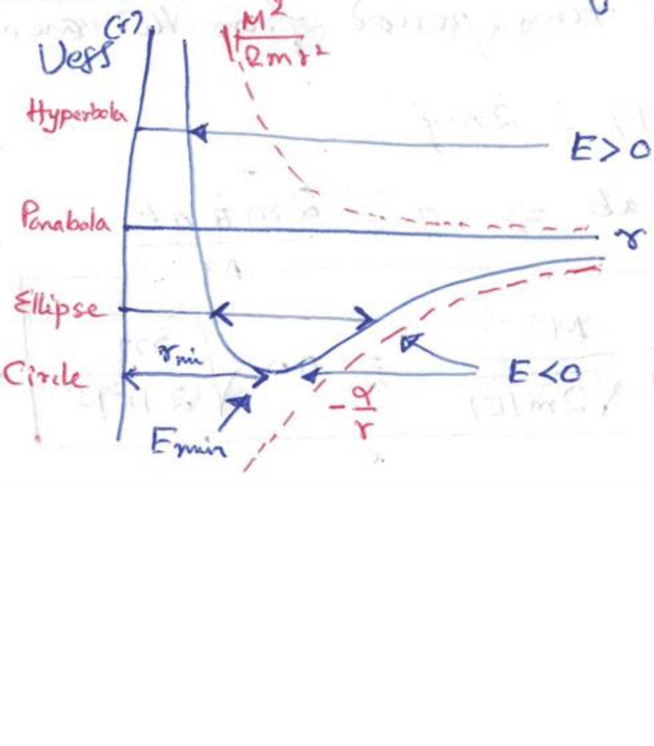
If the energy of the particle $E < 0$ then $e < 1$, the orbit becomes an ellipse.

If $E = 0$, $e = 1 \Rightarrow$ Parabola

If $E > 0$, $e > 1 \Rightarrow$ Hyperbola

If $E = -\dfrac{m\alpha^2}{2M^2}$, $e = 0$

$\Rightarrow$ circle.



But we are in the equivalent-problem. What happens to the two ~~bodies~~ bodies? What kind of motion do they execute?

$$r_1 = \frac{m_2 r}{m_1+m_2} \quad , \quad r_2 = - \frac{m_1 r}{m_1+m_2}$$

They also move in conic sections — they keep the centre of mass as one of the foci.

For elliptical orbits one can get the semi-major and semi minor axes as follows.

$$\left.\boxed{\begin{aligned} a &= \frac{p}{1-e^2} = \frac{\alpha}{2|E|} \\ b &= \frac{p}{\sqrt{1-e^2}} = \frac{M}{\sqrt{2m|E|}} \end{aligned}}\right\} \quad -(45)$$

We can also find the perihelion and aphelion from the centre of the force (i.e. a focus).

$$\left.\begin{aligned} r_{min}\,(\text{perihelion}) &= \frac{p}{1+e} = a(1-e) \\ r_{max}\,(\text{aphelion}) &= \frac{p}{1-e} = a(1+e) \end{aligned}\right\} \quad — \quad (46)$$

These are the roots of the equation $U_{eff}(r) = E$.

We can also find the time period from the area law.

$$M = 2m\dot{f} \Rightarrow MT = 2mf$$

For an ellipse $f = \pi ab \Rightarrow T = \dfrac{2m\pi ab}{M}$

or
$$\boxed{T = \frac{\cancel{2}m\pi}{\cancel{M}} \frac{\alpha}{\cancel{2}|E|} \frac{\cancel{M}}{\sqrt{2m|E|}} = \pi\alpha\sqrt{\frac{m}{2|E|^3}}} \quad — \quad (47)$$

We can consider some more examples. We can determine the time period of a pendulum from its energy.

$$E = \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\cos\theta = -mgl\cos\theta_0 \quad \text{where } \theta_0 = \theta_{max}$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{l}(\cos\theta - \cos\theta_0)$$

$$\therefore \; T = 4\sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} \quad \text{--- (48)}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2}}}$$

Let $\sin\alpha = \frac{\sin\theta/2}{\sin\theta_0/2}$ or $\cos\alpha \, d\alpha = \frac{1}{\sin\theta_0/2} \cos\theta/2 \cdot \frac{d\theta}{2}$

or $\sqrt{1 - \frac{\sin^2\theta/2}{\sin^2\theta_0/2}} \, d\alpha = \frac{1}{\sin\theta_0/2} \cos\theta/2 \, \frac{d\theta}{2}$.

or
$$\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2} - \sin^2\theta/_2}} = \frac{2\,d\alpha}{\cos\theta/_2} = \frac{2\,d\alpha}{\sqrt{1-\sin^2\theta/_2}}$$

$$= \frac{2\,d\alpha}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2\alpha}}$$

limits : $\theta = 0 \rightarrow \alpha = 0$

$\theta = \theta_0 \rightarrow \alpha = \pi/2$

$$\therefore T = 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-\sin^2\alpha\sin^2\frac{\theta_0}{2}}} \quad — \quad (49)$$

Integrals of the form $\int_0^{\pi/2}\frac{d\alpha}{\sqrt{1-\beta^2\sin^2\alpha}} = I(\beta)$ is known as the complete elliptic integral of the first kind.

This integral is known.

$$I(\beta) = \frac{\pi}{2}\left(1 + \left(\frac{1}{2}\right)^2 \beta^2 + \left(\frac{1.3}{2.4}\right)^2 \beta^4 + \dots\right) \text{ if } |\beta| < 1. \quad — (50)$$

In our case $\beta = \sin \theta_0/2$

$$\therefore T = 4\sqrt{\frac{l}{g}}\left(1 + \left(\frac{1}{2}\right)^2 \sin^2 \theta_0/_2 + \left(\frac{1 \times 3}{2 \times 4}\right)^2 \sin^4 \frac{\theta_0}{2} + \dots\right)$$

If θ_0 is small

$$\sin\frac{\theta_0}{2} = \frac{\theta_0}{2} - \frac{\theta_0^3}{2^3 3!} + \frac{\theta_0^5}{2^5 5!} - \dots$$

If we want just the lowest order correction

$$\boxed{T = 4\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16} + \dots\right)} \quad — \quad (51)$$

We saw that the motion of two bodies interacting only with each other by central forces could be reduced to an equivalent one-body problem. Show that such a reduction is also possible for bodies moving in an external uniform gravitational field (Prob. 8.1 of M&T).

Let the gravitational field act along the z direction.
In this case the potential term

$$U + U_g^{(1)} + U_g^{(2)} = U(r) \cancel{+} - m_1 g z_1 - m_2 g z_2$$

$$\therefore L = \frac{1}{2} m \dot{r}^2 - U(r) + m_1 g\left[\frac{m_2}{m_1+m_2} z + \vec{R}\cdot\hat{k}\right] + m_2 g\left[-\frac{m_1}{m_1+m_2} z + \vec{R}\cdot\hat{k}\right] + \frac{1}{2}(m_1+m_2)\dot{R}^2$$

$$= \frac{1}{2} m \dot{r}^2 - U(r) + \frac{1}{2}(m_1+m_2)\dot{R}^2 + \left(\cancel{\mu g z} - \cancel{\mu g z}\right) + (m_1+m_2) g \vec{R}\cdot\hat{k}$$

So if we move the origin to the centre of mass, $\vec{R} = \dot{\vec{R}} = 0$ we get back the old Lagrangian.