Strassen's Matrix Multiplication

- Matrix multiplication
- Divide and conquer
- Strassen's ↓ multiplication approach has taken

Two matrices A and B,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \text{ then}$$

$$C = C_1 \quad A_{ij} \times B_{jk} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$= [$

No. of columns of A should to be equal to no. of rows of B, then it is compatible for multiplication

$$C_{ik} = \sum_{j=1}^{n} A_{ij} \cdot B_{jk}$$

For any matrices more than 2×2 and in the order of power 2, the ~~suppose~~ $n \times n$ matrices are splitted recursively $\frac{n}{2} \times \frac{n}{2}$.

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$
$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$
$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$
$$C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

{ 8-Multiplication & 4 Additions-Fixed }

```
x = 0
for(i=0; i≤n; i++)
{
    for(j=0; j≤n; j++)
    {
        for(k=0; k≤n; k++)
        {
            x = A[i,k] * B[k,j] + x;
        }
        c[i,j] = x;  x = 0;
    }
}
```

A(not Algo)

Complexity

$$T(n) = \begin{cases} A & n \leq 2 \\ 8T\left(\frac{n}{2}\right) + cn^2, & n > 2 \end{cases}$$

Solving Recurrence Relation

$$A\left(\frac{n}{2}\right) = 8\left[8T\left(\frac{n}{4}\right) + c\cdot\left(\frac{n}{2}\right)^2\right] + cn^2$$
$$= 64T\left(\frac{n}{4}\right) + 2c\cdot\frac{n^2}{4} + cn^2$$
$$= 64T\left(\frac{n}{4}\right) + 3cn^2$$

$$A\left(\frac{n}{4}\right) = 64\left(8T\left(\frac{n}{8}\right) + c\cdot\frac{n^4}{16}\right) + 3cn^2$$
$$= 512T\left(\frac{n}{8}\right) + 4cn^4 + 3cn^2$$

⋮

$$A(T) = 8^k T\left(\frac{n}{2^k}\right) +$$

On solving this recurrence relation, it $T(n) = O(n^3)$

Multiplication is more expensive than addition in terms of computation. So, if we replace multiplication with addition, we can hope a better matrix multiplication algorithm. So, Strassen's matrix multiplication is better way.

Strassen's Matrix Multiplication

- Matrix multiplication
- Divide-and-conquer
- Strassen's multiplication approach has taken

Two matrices A and B,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}, \text{ then}$$

$$C = \sum A_{ij} \times B_{jK} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} * \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

No. of columns of A should to be equal to no. of rows of B, then it is compatible for multiplication

$$C_{iK} = \sum_{j=1}^{n} A_{ij} \cdot B_{jK}$$

$$C_{11} = A_{11}B_{11} + A_{12}B_{21}$$
$$C_{12} = A_{11}B_{12} + A_{12}B_{22}$$
$$C_{21} = A_{21}B_{11} + A_{22}B_{21}$$
$$C_{22} = A_{21}B_{12} + A_{22}B_{22}$$

{8-multiplication & 4 Additions - Fixed}

For any matrices more than 2x2 and in the order of power 2, the ~~suppose~~ $n \times n$ matrices are splitted recursively $\frac{n}{2} \times \frac{n}{2}$.

Complexity

$$T(n) = \begin{cases} A & n \le 2 \\ 8T\left(\frac{n}{2}\right) + cn^2, & n > 2 \end{cases}$$

```
X = 0
for(i = 0; i ≤ n; i++)
{
    for(j = 0; j ≤ n; j++)
    {
        for(K = 0; K ≤ n; K++)
        {
            X = A[i,K] * B[K,j] + X;
        }
        c[i,j] = X; X = 0;
    }
}
```

Solving Recurrence Relation

$$A\left(\frac{n}{2}\right) = 8\left[8T\left(\frac{n}{4}\right) + c\cdot\left(\frac{n}{2}\right)^2\right] + cn^2$$
$$= 64T\left(\frac{n}{4}\right) + 2c\cdot\frac{n^2}{4} + cn^2$$
$$= 64T\left(\frac{n}{4}\right) + 3cn^2$$

$$A\left(\frac{n}{4}\right) = 64\left(8T\left(\frac{n}{8}\right) + c\cdot\frac{n^4}{16}\right) + 3cn^2$$
$$= 512T\left(\frac{n}{8}\right) + 4cn^4 + 3cn^2$$
$$\vdots$$
$$A(T) = 8^K T\left(\frac{n}{2^K}\right) +$$

On solving this recurrence relation, it $T(n) = O(n^3)$

Multiplication is more expensive than addition in terms of computation. So, if we replace multiplication with addition, we can hope a better matrix multiplication algorithm. So, Strassen's matrix multiplication is better way.

Strassen's method reduces the no. of multiplication to 7 and 18 additions or subtractions. So, For C_{ij}, 7 multiplication works as follows:

$$P = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$$
$$Q = (A_{21} + A_{22}) \cdot B_{11}$$
$$R = A_{11} \cdot (B_{12} - B_{22})$$
$$S = A_{22} \cdot (B_{21} - B_{11})$$
$$T = (A_{11} + A_{12}) \cdot B_{22}$$
$$U = (A_{21} - A_{11}) \cdot (B_{11} + B_{12})$$
$$V = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$$

These matrices P, Q, R, S, T, U, V used to calculate C_{ij}

$$C_{11} = P + S - T + V$$
$$C_{12} = R + T$$
$$C_{21} = Q + S$$
$$C_{22} = P + R - Q + U$$

* Strassen matrix multiply the base (2x2) matrices to accomplished in 7 multiplications and 18 additions or subtractions.

Proof for C_{11}

$$C_{11} = P + S - T + V$$
$$= A_{11}B_{11} + A_{11}B_{22} + A_{22}B_{11} + A_{22}B_{22}$$
$$+ A_{22}B_{11} + A_{22}B_{11} + A_{11}B_{12} - A_{11}B_{22}$$
$$+ A_{22}B_{21} - A_{22}B_{11} + A_{11}B_{22} + A_{12}B_{22}$$
$$+ A_{21}B_{11} - A_{11}B_{11} + A_{21}B_{12} - A_{11}B_{12}$$
$$+ A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$$

$$A_{11}B_{11} + A_{22}B_{11} + A_{11}B_{22} + A_{22}B_{22}$$
$$+ A_{22}B_{21} - A_{22}B_{11} - A_{11}B_{22} - A_{12}B_{22}$$
$$+ A_{12}B_{21} + A_{12}B_{22} - A_{22}B_{21} - A_{22}B_{22}$$

$$= A_{11}B_{11} + A_{12}B_{21}$$

This is true.

For Strassen's Matrix

$$T(n) = 7T(n/2) + 18n^2, \quad n > 2$$

Each step, we divide the no. into half, i.e $n = n/2$

$$T(n) = 7\left[7T\left(\frac{n}{4}\right) + 18\left(\frac{n}{2}\right)^2\right] + 18n^2$$

$$= 7^2 T\left(\frac{n}{2^2}\right) + \frac{7}{4} 18n^2 + 18n^2$$

As assumed, n is power of 2, i.e $n = 2^k$

$$T(n) = 7^k T\left(\frac{n}{2^k}\right) + 18n^2\left(1 + \frac{7}{4} + \left(\frac{7}{4}\right)^2 + \cdots + \left(\frac{7}{4}\right)^{k-1}\right)$$

$$= 7^k T(1) + 18n^2\left(\frac{\left(\frac{7}{4}\right)^{k-2} - 1}{\frac{7}{4} - 1}\right)$$

$$= 7^{\log_2 n} \cdot b + 18 \times \frac{4}{3} n^2 \left(\frac{7}{4}\right)^{\log_2 n}$$

$$\left[\because n = 2^k \Rightarrow \log_2 n = \log_2 2^k \Rightarrow k = \log_2 n\right]$$

$$= b \cdot 7^{\log_2 n} +$$

Master method

$a = 7,\ b = 2,\ f(n) = 18n^2$

$T(n) = n^{\log_2 7} \cdot U(n)$

$U(n)$ depends on $H(n)$

$$= \frac{18n^2}{n^{\log_2 7}} = \frac{18n^2}{n^{2.81}}$$

$$= 18\left[n^{2-2.81}\right]$$

$\boxed{r < 0}$, $U(n) = 1$

$T(n) = O(n^{2.81})$

$$T(n) = 7T\left(\frac{n}{2}\right) + n^2$$

$$= n^2 + 7\left(7T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^2\right) = n^2 + \frac{7}{2^2}n^2 + 7^2 T\left(\frac{n}{2^2}\right)$$

$$= n^2 + \frac{7}{2^2}n^2 + 7^2\left(7T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^2\right)$$

$$= n^2 + \frac{7}{2^2}n^2 + \left(\frac{7}{2^2}\right)^2 n^2 + 7^3 T\left(\frac{n}{2^3}\right)$$

$$= n^2 + \frac{7}{2^2}n^2 + \left(\frac{7}{2^2}\right)^2 n^2 + \left(\frac{7}{2^2}\right)^3 n^2 + \cdots + \left(\frac{7}{2^2}\right)^{\log n - 1} n^2 + 7^{\log n}$$

$$= \sum_{i=0}^{\log_n - 1} \left(\frac{7}{2^2}\right)^i n^2 + 7^{\log n}$$

$$= n^2 \cdot \Theta\left(\left(\frac{7}{2^2}\right)^{\log n - 1}\right) + 7^{\log n}$$

$$= n^2 \cdot \Theta\left(\frac{7^{\log n}}{(2^2)^{\log n}}\right) + 7^{\log n}$$

$$= n^2 \cdot \Theta\left(\frac{7^{\log n}}{n^2}\right) + 7^{\log n} = \Theta(7^{\log n})$$

Now we have following

$$7^{\log n} = 7^{\frac{\log_7 n}{\log_7 2}} = \left(7^{\log_7 n}\right)^{(1/\log_7 2)} = n^{(1/\log_7 2)}$$

$$= n^{\frac{\log_2 7}{\log_2 2}} = n^{\log 7}, \text{ or in general}$$

$$a^{\log_k n} = n^{\log_k a}. \text{ So the solution is}$$

$$T(n) = \Theta(n^{\log 7}) = \Theta(n^{2.81})$$

Example

1. Given two matrices A and B of sizes 2×2

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 4 & 4 \end{bmatrix}$$

multiply the two matrices using Strassen's method

Sol^n: Here $n = 2$ for matrices A and B.

7 submatrices P, Q, R, S, T, U, V are calculated as:

$P = (A_{11} + A_{22})(B_{11} + B_{22}) = (2+6)(1+4) = 40$

$Q = (A_{21} + A_{22}) \cdot B_{11} = (5+6) \times 1 = 11$

$R = A_{11} \cdot (B_{12} - B_{22}) = 2 \times (2-4) = -4$

$S = A_{22} \cdot (B_{21} - B_{11}) = 6 \times (4-1) = 18$

$T = (A_{11} + A_{12}) \cdot B_{22} = (2+3) \cdot 4 = 20$

$U = (A_{21} - A_{11}) \cdot (B_{11} + B_{12}) = (5-2) \cdot (1+2) = 9$

$V = (A_{12} - A_{22}) \cdot (B_{21} + B_{22}) = (3-6) \cdot (4+4) = -24$

Elements of Result matrices

(58 - 44)

$C_{11} = P + S - T + V = 40 + 18 - 20 + (-24) = 14$

$C_{12} = R + T = -4 + 20 = 16$

$C_{21} = Q + S = 11 + 18 = 29$

49-15=34

$C_{22} = P + R - Q + U = 40 + (-4) - 11 + 9 = 34$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} 14 & 16 \\ 29 & 34 \end{bmatrix}$$

Example-2

$A = \begin{bmatrix} 1 & 3 \\ 5 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 4 \\ 6 & 2 \end{bmatrix}$

Matrix Multiplication $C = ?$ using Strassen's Matrix.

Solⁿ:

$P = (A_{11}+A_{22})(B_{11}+B_{22}) = (1+5)(8+2) = 60$

$Q = (A_{21}+A_{22})\,B_{11} = (5+5)\cdot 8 = 80$

$R = A_{11}\cdot(B_{12}-B_{22}) = 1\cdot(4-2) = 2$

$S = A_{22}\cdot(B_{21}-B_{11}) = 5\cdot(6-8) = -10$

$T = (A_{11}+A_{12})\cdot B_{22} = (1+3)\cdot 2 = 8$

$U = (A_{21}-A_{11})\cdot(B_{11}+B_{12}) = (5-1)\cdot(8+4) = 4\times 12 = 48$

$V = (A_{12}-A_{22})(B_{21}+B_{22}) = (3-5)\cdot(6+2) = -16$

Elements of Resultants matrix

60-34

$C_{11} = P+S-T+V = 60-10-8+(-16) = 26$

$C_{12} = R+T = 2+8 = 10$

$C_{21} = Q+S = 80-10 = 70$

$C_{22} = P+R-Q+U = 60+2-80+48 = 110-80 = 30$

$$C = \begin{bmatrix} 26 & 10 \\ 70 & 30 \end{bmatrix}$$

Example 3

$$A = \begin{bmatrix} 4 & 2 & 0 & 1 \\ 3 & 1 & 2 & 5 \\ 3 & 2 & 1 & 4 \\ 5 & 2 & 6 & 7 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 3 & 2 \\ 5 & 4 & 2 & 3 \\ 1 & 4 & 0 & 2 \\ 3 & 2 & 4 & 1 \end{bmatrix}$$

If size of matrix is >2, then divide the matrix in such a way that each sub-matrix contains four elements, and in case of less than ~~to~~ four elements pad it upto 4-elements.

$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$, where

$A_{11} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix}$, $A_{21} = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix}$

$A_{22} = \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix}$

$B_{11} = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$

$B_{12} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

$B_{21} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

$B_{22} = \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$

$A_{11} = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$

$A_{12} = \begin{bmatrix} 0 & 2 \\ 2 & 5 \end{bmatrix}$

$A_{21} = \begin{bmatrix} 3 & 2 \\ 6 & 2 \end{bmatrix}$

$P = (A_{11} + A_{22})(B_{11} + B_{22})$

$Q = (A_{21} + A_{22}) * B_{11}$

$R = A_{11} * (B_{12} - B_{22})$

$S = A_{22} \cdot (B_{21} - B_{11})$

$T = (A_{11} + A_{12}) * B_{22}$

$U = (A_{21} - A_{11}) * (B_{11} + B_{12})$

$V = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$

$C_{11} = P + S - T + V$

$C_{12} = R + T$

$C_{21} = Q + S$

$C_{22} = P + R - Q + U$

$P = (A_{11} + A_{22})(B_{11} + B_{22})$

$= \left(\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \right) \cdot \left(\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix} \right)$

$= \begin{pmatrix} 5 & 6 \\ 9 & 8 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 9 & 5 \end{pmatrix}$

$= \begin{bmatrix} 10+54 & 15+30 \\ 18+72 & 27+40 \end{bmatrix} = \begin{bmatrix} 64 & 45 \\ 90 & 67 \end{bmatrix}$

$$Q = (A_{21} + A_{22}) \cdot B_{11}$$

$$= \left(\begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \right) \left(\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \right) = \begin{bmatrix} 4 & 6 \\ 11 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 8+30 & 4+24 \\ 22+45 & 11+36 \end{bmatrix} = \begin{bmatrix} 38 & 28 \\ 67 & 47 \end{bmatrix}$$

$$R = A_{11} \left(B_{12} - B_{22} \right) = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \left(\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 0 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 12-4 & 0+4 \\ 9-2 & 0+2 \end{bmatrix} = \begin{bmatrix} 8 & 4 \\ 7 & 2 \end{bmatrix}$$

$$S = A_{22} \cdot \left(B_{21} - B_{11} \right) = \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ -2 & -2 \end{bmatrix} = \begin{bmatrix} -1-8 & 3-8 \\ -6-14 & 18-14 \end{bmatrix} = \begin{bmatrix} -9 & -5 \\ -20 & 4 \end{bmatrix}$$

$$T = \left(A_{11} + A_{12} \right) \cdot B_{22} = \left(\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix} \right) \cdot \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4 & 2+1 \\ 3+2 & 1+5 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 0+56 & 8+14 \\ 0+32 & 4+12 \end{bmatrix}} = \cancel{\begin{bmatrix} 56 & 22 \\ 32 & 12 \end{bmatrix}} = \begin{bmatrix} 0+12 & 8+3 \\ 0+24 & 10+6 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 11 \\ 24 & 16 \end{bmatrix}$$

$$U = \left(A_{21} - A_{11} \right) \cdot \left(B_{11} + B_{12} \right) = \left(\begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \right) \cdot$$

$$\left(\begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right) = \begin{bmatrix} -1 & 0 \\ 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & 3 \\ 7 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -5+0 & -3+0 \\ 10+7 & 6+7 \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ 17 & 13 \end{bmatrix}$$

$$V = \left(A_{12} - A_{22} \right) \cdot \left(B_{21} + B_{22} \right) = \left(\begin{bmatrix} 0 & 1 \\ 2 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 6 & 7 \end{bmatrix} \right) \cdot$$

$$\left(\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 4 & 1 \end{bmatrix} \right) = \begin{bmatrix} -1 & -3 \\ -4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 \\ 7 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1-21 & -6-9 \\ -4-14 & -24-6 \end{bmatrix} = \begin{bmatrix} -22 & -15 \\ -18 & -30 \end{bmatrix}$$

$$C_{11} = P + S - T + V$$

$$= \begin{bmatrix} 64 & 45 \\ 90 & 67 \end{bmatrix} + \begin{bmatrix} -9 & -5 \\ -20 & 4 \end{bmatrix} - \begin{bmatrix} 12 & 11 \\ 24 & 16 \end{bmatrix} + \begin{bmatrix} -22 & -10 \\ -18 & -30 \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 14 \\ 24 & 25 \end{bmatrix}$$

$$C_{12} = R + T = \begin{bmatrix} 8 & 4 \\ 7 & 2 \end{bmatrix} + \begin{bmatrix} 12 & 11 \\ 24 & 16 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 15 \\ 31 & 18 \end{bmatrix}$$

$$C_{21} = Q + S = \begin{bmatrix} 38 & 28 \\ 67 & 47 \end{bmatrix} + \begin{bmatrix} -9 & -5 \\ -20 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 23 \\ 47 & 51 \end{bmatrix}$$

$$C_{22} = P + R - Q + U$$

$$= \begin{bmatrix} 64 & 45 \\ 90 & 67 \end{bmatrix} + \begin{bmatrix} 8 & 4 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 38 & 28 \\ 67 & 47 \end{bmatrix} + \begin{bmatrix} -5 & -3 \\ 17 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 29 & 18 \\ 47 & 35 \end{bmatrix}$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 21 & 14 & 20 & 15 \\ 24 & 25 & 31 & 18 \\ 29 & 23 & 29 & 18 \\ 47 & 51 & 47 & 35 \end{bmatrix}$$