Dynamic Programming

It is a technique for solving a problems in minimum or maximum finding for the particular problem. It applies bottom-up approach, means it solves a smaller sub-problem, store its results, and reuse its previous results for solving a larger sub-problem. The nature of the problem is frequently used in solving the problem in large.

Applications

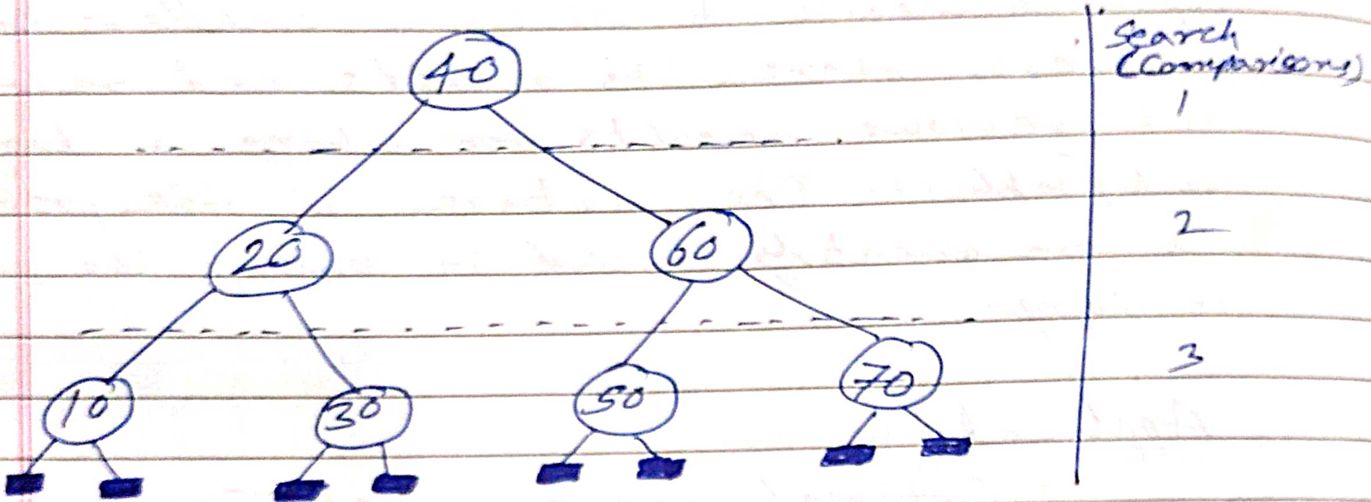
- Optimal Binary Search Tree
- 0/1 Knapsack Problem
- All-pairs shortest Path Problem
- Reliability Design
- Travelling Salesman Problem
- Matrix Chain multiplication
- Longest Common Subsequence
- Mathematical Optimization.

Optimal Binary Search Tree (OBST)

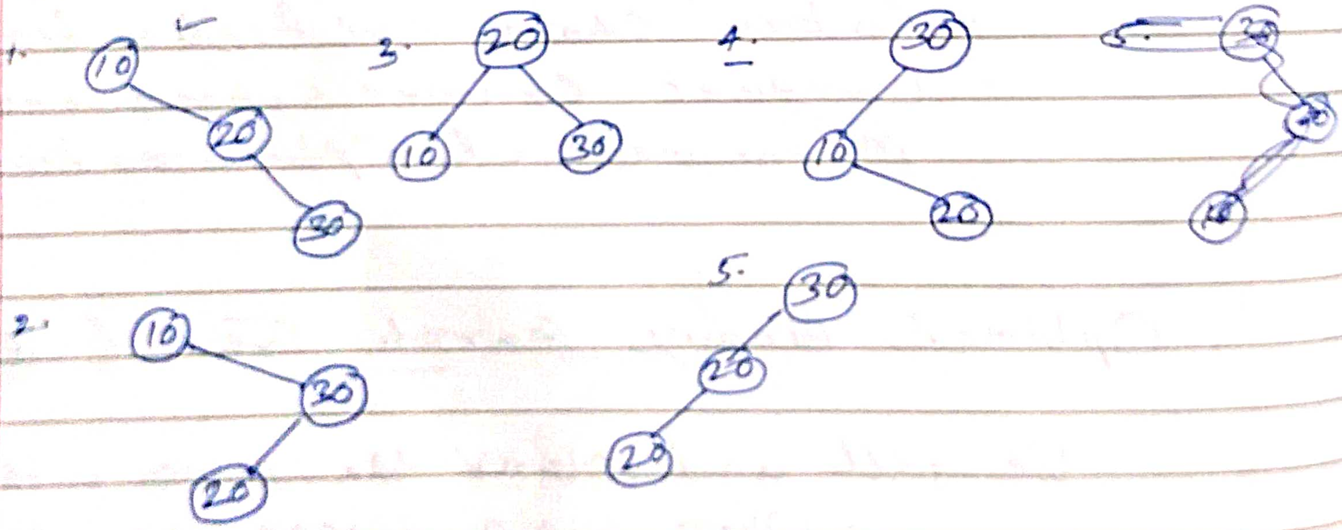
We will understand the same with:

- What is a Binary search Tree
- Cost of searching
- Probability of search / What is OBST
- Dynamic Programming Approach
- Formula / Deriving
- Solving a Problem
- Constructing a OBST

Binary Search Tree (BST) or Binary sorted Tree is one the tree where left ~~root~~ childs of roots are smallar and rights childs of ~~roots~~ are greater than of roots.



Suppose we have keys 10, 20, 30, there are following ways BST can be created.



The cost of searchings for note 1 in 1, 2, 4 & 5 needs 3 comparisons where in 3 cost of searching is 2 comparisons. One can see this one is best, here.

The search depends upon height of the tree, if the height is less, comparisons will be less.

How many possibe ways to create a tree that depends upon, this formula

$$\frac{{}^{2n}C_n}{n+1}, \text{ where } n = \text{no. of nodes/ elements}$$

In previous example, $n = 3$

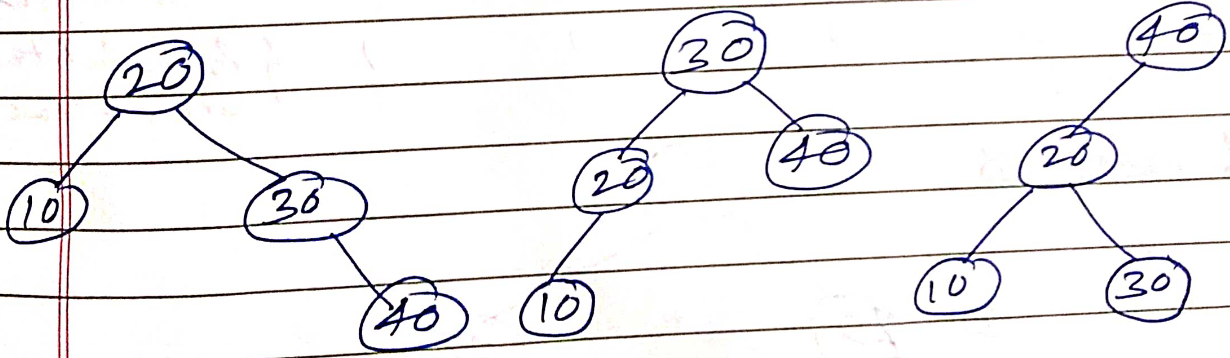
$$\frac{{}^{2\times 3}C_3}{3+1} = \frac{{}^{6}C_3}{4} = \frac{\frac{\lfloor 6}{\lfloor 3 \, \lfloor 6-3}}{4} = \frac{\cancel{\lfloor 6}\,\cancel{6}\times5\times4\times\cancel{\lfloor 3}}{\cancel{\lfloor 3}\times\cancel{3}\times\cancel{2}} = 5$$

Another example

Suppose keys are 10, 20, 30, 40, then how many tree can be possible

$$= \frac{{}^{2n}C_n}{n+1} = \frac{{}^{2\times 4}C_4}{4+1} = \frac{\frac{\lfloor 8}{\lfloor 4 \, \lfloor 4}}{5} = \cancel{\frac{8\times7\times6\times5}{1}}$$

$$= \frac{8\times 7\times \overset{2}{\cancel{6}}\times\cancel{5}\times\cancel{\lfloor 4}}{\cancel{\lfloor 4}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times 1}\times\frac{1}{\cancel{5}} = 14$$



Real-Life Example

1. We in general visit markets for buying vegetables, fruits or grocessory's.

I give one of own example, how I efficiently should to purchase vegetables & fruits, so carry less weight during purchasing.

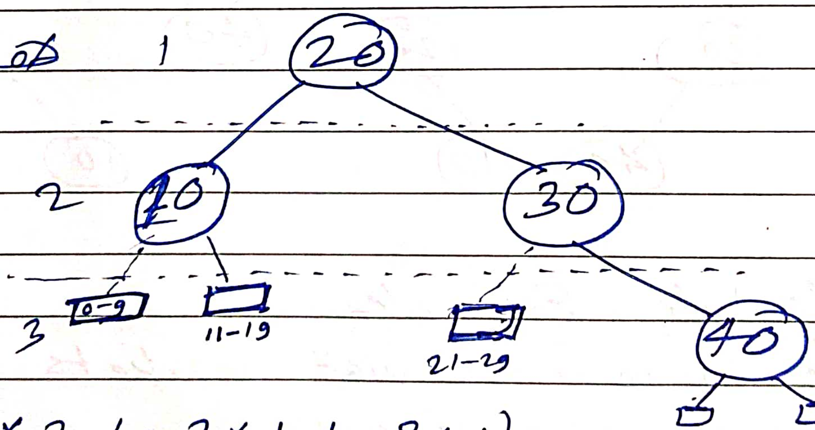
We create a tree, according to items we ~~purp~~ purchase. or we have in list (ours). How Items are pharchasing that depends upon the frequency of item, how I am getting the same. If the item is present in the availability, ~~depends upon~~ it is known by success but if the item doesn't it is ~~...~~ named by unsuccessful search.

One example

	10	20	30	40	
p_i	2	3	1	1	{Probability of successful}

q_i 2   3   1   1   1   {Probability of unsuccessful}

~~Total cost of~~



$(1\times 2 + 2\times 2 + 2\times 1 + 3\times 1) + \underline{2\times 2 + 2\times 3 + 2\times 1 + 3\times 1 + 3\times 1}$

Success search cost

Unsuccessful search Cost

for ith item

$$cost[0,j] = \sum_{0\le i<n} p_i * \text{level of Keys } (a_i) + \sum_{0\le i<n} q_i \times \text{level of External Node } (E_{i-1})$$

What is Optimal BST (OBST)

An OBST is a problem in which we have to generate all BST whose cost is minimum. Creating all BST's and evaluating its costs are time consuming for finding that contains the minimum cost.

Without generating all BST's & finding the minimum cost, Dynamic ~~problem~~ Programming is helful in it. We will try to generate all tree, with costs, but not directly we will do the same. In the result minimum cost finding, that will be beauty of the Dynamic Programming.

Three formula's helpus:

1. $w[i,j] = w[i, j-1] + \underbrace{p[j]}_{\text{Successful Probability}} + \underbrace{Q[j]}_{\text{Unsuccessful Probability}}$

2. $c[i,j] = \min\limits_{i<K\leq j} \{ c[i, K-1] + c[K, j] \} + w[i,j]$

3. $r[i,j] = K$

$w[i,j] = w[0,-1] + p(0) + q(0) = q[j]$

Initial Assumptions

If $i == j$

$w[i,j] = Q[j]$

$c[i,j] = 0$

$r[i,j] = 0$

$i == j$, if

i, j 0 0

Suppose we want to find minimum cost for given nodes, such as

	0	1	2	3	4
Key		10	20	30	40
Probability of Success ← p_i		3	3	1	1
Probability of Unsuccess q_i	2	3	1	1	1

$i \rightarrow$, $j \downarrow$

j \ i	0	1	2	3	4
4	w[0,4] = 16 c[0,4] = 32 r[0,4] = 2	w[1,4] = 11 c[1,4] = 19 r[1,4] = 2	w[2,4] = 5 c[2,4] = 8 r[2,4] = 3	w[3,4] = 2 c[3,4] = 3 r[3,4] = 4	w[4,4] = 1 c[4,4] = 0 r[4,4] = 0
3	w[0,3] = 14 c[0,3] = 25 r[0,3] = 2	w[1,3] = 9 c[1,3] = 12 r[1,3] = 2	w[2,3] = 3 c[2,3] = 3 r[2,3] = 3	w[3,3] = 1 c[3,3] = 0 r[3,3] = 0	
2	w[0,2] = 12 c[0,2] = 19 r[0,2] = 1	w[1,2] = 7 c[1,2] = 7 r[1,2] = 2	w[2,2] = 1 c[2,2] = 0 r[2,2] = 0		
1	w[0,1] = 8 c[0,1] = 8 r[0,1] = 1	w[1,1] = 3 c[1,1] = 0 r[1,1] = 0			
0	w[0,0] = 2 c[0,0] = 0 r[0,0] = 0				

Initial Assumption ~~w[0,0]~~ (i,j) i = = j or for j-i = 0 Difference

$w[0,0] = q_0 = 2, w[1,1] = q_1 = 3, w[2,2] = q_2 = 1,$
$w[3,3] = q_3 = 1, w[4,4] = q_4 = 1$

$c[0,0] = c[1,1] = c[2,2] = c[3,3] = c[4,4] = 0$

$r[0,0] = r[1,1] = r[2,2] = r[3,3] = r[4,4] = 0$

For j-i = 1 Difference [0,1], [1,2], [2,3], [3,4]

(1) $w[0,1] = w[0,0] + p(j) + q(j) = 2 + 3 + 3 = 8$

$c[0,1] = \min_{0<k\leq 1} \{c[0,0] + c[1,1]\} + w[0,1] = 0 + 0 + 8$

$r[0,1] = 1$ } $k=1$

② $w[1,2] = w[1,1] + p(2) + q(2) = 3+3+1 = 7$

$c[1,2] = \min_{1<k\le 2} \{c[1,1] + c[2,2]\} + w[1,2] = 0+0+7 = 7$

$k=2$

③ $r[1,2] = 2$

③ $w[2,3] = w[2,2] + p(3) + q(3) = 1+1+1 = 3$

$c[2,3] = \min_{2<k\le 3} \{c[2,2] + c[3,3]\} + w[2,3] = 0+0+3 = 3$

$k=3$

$r[2,3] = 3$

④ $w[3,4] = w[3,3] + p(4) + q(4) = 1+1+1 = 3$

$c[3,4] = \min_{3\le k\le 4} \{c[3,3] + c[4,4]\} + w[3,4] = 0+0+3$
$= 3$

$k=4$

$r[3,4] = 4$

For $i,j = 2$ Difference

$[0,2], [1,3], [2,4]$

i. $w[0,2] = w[0,1] + p(2) + q(2) = 8+3+1 = 12$

$c[0,2] = \min_{0<k\le 2} \{c[0,0] + c[1,2]\} + w[0,2] = \cancel{8+0+12=20}$
$= 0+7+12 = 19$

$k=1$

$k=2 \quad \{c[0,1] + c[2,2]\} + w[0,2] = 8+0+12 = 20$

$r[0,2] = 1$

ii. $w[1,3] = w[1,2] + p(3) + q(3) = 7+1+1 = \cancel{14}\ 9$

$c[1,3] = \min_{1<k\le 3} \{c[1,1] + c[2,3]\} + w[1,3] = 0+3+9$
$= 12$

$k=2$

$\min_{1<k\le 3} \{c[1,2] + c[3,3]\} + w[1,3] = 7+0+9$
$= 16$

$k=3$

$r[1,3] = 2$

$\{9, 12, 2\}$
$\ \ w\ \ c\ \ r$

iii. $w[2,4] = w[2,3] + p(4) + q(4) = 3+1+1 = 5$

$c[2,4] = \min_{2<k\le 4} \{c[2,2] + c[3,4]\} + w[2,4] = 0+3+5 = 8$

$k=3$

$k=4 \quad \{c[2,3] + c[4,4]\} + w[2,4] = 3+0+5 = 8$

$r[2,4] = 3$

For $i,j = 3$ Difference

we $[0,3]$ & $[1,4]$

$[0,3]$

$w[0,3] = w[0,2] + p(3) + q(3) = 12+1+1 = 14$

$c[0,3] = \min_{0<k\leq 3} \{c[i,k-1] + c[k,j]\} + w[i,j]$

$k=1 \quad \{c[0,0] + c[1,3]\} + 14 = 0+12+14 = 26$

✓ $k=2 \quad \{c[0,1] + c[2,3]\} + 14 = 8+3+14 = 25$

$k=3 \quad \{c[0,2] + c[3,3]\} + 14 = 19+0+14 = 33$

$r[0,3] = 2$

$[1,4]$

$w[1,4] = w[1,3] + p(4) + q(4) = 9+1+1 = 11$

$c[1,4] = \min_{1<k\leq 4} \{c[1,1] + c[2,4]\} + w[1,4] = 0+8+11 = 19$

✓ $k=2$

$k=3 \quad \{c[1,2] + c[3,4]\} + w[1,4] = 7+3+11 = 20$

$k=4 \quad \{c[1,3] + c[4,4]\} + w[1,4] = 12+0+11 = 23$

$r[1,4] = 2$

$\boxed{B}$ For $i,j = 4$ Difference

$[0,4]$ Only

$w[0,4] = w[0,3] + p(4) + q(4) = 14+1+1 = 16$

$c[0,4] = \min_{0<k\leq 4} \{c[0,0] + c[1,4]\} + w[0,4] = 0+19+16 = 35$

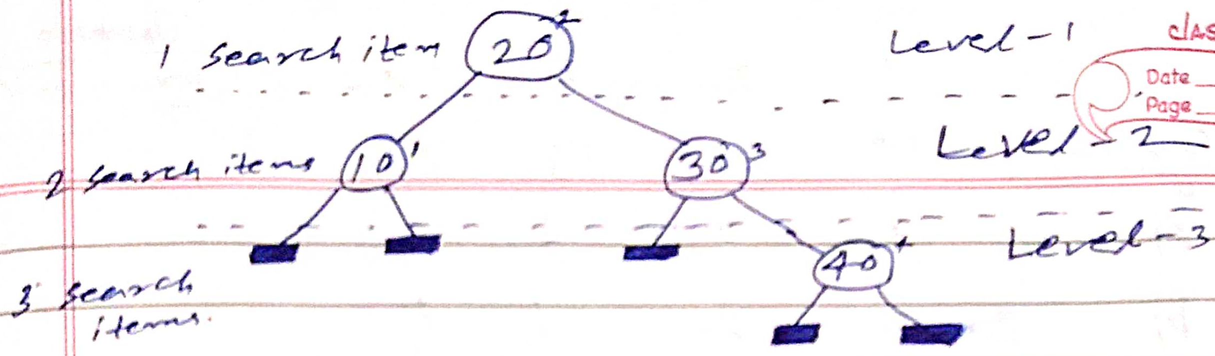
$k=1$

$k=2 \quad \{c[0,1] + c[2,4]\} + w[0,4] = 8+8+16 = 32$

$k=3 \quad \{c[0,2] + c[3,4]\} + w[0,4] = 19+3+16 = 3$

$k=4 \quad \{c[0,3] + c[4,4]\} + w[0,4] = 25+0+16 = 41$

$r[0,4] = 2$



No. of search(s)

No. of Unsuccessful search

$\cancel{1\times 3}$ $1\times 2 + 2\times 3 + 2\times 1 + 3\times 1 + 2\times 2 + 2\times 3 + 2\times 1 + 3\times 1 + 3\times 1$

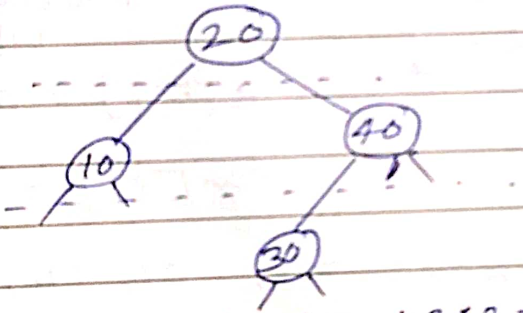
Search success possibilities (Frequency)

Unsuccessful possibilities (Frequency)

$\boxed{= 32}$

Total cost

or



$(1\times 3 + 2\times 3 + 2\times 1 + 3\times 1) + (2\times 2 + 2\times 3 + 3\times 1 + 3\times 1 + 2\times 1)$

Cost will be ~~not~~ same in both ~~those~~ tree

Q-2

Keys	10	20	30	40
p_i	3	3	1	1

j \ i	0	1	2	3	4
4	$w[0,4]=8$ $c[0,4]=14$ $r[0,4]=2$	$w[1,4]=5$ $c[1,4]=8$ $r[1,4]=2$	$w[2,4]=2$ $c[2,4]=3$ $r[2,4]=3$	$w[3,4]=1$ $c[3,4]=1$ $r[3,4]=4$	$w[4,4]=0$ $c[4,4]=0$ $r[4,4]=0$
3	$w[0,3]=7$ $c[0,3]=11$ $r[0,3]=2$	$w[1,3]=4$ $c[1,3]=5$ $r[1,3]=2$	$w[2,3]=1$ $c[2,3]=1$ $r[2,3]=3$	$w[3,3]=0$ $c[3,3]=0$ $r[3,3]$ 0	
2	$w[0,2]=6$ $c[0,2]=9$ $r[0,2]=1$	$w[1,2]=3$ $c[1,2]=3$ $r[1,2]=2$	$w[2,2]=0$ $c[2,2]=0$ $r[2,2]=0$		
1	$w[0,1]=3$ $c[0,1]=3$ $r[0,1]=1$	$w[1,1]=0$ $c[1,1]=0$ $r[1,1]=0$			
0	$w[0,0]=0$ $c[0,0]=0$ $r[0,0]=0$				

Initial Assumption

$$w[i,j] = w[i,j-1] + p(j) + q(j)$$

$$c[i,j] = \min_{i<k\leq j} \{c[i,k-1] + c[k,j]\} + w[i,j]$$

$$r[i,j] = k$$

i,j Difference 1 (01, 12, 23, 34)

01

$w[0,1] = w[0,0] + p(1) = 0+3 = 3$

$c[0,1] = \min_{0<k\leq 1} \{c[0,0] + c[1,1]\} + w[0,1]$
$k=1$
$= 0+0+3 = 3$

$k=1$

12

$w[1,2] = w[1,1] + p(2) = 3$

$c[1,2] = \min_{1<k\leq 2} \{c[1,1] + c[2,2]\} + w[1,2] = 3$
$k=2$

23 $w[2,3] = w[2,2] + p(3) = 1$

$w[2,3] = \min_{2<k\leq 3} \{c[2,2] + c[3,3]\} + w[2,3] = 1$
$k=3$

3,4 $w[3,4] = w[3,3] + p(4) = 1$

$w[3,4] = \{c[3,3] + c[4,4]\} + w[3,4] = 1$
$k=4$

i,j Difference = 2 {(0,2), (1,3), (2,4)}

0,2 $w[0,2] = w[0,1] + p(2) = 3+3 = 6$

$c[0,2] = \min_{0<k\le 2} \{c[0,0] + c[1,2]\} + w[0,2] = 0+3+6 = 9$
$k=1$

$k=2 \quad \{c[0,1] + c[2,2]\} + w[0,2] = 3+0+6 = 9$

1,3 $w[1,3] = w[1,2] + p(3) = 3+1 = 4$

$c[1,3] = \min_{1<k\le 3} \{c[1,1] + c[2,3]\} + 4 = 0+1+4 = 5$
$k=2$

$k=3 \quad \{c[1,2] + c[3,3]\} + 4 = 3+0+4 = 7$

2,4 $w[2,4] = w[2,3] + p(4) = 1+1 = 2$

$c[2,4] = \min_{2<k\le 4} \{c[2,2] + c[3,4]\} + 2 = 1+2 = 3$
$k=3$

$k=4 \quad \{c[2,3] + c[4,4]\} + 2 = 1+2 = 3$

i,j Difference = 3 {(0,3) & (1,4)}

0,3 $w[0,3] = w[0,2] + p(3) = 6+1 = 7$

$c[0,3] = \min_{0<k\le 3} \{c[0,0] + c[1,3]\} + 7 = 0+5+7 = 12$
$k=1$

$k=2 \quad \{c[0,1] + c[2,3]\} + 7 = 3+1+7 = 11$

$k=3 \quad \{c[0,2] + c[3,3]\} + 7 = 9+0+7 = 16$

1,4 $w[1,4] = w[1,3] + p(4) = 4+1 = 5$

$c[1,4] = \min_{1<k\le 4} \{c[1,1] + c[2,4]\} + 5 = 0+3+5 = 8$
$k=2$

$k=3 \quad \{c[1,2] + c[3,4]\} + 5 = 3+1+5 = 9$

$k=4 \quad \{c[1,3] + c[4,4]\} + 5 = 5+0+5$

0,4 $w[0,4] = w[0,3] + p(4) = 7+1 = 8$

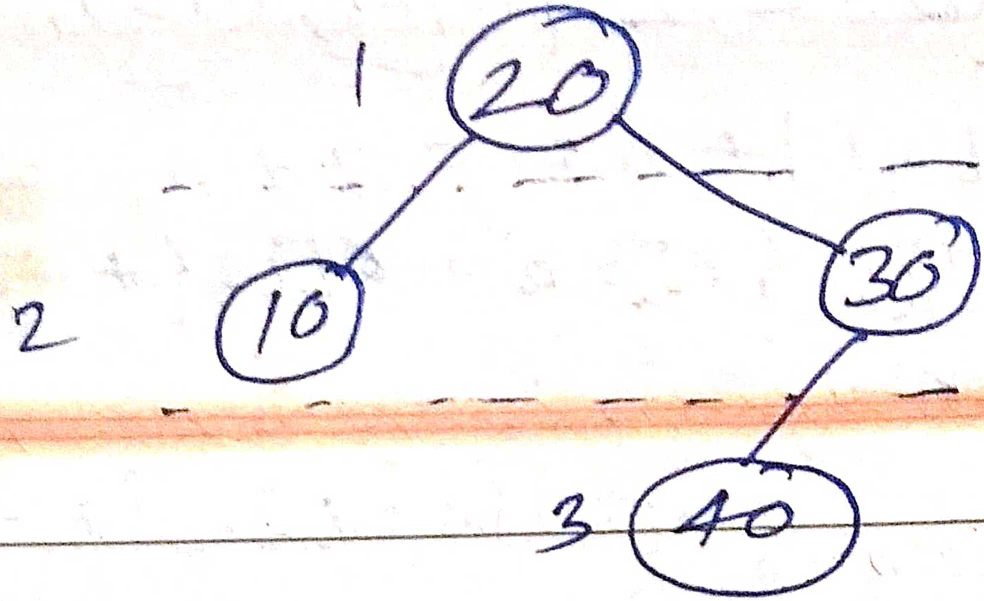
$c[0,4] = \min_{0<k\le 4} \{c[0,0] + c[1,4]\} + 8 = 0+8+8 = 16$
$k=1$

$k=2 \quad \{c[0,1] + c[2,4]\} + 8 = 3+3+8 = 14$

$\{c[0,2] + c[3,4]\} + 8 = 9+1+8 = 18$

$k=3 \quad \{c[0,2] + c[3,4]\} + 8 = 11+0+8 = 19$

$\{c[0,3] + c[4,4]\} + 8$



$1 \times 3 + 2 \times 3 + 2 \times 1 + 3 \times 1$

$3 + 6 + 2 + 3 = \underline{\underline{14}}$