Inflectional Instability of Linearized Incompressible Euler Equations via Linear Partial Inequality Tests

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***Abstract*— In this paper, we consider inflectional instability of linearized incompressible Euler equations in a continuous time formulation. Inflectional instability is a linear inviscid instability mechanism that plays important role in dynamics of transitional and turbulent fluid flows. According to Rayleigh theorem, it occurs when an inflectional point develops in the mean velocity profile. Rayleigh theorem [1], as most of the studies concerning stability analysis of fluid flows, relies on eigenvalue decomposition that restricts analysis to a class of perturbations known as normal models (waves with time-dependent amplitude). In addition to the Rayleigh condition, a second, Fjørtoft condition [2], has also been developed, likewise based on the normal mode assumption. Both these conditions represent necessary conditions for instability assuming a normal mode decomposition. The current paper revisits linear stability analysis of inviscid incompressible flows, but instead develops a generalized, continuous in time analysis framework that does not require a normal mode decomposition or any other assumption on the form of perturbations. The framework transforms a governing Partial Differential Equation (PDE) problem into a Partial Integral Equation (PIE) and then analyzes stability of a continuous in time PIE by invoking Lyapunov-based methods verified through Linear Partial Inequality (LPI) tests. The developed stability test provides a stricter condition for stability than the normal mode analysis since it explicitly searches for a Lyapunov function, which, if found, guarantees stability. The results of this paper show that, even when the Rayleigh and Fjørtoft criteria for instability are not satisfied (i.e. a system has all stable eigenvalues), an appropriate Lyapunov function cannot always be found, meaning that these profiles can be unstable to generalized perturbations. This suggests that, in general, neither Rayleigh nor Fjørtoft criteria can be used as sufficient conditions for stability when the Lyapunov stability of a continuous-time formulation is considered.**

I. INTRODUCTION

While turbulence is a non-linear phenomenon, linear mechanisms are known to play an important role in a development and evolution of turbulent flows. One of the most intriguing linear instabilities of fluid flows is the inflectional instability, which occurs when the mean (or base) velocity profile has an inflection point. Inflectional instability is a type of an inviscid instability, which occurs when the Reynolds number of the flow is infinitely high, or, equivalently, viscosity is infinitely small [3], [4]. Inflectional instability is in the heart of many classical fluid dynamics phenomena, such as Kelvin-Helmholtz instability characterized by a roll up and a subsequent breakdown of vortices in the shear layers [5], instability in the separated regions of the flow that involve inflection points [6], and inflectional instability of streamwise velocity streaks in turbulent boundary layers that is known to play a crucial role in self-sustaining mechanisms of wall-bounded turbulence [7], [8].

In analyzing linear stability of fluid flows, a typical approach has been to consider an eigenvalue decomposition of the linearized Euler or Navier-Stokes equations, where perturbations in the form of the normal modes (traveling waves with complex wave speed) are assumed. Substituting this specific form of perturbations into the equation transforms a time-dependent PDE into an eigenvalue problem, with the complex wave speed, $c = c_r + ic_i$ (equivalently, wave frequency, $\omega = \omega_r + i\omega_i$) serving as an eigenvalue, whose imaginary part determines stability in this normal mode analysis [3], [4]. When inviscid terms are zero in the Navier-Stokes equations, this eigenvalue problem yields a well-known Rayleigh equation [1]. The limitations of the normal-mode approach are: 1) it restricts analysis to a certain type of perturbations (normal modes), 2) it lacks a verifiable proof that guarantees a system stability in the Lyapunov sense.

Based on the normal-mode assumption (thus basing their analysis on the Rayleigh equation), Rayleigh [1] proved an important theorem stating that in order for the system to be inviscidly unstable (in an eigenvalue sense), there needs to be an inflection point in the baseline (mean) velocity profile. In other words, he proved that if there is no inflection point in the profile, all system eigenvalues have to be real ($c_i = 0$). The presence of an inflection point in the profile can be stated as the first necessary condition for eigenvalue instability. There can however be profiles with inflection points that are still linearly stable. Almost a century later, Fjørtoft [2] refined the Rayleigh criterion and proposed the second necessary criterion for instability (in the eigenvalue sense) that, in addition to checking for the inflection point, also checks the sign of a product of velocity with its second derivative. In the context of linear finite-dimensional (ODE) systems, the eigenvalues of a differential operator unequivocally determine the system stability [9]. However, for the infinite-dimensional (PDE) systems, a relation between the eigenvalues of the time-differential operator to a stability of the full spatio-temporal system is less clear. Relevant work on understanding stability properties of PDE systems which do not rely on discretization or eigenmode decomposition include backstepping [10], semigroup theory [11], [12], energy methods [13], and sum-of-squares approaches [14]. While these and similar contributions provide a valuable insight into an analysis and control of PDE problems, the

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methods are typically designed for specific systems and are rarely generalizable. Additionally, to the authors' knowledge, these approaches have not been applied to analyze inviscid instabilities in incompressible Euler systems associated with inflection points.

To revisit both Rayleigh and Fjørtoft criteria in the context of continuous in time (not eigenvalue-decomposed) inviscid incompressible flow models, we have developed a stability analysis framework that transforms the underlying PDE model into an equivalent Partial Integral Equation (PIE) representation [15], [16]. The critical property of the PIE representation is that it does not explicitly require boundary constraints on its solution variables. This permits, with some modifications, stability analysis via standard techniques developed for Ordinary Differential Equations, such as Linear Matrix Inequalities [17], which, in case of PIEs, transform to Linear Partial Inequalities (LPI) [15]. Verifying feasibility of the LPI condition allows us to prove a Lyapunov stability of the PIE, and, by their equivalence, of the original PDE system. In this contribution, we extend the continuous-time formulation to the problems of inviscid instability of generalized plane parallel shear flows and investigate velocity profiles with and without inflection points. To position the current findings in the context of the Rayleigh and Fjørtoft criteria, we consider three groups of profiles: 1) R-stable: $U(y)$ that do not satisfy Rayleigh criterion for eigenvalue instability (no inflection point), 2) F-stable: $U(y)$ that do satisfy Rayleigh criterion for eigenvalue instability (have inflection point, but do not satisfy Fjørtoft criterion), (3) RF-unstable: $U(y)$ that satisfy both Rayleigh and Fjørtoft criteria for eigenvalue instability (have inflection point and Fjørtoft criterion is satisfied).

II. PROBLEM FORMULATION

We consider a two-dimensional, inviscid, incompressible fluid in a domain $(x, y, t) \in [0, L] \times [-h, h] \times [0, \infty]$, governed by the Euler equations,

$$\begin{gathered}\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p, \\ \nabla \cdot \mathbf{u} = 0,\end{gathered} \tag{1}$$

where $\mathbf{u} = (u, v)$ is the velocity in x (streamwise) and y (vertical) directions, respectively, and p is the pressure. Boundary conditions are set as periodic in the streamwise direction and a free slip at the vertical plates,

$$v(x, -h, t) = v(x, h, t) = 0. \tag{2}$$

We split the instantaneous variables into a sum of a baseline solution and the perturbations, $\mathbf{u} = \mathbf{U} + \mathbf{u}'$, $p = P + p'$. We assume a parallel mean flow, $\mathbf{U} = \{U(y), 0\}$, and linearize Eq. (1) as

$$\begin{gathered}\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{U} = -\nabla p'. \\ \nabla \cdot \mathbf{u}' = 0.\end{gathered} \tag{3}$$

Equation (3) can be further simplified by taking the curl of the momentum equation (top) and using the continuity equation (bottom) to eliminate the pressure from the system. This yields a single linear PDE that fully describes the 2D linearized incompressible Euler operator,

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) \nabla^2 - \frac{d^2 U}{d y^2} \frac{\partial}{\partial x}\right] \psi = 0, \tag{4}$$

where $\psi(x, y, t)$ denotes the stream function that obeys the relationships, $u' = \partial \psi / \partial y, v' = -\partial \psi / \partial x$. Periodic boundary conditions in a streamwise direction allow for a Fourier transform of Eq. (4) in x, which gives a one-dimensional PDE per a streamwise wave-number $k \in \mathbb{R}$, $k \neq 0$, as

$$\left[\left(\frac{\partial}{\partial t} + i\, k\, U\right) \hat{\Delta}^2 - i\, k \frac{d^2 U}{d y^2})\right] \hat{\psi} = 0. \tag{5}$$

We use the notation of $\hat{\psi}(y, t)$ for the Fourier coefficient of the stream function for the wave-number k, i is an imaginary unit, and $\hat{\Delta}^2 = \partial^2 / \partial y^2 - k^2$ is the one-dimensional differential operator. Boundary conditions (2) transform into the boundary conditions on the stream function as

$$B(\hat{\psi}) : \hat{\psi}(-h, t) = \hat{\psi}(h, t) = 0. \tag{6}$$

The PDE equation (5), together with the boundary conditions (6), and some suitable initial conditions $\hat{\psi}(y, 0) = \hat{\psi}_0(y)$, represents the second-order initial-boundary value problem.

III. NORMAL-MODE DECOMPOSITION

Before presenting stability analysis of Eq. (5) in a continuous-time domain, which is the focus of the current paper, we briefly review the normal-mode approach [3], [4], which was in the heart of both Rayleigh's [1] and Fjørtoft's [2] studies. In the normal-mode approach, the solutions on the perturbation stream function are sought in a form of time-harmonic modes with complex frequency (normal modes) as $\hat{\psi}(y, t) = \tilde{\psi}(y) \exp(-i\, \omega\, t)$. This form of the perturbation is introduced into Eq. (5) which subsequently transforms into the Rayleigh equation [1]:

$$(U - c)(\tilde{\psi}_{yy} - k^2 \tilde{\psi}) = U_{yy} \tilde{\psi}, \tag{7}$$

where we use a short-hand notation f_{yy} to denote the second derivative in y, and the wave speed $c = \omega / k$ has been introduced, which is a complex quantity.

IV. STABILITY ANALYSIS

We can put forward some useful definitions of stability.

A. STABILITY DEFINITIONS

We define the norms

$$\|f(y, t)\|_{L_2} = \left(\int_{-h}^{h} \|f(y, t)\|^2 d\, y\right)^{1/2}, \tag{8}$$

and

$$\|f(y, t)\|_{H_2} = \|f(y, t)\|_{L_2} + \|f_y(y, t)\|_{L_2} + \|f_{yy}(y, t)\|_{L_2}. \tag{9}$$

It is clear that $\|f(y, t)\|_{L_2} \leq \|f(y, t)\|_{H_2}$.

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Definition 1: The PDE (5) with boundary conditions (6) is stable in L_2 in the sense of Lyapunov if there exist constants $\delta, \epsilon > 0$ such that for any $\hat{\psi}(y,0) \in B(\hat{\psi})$: $\|\hat{\psi}(y,0)\|_{H_2} < \delta$, a solution $\hat{\psi}(y,t)$ of the PDE (5) with (6) satisfies

$$\|\hat{\psi}(y,t)\|_{L_2} < \epsilon, \quad \forall t \geq 0 \tag{10}$$

We also introduce a definition of a "normal-mode" (or "eigenvalue") - stability of the PDE (5):

Definition 2: The PDE (5) with boundary conditions (6) is said to be an "eigenvalue-stable" if there exist constants δ, ϵ such that for any solution of (5) of the form $\hat{\psi}(y,t) = \tilde{\psi}(y)\exp(-i\,\omega\,t)$, with $\tilde{\psi}(\pm h) = 0$, $\omega \in \mathbb{C}$, and $\|\tilde{\psi}(y)\|_{H_2} < \delta$, we have

$$\|\hat{\psi}(y,t)\|_{L_2} < \epsilon, \quad \forall t \geq 0 \tag{11}$$

We can easily show that the following lemma holds.

Lemma 1: The PDE (5) with boundary conditions (6) is stable according to the *Definition 2* if and only if the Rayleigh equation (7) admits strictly real eigenvalues.

Proof: If all eigenvalues of (7) are such that $c_i = 0$ therefore, $\omega_i = 0$, since $\omega = k\,c$, and $k \in \mathbb{R}$. This means that all the normal-mode solutions $\hat{\psi}(y,t) = \tilde{\psi}(y)\exp(-i\,\omega\,t)$ stay bounded according to (11) with $\epsilon = \delta$. Conversely, if (11) holds for all the normal-mode solutions, all ω_i, thus c_i in Eq. (7), must be non-positive. Since Eq. (7) has real coefficients, if c is an eigenvalue, so is its complex conjugate c^*. Therefore, for $c_i \leq 0$ we must have that $c_i = 0$, thus proven. ∎

The following theorem proves that the Lyapunov stability given by the *Definition 1* is a stronger condition than the eigenvalue stability given by the *Definition 2*.

Theorem 1: If the PDE (5) is stable according to the *Definition 1*, it is also stable according to the *Definition 2*. Conversely, if (5) is unstable according to the *Definition 2*, it is also unstable according to the *Definition 1*.

Proof: Suppose the PDE (5) is stable according to the *Definition 1*. Consider a solution to this PDE in the form $\hat{\psi}(y,t) = \tilde{\psi}(y)\exp(-i\,\omega\,t)$. According to the *Definition 1*, the Equation (10) is valid, therefore (11) is valid, and the PDE is stable according to the *Definition 2*.

Now suppose (5) is unstable according to the *Definition 2*. Then for some solution $\hat{\psi}(y,t) = \tilde{\psi}(y)\exp(-i\,\omega\,t)$ of this PDE, the condition (11) is violated. For the same solution, the condition (10) is also violated, and the system is unstable according to the *Definition 1*.

∎

Theorem 1 implies that even if the PDE system (5) is eigenvalue-stable, that is all eigenvalues of the Rayleigh equation are real, it does not necessarily mean that the continuous-time PDE system (5) is stable to any general perturbations.

B. EIGENVALUE STABILITY

We now state two important theorems that were previously established for the eigenvalue stability of the linearized iincompressible Euler system.

Theorem 2: Rayleigh's criterion [1] (first necessary condition for eigenvalue instability). For $c_i \neq 0$ in Eq. (7), it is necessary that U_{yy} changes sign somewhere in $[-h,h]$, i.e. $U(y)$ must have an inflection point in $[-h,h]$. Conversely, if U_{yy} has the same sign in $[-h,h]$ (no inflection point), all eigenvalues of Eq. (7) must be real ($c_i = 0$).

Proof: Multiplying Eq. (7) by the complex conjugate of $\tilde{\psi}^*$ and integrating over $[-h,h]$ yields

$$\int_{-h}^{h} \tilde{\psi}^* \left(\tilde{\psi}_{yy} - k^2\tilde{\psi} - \frac{U_{yy}}{U-c}\tilde{\psi}\right) dy = 0, \tag{12}$$

which can be manipulated as

$$\int_{-h}^{h} \frac{U_{yy}(U-c_r)}{\|U-c\|^2}\|\tilde{\psi}\|^2 dy + ic_i \int_{-h}^{h} \frac{U_{yy}}{\|U-c\|^2}\|\tilde{\psi}\|^2 dy = -\int_{-h}^{h} \left(\|\tilde{\psi}_y\|^2 + k^2\|\tilde{\psi}\|^2\right) dy. \tag{13}$$

Considering the imaginary part of Eq. (13),

$$c_i \int_{-h}^{h} \frac{U_{yy}}{\|U-c\|^2}\|\tilde{\psi}\|^2 dy = 0, \tag{14}$$

it immediately follows that if U_{yy} is of the same sign over $[-h,h]$, then $c_i = 0$, and for $c_i \neq 0$ to exist, U_{yy} must change sign over the interval. ∎

Theorem 3: Fjørtoft's criterion [2] (second necessary condition for eigenvalue instability). For the eigenvalue instability of Eq. (5), the Rayleigh criterion needs to be satisfied (there must be an inflection point, y_c, in the velocity profile), and there must exist some region in $[-h,h]$, where the following condition holds:

$$U_{yy}(U - U_c) < 0, \tag{15}$$

where $U_c = U(y_c)$.

Proof: Considering the real part of Eq. (13), we have

$$\int_{-h}^{h} \frac{U_{yy}(U-c_r)}{\|U-c\|^2}\|\tilde{\psi}\|^2 dy < 0. \tag{16}$$

If flow is unstable, than there exists at least one $c_i \neq 0$. Multiplying Eq. (14) by $(c_r - U_c)/c_i$, since $c_i \neq 0$, and adding it to Eq. (16), we obtain

$$\int_{-h}^{h} \frac{U_{yy}(U-U_c)}{\|U-c\|^2}\|\tilde{\psi}\|^2 dy < 0, \tag{17}$$

which proof the theorem. ∎

C. LYAPUNOV STABILITY

We now develop the stability analysis framework for the original, continuous in time, PDE problem given by Eq. (5). Following [16], we cast Eq. (5) into its equivalent Partial Integral Equation representation.

1) REPRESENTATION OF A PDE AS A PARTIAL-INTEGRAL EQUATION: We first rewrite Eq. (5) in its state-space form

$$\hat{\Delta}^2\dot{\hat{\psi}} = ikU_{yy}\hat{\psi} - ik\,U\hat{\Delta}^2\hat{\psi}, \tag{18}$$

where the notation $\dot{\hat{\psi}} = \partial\,\hat{\psi}/\partial\,t$ is used. Splitting the complex-valued stream function into is real and imaginary

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components, $\hat{\psi} = \hat{\psi}_r + i\,\hat{\psi}_i$, and performing some straightforward manipulations of Eq. (18) leads to

$$\begin{bmatrix} \dot{\hat{\psi}}_R \\ \dot{\hat{\psi}}_I \end{bmatrix} - \frac{1}{k^2} \begin{bmatrix} \dot{\hat{\psi}}_{R\,yy} \\ \dot{\hat{\psi}}_{I\,yy} \end{bmatrix} = (\frac{1}{k} U_{yy} + k\,U)\,S \begin{bmatrix} \hat{\psi}_R \\ \hat{\psi}_I \end{bmatrix} - \frac{U}{k}\,S \begin{bmatrix} \hat{\psi}_{R\,yy} \\ \hat{\psi}_{I\,yy} \end{bmatrix}, \tag{19}$$

where $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ denotes a 2×2 skew-identity matrix. The set of boundary conditions (6) can now be redefined as

$$B(\hat{\psi}_r, \hat{\psi}_i) : \begin{cases} \hat{\psi}_r(-h,t) = \hat{\psi}_r(h,t) = 0 \\ \hat{\psi}_i(-h,t) = \hat{\psi}_i(h,t) = 0 \end{cases} \tag{20}$$

A PDE in its state-space form given by Eq. (19), together with the boundary conditions (20), will now be transformed into its equivalent Partial Integral Equation (PIE) representation. The transformation procedure consists of the following three steps [15], [16].

Step 1. First, we must choose the fundamental (or PIE) state, which typically coincides with the set of the highest-order spatial derivatives of the solution variables appearing in the PDE, as $\hat{\psi}^{\text{PIE}}(y,t) = [\hat{\psi}_{R\,yy},\, \hat{\psi}_{I\,yy}]^T$ in our case. A significance of the PIE state is that, since $\hat{\psi}^{\text{PIE}}(y,t) \in L_2[-h,h]$, it does not need to satisfy boundary conditions.

Step 2. Next, a unique transformation can be written between the original PDE state, $\hat{\psi}(y,t) = [\hat{\psi}_R,\, \hat{\psi}_I]^T$, and the PIE state as

$$\hat{\psi}(y,t) = \mathcal{T} \hat{\psi}^{\text{PIE}}(y,t), \tag{21}$$

with $\mathcal{T} = \mathcal{T}_{\{R_0,R_1,R_2\}}$ being a partial-integrator (PI) operator defined in the following [15].

Definition 3: A partial integral operator $\mathcal{P} = \mathcal{P}_{\{R_0,R_1,R_2\}}$ acting on a fundamental state function $\hat{\psi}^{\text{PIE}}(y,t) \in L_2[-h,h]$ is defined as a three-component operator, according to,

$$\mathcal{P}\hat{\psi}^{\text{PIE}}(y,t) = \mathcal{P}_{\{R_0,R_1,R_2\}}\hat{\psi}^{\text{PIE}}(y,t) = R_0(y)\hat{\psi}^{\text{PIE}}(y,t) + \int_{-h}^{y} R_1(y,s)\hat{\psi}^{\text{PIE}}(s,t)\,d\,s + \int_{y}^{h} R_2(y,s)\hat{\psi}^{\text{PIE}}(s,t)\,d\,s, \tag{22}$$

where $\{R_0(y), R_1(y,s), R_2(y,s)\}$ are the matrices with the entries that are polynomials in the variables y and s [15].

The transformation (21) can be obtained through a multiple application of the fundamental theorem of calculus while taking into account the boundary conditions.

Step 3. The final step in the transformation is to substitute the PDE-to-PIE state map given by (21) into the original PDE system (19) to arrive at the following PIE equation:

$$\mathcal{M}\dot{\hat{\psi}}^{\text{PIE}} = \mathcal{A}\hat{\psi}^{\text{PIE}}, \tag{23}$$

with

$$\mathcal{M} = \mathcal{T} - \frac{1}{k^2}\mathcal{I}, \quad \mathcal{A} = \left((\frac{1}{k} U_{yy} + k\,U)\mathcal{T} - \frac{U}{k}\mathcal{I} \right) S, \tag{24}$$

where the PI identity operator has been defined as $\mathcal{I} = \mathcal{P}_{\{I,0,0\}}$, with I being a 2×2 identity matrix. The exact structure of the operator $\mathcal{T}$ will depend on the domain size and the boundary conditions [18]. For the boundary conditions considered here, and the domain geometry of $[-h,h]$, the $\mathcal{T}$ operator is given by $\mathcal{T}$=$\mathcal{T}_{\{0,R_1,R2\}}$, with

$$R_1 = \frac{1}{2}(\frac{y}{h} - 1)(\frac{s}{h} + 1)I, \;\; R_2 = \frac{1}{2}(\frac{y}{h} + 1)(\frac{s}{h} - 1)I. \tag{25}$$

We further note that, as long as the mean flow profile $U(y)$ is polynomial, the operator $\mathcal{A}$ can also be written as a PI operator [15], [18]. In case of a non-polynomial flow profile, its expansion into a sum of polynomial series can be recommended. The conversion of a PDE to a PIE can be performed automatically in the open-source software PIETOOLS [18], [19].

2) ON THE EQUIVALENCE OF EIGENVALUES BETWEEN PDE AND PIE SYSTEMS: Before proceeding with the Lyapunov stability analysis of the PIE representation, we first remark on the eigenvalue properties of the PIE system. Assuming a normal mode decomposition of the PIE state, $\hat{\psi}^{\text{PIE}}(y,t) = \tilde{\psi}^{\text{PIE}}(y)\exp(-i\omega t)$ and substituting this solution form into the PIE system (23) gives the following Rayleigh's equation analog for the PIE system:

$$(U - c)(\tilde{\psi}^{\text{PIE}} - k^2\mathcal{T}\tilde{\psi}^{\text{PIE}}) = U_{yy}\mathcal{T}\tilde{\psi}^{\text{PIE}}, \tag{26}$$

which could also be obtained via substituting the transformation (21) directly into the Rayleigh equation (7). Furthermore, the "PIE-Rayleigh" equation (26) can be manipulated in the same way, via multiplying both sides of the equation by the complex conjugate of $\mathcal{T}\tilde{\psi}^{*\text{PIE}}$ and integrating over the domain as

$$\int_{-h}^{h} \mathcal{T}\tilde{\psi}^{*\text{PIE}} \left(\tilde{\psi}^{\text{PIE}} - k^2\mathcal{T}\tilde{\psi}^{\text{PIE}} - \frac{U_{yy}}{U - c}\mathcal{T}\tilde{\psi}^{\text{PIE}} \right) dy = 0. \tag{27}$$

By further manipulation, it can be reduced to a form

$$\int_{-h}^{h} \frac{U_{yy}(U - c_r)}{\|U - c\|^2} \|\mathcal{T}\tilde{\psi}^{\text{PIE}}\|^2 dy + ic_i \int_{-h}^{h} \frac{U_{yy}}{\|U - c\|^2} \|\mathcal{T}\tilde{\psi}^{\text{PIE}}\|^2 dy = -\int_{-h}^{h} \left(\|\mathcal{T}_1\tilde{\psi}^{\text{PIE}}\|^2 + k^2\|\mathcal{T}\tilde{\psi}^{\text{PIE}}\|^2 \right) dy, \tag{28}$$

where $\mathcal{T}_1$ is a partial integral operator which relates the PIE state $\hat{\psi}(y,t)$ to the first derivative of the PDE state,

$$\hat{\psi}_y(y,t) = \mathcal{T}_1\hat{\psi}^{\text{PIE}}(y,t). \tag{29}$$

Since Eq. (28) contains quadratic terms in the same positions as Eq. (13), the same arguments that proved the Rayleigh's and Fjørtoft's criteria for the eigenvalues [1], [2] for the PDE system can be equivalently extended to the PIE system. This establishes that the eigenvalues of the PDE and the equivalent PIE system obey the same properties,

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3) STABILITY ANALYSIS OF PIE USING LINEAR PARTIAL INEQUALITIES: To present the methodology for a stability analysis of the PDE (5) in the PIE form (23), we first extend the definition of stability to a PIE system.

Definition 4: The PIE system (23) is Lyapunov-stable in L_2 if there exist constants δ, $\epsilon > 0$ such that for any $\|\hat{\psi}^{\text{PIE}}(y,0)\|_{L_2} < \delta$, a solution $\hat{\psi}^{\text{PIE}}(y,t)$ of the PIE satisfies

$$\|\mathcal{M}\hat{\psi}^{\text{PIE}}(y,t)\|_{L_2} < \epsilon, \;\; \forall t \geq 0 \tag{30}$$

To prove stability of a PIE, a Lyapunov-based approach can be derived, which is stated in the following theorem.

Theorem 4: Suppose there exist constants δ, $\alpha > 0$, and a self-adjoint coercive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that $\mathcal{P} = \mathcal{P}^*$, $\left\langle \hat{\psi}^{\text{PIE}}, \mathcal{P}\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} \geq \alpha \|\hat{\psi}^{\text{PIE}}\|^2_{L_2}$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A} \leq 0, \tag{31}$$

where $\mathcal{M}$, $\mathcal{A}$ are as defined in Eq. (24). Then any solution of the PIE system (23) with $\|\hat{\psi}^{\text{PIE}}(y,0)\|_{L_2} < \delta$ satisfies

$$\|\mathcal{M}\hat{\psi}^{\text{PIE}}(y,t)\|_{L_2} < \epsilon, \;\; \forall t \geq 0, \tag{32}$$

with $\epsilon = \xi_M \delta$, where $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$.

Proof: Suppose $\hat{\psi}^{\text{PIE}}(y,t)$ solves the PIE system (23) for some $\hat{\psi}^{\text{PIE}}(y,0)$ satisfying

$$\|\hat{\psi}^{\text{PIE}}(y,0)\|_{L_2} < \delta. \tag{33}$$

Consider the candidate Lyapunov function defined as

$$V(\hat{\psi}^{\text{PIE}}) = \left\langle \mathcal{M}\hat{\psi}^{\text{PIE}}, \mathcal{P}\mathcal{M}\,\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} \geq \alpha \|\mathcal{M}\,\hat{\psi}^{\text{PIE}}\|^2_{L_2}. \tag{34}$$

The derivative of V along the solution trajectory $\hat{\psi}^{\text{PIE}}(y,t)$ is

$$\begin{aligned} \dot{V}(\hat{\psi}^{\text{PIE}}) &= \\ \left\langle \mathcal{M}\dot{\hat{\psi}}^{\text{PIE}}, \mathcal{P}\mathcal{M}\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} + \left\langle \mathcal{M}\hat{\psi}^{\text{PIE}}, \mathcal{P}\mathcal{M}\,\dot{\hat{\psi}}^{\text{PIE}} \right\rangle_{L_2} &= \\ \left\langle \mathcal{A}\hat{\psi}^{\text{PIE}}, \mathcal{P}\mathcal{M}\,\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} + \left\langle \mathcal{M}\hat{\psi}^{\text{PIE}}, \mathcal{P}\mathcal{A}\,\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} &= \\ \left\langle \hat{\psi}^{\text{PIE}}, (\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A})\,\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} &\leq 0 \end{aligned} \tag{35}$$

Utilizing (35) together with (33) and (34) proves the theorem. ■

4) EQUIVALENCE TO PDE STABILITY: We finally establish the equivalence between stability of the PIE equation (23) and the original PDE equation (5).

Theorem 5: Suppose there exist constants $\delta, \alpha > 0$, and a self-adjoint coercive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that $\mathcal{P} = \mathcal{P}^*$, $\left\langle \hat{\psi}^{\text{PIE}}, \mathcal{P}\hat{\psi}^{\text{PIE}} \right\rangle_{L_2} \geq \alpha \|\hat{\psi}^{\text{PIE}}\|^2_{L_2}$, and

$$\mathcal{A}^*\mathcal{P}\mathcal{M} + \mathcal{M}^*\mathcal{P}\mathcal{A} \leq 0, \tag{36}$$

where $\mathcal{M}$, $\mathcal{A}$ are as defined in Eq. (24). Then there exists a constant $\sigma > 0$, such that any solution of the PDE system (5) with boundary conditions (6) and $\|\hat{\psi}(y,0)\|_{H_2} < \delta$ satisfies

$$\|\hat{\psi}(y,t)\|_{L_2} < \sigma, \;\; \forall t \geq 0 \tag{37}$$

Proof: We first note that, since $\|\hat{\psi}(y,0)\|_{H_2} < \delta$, $\|\hat{\psi}(y,0)\|_{H_2} = \|\hat{\psi}(y,0)\|_{L_2} + \|\hat{\psi}_y(y,0)\|_{L_2} + \|\hat{\psi}_{yy}(y,0)\|_{L_2}$ and $\|\cdot\|_{L_2} \geq 0$, we have $\|\hat{\psi}^{\text{PIE}}(y,0)\|_{L_2} = \|\hat{\psi}_{yy}(y,0)\|_{L_2} < \delta$.

Denote $\xi_M = \|\mathcal{M}\|_{\mathcal{L}(L_2)}$, $\xi_T = \|\mathcal{T}\|_{\mathcal{L}(L_2)}$. Since the PIE operator $\mathcal{M} = \mathcal{P}_{\{-I/k^2,0,0\}} + \mathcal{T}_{\{0,R_1,R_2\}}$ contains an invertible multiplicative part, $\mathcal{M}$ is invertible with a well-defined inverse (see [18] for a proof). Let's call its inverse $\mathcal{M}^{-1}$ and define its upper bound as $\xi'_M = \|\mathcal{M}^{-1}\|_{\mathcal{L}(L_2)}$. Denoting $\hat{\hat{\psi}}(y,t) = \mathcal{M}\hat{\psi}^{\text{PIE}}(y,t)$, and given that $\mathcal{M}$ is invertible, we can write $\hat{\psi}^{\text{PIE}}(y,t) = \mathcal{M}^{-1}\hat{\hat{\psi}}(y,t)$. Note that $\|\hat{\hat{\psi}}(y,t)\|_{L_2} < \xi_M\delta, \;\; \forall t \geq 0$ according to Theorem 4.

Now, considering that $\hat{\psi}(y,t) = \mathcal{T}\hat{\psi}^{\text{PIE}}(y,t)$ according to (21), we have

$$\begin{aligned} \|\hat{\psi}(y,t)\|_{L_2} =& \|\mathcal{T}\hat{\psi}^{\text{PIE}}(y,t)\|_{L_2} = \|\mathcal{T}\mathcal{M}^{-1}\hat{\hat{\psi}}(y,t)\|_{L_2} \\ &< \xi_T\xi'_M\xi_M\delta, \end{aligned} \tag{38}$$

which completes the proof. ■

V. RESULTS

We now present the results of the stability analysis of the linearized incompressible Euler equation with the developed LPI formulation using different mean velocity profiles, $U(y)$. Without loss of generality, we consider the vertical extent of the domain as $[-1,1]$, i.e. we set $h = 1$. Three groups of mean velocity profiles are considered:

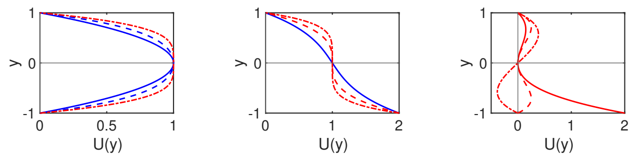
1) R-stable (no inflection point):
   a) $U(y) = 1 - y^2$
   b) $U(y) = (2 - y^2 - y^4)/2$
   c) $U(y) = 1 - y^4$

   These profiles do not satisfy the Rayleigh criterion for instability (no inflection point). Therefore, they are eigenvalue-stable.
2) F-stable (inflection point exists but Fjørtoft's criterion is not satisfied):
   a) $U(y) = 1 - (y + y^3)/2$
   b) $U(y) = 1 - y^3$
   c) $U(y) = 1 - y^5$

   These profiles satisfy the Rayleigh criterion for instability (inflection point exists), but they do not satisfy Fjørtoft's criterion. Therefore, they are eigenvalue-stable.
3) RF-unstable (inflection point exists and Fjørtoft's criterion is satisfied):
   a) $U(y) = y^2 - y^3$
   b) $U(y) = y^2 - y^4$
   c) $U(y) = y - y^3$

   These profiles satisfy the Rayleigh criterion for instability (inflection point exists), and they satisfy Fjørtoft's criterion. Therefore, they may be unstable.

Three groups of flow profiles are visualized in Figure 1. The first group corresponds to typical velocity profiles established in a channel between two parallel plates, with the profile (1a) describing a laminar fully-developed Poiseuille flow, and the other two are of increasing bluntness, which is typically associated with turbulent flow conditions as

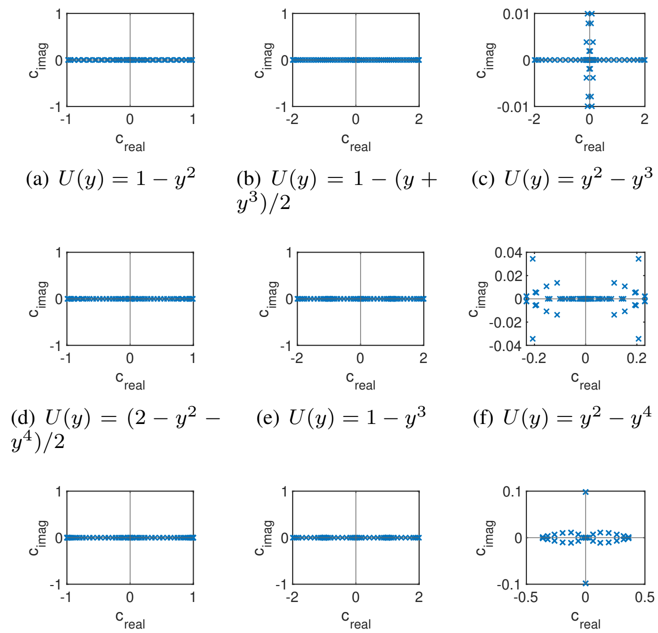


(a) R-stable (b) F-stable (c) RF-unstable

Fig. 1: $U(y)$ velocity profiles used for stability tests. Profiles that are proven to be stable in a continuous analysis via LPI tests are shown in blue, all the others are shown in red. In each of three groups: the profiles labeled with the letter (a) are plotted with solid lines, (b) with dashed lines, and (c) with dash-dotted lines.

TABLE I: Results of eigenvalue and LPI stability analysis.

Profiles	Eigenvalues, $\max(c_i)$	LPI stability SeDuMi / Mosek
R-stable:		
(1a) $U(y) = 1 - y^2$	0	Stable
(1b) $U(y) = (2 - y^2 - y^4)/2$	0	Stable
(1c) $U(y) = 1 - y^4$	0	Unstable
F-stable:		
(2a) $U(y) = 1 - (y + y^3)/2$	0	Stable
(2b) $U(y) = 1 - y^3$	0	Unstable
(2c) $U(y) = 1 - y^5$	0	Unstable
RF-unstable:		
(3a) $U(y) = y^2 - y^3$	0.009	Unstable
(3c) $U(y) = y^2 - y^4$	0.034	Unstable
(3b) $U(y) = y - y^3$	0.098	Unstable



(a) $U(y) = 1 - y^2$ (b) $U(y) = 1 - (y + y^3)/2$ (c) $U(y) = y^2 - y^3$

(d) $U(y) = (2 - y^2 - y^4)/2$ (e) $U(y) = 1 - y^3$ (f) $U(y) = y^2 - y^4$

(g) $U(y) = 1 - y^4$ (h) $U(y) = 1 - y^5$ (i) $U(y) = y - y^3$

Fig. 2: Eigenvalues of the PIE system (23) as computed in PIESIM for $k = 1$. From left to right: R-Stable, F-stable, and RF-unstable profiles.

Reynolds number increases [20]. The second group resembles a flow established between two parallel moving plates, but deviating from a laminar Couete flow solution (which is linear). The third group exemplifies strongly unstable profiles that resemble the wake flows (dashed line), flows with a recirculation region (negative velocity, such as in a separation bubble, dash-dotted line) and a wake region interacting with a bottom moving plate or a jet (solid line).

Stability tests were executed in PIETOOLS 2024 software [18] created specifically for manipulation of Partial Integral Equations [19]. In PIETOOLS, an LPI feasibility test as stated in Theorem 4 is executed via searching for a positive PI operator $\mathcal{P}_{\{R_0,R_1,R_2\}}$ such that the inequality (31) is satisfied. The problem is cast as a convex optimization problem in MATLAB and is solved using standard semi-definite programming (SDP) solvers. In this work, both SeDuMi and Mosek were used as the SDP solvers, giving equivalent results. To check the eigenvalue properties of the PIE system, the PIESIM package of PIETOOLS was utilized [21]. PIESIM performs a high-order discretization of initial boundary-value and boundary-value problems defined by PI operators utilizing Chebyshev Galerkin method [21]. Since PIE problem does not involve imposition of boundary conditions on its solution state (they are implicitly encoded into the structure of the PIE operators) the use of Cheyshev polynomials of the first kind to approximate the PIE state is permissible regardless of the boundary conditions of the underlying PDE problem. In this work, $N = 32$ Chebyshev polynomial modes were utilized to calculate discrete eigenvalues of the PIE system given by Eq. (26).

The value of the wavenumber $k = 1$ was used in the current analysis. The sensitivity of results to the wavenumber value will be investigated elsewhere. The results of the eigenvalue analysis and the LPI stability tests are summarized in Table I. Figure 2 additionally provides the plots of eigenvalues of the PIE system for all the velocity profiles as computed in PIESIM.

VI. CONCLUSIONS

The results indicate that, as expected, all R-stable and F-stable profiles demonstrate eigenvalue stability (as can be theoretically proven using Rayleigh's [1] and Fjørtoft's [2] theorems for PDE and PIE systems). RF-unstable profiles investigated here confirm the presence of unstable eigenvalues, although this, in general, is not guaranteed, since Rayleigh's and Fjørtoft's conditions only represent necessary and not sufficient conditions for the eigenvalue instability.

The LPI stability tests, however, indicate that not all R-stable and F-stable profiles are provably stable in the Lyapunov sense. The existence of a Lyapunov function guarantees the system stability and provides a sufficient condition for stability, which eigenvalue-based analysis is unable to provide. We are able to show that classical Poiseuille flow profile is indeed provably stable in a linearized formulation (in the absence of viscosity) to generic infinitesimal perturbations and not just in the eigenvalue sense. However, as profiles become more blunt (as would be expected, for example,

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in turbulent flows with increasing Reynolds number [20]), the instabilities worsen. Indeed, while the second profile in group 1, $U(y) = (2-y^2-y^4)/2$, is still Lyapunov-stable, the last one, which is also the most blunt, with $U(y) = 1 - y^4$, is found inviscidly unstable by the LPI tests even in the absence of inflection point. The same can be said about F-stable profiles: only the first out of three profiles (expectedly, the least blunt) was proven stable, while the other two indicated instabilities in a continuous test. Therefore, neither Rayleigh nor Fjørtoft criteria are able to provide a sufficient condition for stability considering a time-continuous formulation. The blunt profiles between two moving plates are known to correspond to either an impulsively starting laminar solution [22], or a fully-developed turbulent solution [23], [24]. Indication of inviscid inflectional instabilities for these profiles (even without satisfying Fjørtoft criterion) points towards a potential importance of linear inviscid mechanisms in transitional and turbulent dynamics of these flows at high Reynolds numbers which was previously overlooked. Regarding the three investigated profiles that are marked as potentially unstable by Rayleigh's and Fjørtoft's conditions (and indeed possess unstable eigenvalues), were confirmed to be unstable by the LPI tests showing reliability of the presented PIE-based analysis framework as a tool for analyzing Lyapunov stability of the PDE systems in a continuous time formulation.

A further extension of this work will consider an application of LPI stability tests to nonlinear PDE systems, stability analysis in two and higher spatial dimensions (that is without invoking a Fourier transformation in x) and systems in cylindrical coordinates (such as pipe flows). Additionally, future work will develop stabilizing controllers for unstable fluid flow systems by posing the controller synthesis problem as an LPI feasibility test [25].

VII. ACKNOWLEDGEMENTS

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