UNIT - 'VI' 5/8/11

VECTOR INTEGRAL CALCULUS

Line integral:

Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

then $\int_c \bar{F} \cdot d\bar{r}$ is called line integral

closed curve, smooth curve

along the smooth or closed curve 'c'.

Circulation:

If $\bar{v}$ represents the velocity of a fluid particle and 'c' is a closed curve, then integral $\oint_c \bar{v} \cdot d\bar{r}$ is called the circulation of $\bar{v}$ around the curve 'c'.

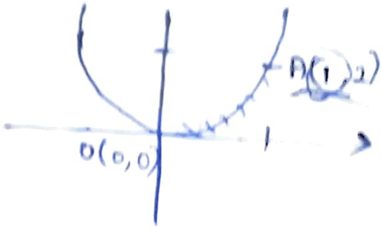
Work done by a force:

If $\bar{F}$ represents the force vector acting on a particle and 'c' is a ~~closed~~ open curve, then the integral $\int_c \bar{F} \cdot d\bar{r}$ is called ~~circulation of $\bar{v}$ around~~ work done by a force along the curve 'c'.

Problems:

1) If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, evaluate $\int_c \bar{F} \cdot d\bar{r}$ where 'c' is the curve $y = 2x^2$ in the xy-plane from $(0,0)$ to $(1,2)$. $x = \sqrt{\frac{y}{2}}$

Sol: Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$



$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$

$\bar{F} \cdot d\bar{r} = f_1 dx + f_2 dy + f_3 dz$

In xy plane, $z = 0 \Rightarrow dz = 0$

$\Rightarrow \bar{F} \cdot d\bar{r} = f_1 dx + f_2 dy$

$$\Rightarrow \int_C \bar{F}\cdot\bar{dr} = \int_0^A 3xy\,dx - y^2\,dy$$

let $y = 2x^2$

$dy = 4x\,dx$

$$\Rightarrow \int_0^A 3(2x^2)x\,dx - 4x^4(4x)\,dx$$

$$\Rightarrow \int_{x=0}^1 6x^3\,dx - 16x^5\,dx$$

$$\Rightarrow 6\left(\frac{x^4}{4}\right)_0^1 - 16\left(\frac{x^6}{6}\right)_0^1$$

$$\Rightarrow \frac{6}{4}(1) - \frac{16}{6}(1) = \frac{36-64}{24} = \frac{-28}{24} = -\frac{7}{6}$$

8) Find the workdone by the force $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$ is moving particle from the point $(0,0,0)$ to $(1,1,1)$ along the curve C: $x = t,\ y = t^2,\ z = t^3$

Sol: Let $A(0,0,0)$ (x,y,z), $B(1,1,1)$ (x,y,z)

for $x = t$, $\begin{cases} x = 0 \Rightarrow t = 0 \\ x = 1 \Rightarrow t = 1 \end{cases}$

for $y = t^2$, $\begin{cases} y=0 \Rightarrow t=0 \\ y=1 \Rightarrow t^2=1 \\ \quad t = \pm 1 \Rightarrow \boxed{t=1} \end{cases}$

for $z = t^3$, $\begin{cases} z=0 \Rightarrow t=0 \\ z=1 \Rightarrow t^3=1 \\ \quad \boxed{t=1} \end{cases}$

$\therefore t \to 0$ to 1

$\therefore$ Workdone by force $= \int_C \bar{F} \cdot d\bar{r}$

$$= \int_A^B (3x^2 - 6yz)dx + (2y + 3zx)dy + (1 - 4xyz^2)dz$$

$$= \int_{t=0}^{1} (3t^2 - 6t^2t^3)dt + (2t^2 + 3t^4)(2t\,dt) + (1 - 4t\,t^2t^6)(3t^2dt)$$

$$= \int_0^1 (-12t^{11} + 4t^3 + 6t^2)dt$$

$$= \left[-t^{12} + t^4 + \frac{\cancel{6}^{2}t^3}{\cancel{3}}\right]_0^1$$

$$= -1 + 1 + 2 \Rightarrow 2$$

Evaluation of line integral.

1) Find the circulation of $\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$ along the circle $x^2 + y^2 = 4$ in the xy-plane.

Sol: let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$

$$= (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$$

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

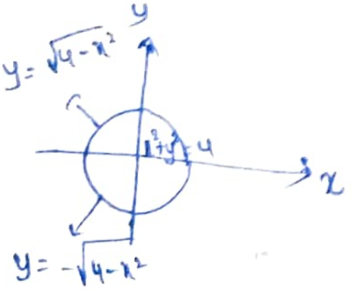
$$\bar{F} \cdot d\bar{r} = F_1 dx + F_2 dy + F_3 dz$$

$$\bar{F} \cdot d\bar{r} = (2x - y + 2z)dx + (x + y - z)dy + (3x - 2y - 5z)dz$$

In xy-plane: $z = 0 \Rightarrow dz = 0$

$$\bar{F} \cdot d\bar{r} = (2x - y)dx + (x + y)dy + 0$$

$\therefore$ Circulation $= \oint_C \bar{F} \cdot d\bar{r} = \oint_C (2x - y)dx + (x + y)dy$



$$= \int_{\theta=0}^{2\pi} [4\cos\theta - 2\sin\theta](-2\sin\theta)d\theta + [2\cos\theta + 2\sin\theta][2\cos\theta]d\theta$$

$$= \int_0^{2\pi} [4 - 4\sin\theta\cos\theta]d\theta$$

$x^2 + y^2 = 4$

$x = 2\cos\theta$, $y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta$

$dy = 2\cos\theta\, d\theta$

$\theta \to 0$ to 2π

$$= \int_0^{2\pi} [4 - 2\sin 2\theta] d\theta$$

$$= \left[4\theta - 2\left(\frac{-\cos 2\theta}{2}\right)\right]_0^{2\pi}$$

$$= 8\pi + \cos 4\pi - [\cos 0] = 8\pi + 1 - 1 = 8\pi$$

2) Find a workdone in moving a particle in the force field $\bar{F} = 3x^2\bar{i} + \bar{j} + z\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$

Sol: $\bar{F} \cdot \bar{dr} = F_1 dx + F_2 dy + F_3 dz = 3x^2 dx + dy + z dz$

Work done $= \int_C \bar{F} \cdot \bar{dr} = \int_{O(0,0,0)}^{A(2,1,3)} 3x^2 dx + dy + z dz$

$$= \left[x^3 + y + \frac{z^2}{2}\right]_{(0,0,0)}^{(2,1,3)}$$

$$= 2^3 + 1 + \frac{3^2}{2} - [0+0+0]$$

$$= 8 + 1 + \frac{9}{2} = \frac{27}{2}$$

3) Prove that the force field $\bar{F} = 2xyz^3\bar{i} + x^2z^3\bar{j} + 3x^2yz^2\bar{k}$, is conservative. Find the workdone by moving a particle from $(1,-1,2)$ to $(3,2,-1)$ in this force field.

Sol: conservative $\Rightarrow$ curl $\bar{F} = 0$

$$= \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^3 & x^2z^3 & 3x^2yz^2 \end{vmatrix} = 0$$

Hence $\bar{F}$ is conservative.

Let $\bar{F} = \nabla\phi \Rightarrow \left(d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz \right)$

Workdone $= \int_C \bar{F}\cdot d\bar{r} = \int_{A(1,-1,2)}^{B(3,2,-1)} F_1dx + F_2dy + F_3dz$

$$= \int_{(1,-1,2)}^{(3,2,-1)} 2xyz^3dx + x^2z^3dy + 3x^2yz^2dz$$

$$= \int_{(1,-1,2)}^{(3,2,-1)} d(x^2yz^3)$$

$$= \left[x^2yz^3\right]_{(1,-1,2)}^{(3,2,-1)}$$

$$= \left[3^2(2)(-1)^3\right] - \left[1^2(-1)2^3\right]$$

$$= -18 + 8 = -10$$

(or) $\frac{x-1}{3-1} = \frac{y+1}{2+1} = \frac{z-2}{-1-2} = t \Rightarrow x = 1+2t,\ y = -1+3t,\ z = 2-3t$

$dx = 2dt$, $dy = 3dt$, $dz = -3dt$

$t \to 0$ to $t = 1$

H.W: Eval. $\int_C yz\,dx + (xz+1)dy + xy\,dz$ where C is the st. line joining the line from $(0,0,0)$ to $(2,1,4)$ Ans: $(xyz+y)_0^A = 9$

4) If $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$, find the circulation of $\bar{F}$ round the curve 'c' is the circle $x^2+y^2=1$, $z=0$.

Sol:

$$\bar{F}\cdot d\bar{r} = y\,dx + z\,dy + x\,dz$$

$$z=0 \Rightarrow dz=0 \Rightarrow y\,dx + z\,dy \Rightarrow y\,dx$$

$x^2+y^2=1$

$x = \cos\theta$

$y = \sin\theta$

circulation $= \oint_c \bar{F}\cdot d\bar{r} = \oint_c y\,dx$

$dx = -\sin\theta\, d\theta$

$$\Rightarrow -\int_0^{2\pi} \sin^2\theta\, d\theta$$

$$\Rightarrow \left[-\frac{1}{2}x + \frac{1}{4}\sin 2x\right]_0^{2\pi}$$

$$\Rightarrow -\frac{1}{2}(2\pi] + 0 - [0] = -\pi$$

5) Find $\int_c \bar{F}\cdot d\bar{r}$, $F = x^2y\bar{i} + y\bar{j}$ and the curve $y^2=4x$ in the xy-plane from $(0,0)$ to $(4,4)$

Sol:

$$\bar{F}\cdot d\bar{r} = x^2y\,dx + y\,dy$$

$$z=0 \Rightarrow x^2y\,dx + y\,dy$$

$$y^2 = 4x \Rightarrow x = \frac{y^2}{4} \Rightarrow dx = \frac{2y\,dy}{4} = \frac{y\,dy}{2}$$

$$\Rightarrow \int_0^4 \frac{y^4}{16}\cdot \frac{y^2 dy}{2} + y\,dy = \left[\frac{y^7}{32\times 7} + \frac{y^2}{2}\right]_0^4$$

H.W.

If $F = (3x^2+6y)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$, eval $\int_c \bar{F}\cdot d\bar{r}$ where c is St. line joining $(0,0,0)$ to $(1,1,1)$ (A: 13/3)

H.W. eval $\int_c x\,dy - y\,dx$ around the circle $x^2+y^2=1$ // Ans: 2π

8) $F = (x-3y)\bar{i} + (y-2x)\bar{j}$, find $\int_C F \cdot dr$

where C is the closed curve in xy plane,

$x = 2\cos t$, $y = 3\sin t$ from $t = 0$ to 2π

Sol:

$$\bar{F} \cdot d\bar{r} = (x-3y)dx + (y-2x)dy$$

$$z = 0 \rightarrow (x-3y)dx + (y-2x)dy$$

$$x = 2\cos t \Rightarrow dx = -2\sin t\, dt$$

$$y = 3\sin t \Rightarrow dy = 3\cos t\, dt$$

$$\int_0^{2\pi} (2\cos t - 9\sin t) - 2\sin t\, dt +$$

$$(3\sin t - 4\cos t)\, 3\cos t\, dt$$

$$\int_0^{2\pi} -4\cos t \sin t\, dt + 18\sin^2 t\, dt$$

$$+ 9\sin t\cos t\, dt - 12\cos^2 t\, dt$$

$$\int_0^{2\pi} 5\sin t\cos t\, dt + 18\sin^2 t\, dt - 12\cos^2 t\, dt$$

$$\left(-\frac{5\cos^2 x}{2} + 18\left[\frac{t}{2} + \frac{\sin 2t}{4}\right] - 12\left(\frac{\sin 2t}{4} + \frac{t}{2}\right)\right)_0^{2\pi}$$

$$\Rightarrow -\frac{5}{2} + 18\left[\frac{2\pi}{2}\right] - 12\left[\frac{2\pi}{2}\right] - \left[-\frac{5}{2}\right]$$

$$= 18\pi - 12\pi = 6\pi$$

7) Find the workdone by force $\bar{F} = (x^2 - yz)\bar{i} + (y^2 - zx)\bar{j} + (z^2 - xy)\bar{k}$ in taking particle from $(1,1,1)$ to $(3,-5,7)$

Sol:

$$\bar{F} \cdot d\bar{r} = (x^2 - yz)dx + (y^2 - 2x)dy + (z^2 - xy)dz$$

$$\int_{(1,1,1)}^{(3,-5,7)} (x^2 - yz)dx + (y^2 - 2x)dy + (z^2 - xy)dz$$

$$\Rightarrow \left[\frac{x^3}{3} - xyz + \frac{y^3}{3} - 2xyz + \frac{z^3}{3} - xyz\right]_{(1,1,1)}^{(3,-5,7)}$$

$$\Rightarrow 9 + 3(5)(7)(2) - \frac{5^3}{3} + 10(3) + \frac{7^3}{3} - \left[\frac{1}{3} - 2 + \frac{1}{3} + \frac{1}{3} - 2\right]$$

$$\Rightarrow 9 + 210 - \frac{5^3}{3} + 30 + \frac{7^3}{3} + 3 = 252 - \frac{125}{3} + \frac{343}{3}$$

$$\frac{974}{3}$$

$$\Rightarrow \left[\frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - (xyz)\right]_{1,1,1}^{3,-5,7}$$

$$\frac{3^3}{3} - \frac{5^3}{3} + \frac{7^3}{3} + 3(5)(7) - [1 - 1]$$

$$= \frac{27}{3} - \frac{125}{3} + \frac{343}{3} + 105 = \frac{560}{3}$$

8) Find the work done by the force $F = 2\bar{i} + x\bar{j} + y\bar{k}$, when it moves a particle along the arc of the curve $\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} + t\bar{k}$ from $t=0$ to $t=2\pi$ (Ans: $-\pi$)

Surface integrals

Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ then $\int_S \bar{F}\cdot\bar{n}\,ds =$

$$\iint_S f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy$$

is called Surface integral over 'S'.

NOTE: Let R_1 be the projection of 's' on 'xy' plane. Then $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_1} \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

Illy in yz and zx planes we can write

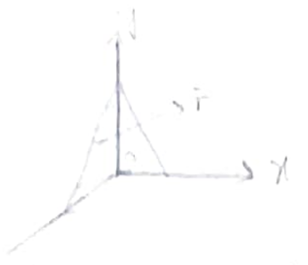
$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_2} \bar{F}\cdot\bar{n}\,\frac{dy\,dz}{|\bar{n}\cdot\bar{i}|}$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_{R_3} \bar{F}\cdot\bar{n}\,\frac{dz\,dx}{|\bar{n}\cdot\bar{j}|}$$

Problems:

1) Evaluate $\int_S \bar{F}\cdot\bar{n}\,ds$ where $\bar{F} = 18z\,\bar{i} - 12\bar{j} + 3y\,\bar{k}$ and 's' is the part of the surface of the plane $2x+3y+6z=12$ located in first octant.

Sol: Let $\phi = 2x+3y+6z-12$



Normal to 'ϕ' is $\nabla\phi$ (or) grad 'ϕ'

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = \bar{i}(2) + \bar{j}(3) + \bar{k}(6)$$

$$\therefore \nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \text{unit normal} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{\sqrt{36+9+4}} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$$

Let 'R' be the projection of 's' on xy - plane

then $\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

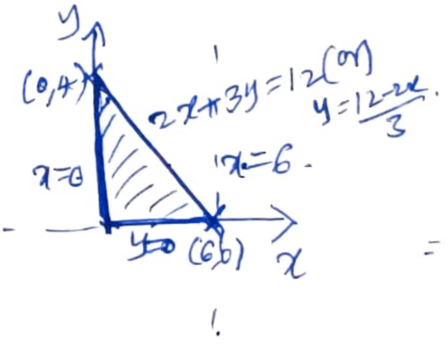
$$\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}, \quad \bar{n} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$$

$$\bar{F}\cdot\bar{n} = \frac{1}{7}(36z - 36 + 18y)$$

$$\bar{n}\cdot\bar{k} = \frac{6}{7}$$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$$

$$= \iint_S \frac{1}{7}(36z - 36 + 18y)\frac{dx\,dy}{6/7}$$



$$= \int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} [(12 - 2x - 3y) - 6 + 3y]\,dx\,dy$$

$$= \iint [6 - 2x]\,dx\,dy$$

$$= \int_{x=0}^{6} 2(3-x)[y]_0^{\frac{12-2x}{3}}$$

($\because$ xy plane, $z=0$

$2x+3y=12$

if $y=0 \Rightarrow x=6$

$x \rightarrow 0$ to 6

$2x+3y=12$

$y = \frac{1}{3}(12-2x)$

$y \rightarrow 0$ to $\frac{1}{3}(12-2x)$

$$= \int_{x=0}^{6} 2(3-x)\left(\frac{12-2x}{3}\right) dx$$

$$= \frac{4}{3}\int_{x=0}^{6} [18 - 9x + x^2]\, dx$$

$$\frac{4}{x}\left[18 - \frac{9x^2}{2} + \frac{x^3}{3}\right]_{x=0}^{6}$$

$$= \frac{4}{6}\left[18 - \frac{9(36)}{2} + \frac{6^3}{3}\right]$$

$$= 24$$

8) If $\bar{F} = (x + y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$, Evaluate $\int_S \bar{F}\cdot\bar{n}\, ds$ where s is the surface of plane $2x + y + 2z = 6$ in first octant. (Ans: 81)

Sol: $\phi = 2x + y + 2z = 6$

$$\nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z} = 2\bar{i} + \bar{j} + 2\bar{k}$$

$$n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i} + \bar{j} + 2\bar{k}}{\sqrt{9}} = \frac{1}{3}(2\bar{i} + \bar{j} + 2\bar{k})$$

$$\bar{F} = (x + y^2)\bar{i} - 2x\bar{j} + 2yz\bar{k}$$

$$n = \frac{1}{3}(2\bar{i} + \bar{j} + 2\bar{k})$$

$$\bar{F}\cdot\bar{n} = \frac{1}{3}\left(\cancel{2x} + 2y^2 - \cancel{2x} + 4yz\right)$$

$$= \frac{1}{3}\,2y(y - z)$$

$$\bar{n}\cdot\bar{k} = \frac{2}{3}$$

$$\iint_S \bar{F}\cdot\bar{n}\,\frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = \iint \frac{1}{3}(2y^2 + 4yz)\,\frac{dx\,dy}{2/3}$$

$$= \iint y^2 + 2yz\,dx\,dy$$

$2x + y + 2z = 6$

$z = 0 \Rightarrow 2x + y = 6$

If $y = 0$

$x = 3$

$x \to 0$ to 3

$2x + y = 6$

$y = 6 - 2x$

$y \to 0$ to $6 - 2x$

$$= \iint y^2 + y(6 - 2x - y)\,dx\,dy$$

$$= \iint \cancel{y^2} + 6y - 2xy - \cancel{y^2}\,dx\,dy$$

$$\Rightarrow \iint \left(\frac{6y^2}{2} - \frac{2xy^2}{\cancel{2}}\right)dx\,dy$$

$$\Rightarrow \int_0^3 \cancel{6}^{\,3}\frac{(6-2x)^2}{\cancel{2}} - x(6-2x)^2\,dx$$

$$\Rightarrow \int_0^3 3(36 + 4x^2 - 24x) - x(36 + 4x^2 - 24x)\,dx$$

$$= \left[108x + \frac{12x^3}{3} - \frac{72x^2}{2} - \frac{36x^2}{2} - \frac{4x^3}{3} + \frac{24x^3}{3}\right]_0^3$$

$$= \left[108x - \frac{32x^3}{3} - \frac{108x^2}{2}\right]_0^3$$

$$= 324 - 288 - 486$$

$$\Rightarrow \iint y(6-2x)\,dx\,dy$$

$$\Rightarrow \int_{x=0}^{3} (6-2x)\left[\frac{y^2}{2}\right]_0^{6-2x}$$

$$\Rightarrow \frac{1}{2}\int_{x=0}^{3} 6-2x\,(6-2x)^2$$

$$\Rightarrow \frac{\cancel{2}}{\cancel{2}}\int_{x=0}^{3} (3-x)(36+4x^2-24x)$$

$$\Rightarrow \int_{x=0}^{3} 3x^2-18x+27-x^3+6x^2-9x$$

$$\Rightarrow 4\left[x^3-9x^2+27x-\frac{x^4}{4}+2x^3-\frac{9x^2}{2}\right]_0^3$$

$$\Rightarrow 4\left(27-81+81-\frac{81}{4}+2(27)-\frac{81}{2}\right)$$

$$\Rightarrow 81$$

Volume integrals.

1) Let $\bar{F} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ and $dv = dx\,dy\,dz$

The volume integral given by

$$\int_V \bar{F}\,dv = \iiint (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})\,dx\,dy\,dz$$

$\Rightarrow$ If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2\bar{k}$, Evaluate $\int_V \bar{F}\,dv$, where V is the region bounded by the surfaces $x=0$, $x=2$, $y=0$, $y=6$ and $z=x^2$ and $z=4$

Sol:-

$$\int_V \bar{F}\cdot dv = \iiint (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\,dx\,dy\,dz$$

$$= \iiint (2xz\,\bar{i} - x\bar{j} + y^2\bar{k})\,dx\,dy\,dz$$

$$= \bar{i}\int_0^2\int_0^6\int_{x^2}^4 2xz\,dx\,dy\,dz - \bar{j}\int_0^2\int_0^6\int_{x^2}^4 x\,dx\,dy\,dz$$

$$+ \bar{k}\int_0^2\int_0^6\int_{x^2}^4 y^2\,dx\,dy\,dz$$

$$= \bar{i}\int_0^2 2x\int_{y=0}^6 \left(\frac{z^2}{2}\right)_{z=x^2}^4 dy\,dx - \bar{j}\int_0^2 x\int_{y=0}^6 (z)_{x^2}^4\,dy\,dx$$

$$+ \bar{k}\int_{x=0}^2\int_{y=0}^6 y^2(z)_{x^2}^4\,dy\,dx$$

$$= \bar{i}\int_{x=0}^{2} x \int_{y=0}^{6} 16-x^2\,dy\,dx - \bar{j}\int_{0}^{2} x \int_{y=0}^{6} (4-x^2)\,dy\,dx$$

$$+ \bar{k}\int_{0}^{2}\int_{y=0}^{6} y(4-x^2)\,dy\,dx$$

$$\Rightarrow \bar{i}\int_{x=0}^{2} 16x-x^5\,(6-0) - \bar{j}\int_{x=0}^{2} 4x-x^3\,(6-0)\,dx$$

$$+ \frac{\bar{k}}{3}\int_{x=0}^{2} (4-x^2)(6^3-0)\,dx$$

$$= 6\bar{i}\left[16\frac{x^2}{2} - \frac{x^6}{6}\right]_0^2 - 6\bar{j}\left[\frac{4x^2}{2} - \frac{x^4}{4}\right]_{x=0}^{2}$$

$$+ \frac{\bar{k}}{3}(6^3)\left[4x - \frac{x^3}{3}\right]_{x=0}^{2}$$

$$\therefore \int_V \bar{F}\cdot dV = 128\bar{i} - 24\bar{j} + 384\bar{k}$$

Surface Integral.

3) Evaluate $\int \bar{F}\cdot\bar{n}\,ds$, $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and 's' is the surface $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$

Sol: Let $\phi = x^2 + y^2 - 16$

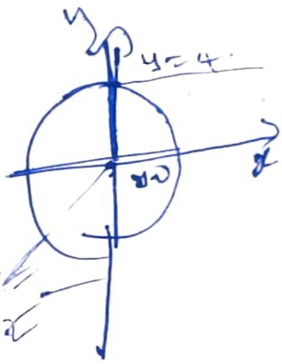
$$\text{normal} = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$= \bar{i}(2x) + \bar{j}(2y) + \bar{k}(0) = 2x\bar{i} + 2y\bar{j}$$

$\bar{n}$: unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2x\bar{i} + 2y\bar{j}}{\sqrt{4x^2+4y^2}} = x\bar{i} + y\bar{j}/4$

Let 'R' be the projection of 's' on yz-plane.

then $\iint_S \bar{F}\cdot\bar{n}\,ds = \iint_R \bar{F}\cdot\bar{n}\,\dfrac{dy\,dz}{|\bar{n}\cdot\bar{j}|}$



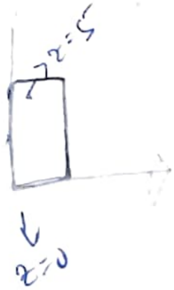
$$\bar{F}\cdot\bar{n} = \frac{zx}{4} + \frac{xy}{4}, \quad \bar{n}\cdot\bar{j} = \frac{x}{4}$$

$$= \int_{y=0}^{4}\int_{z=0}^{5} \frac{x}{4}(z+y)\frac{dy\,dz}{x/4}$$

[$\because x^2+y^2 = 16$

$x=0 \Rightarrow y = 4$

$y \to 0$ to 4.



$$\Rightarrow \int_{y=0}^{4}\int_{z=0}^{5} (z+y)\,dy\,dz$$

$$= \int_{z=0}^{5}\left[zy + \frac{y^2}{2}\right]_{y=0}^{4} dz$$

$$= \int_0^5 \left[4z + \frac{4^2}{2}\right] dz$$

$$\Rightarrow 4\left[\frac{z^2}{2}\right]_0^5 + 8[z]_0^5$$

$$\Rightarrow 4\left[\frac{25}{2}\right] + 40 \Rightarrow 90$$

4) If $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$, Evaluate $\int_s \bar{F}\cdot\bar{n}\, ds$ over the surface $x^2+y^2+z^2=1$ in the first octant (Ans: $\frac{3}{8}$)

Sol:- $\phi = x^2+y^2+z^2 = 1$

$$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{\sqrt{4x^2+4y^2+4z^2}}$$

$$= \frac{2x\bar{i}+2y\bar{j}+2z\bar{k}}{2} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$\bar{n} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$\bar{F}\cdot\bar{n} = xyz + xyz + xyz = 3xyz$$

in xy plane.

$$\iint \bar{F}\cdot\bar{n}\,\frac{dxdy}{|\bar{n}\cdot\bar{k}|} = \iint 3xyz\cdot\frac{dxdy}{\cancel{z}}$$

$x^2+y^2+z^2=1$

$z=0 \Rightarrow x^2+y^2=1$

$y=0 \Rightarrow \boxed{x=1}$

$x \to \pm 1$

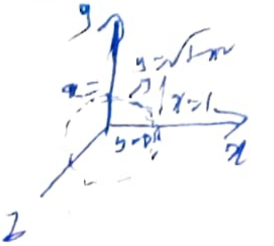
$x \to 0$ to $+1$

$y = \sqrt{1-x^2}$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} 3xy\, dx\, dy$$

$$\Rightarrow \int_0^1 3x\left[\frac{y^2}{2}\right]_0^{1-x^2} dx$$

$$= \int_0^1 \frac{3x(1-x^2)}{2}\, dx$$



$$\frac{3}{2}\int_0^1 (x^2 - x^3)\,dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4} - [0]\right]$$

$$= \frac{3}{2}\left[\frac{1}{4}\right] = \frac{3}{8}$$

Vector integral theorems:

1) Green's theorem:

Statement: If 's' is a closed region in xy-plane bounded by a single 'closed curve' 'c' and if 'm' and 'n' are continuous functions of x and y having contineous derivatives in R, then

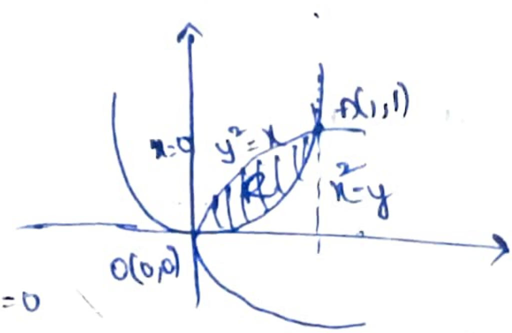
$$\oint_c M\,dx + N\,dy = \iint_s \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where 'c' is traversed in the positive (anti-clock-wise) direction.

Problems:

1) Verify Green's theorem in plane for $\oint_c (3x^2 - 8y^2)dx + (4y - 6xy)dy$, where 'c' is the region bounded by $y = \sqrt{x}$ and $y = x^2$.

Sol: put $y = \sqrt{x}$ in $y = x^2$



we get $\sqrt{x} = x^2$

$x = x^4 \Rightarrow x(1 - x^3) = 0$

$x = 0, x = 1 \Rightarrow y = 0, y = 1$

$O(0,0)$, $A(1,1)$

Using green's theorem: $\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$

$$\frac{\partial M}{\partial y} = -16y \quad , \quad \frac{\partial N}{\partial x} = -6y$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -6y + 16y = 10y$$

$$RHS = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{1} 10\left[\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_{x=0}^{1} [x - x^4]\,dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_{x=0}^{1} = 5\left[\frac{1}{2} - \frac{1}{5}\right]$$

$$= \frac{3}{2}$$

Verification:

$$LHS = \oint_C M\,dx + N\,dy = \int_{OA} M\,dx + N\,dy + \int_{AO} M\,dx + N\,dy$$

$$= I_1 + I_2$$

along OA : $y = x^2 \Rightarrow dy = 2x\,dx \Rightarrow x \to 0$ to 1

let $I_1 = \int_{OA} M\,dx + N\,dy = \int (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$

$$= \int_{x=0}^{1} [3x^2 - 8x^4]\,dx + [4x^2 - 6x^3](2x\,dx)$$

$$= \int_{x=0}^{1} [3x^2 - 8x^4 + 8x^3 - 12x^4]\,dx$$

$$\Rightarrow \left[\frac{3x^3}{3} - \frac{8x^5}{5} + \frac{8x^4}{4} - \frac{12x^5}{5}\right]_0^1 = -1$$

$$I_2 = \oint_C M\,dx + N\,dy = \int_{AO} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{y=\phi}^{0} [6y^5 - 16y^3 + 4y - 6y^3]\,dy$$

$$= \int_{y=1}^{0} [6y^5 - 22y^3 + 4y]\,dy$$

$$\Rightarrow \left[y^6 - 22\frac{y^4}{4} + 4\frac{y^2}{2}\right]_{y=1}^{0}$$

$$I_2 = 5/2$$

$$LHS = -1 + 5/2 = 3/2$$

2) Evaluate by Green's theorem: $\oint_C (\underset{M}{y - \sin x})\,dx + \underset{N}{\cos x}\,dy$

where 'c' is triangle enclosed by the lines

$y = 0$, $x = \pi/2$, $\pi y = 2x$

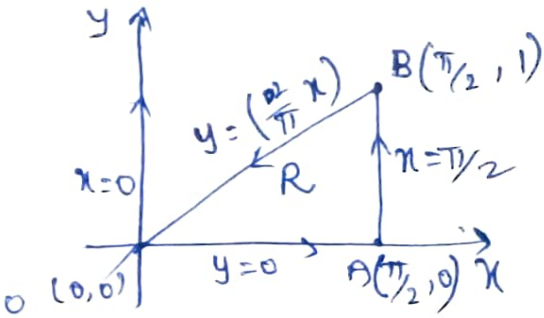
Sol:

$x = \pi/2$, $\pi y = 2x$

$\Rightarrow \pi y = \left(\frac{2\pi}{2}\right)$

$\Rightarrow y = 1$

$A(\pi/2, 1)$



$M = y - \sin x, \quad N = \cos x$

$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$

Using green's theorem: $\oint_C M dx + N dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$

$$\oint_C (y - \sin x) dx + \cos x dy = \iint [-\sin x - 1] dx dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{\frac{2}{\pi}x} (-\sin x - 1) dx (dy)$$

$$= \int_{x=0}^{\pi/2} (-\sin x - 1)\left(\frac{2}{\pi}x\right) dx$$

$$= -\frac{2}{\pi} \int_{x=0}^{\pi/2} (x \sin x + x) dx$$

$$= -\frac{2}{\pi} \left\{ \int_0^{\pi/2} x \sin x + \int_0^{\pi/2} x \, dx \right\}$$

$\left(\because \int uv = u \int v - u' \iint v + u'' \iiint v - \cdots\right)$

$= -\frac{2}{x}$

$$= -\frac{2}{\pi} \left\{ x(-\cos x) - (1)(-\sin x) + 0 + \frac{x^2}{2} \right\}_0^{\pi/2}$$

$$= -\frac{2}{\pi} \left\{ \left(0 + 1 + \frac{\pi^2}{8}\right) - (0 + 0 + 0) \right\}$$

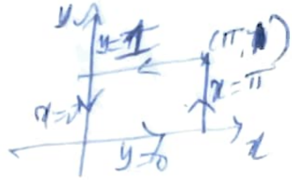
$$= -\frac{2}{\pi} - \frac{\pi}{4}$$

$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$$

⇒ Eval. by Green's th. $\oint_C (x^2-\cosh y)dx + (y+\sin x)dy$

Where 'c' is the rectangle with vertices $(0,0), (\pi,0), (\pi,1), (0,1)$

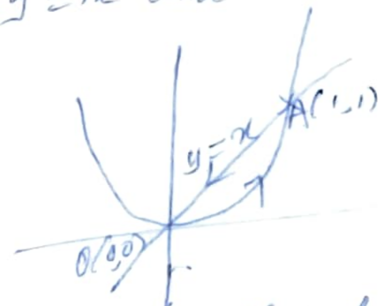
Ans: $\pi(\cosh 1 - 1)$



⇒ verify Green's th. for $\int_C [(xy+y^2)dx + x^2 dy]$,

Where 'c' is bounded by $y=x$ and $y=x^2$.

Ans: $-\frac{1}{20}$



⇒ Apply Green's th. to eval. $\int_C (2xy-x^2)dx + (x^2+y^2)dy$,

where 'c' is bounded by $y=x^2$ and $y^2=x$

Ans: 0

⇒ verify Green's th. for $\int_C (3x^2-8y^2)dx + (4y-6xy)dy$

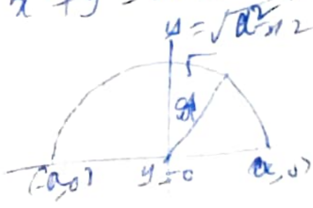
Where 'c' is the region bounded by $x=0, y=0, x+y=1$

Ans: $\frac{5}{3}$.

⇒ using Green's th. eval. $\oint_C (2x^2-y^2)dx + (x^2+y^2)dy$,

where 'c' is the boundary in xy-plane of arc enclosed by x-axis and the semi-circle $x^2+y^2=a^2$ in the upper half of the xy-plane

Ans: $\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$ (or) $\int_{\theta=0}^{\pi}\int_{r=0}^{a} (\quad) r\,dr\,d\theta = \frac{4}{3}a^3$.



⇒ Eval. $\oint_C (x^2 - xy^3)\,dx + (y^2 - 2xy)\,dy$ using Green's th.

C : (0,0), (2,0), (2,2), (0,2)

(A : 8)

⇒ Eval $\oint_C x^2(1+y)\,dx + (y^3 + x^3)\,dy$ " " "

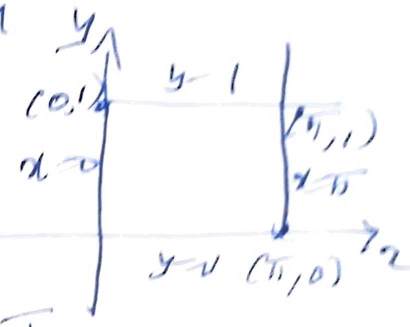
C : y = ±1 and x = ±1

A. 3/2

(1)

⇒ $M = x^2 \cosh y$, $N = y + \sin x$

$\frac{\partial M}{\partial y} = -\sinh y$, $\frac{\partial N}{\partial x} = \cos x$.



R.H.S. $= \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\,dx\,dy$

$= \int_{x=0}^{\pi} \left[y\cos x + \cosh y \right]_{y=0}^{1} dx = \int \left[\cos x + \cosh(1) - 1 \right] dx$

$= \left[\sin x + (\cosh(1) - 1)x \right]_{x=0}^{\pi} = 0 + \pi(\cosh(1) - 1)$

$= \pi\left[\cosh(1) - 1\right]$

(2)

III) Stokes theorem

Statement: Let 's' be a open surface bounded by a closed, non-intersecting curve 'c'. If $\bar{F}$ is any differtiable vector point function then $\oint_c \bar{F}\cdot\bar{dr} = \int_s \text{curl}\bar{F}\cdot\bar{n}\,ds$ where 'c' is traversed in positive direction.

Problems:

1) If $\bar{F} = y\bar{i} + (x-2xz)\bar{j} - xy\bar{k}$, Evaluate $\int_s (\nabla\times\bar{F})\cdot\bar{n}\,ds$ where 's' is the surface of sphere $x^2+y^2+z^2=a^2$ above the xy-plane.

Sol: $\bar{F} = y\bar{i} + (x-2xz)\bar{j} - xy\bar{k} = F_1\bar{i}+F_2\bar{j}+F_3\bar{k}$

where $F_1 = y$, $F_2 = x-2xz$, $F_3 = -xy$

Using stokes theorem $\int_s (\nabla\times\bar{F})\cdot\bar{n}\,ds = \oint_c \bar{F}\cdot\bar{dr}$

$= \oint_c F_1dx + F_2dy + F_3dz$

Above the xy-plane, the sphere is $x^2+y^2=a^2, z=0$

$dz=0$

$\therefore \int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \oint_c ydx + (x-0)dy + 0$

$= \int_c ydx + xdy$

$c: x^2+y^2=a^2$

$x = a\cos\theta$

$y = a\cos\theta$

$\theta \to 0$ to 2π

$$= \int_0^{2\pi} a\sin\theta[-a\sin\theta]\,d\theta + (a\cos\theta)(a\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi} [\cos^2\theta - \sin^2\theta]\,d\theta$$

$$= a^2 \int_0^{2\pi} \cos 2\theta \, d\theta$$

$$= a^2\left[\frac{\sin 2\theta}{2}\right]_0^{2\pi} = \frac{a^2}{2}[0-0] = 0$$

8) Verify stokes theorem: for $\bar{F} = (2x-y)\bar{i} - yz^2\bar{j} - y^2z\bar{k}$ where 's' is the upper half of the surface of the sphere $x^2+y^2+z^2=1$ and 'c' is its boundary.

Sol: Let the boundary 'c' of 's' is a circle in xy-plane.

i.e $x^2+y^2=1$, $z=0$

Using stokes theorem: $\oint_c \bar{F}\cdot d\bar{r}$

$$= \int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

$$\text{LHS} = \oint_c \bar{F}\cdot d\bar{r} = \int_c (2x-y)dx - yz^2dy - y^2z\,dz$$

$(\because z=0$
$dz=0)$

$= \int_C (2x - y)\,dx$

$C: x^2 + y^2 = 1$

$= \int_0^{2\pi} [2\cos\theta - \sin\theta] \cdot -\sin\theta \, d\theta$ $\quad x = \cos\theta, y = \sin\theta$

$\theta \to 0$ to 2π

$LHS = \oint_C \bar{F} \cdot \bar{dr} = \pi$

$RHS = \int_C (\nabla \times \bar{F}) \cdot \bar{n}\, ds$, $\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$

$= \bar{i}\,[-2yz + 2yz] - \bar{j}(0-0) + \bar{k}(1)$

$= \bar{k}$

$\therefore R.H.S = \int_S \bar{k} \cdot \bar{n}\, ds$

$= \iint_S dx\,dy$ ($\because$'s' is projection in xy plane) $\quad$ ($\because$ in xy-plane $ds = \frac{dx\,dy}{|\bar{n} \cdot \bar{k}|}$

$dx\,dy = (\bar{n} \cdot \bar{k})\, ds$

$= \int_{x=-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1)\, dy\, dx = 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$

$= 4 \left[\sqrt{1-x^2}\right]_0^1 = 4[-1] = -4$

(or) polar co-ordinates: $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = |J|\, dr\, d\theta = r\, dr\, d\theta$

$x^2 + y^2 = 1 \Rightarrow r^2 = 1, r = 1$

$r \to 0$ to 1

$$\int_S (\nabla \times \bar{F}) \cdot \bar{n}\, ds = \int_{r=0}^{1} \int_{\theta=0}^{2\pi} r\, dr\, d\theta$$

$$= \int_{r=0}^{1} r\, [\theta]_0^{2\pi}\, dr$$

$$= 2\pi \int_0^1 r\, dr = 2\pi \left[\frac{r^2}{2}\right]_0^1$$

$$= 2\pi \left[\frac{1}{2}\right] = \pi$$

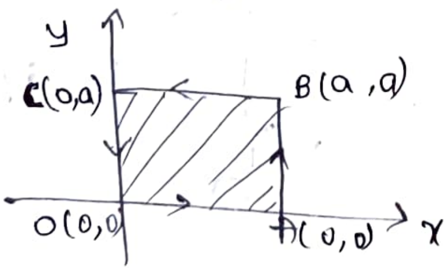
$\therefore$ RHS $= \pi$

$\therefore$ LHS $=$ RHS

** 3) Verify Stokes theorem: for $F = -y^3\bar{i} + x^3\bar{j}$, where 's' is the circular disc $x^2+y^2 \le 1$, $z=0$.

4) Verify Stokes theorem: for $\bar{F} = x^2\bar{i} + xy\bar{j}$ integrated round the square the square in the plan $z=0$ whose sides along the line $x=0, y=0, x=a, y=a$.

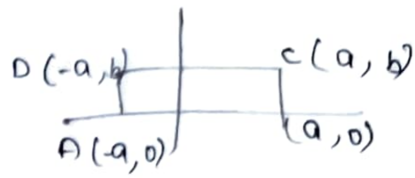
Sol: LHS $= \oint \bar{F} \cdot d\bar{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$



RHS $= \iint (\nabla \times F) \cdot \bar{n}\, ds = \int_0^a \int_0^a (y)\, dx\, dy$

5) Evaluate $\int_c \bar{F} \cdot \bar{dr}$ by Stokes theorem: if $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ where 'c' is the rectangle formed by the lines $x = \pm a$, $y = 0$, $y = b$

(Ans: $-4a^2$)



5.3 GAUSS'S DIVERGENCE THEOREM

Statement: If F is a continuously differentiable vector point in the region V bounded by a closed surface S; then $\iint_S F \cdot N \cdot ds = \iiint_V \nabla \cdot F \cdot dv$. Where, N is the outward drawn unit normal vector to the surface S.

15.4.1 Gauss Divergence Theorem: Cartesian form

Let
$$F = F_1 i + F_2 j + F_3 k: \text{ then}$$
$$\nabla \cdot F = \text{div } F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

i.e.
$$\iint (F_1 dydz + F_2 dzdx + F_3 dxdy) = \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz$$

$$= 2\left(\frac{x^2}{2}\right)_0^1 + \left(\frac{y^2}{2}\right)_0^1 + (z)_0^1 = 2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{2}$$

EXAMPLE 5. Evaluate by divergence theorem

$$\iint_S (x+z)dydz + (y+z)dzdx + (x+y)dxdy$$

where, S is the surface of the sphere $x^2 + y^2 + z^2 = 4$.

Solution: We have $F_1 = x - z;\ F_2 = y - z;\ F_3 = x - y.$

$$\frac{\partial F_1}{\partial x} = 1\ ;\ \frac{\partial F_2}{\partial y} = 1;\ \frac{\partial F_3}{\partial z} = 0.$$

From divergence theorem $\iint_S F_1 dydz + F_2\, dzdx + F_3 dxdy = \int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz.$

Therefore,

$$\iint_S (x+z)dy + (y+z)dz\, cx + (x+y)\, dxdy = \iiint_V 2dx\, dy\, dz = 2\iiint_V dxdydz = 2V.$$

$$= 2(\text{Volume of given sphere})$$

$$= 2 \times \frac{4}{3}\pi(2)^3 = \frac{64\pi}{3}$$

EXAMPLE 6. Evaluate $\iint_S F\cdot N\, ds$; where $F = xi + yj + z^2k$ and S is the surface bounded by the one $x^2 + y^2 = z^2$ in the plane $z = 4$.

Solution: Let V be the volume enclosed by S. Then V is bounded by the surfaces $z = 0$; $z = 4$ and $z^2 = x^2 + y^2$.

We have
$$F = xi + yj + z^2k$$

$$\nabla\cdot F = \text{div } F = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)$$

$$= 1 + 1 + 2z = 2(1+z)$$

By divergence theorem

$$\iint_S F\cdot Nds = \iiint_V \nabla\cdot Fdv = \iiint_V 2(1+z)dz$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z}\int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (1+z)dx\, dy\, dz.$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z}\left(\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}(1+z)\,dx\right)dy\,dz$$

$$= 2\int_{z=0}^{4}\int_{y=-z}^{z}\left(x(1+z)\right)_{\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}}\,dy\,dz$$

$$= 4\int_{z=0}^{4}\int_{y=-z}^{z}\sqrt{z^2-y^2}\,(1+z)dydz$$

$$= 8\int_{z=0}^{4}\int_{y=-z}^{z}(1+z)\sqrt{z^2-y^2}\,dy\,dz$$

$$= 8\int_{z=0}^{4}(1+z)\left[\frac{y}{2}\sqrt{z^2-y^2}+\frac{z^2}{2}\sin^{-1}\frac{y}{z}\right]_0^z dz$$

$$= 8\int_{z=0}^{4}(1+z)\,\frac{z^2}{2}\cdot\frac{\pi}{2}\cdot dz = 2\pi\int_0^4(z^2+z^3)dz$$

$$= 2\pi\left[\frac{z^3}{3}+\frac{z^4}{4}\right]_0^4 = 2\pi\left[\frac{64}{3}+64\right] = \frac{512\pi}{3}$$

EXAMPLE 7. Show that $\int_S (axi + byj + czk)\cdot n\,ds = \frac{4}{3}\pi(a+b+c)$.

where, S is the surface of the sphere $x^2 + y^2 + z^2 = 1$.

Solution: Here, we have $F = axi + byj + czk$

$$\text{div } F = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) = a + b + c$$

S is the surface of the sphere $x^2 + y^2 + z^2 = 1$, radius of the sphere is $r = 1$

By Gauss's theorem

$$\iint_S (axi + byj + czk)\cdot N\,ds = \iiint_V \text{div}(axi + byj + czk)dv$$

$$\iiint_V (a+b+c)dv = (a+b+c)\iiint_V dv$$

$= (a + b + c)v$ (V is the volume of the sphere $x^2 + y^2 + z^2 = 1$)

$$= (a+b+c)\frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi(a+b+c)$$

EXAMPLE 8. Verify divergence theorem for $F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$ taken over the rectangular parallelopiped $0 \le x \le a;\ 0 \le y \le b;\ 0 \le z \le c$.

Solution: Given $F = (x^2 - yz)i + (y^2 - zx)j + (z^2 - xy)k$.

Therefore,
$$\text{div } F = \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy)$$

$$= 2x + 2y + 2z = 2(x + y + z)$$

$$\iiint_V \text{div } F \cdot dv = 2\iiint_V (x + y + z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} (x + y + z)\,dx\,dy\,dz$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}\left[\left(xz + yz + \frac{z^2}{2}\right)\right]_{z=0}^{c} dx\,dy$$

$$= 2\int_{x=0}^{a}\left(\int_{y=0}^{b}\left(cx + cy + \frac{c^2}{2}\right)dy\right)dx$$

$$= 2\int_{x=0}^{a}\left(cxy + c\frac{y^2}{2} + \frac{c^2}{2}y\right)_{y=0}^{b} dx$$

$$= 2\int_{x=0}^{a}\left(bcx + \frac{b^2c}{2} + \frac{bc^2}{2}\right) dx$$

$$= 2\left[bc\frac{x^2}{2} + \frac{b^2c}{2}x + \frac{bc^2}{2}x\right]_{x=0}^{a}$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= \frac{2abc}{2}[a + b + c] = abc[a + b + c] \qquad (1)$$

Verification:

Consider the integral $\iint_S F\cdot N ds$ over the faces of the parallelopiped $0 \le x \le a;\ 0 \le y \le b;\ 0 \le z \le c$.

Over the face PLAN: we have $N = i;\ x = a$

$$\int_{PLAN} F\cdot N\cdot ds = \int_{y=0}^{b}\int_{z=0}^{c} [(a^2 - yz)i + (y^2 - zx)j + (z^2 - ay)k]\,i\,dy\,dz$$

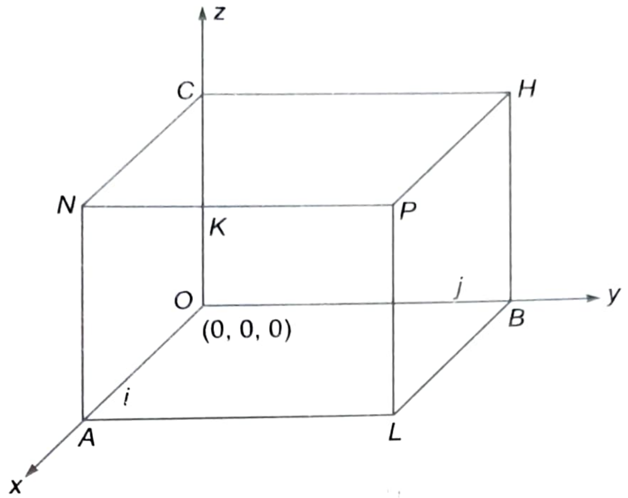


Fig. 15.9

$$= \int_{y=0}^{b}\int_{z=0}^{c} (a^2 - yz)dydz = \int_{z=0}^{0}\left(a^2y - \frac{y^2}{2}z\right)_{y=0}^{b} dz.$$

$$= \int_{z=0}^{c}\left(a^2b - \frac{b^2}{2}z\right) dz = \left(a^2bz - \frac{b^2z^2}{4}\right)_{z=0}^{c} = a^2bc - \frac{b^2c^2}{4}$$

Over the face *BMCO*:

Here $N = -i;\ x = 0$

$$\iint_{BM\ CO} F\cdot N ds = \int_{y=0}^{b}\int_{z=0}^{c} ((0 - yz)i + y^2j + z^2k)(-i)dy\ dz.$$

$$= \int_{y=0}^{b}\int_{z=0}^{c} yz\ dy\ dz = \int_{y=0}^{b} y\cdot dy\int_{z=0}^{c} z\ dz$$

$$= \left(\frac{y^2}{2}\right)_0^b\left(\frac{z^2}{2}\right)_0^c = \frac{b^2c^2}{4}$$

Over the face *BMPL*:

We have $N = j,\ y = b.$

Hence,
$$\iint_{BM\ PL} F\cdot N\cdot ds = \int_{x=0}^{a}\int_{z=0}^{c} [(x^2 - bz)i + (b^2 - zx)j + (z^2 - bx)k]j\ dx\ dz$$

$$= \int_{x=0}^{a}\int_{z=0}^{c} (b^2 - zx)dx\ dz = \int_{x=0}^{a}\left(\int_{z=0}^{c}(b^2 - zx)\ dz\right) dx$$

$$= \int_{x=0}^{a}\left(b^2z - \frac{z^2x}{2}\right)_{z=0}^{c} dx = \int_{x=0}^{a}\left(b^2c - \frac{c^2x}{2}\right) dx$$

$$= \left[b^2cx - \frac{c^2x^2}{4} \right]_{x=0}^{a} = b^2ca - \frac{c^2a^2}{4} = ab^2c - \frac{c^2a^2}{4}$$

Over the face *OANC*:

Here, we have $\qquad N = -j,\ y = 0$

Therefore, $\displaystyle\int_{DN}\int_{AC} F\cdot N\cdot ds = \int_{x=0}^{a}\int_{z=0}^{c} [(x^2 - bz)i + (0 - zx)j + (z^2)k](-j)\,dx\,dz$

$$= \int_{x=0}^{a}\int_{z=0}^{c} zx\,dx\,dz.$$

$$= \int_{x=0}^{a} x\,dx \int_{z=0}^{c} z\,dz = \left(\frac{x^2}{2}\right)_0^a \left(\frac{z^2}{2}\right)_0^b = \frac{a^2c^2}{4}$$

Over the face MCNP:

Here $\qquad N = K;\ Z = C$

$$\int_{MC}\int_{NP} F\cdot N\cdot ds = \int_{x=0}^{a}\int_{y=0}^{b} [(x^2 - cy)i + (y^2 - cx)j + (c^2 - xy)k]k\,dx\,dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} (c^2 - xy)\,dx\,dy = \int_{x=0}^{a} \left(c^2y - \frac{xy^2}{2}\right)_{y=0}^{b} dx.$$

$$= \int_{x=0}^{a} \left(c^2b - \frac{xb^2}{2}\right) dx = \left(c^2dx - \frac{x^2b^2}{4}\right)_{x=0}^{a} = abc^2 - \frac{a^2b^2}{4}$$

Over the face BLAO:

We have $\qquad N = -k;\ z = 0$

$$\iint_S F\cdot Nds = \int_{x=0}^{a}\int_{y=0}^{b} [(x^2 - 0)i + (y^2 - 0)j + (0 - xy)k](-k)\,dx\,dy.$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} xy\cdot dx\cdot dy = \int_{x=0}^{a} x\cdot dx\cdot \int_{y=0}^{b} y\cdot dy$$

$$= \left(\frac{x^2}{2}\right)_0^a \left(\frac{y^2}{2}\right)_0^b = \frac{a^2b^2}{4}$$

Adding all the six integrals, we get

$$\iint_S F\cdot N\cdot ds = a^2bc - \frac{b^2c^2}{4} + \frac{b^2c^2}{4} + ab^2c - \frac{c^2a^2}{4} + \frac{c^2a^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

$$= a^2bc + ab^2c + abc^2 = abc(a + b + c) \qquad (2)$$

From (1) and (2), $\iint_S F \cdot N\, ds = \iiint_V \text{div } F\, dv$

Hence, divergence theorem is verified.

EXAMPLE 9. Use divergence theorem to evaluate $\iint_S F \cdot ds$ where $F = x^3 i + y^3 j + z^3 k$ and S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$.

Solution: We have $F = x^3 i + y^3 j + z^3 k$. Changing to spherical polar coordinates $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ the limits of integration are

$$r = 0 \text{ to } a,\ \theta = 0 \text{ to } \pi,\ \phi = 0 \text{ to } 2\pi$$

and the Jacobian is $|J| = r^2 \sin\theta$.

$$\therefore \qquad \iint_S F \cdot ds = \iiint_V \nabla \cdot F\, dv$$

$$= \iiint_V 3(x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= 3\int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} r^2 (r^2 \sin\theta)\, dr\, d\theta\, d\phi$$

$$= 3\int_{r=0}^{a} \int_{\theta=0}^{\pi} r^4 \sin\theta \left(\int_{\phi=0}^{2\pi} d\phi \right) dr\, d\theta$$

$$= 3\int_{r=0}^{a} \int_{\theta=0}^{\pi} r^4 \sin\theta\, [\phi]_0^{2\pi}\, dr\, d\theta$$

$$= 3(2\pi) \int_{r=0}^{a} \left(\int_{\theta=0}^{\pi} r^4 \sin\theta\, d\theta \right) dr$$

$$= 6\pi \int_{r=0}^{a} r^4 (-\cos\theta)_0^{\pi}\, dr$$

$$= -6\pi \int_{r=0}^{a} r^4 (\cos\pi - \cos 0)\, dr$$

$$= 12\pi \int_{r=0}^{a} r^4\, dr = 12\pi \left(\frac{r^5}{5} \right)_0^a$$

$$= \frac{12}{5} \pi a^5$$

EXAMPLE 10. Using divergence theorem evolute $\iint_S (xdydz + ydzdx + zdxdy)$ where S is the surface $x^2 + y^2 + z^2 = a^2$. (JNTU 2001)

Solution: We have $f = xi + yj + zk$

Therefore, $\text{div } f = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1 + 1 + 1 = 3$

Applying Gauss divergence theorem, we get

$$\iint_S f \cdot N\, ds = \iiint_V \text{div } f\, dv$$

$$= \iiint_V 3 \text{ div} = 3\iiint_V dv$$

$$= 3V$$

$$= 3 \cdot \frac{4}{3}\pi a^3 \quad \left(\text{Since radius of the sphere} = \frac{4}{3}(\text{radius})^3\right)$$

$$= 4\pi a^3$$

EXERCISE 15.2

1. By transforming into triple integral; evaluate $\iint_S (x^3\, dydz + x^2y\, dz\, dx + x^2z\, dxdy)$ where, S is the closed surface consisting of the cylinder $x^2 + y^2 = a^2$ and the circular discs $z = 0$ and $z = b$. (JNTU 2004)

2. Evaluate $\iint_S (x^2)dydz + y^2dzdx + 2z(xy - x - y)dx\, dy$, where S is the surface of the cube $0 \le x \le 1;\ 0 \le y \le 1;\ 0 \le z \le 1$.

3. Evaluate $\iint_S (ax^2 + by^2 + cz^2)ds$ over the unit sphere.

4. Evaluate $\iint_S (y^2z^2i + z^2x^2j + z^2y^2k) \cdot Nds$

   where, S is the part of the unit sphere above the xy-plane and bounded by this plane.
   Hint: $F = y^2z^2i + z^2x^2j + z^2y^2k \Rightarrow \nabla \cdot F = 2zy^2$ limits of integration are

$$0 \le z \le \sqrt{1 - x^2 - y^2}\ ;\ -\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}\ ;\ -1 \le x \le 1$$