Circuits and Transforms

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CONTENTS

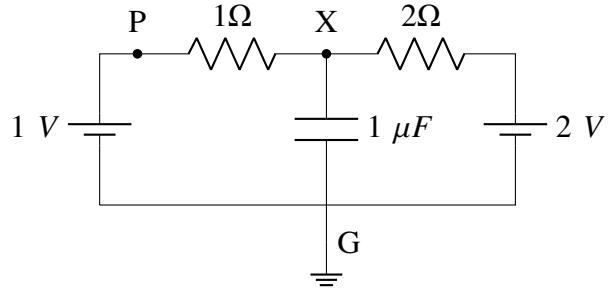*Abstract*—This manual provides a simple introduction to Transforms

Fig. 2.2

1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \qquad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,dt \qquad (1.2)$$

2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.
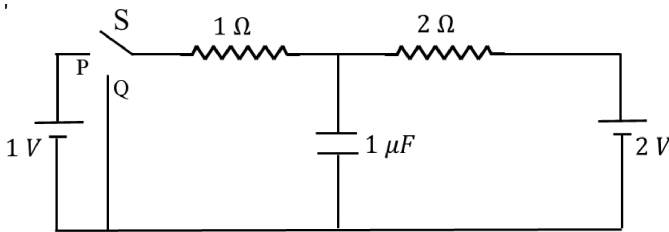

Fig. 2.1

2. Find q_1.
   Solution: The equivalent circuit at steady-state when the switch is at P is shown above.

Assuming the circuit to be grounded at G and the relative potential at point X to be V, we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \qquad (2.1)$$

$$\implies V = \frac{4}{3}\,\text{V} \qquad (2.2)$$

Hence,

$$q_1 = CV = \frac{4}{3}\,\mu C \qquad (2.3)$$

3. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
   Solution: We have,

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_0^{\infty} u(t)e^{-st}dt \qquad (2.4)$$

$$= \int_0^0 \frac{1}{2}e^{-st}dt + \int_0^{\infty} e^{-st}dt \qquad (2.5)$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \qquad (2.6)$$

4. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \qquad (2.7)$$

and find the ROC.
   Solution: Note that by substituting $s := s + a$ in (2.6), and considering $a \in \mathbb{R}$,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty u(t)e^{-(s+a)t}dt \qquad (2.8)$$

$$= \frac{1}{s+a}, \quad \mathfrak{R}(s) > -a \qquad (2.9)$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where
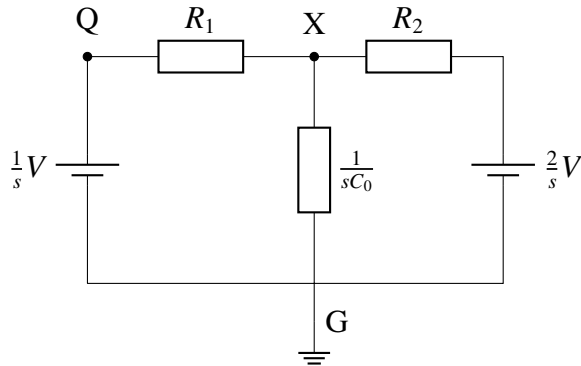

Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \qquad (2.10)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \qquad (2.11)$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution: We see that

$$V_1(s) = \frac{1}{s}V_2(s) = \frac{2}{s} \qquad (2.12)$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0V = 0 \qquad (2.13)$$

$$V\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{1}{s}\left(\frac{1}{R_1} + \frac{2}{R_2}\right) \qquad (2.14)$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right)} \qquad (2.15)$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \qquad (2.16)$$

6. Find $v_{C_0}(t)$. Plot using python.
Solution: Taking the inverse Laplace transform

in (2.16),

$$V(s) \xleftrightarrow{\mathcal{L}} \frac{2R_1 + R_2}{R_1 + R_2}u(t)\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right) \qquad (2.17)$$

$$= \frac{4}{3}\left(1 - e^{-(1.5\times10^6)t}\right)u(t) \qquad (2.18)$$

```python
import numpy as np
from matplotlib import pyplot as plt

def unitstep(t):
    if (t < 0): return 0
    elif (t == 0):return 0.5
    else: return 1

def v1(t):
    if (t >= 0): return 4/3*(1 - np.exp(-t
        *1.5e6))*unitstep(t)
    else: return 0

vc0 = np.vectorize(v1, otypes=['double'])
vc1 = np.loadtxt('v1.txt')
t = np.linspace(0, 1e-5, 1000000)
plt.plot(t, vc0(t))
plt.plot(vc1[:,0], vc1[:,1], '.')
plt.xlabel('t_(s)')
plt.ylabel('$v_{C_0}(t)$_(V)')
plt.grid()
plt.legend(['Simulation', 'Analysis'])
plt.savefig('../figs/2_6.png')
```
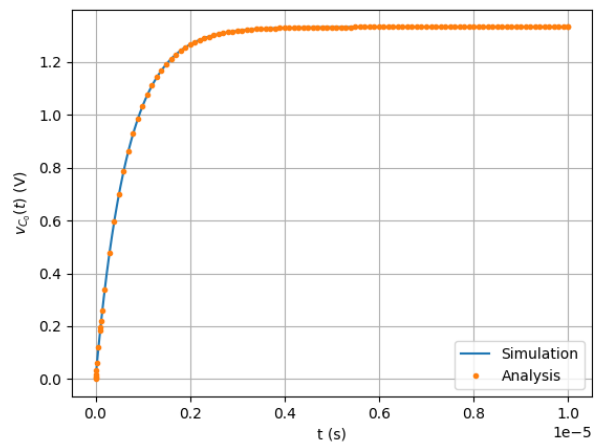
This code gives the following plot:

Fig. 2.4: $v_{C_0}(t)$ before the switch is flipped

7. Verify your result using ngspice.

Solution:

```
Output After Switching
R1 X 0 1
C1 X 0 1u ic=1.33
R2 X 3 2
V2 3 0 dc 2V
.tran 100n 10u uic

.control
run
wrdata v2.txt v(X)
.endc

.end
```

This code simulates the given circuit and the generated output is depicted in Fig. (2.4)

3 INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.
   Solution: The equivalent circuit at steady state when the switch is at Q is shown below.
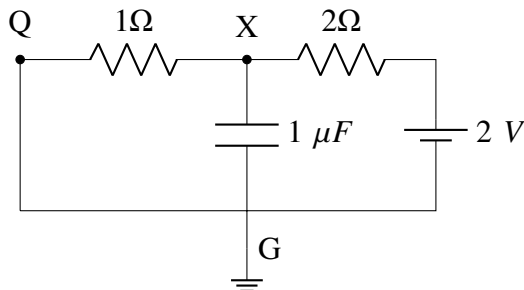

Fig. 3.1

Since capacitor behaves as an open circuit, we use KCL at X.

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \implies V = \frac{2}{3}\,\text{V} \quad (3.1)$$

and hence, $q_2 = \frac{2}{3}\,\mu\text{C}$.

2. Draw the equivalent s-domain resistive circuit w hen S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements.

3. $V_{C_0}(s) = ?$
   Solution: Using KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \quad (3.2)$$

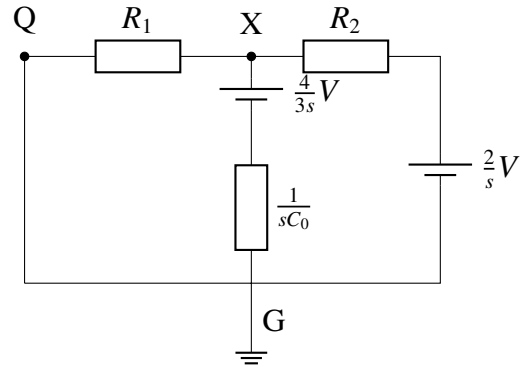$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.3)$$

Fig. 3.2

4. $v_{C_0}(t) = ?$ Plot using python.
   Solution: From (3.3),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right)$$

$$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) + s}\right) \quad (3.4)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}u(t)$$

$$+ \frac{2}{R_2\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \quad (3.5)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \quad (3.6)$$

```python
import numpy as np
from matplotlib import pyplot as plt

def unitstep(t):
    if (t < 0): return 0
    elif (t == 0):return 0.5
    else: return 1

def v1(t):
    if (t >= 0): return 2/3*(1 + np.exp(-t
        *1.5e6))*unitstep(t)
    else: return 0

vc0 = np.vectorize(v1, otypes=['double'])
vc1 = np.loadtxt('v2.txt')
t = np.linspace(0, 1e-5, 1000000)
plt.plot(t, vc0(t))
```

```
plt.plot(vc1[:,0], vc1[:,1], '.')
plt.xlabel('t_(s)')
plt.ylabel('$v_{C_0}(t)$_(V)')
plt.grid()
plt.legend(['Simulation', 'Analysis'])
plt.savefig('../figs/3_4.png')
```
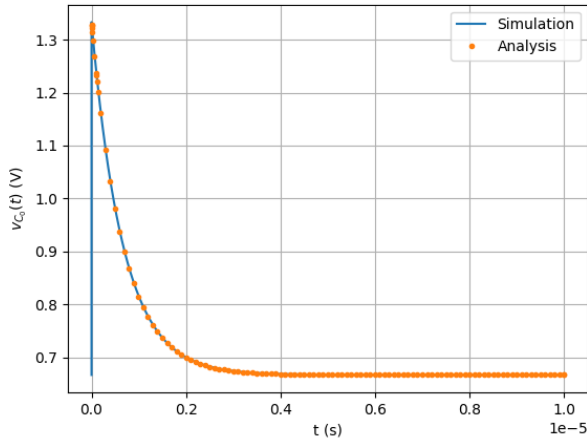

Fig. 3.3: $v_{C_0}(t)$ after the switch is flipped

5. Verify your result using ngspice.
Solution:

```
Output After Switching
R1 X 0 1
C1 X 0 1u ic=1.33
R2 X 3 2
V2 3 0 dc 2V
.tran 100n 10u uic

.control
run
wrdata v2.txt v(X)
.endc

.end
```

This code in ngspice simulates the given circuit and the generated output is depicted in Fig. (3.3).

6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} \text{ V} \qquad (3.7)$$

Using (3.6),

$$v_{C_0}(0+) = \lim_{t \to 0+} v_{C_0}(t) = \frac{4}{3} \text{ V} \qquad (3.8)$$

$$v_{C_0}(\infty) = \lim_{t \to \infty} v_{C_0}(t) = \frac{2}{3} \text{ V} \qquad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations.
Solution: The equivalent circuit in the t-domain is shown below.
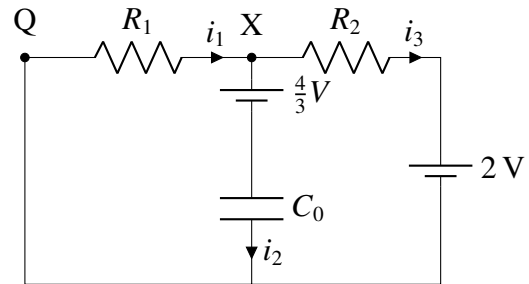

Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \qquad (3.10)$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt = 0 \qquad (3.11)$$

$$\frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt - i_3 R_2 - 2 = 0 \qquad (3.12)$$

$$(3.13)$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \qquad (3.14)$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \qquad (3.15)$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \qquad (3.16)$$

$$(3.17)$$

where $i(t) \overset{\mathcal{L}}{\longleftrightarrow} I(s)$. Note that the capacitor is equivalent to a resistive element of resistance $R_C = \frac{1}{sC_0}$ in the s-domain. Equations (3.14) - (3.16) precisely describe Fig. 3.2.