Digital Signal Processing
Assignment 1

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CONTENTS

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

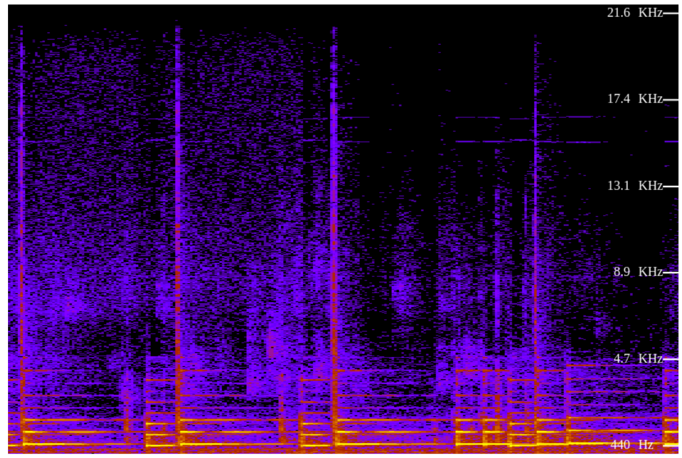
```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution:



There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2∗cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
```

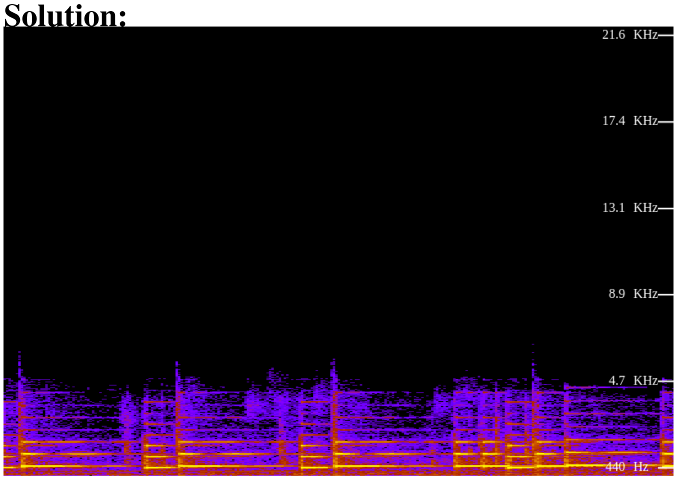
```
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution:



The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.

Solution:

```
import numpy as np
import matplotlib.pyplot as plt

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])

plt.subplot(2, 1, 1)
plt.stem(range(0,6),x)
plt.title('Digital Filter Input-Output')
plt.ylabel('$x(n)$')
plt.grid()
```

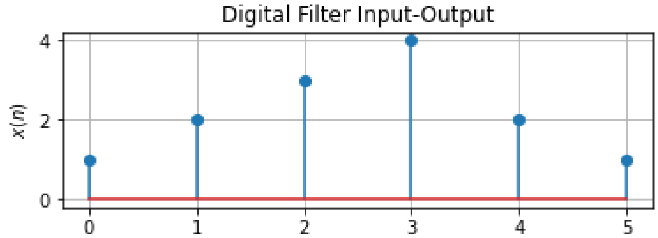


Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2.

```
import numpy as np
import matplotlib.pyplot as plt

x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
k = 20
y = np.zeros(20)


y[0] = x[0]
y[1] = -0.5*y[0]+x[1]

for n in range(2,k-1):
        if n < 6:
                y[n] = -0.5*y[n-1]+x[n]+x
                    [n-2]
        elif n > 5 and n < 8:
                y[n] = -0.5*y[n-1]+x[n-2]
        else:
                y[n] = -0.5*y[n-1]
print(y)

#subplots
plt.subplot(2, 1, 1)
plt.stem(range(0,6),x)
plt.title('Digital Filter Input-Output')
plt.ylabel('$x(n)$')
plt.grid()# minor


plt.subplot(2, 1, 2)
```

```
plt.stem(range(0,k),y)
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor
```

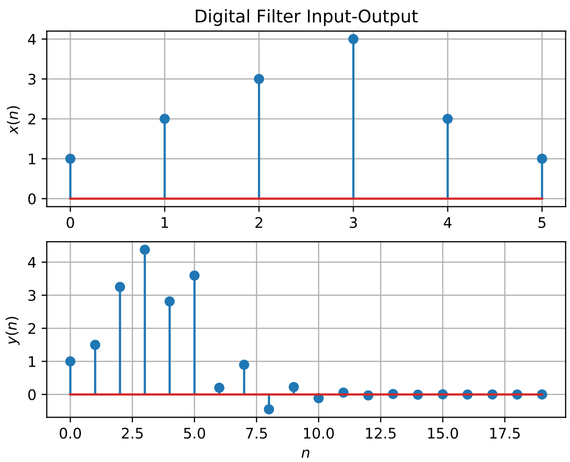


Fig. 3.2

3.3 Repeat the above exercise using a C code.
Solution:

```
#include <stdio.h>
float yn(int* x,int n){
    int y=0;
    if(n==0){y =*p1;}
    else if(n==1){
    y=*(x+1)+yn(x,0);
    }
    else if(n<6){
    y = *(x+n−1) + *(x+n−3) − yn(x,n−1)
        *0.5;
    }
    else {
    y = −yn(x,n−1)*0.5;
    }
    return y;
}

int main(){
FILE* fp;
const int Points;
int x[];
float y[Points];
int i;

x={1,2,3,4,2,1};
Points=20;

for(int i=0;i<number_Of_Points;i++){
y[i]=yn(x,i);
}

fp = fopen("y(n).dat","w");

for(i=0;i<number_Of_Points;i++){
fprintf(fp,"%f\n",y[i]);
}

fclose(fp);
return 0;
}
```

For plotting:

```
import matplotlib.pyplot as plt
import numpy as np


y = np.loadtxt("y(n).dat",dtype = "double")
x = np.array([1,2,3,4,2,1])

# ploting graphs
plt.subplot(211)
plt.stem(np.arange(len(x)),x)
plt.xlabel("n")
plt.ylabel("$x(n)$")
plt.grid()

plt.subplot(212)
plt.stem(np.arange(len(y)),y)
plt.xlabel("n")
plt.ylabel("$y(n)$")
plt.grid()

plt.show()
```

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.

Solution:

$$x(n) = \{1, 2, 3, 4, 2, 1\} \tag{4.7}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.8}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + 4z^{-4} + 2z^{-2} + 1z^{-1} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.16}$$

and from (4.20),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.17}$$

$$= \frac{1}{1-z^{-1}}, \quad |z| > 1 \tag{4.18}$$

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}} \quad |z| > |a| \tag{4.19}$$

Solution:

$$Since, u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.20}$$

$$a^n u(n) = \left\{1, a^1, a^2 ..\right\} \tag{4.21}$$

Z transform of this,

$$= \sum_{n=-\infty}^{\infty} a^n z^{-n} \tag{4.22}$$

$$= 1 + az^{-1} + a^2z^{-2} + \dots \tag{4.23}$$

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-az^{-1}}(|z| > |a|) \tag{4.24}$$

using the formula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.25}$$

Plot $|H\left(e^{\jmath\omega}\right)|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.

Solution: The following code plots Fig. 4.6.

```
import numpy as np
import matplotlib.pyplot as plt



#DTFT
def H(z):
        num = np.polyval([1,0,1],z**(-1))
        den = np.polyval([0.5,1],z**(-1))
        H = num/den
        return H



#Input and Output
omega = np.linspace(-3*np.pi,3*np.pi,1e2)
```

```
#subplots
plt.plot(omega, abs(H(np.exp(1j*omega))))
plt.title('Filter Frequency Response')
plt.xlabel('$\omega$')
plt.ylabel('$|H(e^{\jmath\omega})| $')
plt.grid()# minor

plt.show()
```

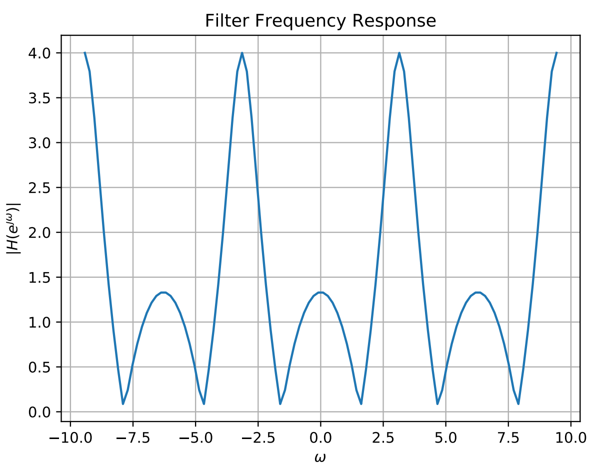


Fig. 4.6: $|H(e^{j\omega})|$

Proving its periodic,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{4.26}$$

for

$$z = e^{jw} \tag{4.27}$$

$$H(z) = H(e^{jw}) = \frac{1+e^{-2jw}}{1+\frac{1}{2}e^{-jw}} \tag{4.28}$$

$$= \frac{1+e^{2jw}}{e^{2jw}+\frac{1}{2}e^{jw}} \tag{4.29}$$

$$\implies \left|H(e^{jw})\right| = \left|\frac{1+e^{2jw}}{e^{2jw}+\frac{1}{2}e^{jw}}\right| \tag{4.30}$$

$$= \frac{\left|1+e^{2jw}\right|}{\left|e^{2jw}+\frac{1}{2}e^{jw}\right|} \tag{4.31}$$

$$= \frac{|1+cos2w+2jsinw| * 2}{|e^{jw}+1|} \tag{4.32}$$

$$= \frac{|cosw+jsinw| * 4cosw}{|2cosw+1+2jsinw|} \tag{4.33}$$

$$= \frac{4cosw}{\sqrt{(2cosw+1)^2+4sin^2w}} \tag{4.34}$$

$$= \frac{4cosw}{\sqrt{5+4cosw}} \tag{4.35}$$

Therefore

$$H(e^{jw}) = \frac{4cosw}{\sqrt{5+4cosw}} \tag{4.36}$$

Consider

$$f(x) = \frac{4cosx}{\sqrt{5+4cosx}} \tag{4.37}$$

then

$$f(x+t) = \frac{4cos(x+t)}{\sqrt{5+4cos(x+t)}} \tag{4.38}$$

$$f(x+t) = \frac{4cos(x+t)}{\sqrt{5+4cos(x+t)}} \tag{4.39}$$

$$= \frac{4(cosxcost - sinxsint)}{\sqrt{5+4(cosxcost - sinxsint)}} \tag{4.40}$$

By comparing (4.37) and (4.38), we get cost=1 and sint=0
This is true for $t = 2k\pi$. This implies that the principal period of this function is 2π.

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$.

$$H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jn\omega} \tag{4.41}$$

$$\implies \left(e^{j\omega}\right) * e^{jk\omega} = \sum_{n=-\infty}^{\infty} h(n)\, e^{-jn\omega} e^{jk\omega} \tag{4.42}$$

$$\implies \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) * e^{jk\omega} d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} h(n)\, e^{-jn\omega} e^{jk\omega} \tag{4.43}$$

$$\implies \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) * e^{jk\omega} d\omega = \sum_{n=-\infty}^{\infty} h(n) \int_{-\pi}^{\pi} e^{-jn\omega} e^{jk\omega} \tag{4.44}$$

NOTE: We know that,

$$\int_{-\pi}^{\pi} e^{-jn\omega} e^{jk\omega} = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.45}$$

Hence,

$$\int_{-\pi}^{\pi} H\left(e^{j\omega}\right) * e^{jk\omega} d\omega = \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} h(n)\, e^{-jn\omega} e^{jk\omega} \tag{4.46}$$

$$\implies \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) * e^{jk\omega} d\omega = 2\pi h(n) \tag{4.47}$$

$$\implies \frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) * e^{jk\omega} d\omega = h(n) \tag{4.48}$$

Therefore,

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H\left(e^{j\omega}\right) e^{j\omega n} d\omega \tag{4.49}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12).
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$

Let $z^{-1} = x$,then, by polynomial long division we get

$$\implies (1 + z^{-2}) = (\frac{1}{2}z^{-1} + 1)(2z^{-1} - 4) + 5 \tag{5.3}$$

$$\implies \frac{(1 + z^{-2})}{\frac{1}{2}z^{-1} + 1} = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.4}$$

$$\implies H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.5}$$

Now, consider $\frac{5}{\frac{1}{2}z^{-1}+1}$
The denominator $\frac{1}{2}z^{-1} + 1$ can be expressed as sum of an infinite geometric progression, which as its first term equal to 1 and common ratio $\frac{-1}{2}z^{-1}$
Therefore, we can write $\frac{5}{\frac{1}{2}z^{-1}+1}$ as $5\left(1 + \left(\frac{-1}{2}z^{-1}\right) + \left(\frac{-1}{2}z^{-1}\right)^2 + \left(\frac{-1}{2}z^{-1}\right)^3 + \left(\frac{-1}{2}z^{-1}\right)^4 + \dots\right)$
Therefore, H(z) can be given by,

$$H(z) = (2z^{-1} - 4) + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.6}$$

$$\tag{5.7}$$

$$= 2z^{-1} - 4 + 5 + \frac{-5}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.8}$$

$$\implies H(z) = 1z^0 + \frac{-1}{2}z^{-1} + \frac{5}{4}z^{-2} + \frac{-5}{8}z^{-3} + \frac{5}{16}z^{-4} + \dots \tag{5.9}$$

Comparing the above expression to (4.1) we get h(n) for n<5 as,

$$h(0) = 1 \tag{5.10}$$

$$h(1) = \frac{-1}{2} \tag{5.11}$$

$$h(2) = \frac{5}{4} \tag{5.12}$$

$$h(3) = \frac{-5}{8} \tag{5.13}$$

$$h(4) = \frac{5}{16} \tag{5.14}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}} + \frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.16}$$

$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (4.19) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
import numpy as np
import matplotlib.pyplot as plt


n = np.arange(10)
fn=(−1/2)∗∗n
hn1=np.pad(fn, (0,2), 'constant',
    constant_values=(0))
hn2=np.pad(fn, (2,0), 'constant',
    constant_values=(0))
plt.stem(np.arange(12), hn1+hn2)
plt.title('Filter␣Impulse␣Response')
plt.xlabel('$n$')
plt.ylabel('$h(n)$')
plt.grid()# minor

plt.show()
```

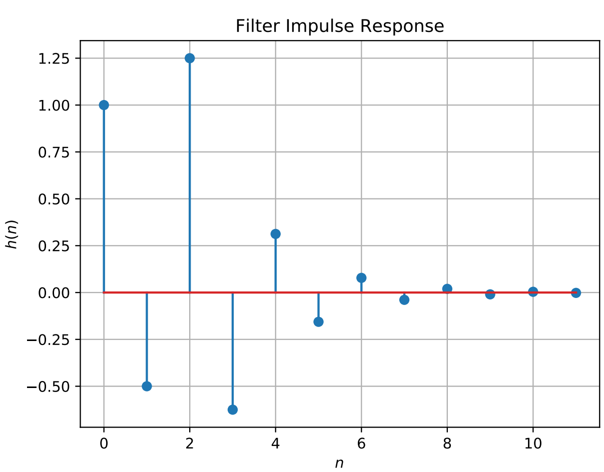


Fig. 5.3: $h(n)$ as the inverse of $H(z)$

From (5.17) we know that

$$h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.18}$$

Implies we can write that

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \ge 2 \end{cases} \tag{5.19}$$

A sequence is said to be bounded when

$$|x_n| \le M, \forall n \in \mathcal{N} \tag{5.20}$$

Now consider (5.19),
For n<0,

$$|h(n)| \le 0 \tag{5.21}$$

For $0 \le n < 2$,

$$|h(n)| = (\frac{1}{2})^n \tag{5.22}$$

$$\implies |h(n)| \le 1 \tag{5.23}$$

For $n \ge 2$,

$$|h(n)| = 5(\frac{1}{2})^n \tag{5.24}$$

$$\implies |h(n)| \le 5 \tag{5.25}$$

From above we can say that,

$$M = max\{0, 1, 5\} \tag{5.26}$$

$$= 5 \tag{5.27}$$

Therefore since M exists and is a real value, we can say that h(n) is bounded.

5.4 Convergent? Justify using the ratio test. **Solution:**
We can check if a sequence os convergent by ratio test. From the defination of ratio test we can say that a sequence is convergent if

$$\lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.28}$$

Here, applying ratio test on (5.17) is same as applying ratio test on (5.19)

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{5(\frac{-1}{2})^{n+1}}{5(\frac{-1}{2})^{n}}\right| \tag{5.29}$$

$$= \left|\frac{-1}{2}\right| \tag{5.30}$$

$$= \frac{1}{2} \tag{5.31}$$

From (5.31) we can clearly say that the sequence h(n) is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.32}$$

Is the system defined by (3.2) stable for the impulse response in (5.15)?

Solution:

From (5.17) we know that,

$$h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.33}$$

Given that, a system is stable when

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.34}$$

$$\implies \sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}(\left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2)) \tag{5.35}$$

$$= 2 * \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n} u(n) \tag{5.36}$$

$$\implies \sum_{n=-\infty}^{\infty} h(n) = 2 * (\frac{1}{1+\frac{1}{2}}) \tag{5.37}$$

$$= \frac{4}{3} < \infty \tag{5.38}$$

Hence, the system is stable.

5.6 Verify the above result using a python code.

```
def u(n):
    if n <0 :
        return 0.0
    else :
        return 1.0
def h(n):
    return u(n)*(-1.0/2)**n + u(n-2)*(-1.0/2)**(n-2)
sum = 0
for i in range(200000):
    sum +=h(i)
print(sum)
```

```
1.3333333333333333
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.39}$$

This is the definition of $h(n)$.

Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if


k = 12
h = np.zeros(k)
h[0] = 1
h[1] = -0.5*h[0]
h[2] = -0.5*h[1] + 1

for n in range(3,k-1):
        h[n] = -0.5*h[n-1]

#subplots
plt.stem(range(0,k),h)
plt.title('Impulse Response Definition')
plt.xlabel('$n$')
plt.ylabel('$h(n)$')
plt.grid()# minor

#If using termux
plt.savefig('../figs/hndef.pdf')
subprocess.run(shlex.split("termux-open ../figs/hndef.pdf"))
#else
#plt.show()
```

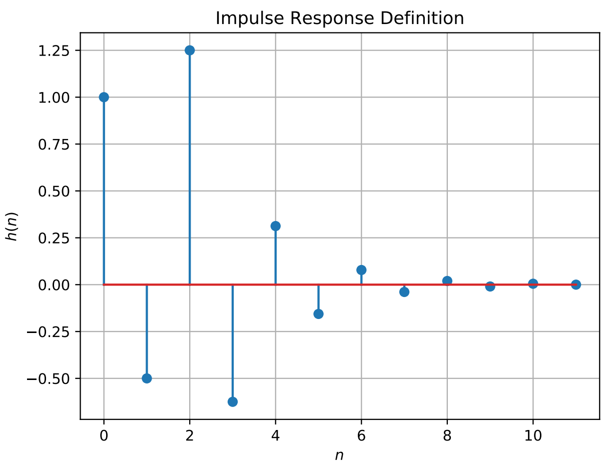


Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{n=-\infty}^{\infty} x(k)h(n-k) \quad (5.40)$$

Comment. The operation in (5.40) is known as *convolution*.

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
import numpy as np
import matplotlib.pyplot as plt
#If using termux
import subprocess
import shlex
#end if



n = np.arange(14)
fn=(−1/2)**n
hn1=np.pad(fn, (0,2), 'constant',
     constant_values=(0))
hn2=np.pad(fn, (2,0), 'constant',
     constant_values=(0))
h = hn1+hn2

nh=len(h)
x=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
nx = len(x)

y = np.zeros(nx+nh−1)

for k in range(0,nx+nh−1):
        for n in range(0,nx):
                if k−n >= 0 and k−n < nh:
                        y[k]+=x[n]*h[k−n]

print(y)
#plots
plt.stem(range(0,nx+nh−1),y)
plt.title('Filter Output using Convolution')
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor

#If using termux
plt.savefig('../figs/ynconv.pdf')
plt.savefig('../figs/ynconv.eps')
subprocess.run(shlex.split("termux−open ../
     figs/ynconv.pdf"))
#else
#plt.show()
```

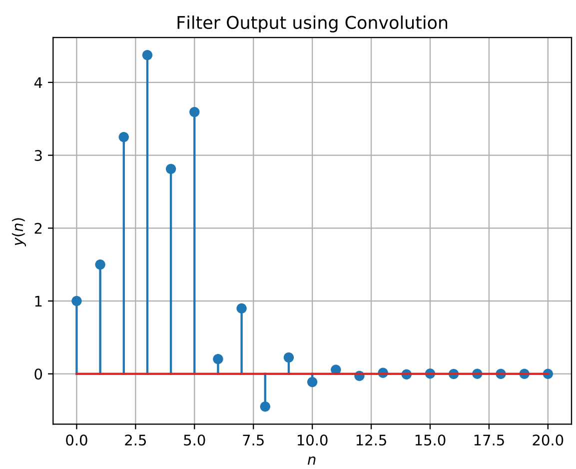


Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Teoplitz matrix.

Solution:

We know that from, (5.40),

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad (5.41)$$

This can also be wrtten as a matrix-vector multiplication given by the expression,

$$y = T(h) * x \quad (5.42)$$

In the equation (5.42), $T(h)$ is a Teoplitz

matrix.
The equation (5.42) can be expanded as,

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} \tag{5.43}$$

$$\mathbf{y} = \begin{pmatrix} h_1 & 0 & . & . & . & 0 \\ h_2 & h_1 & . & . & . & 0 \\ h_3 & h_2 & h_1 & . & . & 0 \\ . & . & . & . & . & . \\ h_{n-1} & h_{n-2} & h_{n-3} & . & . & 0 \\ h_n & h_{n-1} & h_{n-2} & . & . & h_1 \\ 0 & h_n & h_{n-1} & h_{n-2} & . & h_2 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & h_{n-1} \\ 0 & . & . & . & 0 & h_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \tag{5.44}$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.45}$$

Solution:
From the defination of convolution given in (5.40),we know that,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.46}$$

$$\implies y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.47}$$

Replace k with $n-k$. Now since n varies from n=$-\infty$ to n=∞, n-k will also vary from $-\infty$ to ∞. Therefore, we get,

$$y(n) = \sum_{n-k=-\infty}^{\infty} x(n-k)h(k) \tag{5.48}$$

$$= \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.49}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\mathrm{j}2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution:
From 3.1, we know that ,

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{6.2}$$

Here, let, $\omega = e^{-j2\pi k}$. Then,

$$X(k) = 1 + 2\omega^{\frac{1}{5}} + 3\omega^{\frac{2}{5}} + 4\omega^{\frac{3}{5}} + 2\omega^{\frac{4}{5}} + \omega \tag{6.3}$$

Similarly, we know from (5.19),

$$h(n) = \begin{cases} 0 & , n < 0 \\ \left(\frac{-1}{2}\right)^n & , 0 \le n < 2 \\ 5\left(\frac{-1}{2}\right)^n & , n \ge 2 \end{cases} \tag{6.4}$$

Now,again let, $\omega = e^{-j2\pi k}$. Then,

$$H(k) = 1 + \frac{-1}{2}\omega^{\frac{1}{5}} + \frac{5}{4}\omega^{\frac{2}{5}} + \frac{-5}{8}\omega^{\frac{3}{5}} + \frac{5}{16}\omega^{\frac{4}{5}} + \frac{-5}{32}\omega \tag{6.5}$$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.6}$$

Solution:
Now, from (6.3) and (6.5), we know $X(k) and H(k)$. Now, given that,

$$Y(k) = X(k) * H(k) \tag{6.7}$$

$$\begin{aligned} Y(k) = {} & (1 + 2\omega^{\frac{1}{5}} + 3\omega^{\frac{2}{5}} + 4\omega^{\frac{3}{5}} + 2\omega^{\frac{4}{5}} + \omega) * \\ & (1 + \frac{-1}{2}\omega^{\frac{1}{5}} + \frac{5}{4}\omega^{\frac{2}{5}} + \frac{-5}{8}\omega^{\frac{3}{5}} + \frac{5}{16}\omega^{\frac{4}{5}} + \frac{-5}{32}\omega) \end{aligned} \tag{6.8}$$

$$\begin{aligned} Y(k) = {} & 1 + \frac{3}{2}\omega^{\frac{1}{5}} + \frac{13}{4}\omega^{\frac{2}{5}} + \frac{35}{8}\omega^{\frac{3}{5}} + \frac{45}{16}\omega^{\frac{4}{5}} \\ & \frac{115}{32}\omega^{\frac{5}{5}} + \frac{1}{8}\omega^{\frac{6}{5}} + \frac{25}{32}\omega^{\frac{7}{5}} - \frac{5}{8}\omega^{\frac{8}{5}} \\ & - \frac{5}{32}\omega^{5} \end{aligned} \tag{6.9}$$

where, $\omega = e^{-j2k\pi}$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{\mathrm{j}2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.10}$$

Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```python
import numpy as np
import matplotlib.pyplot as plt


N = 14
n = np.arange(N)
fn=(-1/2)**n
hn1=np.pad(fn, (0,2), 'constant',
    constant_values=(0))
hn2=np.pad(fn, (2,0), 'constant',
    constant_values=(0))
h = hn1+hn2

xtemp=np.array([1.0,2.0,3.0,4.0,2.0,1.0])
x=np.pad(xtemp, (0,8), 'constant',
    constant_values=(0))

X = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
        for n in range(0,N):
                X[k]+=x[n]*np.exp(-1j*2*
                    np.pi*n*k/N)
H = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
        for n in range(0,N):
                H[k]+=h[n]*np.exp(-1j*2*
                    np.pi*n*k/N)

Y = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
        Y[k] = X[k]*H[k]

y = np.zeros(N) + 1j*np.zeros(N)
for k in range(0,N):
        for n in range(0,N):
                y[k]+=Y[n]*np.exp(1j*2*np
                    .pi*n*k/N)

#print(X)
y = np.real(y)/N
#plots
plt.stem(range(0,N),y)
plt.title('Filter Output using DFT')
plt.xlabel('$n$')
plt.ylabel('$y(n)$')
plt.grid()# minor
plt.show()
```

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.

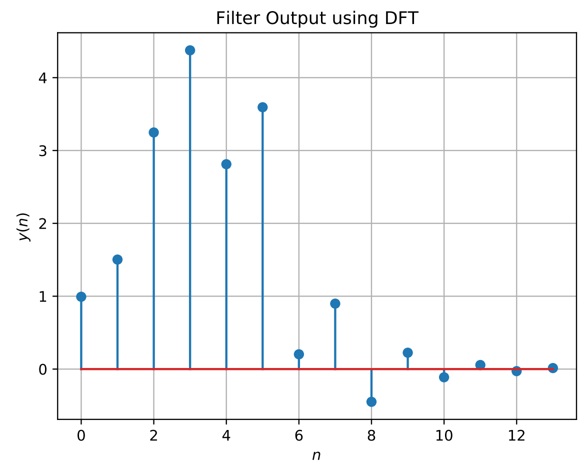


Fig. 6.3: $y(n)$ from the DFT

6.5 Wherever possible, express all the above equations as matrix equations.
6.6 Verify the above equations by generating the DFT matrix in python.

7 Exercises

Answer the following questions by looking at the python code in Problem 2.3.

7.1 The command

```python
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a(m)\, y(n-m) = \sum_{k=0}^{N} b(k)\, x(n-k) \quad (7.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.

7.2 Repeat all the exercises in the previous sections for the above a and b.

7.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

7.4 What is type, order and cutoff-frequency of the above butterworth filter
Solution: The given butterworth filter is low pass with order=2 and cutoff-frequency=4kHz.

7.5 Modifying the code with different input parameters and to get the best possible output.