Math 262, Fall 2022, Final Exam

NAME:____

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This is a review for the final. It includes all the problems that were on the Fall 2022 final (in blue) as well as many other problems for you to practice on. None of the problems on this review will be on the Spring 2023 final. However, these problems should give you an idea of the scope of the final exam. Below is what the front page of your final exam will look like.

Take a moment to remember your obligations as a CSUN student. When you submit your final exam, you will be also affirming that:

1. The answers I give on this exam will be my own work.
2. The only accepted materials to use for this exam are my one-page of notes (two-sided, 8.5×11 maximum) and a scientific calculator that cannot be connected to the internet and is not a phone. I understand that my one-page of notes must be turned in with my exam and that if I use a calculator I must show the steps of my work.
3. Any external assistance (e.g. cell phones/cameras, PDAs, other electronic devices, or conversation with others) is prohibited.
4. Uploading or downloading any portions of this exam to or from the internet, aside from submitting the assignment to Canvas, is prohibited.
5. All suspected violations of the CSUN Code of Student Conduct will be investigated and will be subject to disciplinary actions.
6. I will use a sharp pencil or good pen, making sure the answers I upload can be easily read.
7. I will do the exam on the provided printout.
8. I will ask if I am not sure of anything on the exam.
9. I will turn in the exam at the time it is due: 3:00 pm on Thursday, December 15, 2022 with 15 minutes for uploading to Math 262 Common Resource Canvas site: https://canvas.csun.edu/courses/131776.

Points

Question	1	2	3	4	5	6	7	Total
Possible Points	20	20	15	5	10	20	10	100
Points Earned								

Please note: This exam is printed two-sided so often your next problem will be on the back of the problem you are working on. There are also two pages deliberately left blank on this exam. You may use them for scrap or to continue your work if you need more space.

1. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 1 & 2 & 4 \end{bmatrix}$.

   (a) Calculate the rank of matrix A. Show what method you are using and at least one intermediate step.

   (b) What is the nullity of A? Answer only, no justification required.

   (c) True or False: $\det(A) = 0$. Justify your answer. You can use your work from other parts of this problem to do so.

   (d) True or False? A is invertible. Justify your answer.

   (e) True or False? The column vectors of A are linearly independent. Justify your answer.

   (f) Find all solutions for the following linear system. Write your answer in vector notation.
   $$\left| \begin{array}{ccccccc} x & + & 2y & + & 3z & = & 1 \\ 2x & + & 4y & + & 5z & = & 2 \\ x & + & 2y & + & 4z & = & -2 \end{array} \right|.$$

   (g) **How many solutions** are there to the following system of equations? Justify your answer.
   $$\left| \begin{array}{ccccccc} x & + & 2y & + & 3z & = & 0 \\ 2x & + & 4y & + & 5z & = & 0 \\ x & + & 2y & + & 4z & = & 0 \end{array} \right|.$$

   (h) **How many solutions** are there to the linear system $A\vec{x} = \vec{0}$? Justify your answer.

   (i) True or False: The column vectors of A are linearly independent. Justify your answer.

   (j) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$. Is T onto? Answer only, no justification required.

2. Consider the linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ defined by $L(\vec{x}) = B\vec{x}$, where

$$B = \begin{bmatrix} 1 & 3 & 1 & 1 & 5 \\ -2 & -6 & -1 & 3 & -1 \\ 0 & 0 & 2 & 5 & 8 \\ 2 & 6 & 6 & 3 & 8 \end{bmatrix} \quad \text{where} \quad \operatorname{rref}(B) = \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   In each of the following questions, refer to the information above.

   (a) What is the value of m?

   (b) What is the value of n?

   (c) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be the column vectors that form the matrix B. Are the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent? If yes, why? If no, give one nontrivial linear relation between the column vectors of B.

   (d) Find a basis for $\operatorname{im}(L)$ (the image of L).

   (e) Calculate $\operatorname{rank}(L)$.

   (f) Find a basis for $\ker(L)$ (the kernel of L).

   (g) Calculate $\operatorname{nullity}(L)$. Justify your answer.

   (h) Is L onto? Why or why not?

   (i) Is L one-to-one? Why or why not?

3. Let $\text{proj}_L : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection to the line that contains $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

   (a) Find the projections $\text{proj}_L(\vec{e}_1)$, $\text{proj}_L(\vec{e}_2)$, and $\text{proj}_L(\vec{e}_3)$.
   (b) Find the matrix A of the linear transformation proj_L.
   (c) Find the matrix A^2.
   (d) True or false: $A^3 = A$. Explain why or why not.
   (e) Find the orthogonal projection of $\vec{e_1}$ to the plane $2x - y + 3z = 0$. You can use your work from other parts of this problem to do so.
   (f) Find the reflection of $\vec{e_1}$ in the plane $2x - y + 3z = 0$. You can use your work from other parts of this problem to do so.

4. Recall that $\mathbb{R}^{2\times 2}$ is the linear space of 2×2 matrices.

   (a) Prove that $U = \left\{ \begin{bmatrix} p & q \\ q & p \end{bmatrix} : p + q = 1 \right\}$ is not a subspace of $\mathbb{R}^{2\times 2}$.
   (b) Prove that $V = \left\{ \begin{bmatrix} p & q \\ q & p \end{bmatrix} : pq = 0 \right\}$ is not a subspace of $\mathbb{R}^{2\times 2}$.
   (c) Prove that $W = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a = 2c \right\}$ is a subspace of $\mathbb{R}^{2\times 2}$.
   (d) Find a basis of W.
   (e) What is $\dim(W)$?

5. **Proposition 1** *If $T : V \to W$ a one-to-one linear transformation, and the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent in V, then the vectors $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$ are linearly independent in W*

   Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof.

   Proof: Suppose, to the contrary, that the vectors $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$ are not linearly independent in W. Then there is a nontrivial linear relation among them:

$$c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \ldots + c_n T(\vec{v}_n) = \underline{\hspace{3cm}},$$

   where at least one of the coefficients $c_1, c_2, \ldots, c_n$ is ____________________.
   Since the linear transformation T respects addition and scalar multiplication, then

$$\begin{aligned} T(c_1\vec{v}_1) + T(c_2\vec{v}_2) + \ldots + T(c_n\vec{v}_n) &= \vec{0} \\ T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) &= \vec{0} \end{aligned}$$

   Also, we have $T(\vec{0}) =$ ______ because T is ________________________.
   The combination of the two equations above gives

$$T(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n) = T(\vec{0}).$$

   Since T is one-to-one, this means that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \underline{\hspace{3cm}}.$$

   As the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent, the only linear relation among them is the trivial relation. This implies that the coefficients $c_1, c_2, \ldots, c_n$ are all ________________________, which is a contradiction.

6. **Proposition 2** *If $T: V \to W$ is an onto linear transformation and the vectors $\vec{v}_1, \ldots, \vec{v}_n$ span V, then the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_n)$ span W.*

   Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof.

   Proof: To start, we recall that $W = \operatorname{span}\Big(T(\vec{v}_1), \ldots, T(\vec{v}_n)\Big)$ if every vector in W is a linear combination of the vectors ______________________________.

   Let $\vec{w}$ be an arbitrary vector in W. Since T is onto, then there exists some vector $\vec{u}$ in __________ such that $T(\vec{u}) = \vec{w}$. Because the vectors $\vec{v}_1, \ldots, \vec{v}_n$ span V, then every vector in V is a ______________________________ of $\vec{v}_1, \ldots, \vec{v}_n$. In particular, $\vec{u}$ can be written as

   $$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n.$$

   for some coefficients $c_1, \ldots, c_n$. Now we apply T on both sides of the equation,

   $$T(\vec{u}) = T\Big(c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n\Big).$$

   Because T is _______________, it respects addition and scalar multiplication. So this equation becomes

   $$T(\vec{u}) = c_1T(\vec{v}_1) + c_2T(\vec{v}_2) + \ldots + c_nT(\vec{v}_n).$$

   This shows that $\vec{w} = T(\vec{u})$ is a linear ____________________ of the vectors $T(\vec{v}_1), \ldots, T(\vec{v}_n)$.

7. Recall that $U^{2\times2}$ is the space of 2×2 upper-triangular matrices. Consider the linear transformation $T(M) = M\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$ from $U^{2\times2}$ to $U^{2\times2}$.

   (a) For the matrix $M = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, compute $T(M) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$.

   (b) Find the matrix B of the linear transformation T with respect to the basis

   $$\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

   (c) Calculate $\operatorname{rank}(T)$.

   (d) Is T invertible? Why or why not?

   (e) Is T an isomorphism? Why or why not?

8. Consider the following two functions from $\mathbb{R}^3$ to $\mathbb{R}^3$.

   (a) If S is linear, find a matrix A such that $S(\vec{x}) = A\vec{x}$ for all $\vec{x}$; otherwise prove that S is not linear.

   $$S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 - 4x_3 \\ x_1x_3 \end{bmatrix}$$

   (b) If T is linear, find a matrix B such that $T(\vec{x}) = B\vec{x}$ for all $\vec{x}$; otherwise prove that T is not linear.

   $$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 - 4x_1 \\ x_1 - x_3 \end{bmatrix}$$

9. Consider the matrix $A = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix}$.

   (a) Calculate the characteristic polynomial of A.

   (b) Find all eigenvalues of A, and all associated eigenvectors.

   (c) If A is diagonalizable, then find a diagonal matrix D and an invertible matrix S such that $D = S^{-1}AS$. If A is not diagonalizable, then state why it is not.

10. Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

    (a) Show that the only eigenvalues of A are 1, 2, and 3 and find their algebraic multiplicities. Show all your work.

    (b) The eigenvalues of A are 1, 2, and 3. The eigenspaces associated to eigenvalues $\lambda = 1$ and $\lambda = 2$ are

$$E_1 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}, \quad E_2 = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$$

    Find a basis for the eigenspace E_3 associated to eigenvalue $\lambda = 3$. Show all your work.

    (c) If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable then give an eigenbasis $\mathfrak{B}$ and the diagonalization $[T]_{\mathfrak{B}}$. (You do not need to repeat work you did in parts (a) and (b) of this problem.) If A is not diagonalizable then state why it is not.

    (d) Based on the work above, what is the characteristic polynomial of A?

    (e) Find $\det\left(A^2\right)$. (You can use your work done in previous parts of this problem.)

11. Consider the subspace $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}\right)$ in $\mathbb{R}^4$.

    (a) Find a basis $\beta = (\vec{v}_1, \vec{v}_2)$ of the orthogonal subspace $W^\perp$.

    (b) Find a basis $\gamma = (\vec{u}_1, \vec{u}_1)$ of W such that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal vectors of unit length.

    (c) Find a basis $\delta = (\vec{u}_1, \vec{u}_1)$ of $W^\perp$ such that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal vectors of unit length.

12. Consider the vectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$.

    (a) Calculate the length of $\vec{v}_1$.

    (b) Are $\vec{v}_1$ and $\vec{v}_2$ orthogonal? Justify your answer.

    (c) Perform the Gram-Schmidt process on the vectors $\vec{v}_1$ and $\vec{v}_2$.

    (d) Find the QR-factorization of the matrix $\begin{bmatrix} 1 & 0 \\ 3 & 1 \\ -1 & 2 \end{bmatrix}$.