Math 262, Fall 2021, Review Sheet for Final Exam

This is a review for the final. It includes all the problems that were on the Fall 2020 final (in blue) as well as many other problems for you to practice on. None of the problems on this review will be on the Fall 2021 final. However, these problems should give you an idea of the scope of the final exam. Below is what the front page of your final exam will look like.

NAME:____

Instructor (circle one): Candel, Castro, Kim, Shubin, Yomba,

Take a moment to remember your obligations as a CSUN student. When you submit your final exam, you will be also affirming that:

1. The answers I give on this exam will be my own work.
2. The only accepted materials to use for this exam are my one-page of notes (two-sided, 8.5×11 maximum) and a scientific calculator that cannot be not connected to the internet and is not a phone. I understand that my one-page of notes must be turned in with my exam and that if I use a calculator I must show the steps of my work.
3. Any external assistance (e.g. cell phones/cameras, PDAs, other electronic devices, or conversation with others) is prohibited.
4. Uploading or downloading any portions of this exam to or from the internet, aside from submitting the assignment to Canvas, is prohibited.
5. All suspected violations of the CSUN Code of Student Conduct will be investigated and will be subject to disciplinary actions.
6. I will use a sharp pencil or good pen, making sure the answers I upload can be easily read.
7. I will do the exam on the provided printout.
8. I will ask if I am not sure of anything on the exam.
9. I will turn in the exam at the time it is due: 2:45 pm on Thursday December 16, 2021 with 15 minutes for uploading to Math 262 Common Resource Canvas site: https://canvas.csun.edu/courses/113140.

We believe in you and trust you. Breathe and do your best.

1. Let $M = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

(a) Find M^{-1}.
ANS. Method 1: Use row reduction

$$[M|I] = \left[\begin{array}{cc|cc} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 1 & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-3}{2} & 0 & -\sqrt{3} \end{array}\right] \to \left[\begin{array}{cc|cc} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 1 & 0 \\ 0 & -2 & 1 & -\sqrt{3} \end{array}\right] \to \left[\begin{array}{cc|cc} \sqrt{3} & -1 & 2 & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{array}\right] \to$$

$$\left[\begin{array}{cc|cc} \sqrt{3} & 0 & \frac{3}{2} & \frac{\sqrt{3}}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{array}\right].$$

Method 2: Use the definition of the inverse of a 2×2 matrix to get the same answer

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Method 3: Notice that this is a rotation matrix, rotating vectors in the plane by 30 degrees counter clockwise. Thus the inverse matrix is the one that rotates all vectors in the plane by -30 degrees (that is, by 30 degrees clockwise), so

$$M^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

(b) Solve the system of equations $\left\{ \begin{array}{ccccc} \frac{\sqrt{3}}{2}x & + & \frac{-1}{2}y & = & 0 \\ \frac{1}{2}x & + & \frac{\sqrt{3}}{2}y & = & 1 \end{array} \right\}$.
(As you review, try to do this three different ways: (i) using row reduction of the associated augmented matrix, (ii) using M^{-1}, and (iii) using geometry.)
ANS. Method 1: Use row reduction

$$[M|I] = \left[\begin{array}{cc|c} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 1 \end{array}\right] \to \left[\begin{array}{cc|c} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ \frac{-\sqrt{3}}{2} & \frac{-3}{2} & -\sqrt{3} \end{array}\right] \to \left[\begin{array}{cc|c} \frac{\sqrt{3}}{2} & \frac{-1}{2} & 0 \\ 0 & -2 & -\sqrt{3} \end{array}\right] \to \left[\begin{array}{cc|c} \sqrt{3} & -1 & 0 \\ 0 & 1 & \frac{\sqrt{3}}{2} \end{array}\right] \to$$

$$\left[\begin{array}{cc|c} \sqrt{3} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 1 & \frac{\sqrt{3}}{2} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{\sqrt{3}}{2} \end{array}\right].$$

Method 2: Multiply 1

$$M^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}.$$

Method 3: Using the fact that this is a rotation by 30 degrees, the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ we seek is one that when rotated by 30 degrees clockwise will give the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$.

2. Let $N = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 3 \end{bmatrix}$.

(a) Find N^{-1}. Show what method you are using and at least one intermediate step.

ANS. $[N|\,I\,] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 4 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 1 & 0 & 4 & -1 \end{array}\right]$

$N^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -3 & 1 \\ 0 & 4 & -1 \end{bmatrix}$.

(b) Use your work above to solve the system of equations $\left\{ \begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ & & y & + & z & = & 0 \\ & & 4y & + & 3z & = & 0 \end{array} \right\}$.

ANS. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ because $N^{-1}\vec{0} = \vec{0}$. Or, because rref of N is I.

(c) True or False: The column vectors of N are linearly independent. Justify your answer. You can use your work from other parts of this problem to do so.
ANS. True, because the matrix is invertible. OR, True, because in the rref the matrix is the identity and thus has a leading 1 in every column. Or,...

(d) What is the rank of N? **ANS.** 3

(e) What is the nullity of N? **ANS.** 0

(f) True or False: $\det(N) = 0$. State why or why not – you may use your work above.
ANS. $\det(N) \neq 0$ because N is invertible.

(g) Use your work above to justify **how many solutions** there are to the following system of equations.
$\left\{ \begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ & & y & + & z & = & 2 \\ & & 4y & + & 3z & = & 3 \end{array} \right\}$.
ANS. Exactly one solution because N is invertible. Note, since we are only asking HOW MANY SOLUTIONS and since in the previous parts you will have found rref of N, for this part DO NOT redo the rref for the associated augmented matrix. We don't want to see you redo unnecessary work. On the other hand if this were the only part of the problem or if you were asked for the SOLUTIONS to the system then you would want to do the rref of the associated augmented matrix.

(h) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ via $T(\vec{x}) = N\vec{x}$, is T one to one? Onto? Invertible?
ANS. N is invertible thus T is also invertible, one-to-one, and onto.

(i) How would each part of this problem change if you did an example where N is not invertible?

3. Consider the Linear Transformation. $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 & -1 \\ -2 & 4 & 1 & -1 & 1 \\ 3 & -6 & 3 & 6 & 0 \end{bmatrix} \quad \& \quad \text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

In each of the following refer to the information above.

(a) Determine the value of n. **ANS.** 5

(b) Determine the value of m. **ANS.** 4

(c) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be the column vectors that form the matrix A. Are the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent? If yes, why? If no, give one dependence relation between the column vectors of A.
ANS. No. For example, $2\vec{v}_1 + \vec{v}_2 = \vec{0}$.

(d) Find a basis for the image of T. **ANS.** Basis $\{\vec{v}_1, \vec{v}_3, \vec{v}_5\}$ because the image is spanned by the columns of A and in the rref(A) the columns corresponding to these three vectors are linearly independent.

(e) What is the rank of T? Explain how you determined it. **ANS.** 3.

(f) Find a basis for the kernel of T. **ANS.** $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(g) What is the nullity of T? Explain how you determined it.
ANS. Rank + Nullity = $\dim(\mathbb{R}^n) = 5$. Since the rank of A is 3, the nullity must be 2. Alternatively use basis from previous part.

(h) Is T one to one? Why or why not?
ANS. No because the kernel of A is not trivial because the dimension of the kernel of A (its nullity) is 2 by the previous part.

(i) Is T onto? Why or why not?
ANS. T is not onto because $\text{rank}(L) = 3 < 4 = \dim(\mathbb{R}^4)$.

(j) Is T invertible? Why or why not?
ANS. T is not invertible because it is not onto. OR it is not invertible because it is not one-to-one.

4. Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(a) For the matrix A, find the eigenvalues and their algebraic multiplicities.
ANS. $\det(A - xI) = \det\left(\begin{bmatrix} 2-x & 2 & -2 \\ 0 & 4-x & 0 \\ -2 & 2 & 2-x \end{bmatrix}\right) = (2-x)(4-x)(2-x) - (2)(0)(0) + (-2)(0 - (-2)(4-x))$
$= (4-x)\left((2-x)^2 - 4\right) = (4-x)\left((4 - 4x + x^2 - 4\right) = (4-x)(-4x + x^2) = (4-x)x(-4+x)$
$= (4-x)(x)(-1)(4-x) = -(4-x)^2 x$.
The eigenvalues are $\lambda = 4$ with algebraic multiplicity 2 and $\lambda = 0$ with algebraic multiplicity 1. Notice that even if you get this part wrong or you get stuck, you can still do all the other parts of this problem so do NOT get bogged down here. Set it up, try it once then move on and come back if you have time.

(b) The eigenspace associated to $\lambda = 0$ is

$$E_0 = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Find eigenspace E_4 associated to $\lambda = 4$.

ANS. $E_4 = \ker(A - 4I) = \ker \begin{bmatrix} 2-4 & 2 & -2 \\ 0 & 4-4 & 0 \\ -2 & 2 & 2-4 \end{bmatrix} = \ker \begin{bmatrix} -2 & 2 & -2 \\ 0 & 0 & 0 \\ -2 & 2 & -2 \end{bmatrix} = \ker \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$ span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. The basis is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(c) Suppose you had not computed the characteristic polynomial in part a. Based on the bases for the eigenspaces, how could you have concluded that the characteristic polynomial of A is $-(4-x)^2(x)$ or $(4-x)^2(x)$?

ANS. From the previous part we know that the geometric multiplicities of eigenvalues 4 and 0 are 2 and 1, respectively. Therefore, their algebraic multiplicities have to be at least 2 and 1, respectively. The algebraic multiplicities must sum to a number less than or equal to the degree of the characteristic polynomial and the degree of the characteristic polynomial is 3 because A is a 3×3 matrix. Thus, geometric multiplicities of eigenvalues 4 and 0 must be exactly 2 and 1, respectively. This demonstrates that the characteristic polynomial must be $-(4-x)^2(x)$ or $(4-x)^2(x)$.

(d) If possible, determine a basis $\mathfrak{B}$ for $\mathbb{R}^3$ consisting of eigenvectors for A. If it is not possible explain why not.

ANS. $\mathfrak{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

(e) Is A diagonalizable? If so give the diagonalization $D = [T]_{\mathfrak{B}}$.

ANS. $[T]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$. Notice that the order of the basis must match the order of the diagonal elements!

(f) Find the matrices S and S^{-1} such that $D = S^{-1} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix} S$.

ANS. The matrix S is the matrix whose columns are the eigenbasis vectors, so $S = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then use a calculator to compute $S^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(g) What is $S^{-1} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix} S$.

ANS. $S^{-1} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix} S = [T]_{\mathfrak{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(h) Compute $\det\left(\begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix}^3\right)$.

ANS. Notice that the kernel of A is the same as the eigenspace of 0. Since the eigenspace of 0 is one dimensional, the kernel is non-trivial. That means that A is not invertible so $\det A = 0$. Therefore,

$$\det\left(\begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix}^3\right) = \det\left(\begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix}\right)^3 = 0^3 = 0.$$

5. Let $Z = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = z\right\}$. Prove that Z is a subspace of $\mathbb{R}^3$.

ANS. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in Z$ because $y = z = 0$. Now, given $\begin{bmatrix} a \\ b \\ c \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in Z$ and $r \in \mathbb{R}$, we know that $b = c$ and $y = z$. Consider $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + r\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a + rx \\ b + ry \\ c + rz \end{bmatrix} \in Z$ because $b + ry = c + rz$ because $b = c$ and $y = z$. Thus Z is a subspace of $\mathbb{R}^3$.

6. Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 matrices with real entries. Define a function $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & 1 \\ 1 & a \end{bmatrix}$. Show that T is NOT a linear transformation.

ANS. $T\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ because $1 \neq 0$. Thus T is not a linear transformation.

7. Let $R^{2\times 2}$ be the set of all 2×2 matrices with real entries.

(a) Let $X = \{M \in R^{2\times 2} \mid \det(M) \neq 0\}$. Prove that X is not a subspace of $R^{2\times 2}$.
ANS. Notice that a matrix is in X if and only if its determinant is NOT zero. The zero matrix has determinant 0 and thus it is not in X. Therefore, X is not a subspace of $R^{2\times 2}$.

(b) Let $Y = \{M \in R^{2\times 2} \mid \det(M) = 0\}$. Prove that Y is not a subspace of $R^{2\times 2}$.

ANS. Notice that a matrix is in Y if and only if its determinant IS zero. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ both have determinant zero so they are both in X but $A + B = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$ has determinant -4 and so it is not in Y. Thus Y is not closed under addition and thus is not a subspace of $R^{2\times 2}$.

(c) Let $Z = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times 2} \mid c = b\right\}$. Prove that Z is a subspace of $R^{2\times 2}$.

ANS. Notice that the condition for a matrix to be in Z is that the lower left and the upper right entries must be equal ($b = c$). The zero matrix is in Z because all of its entries are equal. Now suppose that A and B are in Z. Then there are real numbers a, b, c, d, e, f, g, h such that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} e & f \\ h & g \end{bmatrix}$ where $b = c$ and

$f=h$. Given a real number r, consider $A+rB=\begin{bmatrix} a & b \\ c & d \end{bmatrix}+r\begin{bmatrix} e & f \\ h & g \end{bmatrix}=\begin{bmatrix} a+re & b+rf \\ c+rh & d+rg \end{bmatrix}$. Since $b=c$ and $f=h$ we have that $b+rf=c+rh$. Thus $A+rB$ is in Z thus Z is a subspace of $R^{2\times 2}$. Note: all the words and computations in this proof are needed other than the first sentence, so do not cut corners.

(d) Let $Z=\left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in R^{2\times 2} \mid c=b\right\}$. Find a basis for Z and its dimension.

ANS. Notice that the following matrices are all in Z: $A_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Moreover, if $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in Z then $b=c$ so $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a & b \\ b & d \end{bmatrix}=aA_1+dA_2+bA_3$. Thus $\{A_1,A_2,A_3\}$ span Z. Suppose that $xA_1+yA_2+zA_3=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} x & z \\ z & y \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus, $x=y=z=0$ and so $\{A_1,A_2,A_3\}$ are linearly independent. There for $\{A_1,A_2,A_3\}$ is a basis for Z.

Alternative: Notice that we were actually done when we had three linearly independent matrices. That is because finding 3 linearly independent matrices shows that $\dim Z\geq 3$. Moreover, $Z\neq R^{2\times 2}$ because there are matrices that are in $R^{2\times 2}$ but not Z. Thus $\dim Z<\dim(R^{2\times 2})=4$. The only possibility is that $\dim Z=3$. Now we have 3 linearly independent matrices in a 3 dimensional subspace, those three matrices form a basis for Z.

(e) Let

$$W=\left\{M\in\mathbb{R}^{2\times 2} \mid M\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}M\right\}.$$

Prove that W is a subspace of $\mathbb{R}^{2\times 2}$.

ANS. Notice that the condition for a matrix M to be in the set W is that the matrix M commutes with the matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Therefore, the zero matrix is in W because $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Moreover, if A and B are matrices in W then $A\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}A$ and $B\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}B$. Now consider a real number r. Is $A+rB$ in W? Well, we have to check whether $A+rB$ commutes with $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. So start on the lefthand side, and use the properties of matrix multiplication and addition:

$$\begin{aligned}(A+rB)\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} &= A\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + rB\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}A + r\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}B \\ &= \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}A + \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}(rB) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}(A+rB)\end{aligned}.$$

Thus $A+rB$ is in W. Thus, W is a subspace of $R^{2\times 2}$.

8. Let $T:R^{2\times 2}\to R^{2\times 2}$ via $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}$.

(a) Show that T is a linear transformation.

ANS. Given $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$ and a real number r, consider

$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}+r\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)=T\left(\begin{bmatrix} a+re & b+rf \\ c+rg & d+rh \end{bmatrix}\right)=\begin{bmatrix} 2(a+re) & b+rf+c+rg \\ c+rg+b+rf & 2(d+rh) \end{bmatrix}.$

By comparison
$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)+rT\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)=\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}+\begin{bmatrix} 2re & rf+rg \\ rg+rf & 2rh \end{bmatrix}=\begin{bmatrix} 2(a+re) & b+rf+c+rg \\ c+rg+b+rf & 2(d+rh) \end{bmatrix}$.
Thus, $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}+r\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)=T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)+rT\left(\begin{bmatrix} e & f \\ g & h \end{bmatrix}\right)$. So T is a linear transformation.

(b) Find the kernel of T.
Suppose that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus $2a=0$, so $a=0$; $b+c=0$ and $c+b=0$, so $b=-c$; and $2d=0$ so $d=0$. Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix}=c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore, the kernel is equal to $\text{Span}\left\{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\right\}$.

(c) Prove that T is NOT onto.
$T: R^{2\times 2}\to R^{2\times 2}$ and from the previous part the kernel of T is spanned by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus it has dimension 1. We know that Rank + Nullity = the dimension of the domain space = $\dim(R^{2\times 2})=4$. The basis for the kernel shows that Nullity is 1, so the rank of T must be 3. This is less than the dimension of the co-domain space $R^{2\times 2}$ so T is not onto.
Alternatively, you could show that the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not in the image because if it were then for some matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we would have $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. But then $\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix}=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ so $b+c=3$ and $c+b=4$ which is impossible.

9. Let $P_2=\{a_0+a_1t+a_2t^2 \mid a_0,a_1,a_2\in\mathbb{R}\}$. That is, P_2 is the linear space of all polynomials of degree less than or equal to two.

(a) The standard basis for P_2 is $\mathfrak{U}=\{1,t,t^2\}$. Consider a linear transformation $L: P_2\to P_2$ defined by

$$L(f(t))=f(0)+f(2)t^2.$$

Find $[L]_{\mathfrak{U}}$.
ANS. STOP! Begin this problem by really paying attention to what L does to a polynomial $f(t)=a_0+a_1t+a_2t^2$. It asks you to first compute $f(0)$ (plug in 0 for t in $f(t)$) and then compute $f(2)$. That gives you two numbers. You then put those numbers in as the constant term and the coefficient of t^2 in a NEW polynomial $L(f(t))=f(0)+f(2)t^2$. Try this out on the polynomial $f(t)=1+2t+3t^2$ to see that you get $L(f(t))=1+17t^2$. Okay NOW you are ready to start this problem.
Recall that $[L]_{\mathfrak{U}}=\left[[L(1)]_{\mathfrak{U}}\ [L(t)]_{\mathfrak{U}}\ [L(t^2)]_{\mathfrak{U}}\right]$. Now consider $L(1)$. That means that we start with $f(t)=1+0t+0t^2$. This is just the constant function $f(t)=1$. Then the function L tells us to first compute: $f(0)=1+0(0)+0(0)^2=1$ and $f(2)=1+0(2)+0(2)^2=1$. Then,

$$L(1)=L(f(t))=f(0)+f(2)t^2=1+1t^2=1+0t+1t^2.$$

Similarly, to compute $L(t)$, consider $h(t)=t=0+1t+t^2$. Then $h(0)=0+1(0)+1(0)^2=0$ and $h(2)=0+1(2)+0(2)^2=2$, so

$$L(t)=L(h(t))=h(0)+h(2)t^2=0+2t^2=0+0t+2t^2.$$

Finally to compute $L(t^2)$ let $g(t) = t^2 = 0 + 0t + 1t^2$. Then $g(0) = 0$ and $g(2) = 4$. Thus

$$L(t^2) = L(g(t)) = g(0) + g(2)t^2 = 0 + 0t + 4t^2.$$

So $[L(1)]_{\mathfrak{U}} = [1 + 0t + 1t^2]_{\mathfrak{U}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $[L(t)]_{\mathfrak{U}} = [0 + 0t + 2t^2]_{\mathfrak{U}} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$, $[L(t^2)]_{\mathfrak{U}} = [0 + 0t + 4t^2]_{\mathfrak{U}} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$.

So $[L]_{\mathfrak{U}} = \big[[L(1)]_{\mathfrak{U}}\ [L(t)]_{\mathfrak{U}}\ [L(t^2)]_{\mathfrak{U}}\big] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 4 \end{bmatrix}$.

(b) Let $\mathfrak{B} = \{1 - t, t^2, 1 + t\}$. Prove that $\mathfrak{B}$ is a basis for P_2.
ANS. Recall that $\mathfrak{B} = \{1-t, t^2, 1+t\}$ is a basis for P_2 if and only if $\{[1-t]_{\mathfrak{U}}, [t^2]_{\mathfrak{U}}, [1+t]_{\mathfrak{U}}\}$ is a basis for $\mathbb{R}^3$. (This is because the linearl transformation from P_2 to $\mathbb{R}^3$ that writes a polynomial as its vector with respect to a basis is an isomorphism.) So begin by considering $\{[1 - t]_{\mathfrak{U}}, [t^2]_{\mathfrak{U}}, [1 + t]_{\mathfrak{U}}\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. Consider the matrix whose columns are these vectors and put it in rref:

$$\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus $\{[1 - t]_{\mathfrak{U}}, [t^2]_{\mathfrak{U}}, [1 + t]_{\mathfrak{U}}\} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$ and therefore $\mathfrak{B} = \{1 - t, t^2, 1 + t\}$ is a basis for P_2.

(c) Find the change of basis matrix S such that $[L]_{\mathfrak{B}} = S^{-1}[L]_{\mathfrak{U}}S$.
ANS. Recall that $S = \big[[1 - t]_{\mathfrak{U}}[t^2]_{\mathfrak{U}}[1 + t]_{\mathfrak{U}}\big] = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

(d) Is the linear transformation L invertible?
ANS. L is invertible if and only if $[L]_{\mathfrak{U}}$ is an invertible matrix. Looking at $[L]_{\mathfrak{U}}$ from part (a) we see that there is a row of zeros. Therefore it is not invertible and so L is not invertible.

10. Let A and B be $n \times n$ matrices. Prove the following:

(a) $\operatorname{im}(AB) \subseteq \operatorname{im}(A)$.
Proof: Suppose that $\vec{v} \in \operatorname{im}(AB)$. Then there is an $\vec{x} \in \mathbb{R}^n$ such that $(AB)\vec{x} =$ ____________. Now, Let $\vec{y} = B\vec{x}$, then $A\vec{y} = A(B$____________$) =$ ____________. Thus ____.
ANS. Proof: Suppose that $\vec{v} \in \operatorname{im}(AB)$. Then there is an $\vec{x} \in \mathbb{R}^n$ such that $(AB)\vec{x} = \vec{v}$. Now, Let $\vec{y} = B\vec{x}$, then $A\vec{y} = A(B\vec{x}) = \vec{v}$. Thus $\vec{v} \in \operatorname{im}(A)$.

(b) $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.
ANS. By part (a), $\operatorname{im}(AB) \subseteq \operatorname{im}(A)$. This implies that $\dim(\operatorname{im}(AB)) \leq \dim(\operatorname{im}(A)$ which implies that $\operatorname{rank}(AB) \leq \operatorname{rank}(A)$.

11. **Proposition 1** *Let $T : V \to W$ be a linear transformation of linear spaces V and W, suppose that a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$ has the property that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subset W$ is linearly independent. Then, the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$ are linearly independent.*

Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof.

Proof: Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$. Then, applying T to both sides we get

____. Since T is a linear transformation, we get

____. Since $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ are linearly independent

we get that ____.

ANS. Proof: Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$. Then, applying T to both sides we get

$T(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = T(\vec{0})$. Since T is a linear transformation, we get

$a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + a_3T(\vec{v}_3) = \vec{0}$. Since $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ are linearly independent

we get that $a_1 = a_2 = a_3 = 0$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$ are linearly independent..

12. Suppose a $n \times n$ matrix M has the property that $M^2 = M$. Show that the only eigenvalues are 0 and 1. In the spaces, fill in the missing parts to complete the proof.

Proof: Suppose that $\vec{x}$ is an eigenvector for M with eigenvalue λ. Then,

$M\vec{x} =$__________________. Since $M^2 = M$ we have $M^2\vec{x} = M\vec{x}$, so $\lambda^2\vec{x} = \lambda\vec{x}$. Therefore, $\lambda^2\vec{x} - \lambda\vec{x} = \vec{0}$

so (______________________________)$\vec{x} = \vec{0}$.

Because $\vec{x}$ is ______________________________, it follows that $\lambda^2 - \lambda = 0$.

Therefore, $\lambda =$ ______________________________ or $\lambda =$ ______________________________

ANS. Proof: Suppose that $\vec{x}$ is an eigenvector for M with eigenvalue λ. Then,

$M\vec{x} = \lambda\vec{x}$. Since $M^2 = M$ we have $M^2\vec{x} = M\vec{x}$, so $\lambda^2\vec{x} = \lambda\vec{x}$. Therefore, $\lambda^2\vec{x} - \lambda\vec{x} = \vec{0}$

so $(\lambda^2 - \lambda)\vec{x} = \vec{0}$.

Because $\vec{x}$ is not the zero vector, it follows that $\lambda^2 - \lambda = 0$.

Therefore, $\lambda = 0$ or $\lambda = 1$. .

13. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects a vector $\vec{x}$ in $\mathbb{R}^2$ onto the x-axis. Let M be the 2×2 matrix such that $L(\vec{x}) = M\vec{x}$.
Give one eigenvector and associated eigenvalue for M. In this problem it is fine to give a thorough geometric explanation without finding the matrix M.

ANS. E.g. Eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ with eigenvalue $\lambda = 1$. Or, Eigenvector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with eigenvalue $\lambda = 0$.

14. **Proposition 2** *Let $T : V \to W$ be a one-to-one linear transformation of linear spaces V and W, suppose that a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq V$ is linearly independent. Prove that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subseteq W$ is linearly independent.*

Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof or give your own proof.

Proof: Suppose that $a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + a_3T(\vec{v}_3) = \vec{0}$. Then, since T is a linear transformation we get

$$T(\underline{\hspace{8cm}}) = \vec{0}.$$

Since T is one-to-one, we get that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \underline{\hspace{2cm}}.$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set

we get that ____.
Thus $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subseteq W$ is linearly independent.

ANS. Suppose that $a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + a_3T(\vec{v}_3) = \vec{0}$. Then, since T is a linear transformation we get

$$T(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = \vec{0}.$$

Since T is one-to-one, we get that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}.$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent

we get that $a_1 = a_2 = a_3 = 0$. Thus $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subset W$ is linearly independent.

15. Let W be a subspace in $\mathbb{R}^3$ and define $W^\perp = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$. Show that $W^\perp$ is a subspace of $\mathbb{R}^3$.
ANS. First notice that for a vector $\vec{v}$ in $\mathbb{R}^3$ to be in $W^\perp$ it must be that $\vec{v} \cdot \vec{w} = 0$ for ALL vectors $\vec{w} \in W$. Consider whether $\vec{0}$ in $\mathbb{R}^3$ is in $W^\perp$. Take any $\vec{w} \in W$, then we have $\vec{0} \cdot \vec{w} = 0$. Thus $\vec{0} \in W^\perp$. Now suppose that $\vec{v}$ and $\vec{u}$ are in $W^\perp$ and r is a real number. Then for any $\vec{w} \in W$ we have $\vec{v} \cdot \vec{w} = 0$ and $\vec{u} \cdot \vec{w} = 0$. The question is whether $\vec{v} + r\vec{u}$ is in $W^\perp$. To check consider $(\vec{v} + r\vec{u}) \cdot \vec{w}$ for any $\vec{w}$ in W. Now because the dot product is linear, we have $(\vec{v} + r\vec{u}) \cdot \vec{w} = (\vec{v} \cdot \vec{w}) + r(\vec{u} \cdot \vec{w}) = 0 + r0 = 0 + 0 = 0$. Thus, $\vec{v} + r\vec{u}$ is in $W^\perp$ and $W^\perp$ is a subspace of $\mathbb{R}^3$.

16. Find a basis for the subspace of $\mathbb{R}^3$ consisting of all vectors perpendicular to $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

ANS. We want to find vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = 0$. That is we want to find x, y, z such that

$x+3y+5z=0$. Take y and z as free variables. Then x depends on y and z with the equation $x=-3y-5z$. So solutions look like $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3y-5z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}$. This shows that a basis for the set of solutions is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

17. Let W be a subspace of $\mathbb{R}^3$ with basis $\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ find $\text{Proj}_W\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.

ANS. Recall that $\text{Proj}_W(\vec{x}) = (\vec{x}\cdot\vec{v}_1)\vec{v}_1 + (\vec{x}\cdot\vec{v}_2)\vec{v}_2$ so

$$\begin{aligned}
\text{Proj}_W\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) &= \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}\right)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} \\
&= (1)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \left(\frac{2}{\sqrt{2}}\right)\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\end{aligned}$$