Math 262, Fall 2021, Review Sheet for Final Exam

This is a review for the final. It includes all the problems that were on the Fall 2020 final (in blue) as well as many other problems for you to practice on. None of the problems on this review will be on the Fall 2021 final. However, these problems should give you an idea of the scope of the final exam. Below is what the front page of your final exam will look like.

NAME:____

Instructor (circle one): Candel, Castro, Kim, Shubin, Yomba,

Take a moment to remember your obligations as a CSUN student. When you submit your final exam, you will be also affirming that:

1. The answers I give on this exam will be my own work.
2. The only accepted materials to use for this exam are my one-page of notes (two-sided, 8.5×11 maximum) and a scientific calculator that cannot be not connected to the internet and is not a phone. I understand that my one-page of notes must be turned in with my exam and that if I use a calculator I must show the steps of my work.
3. Any external assistance (e.g. cell phones/cameras, PDAs, other electronic devices, or conversation with others) is prohibited.
4. Uploading or downloading any portions of this exam to or from the internet, aside from submitting the assignment to Canvas, is prohibited.
5. All suspected violations of the CSUN Code of Student Conduct will be investigated and will be subject to disciplinary actions.
6. I will use a sharp pencil or good pen, making sure the answers I upload can be easily read.
7. I will do the exam on the provided printout.
8. I will ask if I am not sure of anything on the exam.
9. I will turn in the exam at the time it is due: 2:45 pm on Thursday December 16, 2021 with 15 minutes for uploading to Math 262 Common Resource Canvas site: https://canvas.csun.edu/courses/113140.

We believe in you and trust you. Breathe and do your best.

1. Let $M = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

   (a) Find M^{-1}.

   (b) Solve the system of equations $\left\{ \begin{array}{ccccc} \frac{\sqrt{3}}{2}x & + & \frac{-1}{2}y & = & 0 \\ \frac{1}{2}x & + & \frac{\sqrt{3}}{2}y & = & 1 \end{array} \right\}$.
   (As you review, try to do this three different ways: (i) using row reduction of the associated augmented matrix, (ii) using M^{-1}, and (iii) using geometry.)

2. Let $N = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 4 & 3 \end{bmatrix}$.

   (a) Find N^{-1}. Show what method you are using and at least one intermediate step.

   (b) Use your work above to solve the system of equations $\left\{ \begin{array}{ccccccc} x & + & y & + & z & = & 0 \\ & & y & + & z & = & 0 \\ & & 4y & + & 3z & = & 0 \end{array} \right\}$.

   (c) True or False: The column vectors of N are linearly independent. Justify your answer. You can use your work from other parts of this problem to do so.

   (d) What is the rank of N?

   (e) What is the nullity of N?

   (f) True or False: $\det(N) = 0$. State why or why not – you may use your work above.

   (g) Use your work above to justify **how many solutions** there are to the following system of equations.
   $\left\{ \begin{array}{ccccccc} x & + & y & + & z & = & 1 \\ & & y & + & z & = & 2 \\ & & 4y & + & 3z & = & 3 \end{array} \right\}$.

   (h) If $T : \mathbb{R}^3 \to \mathbb{R}^3$ via $T(\vec{x}) = N\vec{x}$, is T one to one? Onto? Invertible?

   (i) How would each part of this problem change if you did an example where N is not invertible?

3. Consider the Linear Transformation. $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 & -1 \\ -2 & 4 & 1 & -1 & 1 \\ 3 & -6 & 3 & 6 & 0 \end{bmatrix} \quad \& \quad \text{rref}(A) = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   In each of the following refer to the information above.

(a) Determine the value of n.

(b) Determine the value of m.

(c) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ be the column vectors that form the matrix A. Are the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5\}$ linearly independent? If yes, why? If no, give one dependence relation between the column vectors of A.

(d) Find a basis for the image of T.

(e) What is the rank of T? Explain how you determined it.

(f) Find a basis for the kernel of T.

(g) What is the nullity of T? Explain how you determined it.

(h) Is T one to one? Why or why not?

(i) Is T onto? Why or why not?

(j) Is T invertible? Why or why not?

4. Let $A = \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(a) For the matrix A, find the eigenvalues and their algebraic multiplicities.

(b) The eigenspace associated to $\lambda = 0$ is

$$E_0 = \text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

Find eigenspace E_4 associated to $\lambda = 4$.

(c) Suppose you had not computed the characteristic polynomial in part a. Based on the bases for the eigenspaces, how could you have concluded that the characteristic polynomial of A is $-(4-x)^2(x)$ or $(4-x)^2(x)$?

(d) If possible, determine a basis $\mathfrak{B}$ for $\mathbb{R}^3$ consisting of eigenvectors for A. If it is not possible explain why not.

(e) Is A diagonalizable? If so give the diagonalization $D = [T]_{\mathfrak{B}}$.

(f) Find the matrices S and S^{-1} such that $D = S^{-1} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix} S$.

(g) What is $S^{-1} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix} S$.

(h) Compute $\det\left(\begin{bmatrix} 2 & 2 & -2 \\ 0 & 4 & 0 \\ -2 & 2 & 2 \end{bmatrix}^3\right)$.

5. Let $Z = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = z \right\}$. Prove that Z is a subspace of $\mathbb{R}^3$.

6. Let $\mathbb{R}^{2\times 2}$ be the set of all 2×2 matrices with real entries. Define a function $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & 1 \\ 1 & a \end{bmatrix}$. Show that T is NOT a linear transformation.

7. Let $R^{2\times 2}$ be the set of all 2×2 matrices with real entries.

   (a) Let $X = \{M \in R^{2\times 2} \mid \det(M) \neq 0\}$. Prove that X is not a subspace of $R^{2\times 2}$.

   (b) Let $Y = \{M \in R^{2\times 2} \mid \det(M) = 0\}$. Prove that Y is not a subspace of $R^{2\times 2}$.

   (c) Let $Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times 2} \mid c = b \right\}$. Prove that Z is a subspace of $R^{2\times 2}$.

   (d) Let $Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in R^{2\times 2} \mid c = b \right\}$. Find a basis for Z and its dimension.

   (e) Let
$$W = \left\{ M \in \mathbb{R}^{2\times 2} \mid M \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} M \right\}.$$
   Prove that W is a subspace of $\mathbb{R}^{2\times 2}$.

8. Let $T : R^{2\times 2} \to R^{2\times 2}$ via $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 2a & b + c \\ c + b & 2d \end{bmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find the kernel of T.
   Suppose that $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $\begin{bmatrix} 2a & b + c \\ c + b & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus $2a = 0$, so $a = 0$; $b + c = 0$ and $c + b = 0$, so $b = -c$; and $2d = 0$ so $d = 0$. Thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & -c \\ c & 0 \end{bmatrix} = c\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Therefore, the kernel is equal to $\text{Span}\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\}$.

   (c) Prove that T is NOT onto.
   $T : R^{2\times 2} \to R^{2\times 2}$ and from the previous part the kernel of T is spanned by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Thus it has dimension 1. We know that Rank + Nullity = the dimension of the domain space = $\dim(R^{2\times 2}) = 4$. The basis for the kernel

shows that Nullity is 1, so the rank of T must be 3. This is less than the dimension of the co-domain space $R^{2\times 2}$ so T is not onto.

Alternatively, you could show that the matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not in the image because if it were then for some matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we would have $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. But then $\begin{bmatrix} 2a & b+c \\ c+b & 2d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ so $b+c=3$ and $c+b=4$ which is impossible.

9. Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$. That is, P_2 is the linear space of all polynomials of degree less than or equal to two.

   (a) The standard basis for P_2 is $\mathfrak{U} = \{1, t, t^2\}$. Consider a linear transformation $L : P_2 \to P_2$ defined by

   $$L(f(t)) = f(0) + f(2)t^2.$$

   Find $[L]_{\mathfrak{U}}$.

   (b) Let $\mathfrak{B} = \{1-t, t^2, 1+t\}$. Prove that $\mathfrak{B}$ is a basis for P_2.

   (c) Find the change of basis matrix S such that $[L]_{\mathfrak{B}} = S^{-1}[L]_{\mathfrak{U}}S$.

   (d) Is the linear transformation L invertible?

10. Let A and B be $n \times n$ matrices. Prove the following:

    (a) $\mathrm{im}(AB) \subseteq \mathrm{im}(A)$.
    Proof: Suppose that $\vec{v} \in \mathrm{im}(AB)$. Then there is an $\vec{x} \in \mathbb{R}^n$ such that $(AB)\vec{x} =$ __________. Now, Let $\vec{y} = B\vec{x}$, then $A\vec{y} = A(B$__________$) =$ __________. Thus ____.

    (b) $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

11. **Proposition 1** *Let $T : V \to W$ be a linear transformation of linear spaces V and W, suppose that a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$ has the property that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subset W$ is linearly independent. Then, the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subset V$ are linearly independent.*

    Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof.

    Proof: Suppose that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$. Then, applying T to both sides we get

    ____. Since T is a linear transformation, we get

    ____. Since $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ are linearly independent

    we get that ____.

12. Suppose a $n \times n$ matrix M has the property that $M^2 = M$. Show that the only eigenvalues are 0 and 1. In the spaces, fill in the missing parts to complete the proof.

Proof: Suppose that $\vec{x}$ is an eigenvector for M with eigenvalue λ. Then,

$M\vec{x} =$____________________. Since $M^2 = M$ we have $M^2\vec{x} = M\vec{x}$, so $\lambda^2\vec{x} = \lambda\vec{x}$. Therefore, $\lambda^2\vec{x} - \lambda\vec{x} = \vec{0}$

so (________________________________)$\vec{x} = \vec{0}$.

Because $\vec{x}$ is ________________________________, it follows that $\lambda^2 - \lambda = 0$.

Therefore, $\lambda =$ ________________________________ or $\lambda =$ ________________________________

13. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that projects a vector $\vec{x}$ in $\mathbb{R}^2$ onto the x-axis. Let M be the 2×2 matrix such that $L(\vec{x}) = M\vec{x}$.
Give one eigenvector and associated eigenvalue for M. In this problem it is fine to give a thorough geometric explanation without finding the matrix M.

14. **Proposition 2** *Let $T : V \to W$ be a one-to-one linear transformation of linear spaces V and W, suppose that a set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq V$ is linearly independent. Prove that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subseteq W$ is linearly independent.*

Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof or give your own proof.

Proof: Suppose that $a_1T(\vec{v}_1) + a_2T(\vec{v}_2) + a_3T(\vec{v}_3) = \vec{0}$. Then, since T is a linear transformation we get

$$T(\underline{\hspace{8cm}}) = \vec{0}.$$

Since T is one-to-one, we get that

$$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \underline{\hspace{2cm}}.$$

Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent set

we get that ____.
Thus $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subseteq W$ is linearly independent.

15. Let W be a subspace in $\mathbb{R}^3$ and define $W^\perp = \{\vec{v} \in \mathbb{R}^3 \mid \vec{v} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W\}$. Show that $W^\perp$ is a subspace of $\mathbb{R}^3$.

16. Find a basis for the subspace of $\mathbb{R}^3$ consisting of all vectors perpendicular to $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$.

17. Let W be a subspace of $\mathbb{R}^3$ with basis $\{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ find $\mathrm{Proj}_W\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$.