Math 262, Spring 2022, Review Sheet for Final Exam

This is a review for the final. It includes all the problems that were on the Spring 2022 final (in blue) as well as many other problems for you to practice on. None of the problems on this review will be on the Fall 2022 final. However, these problems should give you an idea of the scope of the final exam. Below is what the front page of your final exam will look like.

NAME:____

Instructor (circle one): Hakobyan, Imhoff, Shubin, Toth

Take a moment to remember your obligations as a CSUN student. When you submit your final exam, you will be also affirming that:

1. The answers I give on this exam will be my own work.
2. The only accepted materials to use for this exam are my one-page of notes (two-sided, 8.5×11 maximum) and a scientific calculator that cannot be connected to the internet and is not a phone. I understand that my one-page of notes must be turned in with my exam and that if I use a calculator I must show the steps of my work.
3. Any external assistance (e.g. cell phones/cameras, PDAs, other electronic devices, or conversation with others) is prohibited.
4. Uploading or downloading any portions of this exam to or from the internet, aside from submitting the assignment to Canvas, is prohibited.
5. All suspected violations of the CSUN Code of Student Conduct will be investigated and will be subject to disciplinary actions.
6. I will use a sharp pencil or good pen, making sure the answers I upload can be easily read.
7. I will do the exam on the provided printout.
8. I will ask if I am not sure of anything on the exam.
9. I will turn in the exam at the time it is due: 5:15 pm on Monday, May 16, 2022 with 15 minutes for uploading to Math 262 Common Resource Canvas site: https://canvas.csun.edu/courses/122729.

Please note: This exam is printed two-sided so often your next problem will be on the back of the problem you are working on. There are also two pages deliberately left blank on this exam. You may use them for scrap or to continue your work if you need more space.

1. Let $N = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 0 & -1 & 0 \end{bmatrix}$.

   (a) The matrix N is invertible. Find N^{-1}. Show what method you are using and at least one intermediate step.

   (b) Solve the following system of equations. Write your answer in vector notation. $\left\{ \begin{array}{ccccccc} x & + & 2y & + & z & = & 0 \\ 2x & + & 4y & + & 3z & = & 0 \\ & & -y & & & = & 0 \end{array} \right\}$.

   (c) True or False: The column vectors of N are linearly independent. Justify your answer. You can use your work from other parts of this problem to do so.

   (d) What is the rank of N?

   (e) What is the nullity of N? Answer only, no justification required.

   (f) True or False: $\det(N) = 0$. Justify your answer. You can use your work from other parts of this problem to do so.

   (g) **How many solutions** there are to the following system of equations. Use your work above to justify your answer.
   $\left\{ \begin{array}{ccccccc} x & + & 2y & + & z & = & 1 \\ 2x & + & 4y & + & 3z & = & 2 \\ & & -y & & & = & 3 \end{array} \right\}$.

   (h) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\vec{x}) = N\vec{x}$. Is T one to one? Is it onto? Is it invertible? Answer only, no justification required.

2. Consider the linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ defined by $L(\vec{x}) = B\vec{x}$, where

$$B = \begin{bmatrix} 1 & 1 & -2 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 3 & 0 & 3 \\ 2 & 2 & -4 & -2 & 3 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 & 4 & 6 & -1 & 5 \\ 3 & 3 & -6 & -3 & 5 & 2 & 1 & 3 \end{bmatrix} \quad \& \quad \operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & -2 & -2 & 0 & -2 & 0 & -2 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

   In each of the following refer to the information above.

   (a) What is the value of m.

   (b) Determine the value of n.

   (c) Let $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6, \vec{v}_7, \vec{v}_8\}$ be the column vectors that form the matrix B. Are the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4, \vec{v}_5, \vec{v}_6, \vec{v}_7, \vec{v}_8\}$ linearly independent? If yes, why? If no, give one dependence relation between the column vectors of B.

   (d) Give a basis for $\operatorname{im}(L)$ (the image of L).

   (e) What is $\operatorname{rank}(L)$?

   (f) Find a basis for $\ker(L)$ (the kernel of L).

   (g) Is L onto? Why or why not?

   (h) What is the nullity of L? How do you know?

   (i) Is L one to one? Why or why not?

   (j) Is L invertible? Why or why not?

3. Let $A = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

   (a) Show that the only eigenvalues of A are 0, 1, and 3 and find their algebraic multiplicities. Show all your work.

   (b) The eigenvalues of A are 0, 1, and 3. The eigenspaces associated to eigenvalues $\lambda = 0$ and $\lambda = 3$ are

$$E_0 = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}, \quad E_3 = \text{span}\left\{\begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}\right\}$$

   Find a basis for the eigenspace E_1 associated to eigenvalue $\lambda = 1$. Show all your work.

   (c) If $A = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is diagonalizable, then give an eigenbasis $\mathfrak{B}$ and the diagonalization $[T]_{\mathfrak{B}}$. (You do not need to repeat work you did in parts (a) and (b) of this problem.) If A is not diagonalizable, then state why it is not.

   (d) Based on the work above, what could the characteristic polynomial of A be?

   (e) Is T an invertible linear transformation?

   (f) Compute $\det\left(A^3\right)$.

4. (a) Let $Y = \left\{M \in \mathbb{R}^{2\times 2} \mid M = \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \text{ for some } b \in \mathbb{R}\right\}$. Prove that Y is not a subspace of $\mathbb{R}^{2\times 2}$.

   (b) Let $W = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \mid y = 1\right\}$. Prove that W is not a subspace of $\mathbb{R}^3$.

   (c) Let $Z = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2} \mid c = b\right\}$. Prove that Z is a subspace of $\mathbb{R}^{2\times 2}$.

5. Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ via $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} d & b+c \\ b+c & a \end{bmatrix}$.

   (a) Show that T is a linear transformation.

   (b) Find a basis for the kernel of T.

   (c) Prove that T is not onto.

6. Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_0, a_1, a_2 \in \mathbb{R}\}$. That is, P_2 is the linear space of all polynomials of degree less than or equal to two with standard basis $\mathfrak{A} = \{1, t, t^2\}$. Let $W = \{f(t) \in P_2 \mid f'(0) = 0\}$. You may assume that this is a subspace of P_2.

   (a) Show that $g(t) = t^2 \in W$, and $h(t) = t \notin W$.

   (b) Show that the set $\mathfrak{B} = \{1, t^2\}$ spans W by proving that if a polynomial $f(t) = a_0 + a_1 t + a_2 t^2$ is in W then $a_1 = 0$.

   (c) The set $\mathfrak{B} = \{1, t^2\}$ spans W. Show that $\mathfrak{B} = \{1, t^2\}$ is a basis for W.

   (d) What is $\dim(W)$? You do not need to show work.

7. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+z \\ 0 \\ z+x \end{bmatrix}$$

(a) Find $L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) =$

(b) Assume that L as defined above is a linear transformation and that $\mathfrak{A} = \left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right\}$ is a basis for $\mathbb{R}^3$. Give the definition of $[L]_{\mathfrak{A}}$ and then find it. You do not need to repeat work.

(c) Is L invertible? Why or why not?

(d) The set $\mathfrak{C} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is another basis for $\mathbb{R}^3$.
Find the change of basis matrix $S_{\mathfrak{A}\to\mathfrak{C}}$ such that $[L]_{\mathfrak{A}} = S_{\mathfrak{A}\to\mathfrak{C}}^{-1}[L]_{\mathfrak{C}} S_{\mathfrak{A}\to\mathfrak{C}}$.

8. Let L be the line $y = 2x$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the projection onto the line L. This is a linear transformation. Let M be the 2×2 matrix such that $T(\vec{x}) = M\vec{x}$.
Give one eigenvector and associated eigenvalue for M. It is fine to give a thorough geometric explanation without finding the matrix M.

9. A 3×3 matrix A has the following properties:

(i) A is diagonalizable

(ii) A has rank 1

(iii) The eigenvalues of A are 0 and 3

Answer the following

(a) Find the characteristic polynomial of A

(b) Find the algebraic multiplicity of the eigenvalues of A

(c) Find the geometric multiplicity of the eigenvalues of A

(d) Which of the following matrices have some or all of properties (i), (ii), and (iii) as A?

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

10. Let A be $n \times n$ matrix.

(a) Prove that $\ker(A) \subseteq \ker(A^2)$ by filling in the outline provided.

Proof outline: Suppose that $\vec{v}$ is in $\ker(A)$. Then $A\vec{v} =$ ____________. Now,

$A^2\vec{v} = A(A\vec{v}) = A($____________$) =$ ____________.

Thus ____.

(b) Give an example where $\ker(A) \neq \ker(A^2)$ and show that it works.

(c) Assume that D is a diagonal $n \times n$ matrix. Prove that $\ker(D) = \ker(D^2)$.

11. **Proposition 1** *Let $T : V \to W$ be an onto linear transformation of linear spaces V and W, suppose that the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \subseteq V$ span V. Prove that $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\} \subseteq W$ span W.*

    Below is a "proof" of the above proposition. In the spaces, fill in the missing parts to complete the proof.

    Proof: Suppose that $\vec{w}$ is a vector in W.

    Then because T is onto, there is a vector $\vec{v} \in$ __________ such that $T($__________$) = \vec{w}$.

    Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ span V, there exist coefficients a_1, a_2, and a_3 in $\mathbb{R}$ such that

    $$\vec{v} = \underline{\hspace{8cm}}$$

    Now, since T is a linear transformation, then

    $$\begin{aligned} \vec{w} \;=\; T(\vec{v}) \;&=\; T(\underline{\hspace{8cm}}) \\ &=\; \underline{\hspace{1.5cm}}\,T(\underline{\hspace{1.5cm}}) + \underline{\hspace{1.5cm}}\,T(\underline{\hspace{1.5cm}}) + \underline{\hspace{1.5cm}}\,T(\underline{\hspace{1.5cm}}) \end{aligned}$$

    Therefore $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ span W.

12. Suppose a 3×3 matrix A of rank 2 with eigenvalues 0 and 5. Prove that the eigenspace E_0 associated to eigenvalue 0 has dimension 1.
    Below is an outline. In the spaces, fill in the missing parts or give your own proof.

    Proof: Since $E_0 = \{\vec{v} \in \mathbb{R}^3 \mid A\vec{v} = \underline{\hspace{1.5cm}}\vec{v} = \underline{\hspace{1.5cm}}\}$, then $E_0 = \ker(A)$.

    By hypothesis, rank(A)=2.

    Since rank(A)+nullity(__________) $= \dim(\mathbb{R}^3) =$ __________, we see that

    nullity(A) $=$ __________ $-$ __________ $=$ __________. Thus $\dim(E_0) =$ __________.

13. Consider the subspace $W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 3 \end{bmatrix}\right)$ in $\mathbb{R}^4$.

    (a) Find a basis $\beta = (\vec{v}_1, \vec{v}_2)$ of the orthogonal subspace $W^{\perp}$.

    (b) Find a basis $\gamma = (\vec{u}_1, \vec{u}_1)$ of W such that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal vectors of unit length.

    (c) Find a basis $\delta = (\vec{u}_1, \vec{u}_1)$ of $W^{\perp}$ such that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal vectors of unit length.

14. Perform the Gram-Schmidt process on the sequence of vectors

    (a) $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$.

    (b) Find the QR-factorization of the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -1 & 2 \end{bmatrix}$.